Supplement to "Inference for VARs identified with sign restrictions"

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This Online Appendix accompanies the paper "Inference for VARs Identified with Sign Restrictions" by E. Granziera, H. R. Moon, and F. Schorfheide. In Appendix A, we provide proofs for the theoretical results in Section 4 of the main paper. Additional technical lemmas are stated and proved in Appendix B. Appendix C provides analytical derivations for the Monte Carlo experiment presented in Section 6 of the main text. Appendix D contains additional information about the empirical analysis.

Appendix A: Proofs of main results

To simplify the notation in some of the proofs, we eliminate ρ from the formulas and index the probability distribution by $\phi \in \mathcal{P}$ instead of $\rho \in \mathcal{R}$. Thus we write

$$\inf_{\phi \in \mathcal{P}} \inf_{\theta \in F^\theta(\phi)} P_\phi\{\theta \in CS^\theta(\hat{\phi})\}$$

instead of

$$\inf_{\rho \in \mathcal{R}} \inf_{\theta \in F^\theta(\phi(\rho))} P_\rho\{\theta \in CS^\theta(\hat{\phi})\}.$$

Reduced-form parameter sequences ρ_T and $\phi(\rho_T)$ are simply abbreviated by ϕ_T.

A.1 *Proof of Lemma 1*

To simplify the notation, we omit tildes and write $S_\theta(q)$, and $S(q)$ instead of $\tilde{S}_\theta(q)$ and $\tilde{S}(q)$.

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Convexity: Suppose $\theta_i \in F^\theta(\phi_q, \phi_\theta)$, $i = 1, 2$, and $\theta_1 < \theta_2$. Then there exist q_i with $\|q_i\| = 1$ and $\mu_i \geq 0$ such that

$$S_\theta(q_i)\phi - \theta_i = 0, \qquad S(q_i)\phi - \mu_i = 0. \tag{A.1}$$

We distinguish two cases: $q_1 \neq -q_2$ and $q_1 = -q_2$.

Case (i): Suppose that $q_1 \neq -q_2$. We now verify that for any $\lambda \in [0, 1]$ $\theta = \lambda\theta_1 + (1 - \lambda)(\theta_2) \in F^\theta(\cdot)$. For $\tau \in [0, 1]$, define

$$q(\tau) = \frac{\tau q_1 + (1-\tau)q_2}{\|\tau q_1 + (1-\tau)q_2\|}, \qquad H(\tau) = S_\theta\big(q(\tau)\big)\phi - \theta.$$

The linearity of $S_\theta(q)$ with respect to q and (A.1) implies that

$$\begin{aligned} H(\tau) &= \frac{\tau S_\theta(q_1)\phi}{\|\tau q_1 + (1-\tau)q_2\|} + \frac{(1-\tau)S_\theta(q_2)\phi}{\|\tau q_1 + (1-\tau)q_2\|} - \lambda\theta_1 - (1-\lambda)\theta_2 \\ &= \frac{\tau\theta_1}{\|\tau q_1 + (1-\tau)q_2\|} + \frac{(1-\tau)\theta_2}{\|\tau q_1 + (1-\tau)q_2\|} - \lambda\theta_1 - (1-\lambda)\theta_2. \end{aligned}$$

Using $\|q_i\| = 1$, we obtain

$$\begin{aligned} H(0) &= \theta_2 - \lambda\theta_1 - (1-\lambda)\theta_2 = \lambda(\theta_2 - \theta_1) \geq 0, \\ H(1) &= \theta_1 - \lambda\theta_1 - (1-\lambda)\theta_2 = -(1-\lambda)(\theta_2 - \theta_1) \leq 0. \end{aligned}$$

Since $H(\tau)$ is continuous we deduce that there exists a τ^* such that $H(\tau^*) = 0$. Now consider

$$\begin{aligned} S\big(q(\tau^*)\big)\phi &= \frac{\tau^* S(q_1)\phi}{\|\tau^* q_1 + (1-\tau^*)q_2\|} + \frac{(1-\tau^*)S(q_2)\phi}{\|\tau^* q_1 + (1-\tau^*)q_2\|} \\ &= \frac{\tau^*\mu_1}{\|\tau^* q_1 + (1-\tau^*)q_2\|} + \frac{(1-\tau^*)\mu_2}{\|\tau^* q_1 + (1-\tau^*)q_2\|} \\ &\geq 0. \end{aligned}$$

The first equality follows from the linearity of $S(q)$, the second equality is implied by (A.1), and the inequality follows from $\mu_i \geq 0$. Thus, $\theta \in F^\theta(\phi_q, \phi_\theta)$.

Case (ii): Suppose that $q_1 = -q_2$. The linearity of $S_\theta(q)$ implies that $\theta_1 = -\theta_2$. By assumption, there exists a $q_3 \neq q_1, -q_1$ with the property that $S(q_3)\phi \geq 0$. Let $\theta_3 = S_\theta(q_3)\phi$. By construction, $\theta_3 \in F^\theta(\cdot)$. If $\theta_3 = \theta_1$ ($\theta_3 = \theta_2$), we simply replace q_1 (q_2) by q_3 and follow the steps outlined for Case (i). If $\theta_1 < \theta_3 < \theta_2$, then the Case (i) argument implies that any θ in the intervals $[\theta_1, \theta_3]$ and $[\theta_3, \theta_2]$ and thereby any $\theta = \lambda\theta_1 + (1-\lambda)\theta_2$ is included in the identified set. Finally, if $\theta_3 < \theta_1$ ($\theta_2 < \theta_3$), we deduce from Case (i) that the interval $[\theta_3, \theta_2]$ ($[\theta_1, \theta_3]$) is included in the identified set.

Boundedness: We shall prove a slightly more general result. Suppose that $\theta \in F^\theta(\phi_q, \phi_\theta)$. Since $F^\theta(\phi_q, \phi_\theta)$ is a multivalued set, we assume without loss of generality that $\theta > 0$. So, the sign restriction $\theta \geq 0$ is satisfied if it exists. Define

$$G^\theta(\theta; \phi_q, \phi_\theta) = \min_{q = \|1\|, \mu \geq 0} \|S_\theta(q)\phi_\theta - \theta\|^2 + \|S(q)\phi_q - \mu\|^2$$

such that $G^{\theta}(\theta; \phi_q, \phi_\theta) = 0$ if and only if $\theta \in F^{\theta}(\phi_q, \phi_\theta)$. We now show by contradiction that $F^{\theta}(\phi_q, \phi_\theta)$ has an upper bound.

Suppose, to the contrary, that no such upper bound exists. This guarantees the existence of a series $a_n > 0$ with $a_n \uparrow \infty$ such that $a_n\theta_n \in F^{\theta}(\phi_q, \phi_\theta)$ for each n. Consider the bound

$$G^{\theta}(a_n\theta; \phi_q, \phi_\theta) \geq \min_{q=\|1\|} \left\| S_\theta(q)\phi_\theta - a_n\theta \right\|^2.$$

Since $\|S_\theta(q)\phi_\theta\|$ is a continuous function of q for fixed ϕ_θ and the set of q is a compact unit sphere, there exists a finite constant M such that $\|S_\theta(q)\phi_\theta\| < M$. From this, we deduce that

$$\min_{q=\|1\|} \left\| S_\theta(q)\phi_\theta - a_n\theta \right\|^2 \longrightarrow \infty,$$

which contradicts the requirement $G^{\theta}(\theta; \phi_q, \phi_\theta) = 0$. The existence of a lower bound can be established by considering a sequence $-a_n$. Moreover, $\theta < 0$ can be handled by a straightforward modification of the argument. □

A.2 *Proof of Theorem 1*

Recall the definition $F^q(\Phi_q) = \{q \in \mathbb{S}^n \mid \Phi_q' q \geq 0\}$. Thus, $\{q \in \mathbb{S}^n \mid \Phi_q' q \gg 0\} \subset F^q(\phi)$. The statement of the theorem follows once we have shown that there exists a nonempty, nonsingleton, n-dimensional subset $\mathbb{Q}$ of $\mathbb{S}^n$, such that $\Phi_q q \gg 0$ if $q \in \mathbb{Q}$.

Existence: Suppose Φ_q' is an $r \times n$ matrix. According to Gordan's alternative theorem—see, for instance, Border (2007)—exactly one of the two alternatives holds: (a) there exists an $x \in \mathbb{R}^n$ satisfying $\Phi_q' x^* \gg 0$, or (b) there exists an $r \times 1$ vector $z > 0$ satisfying $\Phi_q z = 0$. Assumption 1(i) rules out alternative (b). Thus, there exists an x^* such that

$$\Phi_q' x^* \gg 0. \tag{A.2}$$

Notice that x^* in (A.2) is not zero. Then, $q^* := \frac{x^*}{\|x^*\|}$ satisfies the requirement $q^* \in \mathbb{Q}$ and $\Phi_q' q^* \gg 0$.

Nonsingleton: We show that $F^q(\Phi_q)$ contains multiple elements using proof by contradiction. For this, we define a function $f_\Phi : \mathbb{S}^n \to \mathbb{R}^r$ as $f_\Phi(q) := \Phi_q' q$ for $q \in \mathbb{S}^n$. Then, $f_\Phi(\cdot)$ is continuous on a compact set $\mathbb{S}^n$.

Suppose that q^* defined in the existence proof is the only element of $F^q(\Phi_q)$, that is, $F^q(\Phi_q) = \{q^*\}$. This implies that $f_\Phi(q) \notin \mathbb{R}_+^r$ for all $q \in \mathbb{S}^n$ with $q \neq q^*$, where $\mathbb{R}_+^r = \{x \in \mathbb{R}^r : x \geq 0\}$. Let $\varepsilon := \|\Phi_q' q^*\|_{\min}$, where the norm $\|x\|_{\min} := \min\{|x_1|, \dots, |x_r|\}$ for $x \in \mathbb{R}^r$. Notice that $\Phi_q' q^* \gg 0$ implies $\varepsilon > 0$. Consider an arbitrary $q \in \mathbb{S}^n$ such that $q \neq q^*$. Then, because $f_\Phi(q) \notin \mathbb{R}_+^r$ but $f_\Phi(q^*) \gg 0$, we have $\|f_\Phi(q) - f_\Phi(q^*)\| \geq \varepsilon$. Because q was arbitrary, given our choice of $\varepsilon > 0$ it is not possible to find a $\delta > 0$ such that $\|f_\Phi(q) - f_\Phi(q^*)\| \leq \varepsilon$ for $\|q - q^*\| \leq \delta$. This contradicts the fact that $f_\Phi(q)$ is continuous at q^*. Therefore, we can deduce the that $F^q(\Phi_q)$ is not a singleton and contains multiple elements. □

A.3 *Proof of Theorem* 2, *part* (i)

Recall the definition of the Hausdorff distance: $d(A,B) = \max\{d(A \mid B),\ d(B \mid A)\}$, where $d(A \mid B) = \sup_{a\in A}\ d(a,B)$ and $d(a,B) = \inf_{b\in B}\ \|a-b\|$. We set $d(A,B)=\infty$ if either A or B is empty. For any $\varepsilon > 0$, define an open ball around set $A \subset \mathbb{R}^n$ as $\mathbb{B}(A,\varepsilon) = \{b \in \mathbb{R}^n : d(b \mid A) < \varepsilon\}$.

The proof of the theorem exploits the continuity of $F^q(\phi)$ with respect to ϕ. The statement of the theorem is a consequence of Lemma 2, Lemma 3, and the continuous mapping theorem. □

Lemma 2. *Suppose that $F(\phi)$ is a nonempty compact-valued continuous correspondence. Then $\phi \longrightarrow \phi^*$ implies that $d(F(\phi), F(\phi^*)) \longrightarrow 0$.*

Proof of Lemma 2. Follows directly from Theorem 17.15 of Aliprantis and Border (2006). □

Lemma 3. *Suppose that Assumption* 1(i) *is satisfied. Then*:

(i) *$F^q(\Phi_q)$ is compact for all Φ_q;*

(ii) *$F^q(\Phi_q)$ is continuous at all Φ_q.*

Proof of Lemma 3. For notational simplicity, we omit the subscript notation q and write Φ_q as Φ. Let $\mathbb{S}^n = \{q \in \mathbb{R}^n : \|q\| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Recall from Theorem 1 that $F^q(\Phi)$ is nonempty.

Part (i): We show that $F^q(\Phi)$ is bounded and closed.

Boundedness: It is straightforward since $F^q(\Phi) \subset \mathbb{S}^n$.

Closedness: Consider any sequence $q_j \in F^q(\Phi)$ such that $q_j \longrightarrow q$, where $\|q_j\| = 1$ and $\|q\| = 1$. Then $0 \leq \Phi' q_j \longrightarrow \Phi' q$, so that it should be $\Phi' q \geq 0$. This implies that $q_0 \in F^q(\Phi)$, as required for closedness.

Part (ii): We show $F^q(\Phi)$ is upper hemi-continuous (UHC) and lower hemi-continuous (LHC) at Φ.

UHC: Since $F^q(\Phi)$ is nonempty and compact-valued, the UHC of $F^q(\Phi)$ at Φ follows if we show that for every sequence $\Phi_j \longrightarrow \Phi$ and $q_j \in F^q(\Phi_j)$, there exists a subsequence q_{j_i} of q_j such that $q_{j_i} \longrightarrow q \in F^q(\Phi)$. (See Border (2010), Proposition 20.) Since $\{q_j\} \subset \mathbb{S}^n$ and $\mathbb{S}^n$ is compact, we can choose a convergent subsequence q_{j_i} such that $q_{j_i} \longrightarrow q$. Then, $0 \leq \Phi'_{j_i} q_{j_i} \longrightarrow \Phi' q$, and it follows that $\Phi' q \geq 0$. This implies that $q \in F^q(\Phi)$, as required.

LHC: $F^q(\Phi)$ is LHC at Φ if and only if for any sequence $\{\Phi_j\}$ with $\Phi_j \longrightarrow \Phi$ and $q \in F^q(\Phi)$, there exists a sequence $q_j \in F^q(\Phi_j)$ with $q_j \longrightarrow q$. We reorder and partition the matrix Φ_0 to $\Phi = [\Phi_1, \Phi_2]$, where $\Phi_1' q = 0$ and $\Phi_2' q \gg 0$.

For a matrix A, we denote the lth column of A as $(A)_l$. By Gordan's alternative theorem (see Border (2007)), Assumption 1(i) implies that there exists a $\xi^* \in \mathbb{R}^m$ such that

$$\Phi_1' \xi^* \gg 0.$$

Let

$$\xi = \frac{1}{\min_l (\Phi_1)_l' \xi^*} \xi^*$$

such that for all l

$$(\Phi_1)_l' \xi > 1.$$

Set $\varepsilon_{j,l} = \|(\Phi_j - \Phi)_l\|$ and $\varepsilon_j = \max_l \{\varepsilon_{j,l}\}$, and define

$$q_j = \frac{q + \varepsilon_j \xi}{\|q + \varepsilon_j \xi\|}.$$

Notice that q_j is well-defined when ε_j is small enough because $q \in \mathbb{S}^n$ and as a result, $q \neq \varepsilon_j \xi$ when ε_j is small and ξ is fixed.

Case (i): Suppose $(\Phi_l)' q = 0$. Then, when j is large so that $(\Phi)_l' \xi - 1 \geq \varepsilon_j \|\xi\|$, we have

$$\begin{aligned}
(\Phi_j)_l' q_j &= (\Phi_{r,j} - \Phi_r)_l' q_j + (\Phi)_l' q_j \\
&= \frac{1}{\|q + \varepsilon_j \xi\|} \big\{ (\Phi_j - \Phi)_l' q + \varepsilon_j (\Phi_j - \Phi)_l' \xi + (\Phi)_l' q + \varepsilon_j (\Phi)_l' \xi \big\} \\
&\geq \frac{1}{\|q + \varepsilon_j \xi\|} \big\{ -\big\|(\Phi_j - \Phi)_l\big\| \|q\| - \varepsilon_j \big\|(\Phi_j - \Phi)_l\big\| \|\xi\| + \varepsilon_n (\Phi)_l' \xi \big\} \\
&\geq \frac{1}{\|q + \varepsilon_j \xi\|} \big(-\varepsilon_j - \varepsilon_j^2 \|\xi\| + \varepsilon_j (\Phi)_l' \xi \big) \\
&= \frac{1}{\|q + \varepsilon_j \xi\|} \varepsilon_j \big((\Phi)_l' \xi - 1 - \varepsilon_j \|\xi\| \big) \\
&\geq 0.
\end{aligned}$$

Case (ii): Suppose $(\Phi)_l' q > 0$. Then, since $\|(\Phi)_l\| \leq M$ (compact parameter set), we have

$$\begin{aligned}
(\Phi_j)_l' q_j &= (\Phi_j - \Phi)_l' q_j + (\Phi)_l' q_j \\
&= \frac{1}{\|q + \varepsilon_j \xi\|} \big\{ (\Phi_j - \Phi)_l' q + \varepsilon_j (\Phi_j - \Phi)_l' \xi + (\Phi)_l' q + \varepsilon_j (\Phi)_l' \xi \big\} \\
&\geq \frac{1}{\|q + \varepsilon_j \xi\|} \big\{ -\big\|(\Phi_j - \Phi)_l\big\| \|q\| - \varepsilon_j \big\|(\Phi_j - \Phi)_l\big\| \|\xi\| + (\Phi)_l' q - \varepsilon_j \big\|(\Phi)_l\big\| \|\xi\| \big\} \\
&\geq \frac{1}{\|q + \varepsilon_j \xi\|} \big((\Phi)_l' q - \varepsilon_j - \varepsilon_j^2 M - \varepsilon_j M^2 \big) \\
&\geq 0,
\end{aligned}$$

when j is large. The last inequality holds since $(\Phi)_l' q > 0$.

From these, we can deduce that

$$\Phi_j' q_j \geq 0.$$

Also, since $\varepsilon_j \longrightarrow 0$, we have

$$q_j \longrightarrow q.$$

Then we have all the required results for the LHC. □

A.4 *Proof of Theorem* 2, *part* (ii)

We closely follow the proofs of Theorem 1 and Lemma 2 of Andrews and Soares (2010). The main modification is to accommodate the reduced rank possibility of $\Sigma(q)$ and $D(q)$. The proof makes use of various lemmas that are stated and proved in Section B below. To simplify the notation, we eliminate ρ from the formulas and index the probability distribution by $\phi \in \mathcal{P}$ instead of $\rho \in \mathcal{R}$. We also skip the subscription notation q and write, for example, ϕ_q, $\hat{\phi}_q$, Λ_{qq}, D_q as ϕ, $\hat{\phi}$, Λ, D, respectively. Thus, we write

$$\inf_{\phi\in\mathcal{P}} \inf_{q\in F^q(\phi)} P_\phi\{q \in CS^q(\hat{\phi})\} \quad \text{instead of} \quad \inf_{\rho\in\mathcal{R}} \inf_{q\in F^q(\phi(\rho))} P_\rho\{q \in CS^q(\hat{\phi}_q)\}.$$

Reduced-form parameter sequences ρ_T and $\phi(\rho_T)$ are simply abbreviated by ϕ_T.

We need to show

$$\liminf_T \inf_{\phi\in\mathcal{P}} \inf_{q\in F^q(\phi)} P_\phi\{q \in CS^q\} \geq 1-\alpha. \tag{A.3}$$

Let

$$\text{Asy}\, CP = \liminf_T \inf_{\phi\in\mathcal{P}} \inf_{q\in F^q(\phi)} P_\phi\{q \in CS^q\}.$$

Then there exists sequences $\{\phi_T, q_T\}$ such that $q_T \in F^q(\phi_T)$ and

$$\text{Asy}\, CP = \liminf_T P_{\phi_T}\{q_T \in CS^q\}.$$

Furthermore, there exists a subsequence of $\{T\}$, $\{T'\} \subset \{T\}$, such that

$$\text{Asy}\, CP = \lim_{T'} P_{\phi_{T'}}\{q_{T'} \in CS^q\}.$$

In what follows, we show that there exists a sub-subsequence, say $\{T''\} \subset \{T'\}$, such that

$$\lim_{T''} P_{\phi_{T''}}\{q_{T''} \in CS^q\} \geq 1-\alpha. \tag{A.4}$$

Then the desired result (A.3) follows and the proof of the theorem is complete.

Define

$$\mu(q,\phi) = S(q)\phi$$

and decompose

$$\Sigma(q) = S(q)\Lambda S(q)' = S(q)LL'S(q)' = D^{1/2}(q)\Omega(q)D^{1/2}.$$

Moreover, let

$$A(q) = L'S'(q)D^{-1/2}(q).$$

To simplify the notation, we suppress the dependence of matrices on ϕ. The matrix $\Omega(q) = A'(q)A(q)$ is a correlation matrix and $D^{1/2}(q)$ is a diagonal matrix of standard deviations. Also, recall that $W(q) = D^{1/2}(q)B(q)D^{1/2}(q)$, where either $B(q) = \Omega^{-1}(q)$ or $B(q) = I$. The proof is completed in three steps.

Step 1: Choosing the subsequence T''. We choose a subsequence T'' from T' along which the subsequent conditions are satisfied. This is done sequentially by choosing a subsequence that satisfies criterion (i), and then, step-by-step choosing subsequences of the subsequences to satisfy the next criterion until all five conditions are satisfied:

(i) $\phi_{T''} \longrightarrow \phi$.

(ii) $r(q_{T''}) = r$, $V(q_{T''}) = V$ for all T''.

(iii) For $j = 1, \dots, r$, the slackness (recall that $\mu_j = [S(q)\phi_q]_j \mathcal{I}\{[S(q)\phi_q]_j \geq 0\}$) in inequality j converges to

$$\sqrt{T''}\mu_j(q_{T''}, \phi_{T''}) \longrightarrow h_j,$$
$$\kappa_{T''}^{-1} D_{jj}^{-1/2}(q_{T''})\sqrt{T''}\mu_j(q_{T''}, \phi_{T''}) \longrightarrow \pi_j$$

such that one of the following is true: (a) $h_j < \infty$ and $\pi_j = 0$; (b) $h_j = \infty$ and $\pi_j < \infty$; (c) $h_j = \infty$ and $\pi_j = \infty$.

(iv) The sequence $A(q_{T''})$ has a full rank limit, denoted by A.

We can satisfy condition (i) because the reduced-form parameter set $\mathcal{R}$ is assumed to be compact (Assumption 1(i)) and the function $\phi(\rho)$ is continuously differentiable (Assumption 1(ii)). As remarked in the main text, this implies that the parameter set for ϕ is also compact. If condition (i) holds, then we obtain:

(v) The convergence $\phi_{T''} \longrightarrow \phi$ implies that $\Lambda(\phi_{T''}) \longrightarrow \Lambda(\phi)$ since $\Lambda(\phi)$ is continuous by Assumption 1(v). Also, $\hat{\Lambda}(\hat{\phi}_{T''}) \xrightarrow{p} \Lambda$ by Assumption 1(v).

Condition (ii) is satisfied since $r(q_{T'})$ and $V(q_{T'})$ are sequences that take only a finite number of discrete values. Condition (iii) is satisfied because the range of the sequences of interest is $[0, \infty]$ and by a similar argument used in the proof of Theorem 1 of Andrews and Soares (2010). Roughly speaking, in Case (iii)(a) the slackness is small and the selection criterion regards the inequality asymptotically as binding. In Case (iii)(c), the slackness is large and the selection criterion regards the inequality as nonbinding and (iii)(b) is an intermediate case. Condition (iv) is satisfied according to Lemma B2. If Condition (iv) is satisfied, then the following conditions also hold ((vii) is a consequence of Lemma B2):

(vi) $\Omega(q_{T''}) \longrightarrow A'A > 0$ and $B(q_{T''}) \longrightarrow B > 0$, where $B = (A'A)^{-1}$ if $B(q) = \Omega^{-1}(q)$ and $B = I$ if $B(q) = I$.

(vii) $\hat{\Omega}(q_{T''}) \xrightarrow{p} A'A > 0$ and $\hat{B}(q_{T''}) \xrightarrow{p} B > 0$, where $B = (A'A)^{-1}$ if $\hat{B}(q) = \hat{\Omega}^{-1}(q)$ and $B = I$ if $\hat{B}(q) = I$.

We now reorder the rows of $S(q_{T''})$ such that $\pi_j = 0$ for rows $j = 1, \dots, r_1$ and $\pi_j > 0$ for rows $j = r_1 + 1, \dots, r$. Along the sequence T'', the last $r_2 = r - r_1$ restrictions correspond to nonbinding moment inequalities. In the subsequent steps, we show that the inequality selection procedure used in the critical-value computation in (28) asymptotically underestimates r_2 (and thereby overestimates r_1), which makes the critical value asymptotically conservative to achieve the uniform coverage requirement.

Step 2: Constructing an upper bound for the critical value $c^\alpha(q)$ in (28). For notational simplicity, we use sequence notation $\{T\}$ for the sub-subsequence $\{T''\}$ in Step 1. Recall the definitions

$$\xi_{j,T}(q_T) = D_{jj}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \hat{\phi}) \quad \text{and} \quad \hat{\xi}_{j,T}(q_T) = \hat{D}_{jj}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \hat{\phi}).$$

Let $\varphi_T(q_T)$ and $\hat{\varphi}_T$ be vectors with elements

$$\varphi_{j,T}(q_T) = \begin{cases} \infty, & \text{if } \xi_{j,T}(q_T) \geq \kappa_T, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{\varphi}_{j,T}(q_T) = \begin{cases} \infty, & \text{if } \hat{\xi}_{j,T}(q_T) \geq \kappa_T, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. Moreover, define $\varphi_T^*(q_T)$ and $\hat{\varphi}_T^*(q_T)$ with elements

$$\varphi_{j,T}^*(q_T) = \begin{cases} \varphi_{j,T}(q_T), & \text{if } \pi_j = 0, \\ \infty, & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{\varphi}_{j,T}^*(q_T) = \begin{cases} \hat{\varphi}_{j,T}(q_T), & \text{if } \pi_j = 0, \\ \infty, & \text{otherwise,} \end{cases}$$

where, according to Case (iii) in Step 1,

$$\pi_j = \lim \kappa_T^{-1} D_{jj}^{-1/2}(q_T)\sqrt{T}\mu_j(q_T, \phi_T).$$

Finally, define the vector π_* with elements

$$\pi_j^* = \begin{cases} 0, & \text{if } \pi_j = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

To characterize the critical values, define the objective function

$$\bar{\mathcal{G}}\big(q_T; A(\cdot), B(\cdot), \varphi(\cdot)\big) = \min_{v \geq -\varphi(q_T)} \big\| A(q_T)' Z_m - v \big\|_{B(q_T)}^2.$$

Note that the notation in the proof is slightly different from the notation in the main text. In (27) of the main text, we defined $\bar{\mathcal{G}}(q; \hat{B}(q), M_{\hat{\xi}}(q))$, which corresponds to $\bar{\mathcal{G}}(q_T; \hat{A}(\cdot), \hat{B}(\cdot), \hat{\varphi}(\cdot))$ in this proof. We let

$$c_T^\alpha\big(A(\cdot), B(\cdot), \varphi(\cdot)\big) = (1-\alpha) \quad \text{quantile of } \bar{\mathcal{G}}\big(q_T; A(\cdot), B(\cdot), \varphi(\cdot)\big). \tag{A.5}$$

To cover the special case $r = r_2 > 0$, that is, all the inequality conditions are nonbinding, we adopt the convention that

$$c_T^\alpha\big(A(\cdot), B(\cdot), \varphi(\cdot)\big) = 0 \tag{A.6}$$

if $\varphi(q_T) = \hat{\varphi}_T^*(q_T)$ or $\varphi(q_T) = \pi^*$. The critical value $c^\alpha(q)$ in (28) in the main text can be expressed as

$$c^\alpha(q_T) = c_T^\alpha\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T(q_T)\big).$$

Notice by definition that

$$\hat{\varphi}_T^*(q_T) \geq \hat{\varphi}_T(q_T).$$

This implies that

$$c_T^{\alpha}\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T^*(q_T)\big) \leq c_T^{\alpha}\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T(q_T)\big). \tag{A.7}$$

Step 3: Establish the asymptotic coverage probability. Along the sequence defined in Step 1 we will show the desired result

$$\text{Asy}\, CP = \lim_T P_{\phi_T}\big\{(q_T) \in CS^q\big\} \geq 1 - \alpha.$$

We consider two different cases: (i) some inequalities are "binding," that is, $r_1 > 0$; (ii) all inequalities are "nonbinding," that is, $r_1 = 0$.

Step 3(i). Suppose that $r_1 > 0$. By Lemma B1 and (A.7), we have

$$\begin{aligned}
\text{Asy}\, CP &= \lim_T P_{\phi_T}\big\{q_T \in CS^q\big\} \\
&= \lim_T P_{\phi_T}\big\{G\big(q_T; \hat{\phi}, \hat{W}(\cdot)\big) \leq c_T^{\alpha}\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T(q_T)\big)\big\} \\
&= \lim_T P_{\phi_T}\big\{G\big(q_T; \hat{\phi}, W(\cdot)\big) + o_p(1) \leq c_T^{\alpha}\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T(q_T)\big)\big\} \\
&\geq \lim_T P_{\phi_T}\big\{G\big(q_T; \hat{\phi}, W(\cdot)\big) + o_p(1) \leq c_T^{\alpha}\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T^*(q_T)\big)\big\}.
\end{aligned}$$

By using an argument similar to that used in showing (A.10) of Andrews and Guggenberger (2009), it can be shown that

$$\begin{aligned}
&G\big(q_T; \hat{\phi}, W(\cdot)\big) + o_p(1) \\
&\quad = \min_{v \geq -D_R^{-1/2}(q_T)\sqrt{T}\mu(q_T, \phi_T)} \big\| A(q_T)' L^{-1} \sqrt{T}(\hat{\phi} - \phi_T) - v \big\|_{B(q_T)}^2 + o_p(1) \\
&\quad \Longrightarrow \min_{v \geq -h} \big\| A' Z_m - v \big\|_B \\
&\quad \leq \min_{v \geq -\pi^*} \big\| A' Z_m - v \big\|_B.
\end{aligned}$$

The last inequality holds because $h \geq \pi^*$. (This is true because $\pi_j = 0$ implies that $h_j < \infty$ and $\pi_j^* = 0$, while $\pi_j > 0$ implies that $h_j = \pi_j^* = \infty$.) According to Lemma B3,

$$c_T^{\alpha}\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T^*(q_T)\big) \xrightarrow{p} c_T^{\alpha}\big(A, B, \pi^*\big).$$

Since $r > r_2$, $c_T^{\alpha}(A, B, \pi^*) > 0$. Also, the distribution function of $\min_{v \geq -\pi^*} \|A' Z_m - v\|_B$ is continuous near the $(1 - \alpha)$th quantile (see p. 6 of Andrews and Soares (2010)). Then we have the required result:

$$\text{Asy}\, CP = \lim_T P_{\phi_T}\big\{q_T \in CS^q\big\}$$

$$\geq \lim_T P_{\phi_T}\{G(q_T; \hat{\phi}, W(\cdot)) \leq c_T^\alpha(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T^*(q_T)) + o_p(1)\}$$
$$\geq P\Big\{\min_{v \geq -\pi^*} \|A'Z_m - v\|_B \leq c_T^\alpha(A, B, \pi^*)\Big\}$$
$$= 1 - \alpha.$$

Step 3(ii). Suppose that $r_1 = 0$. In this case, $h_j = \infty$ and $\pi_j > 0$ for all $j = 1, \ldots, r$. Then we have $\hat{\varphi}_T^*(q_T) = \pi = \infty$ for all T. Recall the definitions that $c_T^\alpha(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T^*(q_T)) = c_T^\alpha(A, B, \pi^*) = 0$. Then, by Lemma B1 and (A.7), we have

$$\begin{aligned}
\text{Asy}\,CP &= \lim_T P_{\phi_T}\{q_T \in CS^q\} \\
&= \lim_T P_{\phi_T}\{G(q_T; \hat{\phi}, \hat{W}(\cdot)) \leq c_T^\alpha(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T(q_T))\} \\
&= \lim_T P_{\phi_T}\{G(q_T; \hat{\phi}, W(\cdot)) + o_p(1) \leq c_T^\alpha(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}_T(q_T))\} \\
&\geq \lim_T P_{\phi_T}\{G(q_T; \hat{\phi}, W(\cdot)) + o_p(1) \leq c_T^\alpha(A, B, \pi^*) = 0\}.
\end{aligned}$$

By using the same argument used in (S1.23) on page 7 of Andrews and Soares (2010), we can deduce that

$$\lim_T P_{\phi_T}\{G(q_T; \hat{\phi}, W(\cdot)) + o_p(1) \leq 0\} \geq 1 - \alpha. \quad \square$$

A.5 *Proof of Theorem* 3, *part* (i)

The proof of the theorem exploits the continuity of $F^q(\phi_q)$ with respect to ϕ_q. The statement of the theorem is a consequence of Lemma 2, Lemma 4, and the continuous mapping theorem. $\square$

LEMMA 4. *Suppose that Assumptions* 1(i) *and* 2(ii) *are satisfied. Then*:

(i) $F^\theta(\Phi_q, \Phi_\theta)$ *is compact for all* (Φ_q, Φ_θ);

(ii) $F^\theta(\Phi_q, \Phi_\theta)$ *is continuous at all* (Φ_q, Φ_θ).

PROOF OF LEMMA 4, PART (I). Since $F^\theta(\Phi_q, \Phi_\theta) \subset \mathbb{R}^k$, for the required result, we show that $F^\theta(\Phi_q, \Phi_\theta)$ is closed and bounded.

Boundedness: The set $\{\theta = f(\Phi_\theta, q) : \|q\| = 1\}$ is compact because $f(\cdot)$ is continuous in both of its arguments by Assumption 2(i) and the domain of q, $\mathbb{S}^n$, is compact. Since $F^\theta(\Phi_q, \Phi_\theta) \subset \{\theta = f(\Phi_\theta, q) : \|q\| = 1\}$, we deduce that $F^\theta(\Phi_q, \Phi_\theta)$ is bounded.

Closedness: Consider any sequence $\theta_j \in F^\theta(\Phi_q, \Phi_\theta)$, $j = 1, 2, \ldots$, such that $\theta_j \longrightarrow \theta$. We show that $\theta \in F^\theta(\Phi_q, \Phi_\theta)$, that is, we need to find a $q \in F^q(\Phi_q)$ such that $\theta = f(\Phi_\theta, q)$. Then the desired result follows.

For $\theta_j \in F^\theta(\Phi_q, \Phi_\theta)$, by definition we can choose a $q_j \in F^q(\Phi_q)$ such that $\theta_j = f(\Phi_\theta, q_j)$. Since $\{q_j\} \in \mathbb{S}^n$ and $\mathbb{S}^n$ is compact, we can choose a convergent subsequence q_{j_i} such that $q_{j_i} \longrightarrow q$. Then it follows from the continuity of $f(\cdot)$ that $f(\Phi_\theta, q_{j_i}) \longrightarrow$

$f(\Phi_\theta, q)$. Since the subsequence θ_{j_i} also converges to θ, we have $f(\Phi_\theta, q) = \theta$. By definition of $F^\theta(\Phi_q, \Phi_\theta)$, then we have $\theta \in F^\theta(\Phi_q, \Phi_\theta)$, as required for closedness.

Part (ii). According to Assumption 2(i), the function $f(\Phi_\theta, q)$ is continuous. Then the product correspondence

$$\tilde{F}^q(\Phi_q, \Phi_\theta) = F^q(\Phi_q) \times \Phi_\theta$$

is continuous by Proposition 34 of Border (2010). Notice that the correspondence $F^\theta(\Phi_q, \Phi_\theta)$ is a composite of $f(\cdot)$ and $\tilde{F}^q(\cdot)$:

$$F^\theta(\Phi_q, \Phi_\theta) = \bigcup_{q \times \Phi_\theta \in \tilde{F}^q(\Phi_q, \Phi_\theta)} f(\Phi_\theta, q).$$

Since both $f(\cdot)$ and $\tilde{F}^q(\cdot)$ are continuous, by Theorem 12.23 of Aliprantis and Border Aliprantis and Border (2006), $F^\theta(\Phi_q, \Phi_\theta)$ is continuous. □

A.6 *Proof of Theorem* 3, *part* (ii)

Let $F^{\theta,q}(\phi_q, \phi_\theta) = \{\theta \in \Theta, q \in \mathbb{S}^n : q \in F^q(\phi_q), \theta = f(\Phi_\theta, q)\}$. Notice that

$$\begin{aligned}
&\liminf_T \inf_{\rho \in \mathcal{R}} \inf_{\theta \in F^\theta(\phi_q(\rho), \phi_\theta(\rho))} P_\rho\{\theta \in CS^\theta(\hat{\phi}_q, \hat{\phi}_\theta)\} \\
&\quad \geq \liminf_T \inf_{\rho \in \mathcal{R}} \inf_{(\theta,q) \in F^{\theta,q}(\phi_q(\rho), \phi_\theta(\rho))} P_\rho\{q \in CS^q(\hat{\phi}_q),\ \theta \in CS_q^\theta(\hat{\phi}_\theta)\} \\
&\quad \geq \liminf_T \inf_{\rho \in \mathcal{R}} \inf_{q \in F^q(\phi_q(\rho))} P_\rho\{q \in CS^q(\hat{\phi}_q)\} \\
&\qquad + \liminf_T \inf_{\rho \in \mathcal{R}} \inf_{(\theta,q) \in F^{\theta,q}(\phi_q(\rho), \phi_\theta(\rho))} P_\rho\{\theta \in CS_q^\theta(\hat{\phi}_\theta)\} - 1.
\end{aligned}$$

Recall that $\theta = f(\Phi_\theta, q)$. According to Theorem 2(ii)

$$\liminf_T \inf_{\rho \in \mathcal{R}} \inf_{q \in F^q(\phi_q(\rho))} P_\rho\{q \in CS^q(\hat{\phi}_q)\} \geq 1 - \alpha_1 \tag{A.8}$$

and

$$\liminf_T \inf_{\rho \in \mathcal{R}} \inf_{(\theta,q) \in F^{\theta,q}(\phi_q(\rho), \phi_\theta(\rho))} P_\rho\{\theta \in CS_q^\theta(\hat{\phi}_\theta)\} \geq 1 - \alpha_2 \tag{A.9}$$

holds according to Assumption 2. □

Appendix B: Additional technical lemmas

Throughout this section, we use the following notation. When A is a matrix, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the largest and the smallest eigenvalues of A, respectively. We denote A_k as the kth column vector of A; A^j as the jth row vector of A; and A_{jk} as the (j,k)th element of A. Throughout the proofs, we sometimes omit the ϕ_T argument from the asymptotic covariance matrix $\Lambda = LL'$ and use the notation Λ_T, $\hat{\Lambda}_T$, L_T, and $\hat{L}_T$ for simplicity. We also often omit the q_T argument for some of the matrices that depend on q_T and simply write, say, S_T, D_T, $\hat{D}_T$, A_T, $\hat{A}_T$, Ω_T, and $\hat{\Omega}_T$ for short.

LEMMA B1. *Suppose that Assumption* 1 *is satisfied. Consider the sequence* $\{(\phi_T, q_T)\}$ *with* $q_T \in F^q(\phi_T)$ *that satisfies conditions* (i) *and* (iv) *in the proof of Theorem* 2(ii). *Then*

$$G\big(q_T; \hat{\phi}, \hat{W}(\cdot)\big) - G\big(q_T; \hat{\phi}, W(\cdot)\big) = o_p(1).$$

PROOF OF LEMMA B1. According to condition (ii) in the proof of Theorem 2(ii) $V(q_T) = V$ for all T. If $V = 0$, that is, $S(q_T) = 0$ for all T, it is trivial to deduce the required result because by definition

$$G\big(q_T; \hat{\phi}, \hat{W}(\cdot)\big) = G\big(q_T; \hat{\phi}, W(\cdot)\big) = 0.$$

Now suppose that $V \neq 0$. Notice that $S_T\phi_T \geq 0$ and $D_T^{-1/2}$ and $\hat{D}_T^{-1/2}$ are well-defined since S_T is a full (row) rank matrix and $\Lambda_T, \hat{\Lambda}_T > 0$. We now consider the two cases (i) $B_T = \hat{B}_T = I$ and (ii) $B_T = \Omega_T^{-1}$ and $\hat{B}_T = \hat{\Omega}_T^{-1}$ separately.

Case (i) $B_T = \hat{B}_T = I$. Write

$$\begin{aligned} G\big(q_T; \hat{\phi}, \hat{W}(\cdot)\big) &= \min_{\mu \geq 0} T\big\|\hat{D}_T^{-1/2}S_T\hat{\phi} - \hat{D}_T^{-1/2}V_T\mu\big\|^2 \\ &= \min_{v \geq -\sqrt{T}\hat{D}_T^{-1/2}\mu(q_T,\phi_T)} \big\|\hat{D}_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2 \end{aligned}$$

and

$$G\big(q; \hat{\phi}, W(\cdot)\big) = \min_{v \geq -\sqrt{T}D_T^{-1/2}\mu(q,\phi)} \big\|D_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2,$$

where $\mu(q_T, \phi_T) = S_T\phi_T$. Define

$$\begin{aligned} v_T(\hat{\Lambda}_T) &= \operatorname*{argmin}_{v \geq -\sqrt{T}\hat{D}_T^{-1/2}\mu(q_T,\phi_T)} \big\|\hat{D}_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2, \\ v_T(\Lambda_T) &= \operatorname*{argmin}_{v \geq -\sqrt{T}D_T^{-1/2}\mu(q,\phi)} \big\|D_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2. \end{aligned}$$

Recall that $A' = D^{-1/2}SL$ and, therefore, $D^{-1/2}S = A'L^{-1}$. Then

$$\begin{aligned} &G\big(q_T; \hat{\phi}, \hat{W}(\cdot)\big) - G\big(q_T; \hat{\phi}, W(\cdot)\big) \\ &\quad \leq \big\|\hat{D}_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big\|^2 - \big\|D_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big\|^2 \\ &\quad \leq \big\|\hat{D}_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T) - D_T^{-1/2}S_T\sqrt{T}(\hat{\phi} - \phi_T)\big\|^2 \\ &\quad \leq \big\|(\hat{A}_T - A_T)'\hat{L}_T^{-1}\sqrt{T}(\hat{\phi} - \phi_T)\big\| + \big\|A_T'\big(\hat{L}_T^{-1} - L_T^{-1}\big)\sqrt{T}(\hat{\phi} - \phi_T)\big\| \\ &\quad = o_p(1). \end{aligned}$$

The last equality holds because $\hat{A}_T - A_T = o_p(1)$, $A_T = O(1)$, $\hat{L}_T^{-1} - L_T^{-1} = o_p(1)$, $L_T^{-1} = O(1)$, $\sqrt{T}(\hat{\phi} - \phi_T) = O_p(1)$, and $\hat{B}_T \xrightarrow{p} B > 0$ according to Lemma B2 and Assumption 1(v)–(vi).

Case (ii): $B_T = \Omega_T^{-1}$ and $\hat{B}_T = \hat{\Omega}_T^{-1}$. In this case, we can write

$$\begin{aligned} G\big(q_T; \hat{\phi}, \hat{W}(\cdot)\big) &= \min_{\mu \geq 0} T \, \|S_T \hat{\phi} - V_T \mu\|^2_{\hat{\Sigma}_T^{-1}} \\ &= \min_{v \geq -\sqrt{T}\mu(q_T, \phi_T)} \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2_{\hat{\Sigma}_T^{-1}} \end{aligned}$$

and

$$G\big(\theta, q; \hat{\phi}, W(\cdot)\big) = \min_{v \geq -\sqrt{T}\mu(q_T, \phi_T)} \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2_{\Sigma_T^{-1}},$$

where $\mu(q_T, \phi_T) = S_T \phi_T$. Define

$$\begin{aligned} v_T(\hat{\Lambda}_T) &= \operatorname*{argmin}_{v \geq -\sqrt{T}\mu(q_T, \phi_T)} \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2_{\hat{\Sigma}_T^{-1}}, \\ v_T(\Lambda_T) &= \operatorname*{argmin}_{v \geq -\sqrt{T}\mu(q_T, \phi_T)} \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v\big\|^2_{\Sigma_T^{-1}}. \end{aligned}$$

Then

$$\begin{aligned} & G\big(\theta_T, q_T; \hat{\phi}, \hat{W}(\cdot)\big) - G\big(\theta_T, q_T; \hat{\phi}, W(\cdot)\big) \\ & \quad \leq \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big\|^2_{\hat{\Sigma}_T^{-1}} - \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big\|^2_{\Sigma_T^{-1}} \\ & \quad = \big[S_T \sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big]' \Sigma_T^{-1/2} \big[\Sigma_T^{1/2} \hat{\Sigma}_T^{-1} \Sigma_T^{1/2} - I_r\big] \\ & \qquad \times \Sigma_T^{-1/2} \big[S_T \sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big] \\ & \quad \leq \big\|S_T \sqrt{T}(\hat{\phi} - \phi_T) - v_T(\Lambda_T)\big\|^2_{\Sigma_T^{-1}} \big\|\Sigma_T^{1/2} \hat{\Sigma}_T^{-1} \Sigma_T^{1/2} - I_r\big\| \\ & \quad = I \times II, \quad \text{say.} \end{aligned}$$

For term I, notice that since $\mu(q_T, \phi_T) \geq 0$, we have

$$I = \min_{v \geq -\sqrt{T}\mu(q_T, \phi_T)} \big\|S_T \sqrt{T}(\hat{\phi} - \phi) - v\big\|^2_{\Sigma_T^{-1}} \leq \big\|A_T{}' L_T^{-1} \sqrt{T}(\hat{\phi} - \phi_T)\big\|^2_{\Omega_T^{-1}} = O_p(1).$$

The last equality holds since $A_T{}' L_T^{-1} \sqrt{T}(\hat{\phi} - \phi_T) = O_p(1)$ and $\Omega_T^{-1} = (A_T' A_T)^{-1} \longrightarrow (A' A) > 0$ by condition (vi) in the proof of Theorem 2(ii). Since $\Sigma_T = S_T L_T L_T' S_T'$, term II can be bounded as follows:

$$\begin{aligned} II &= \big\|\Sigma_T^{1/2} \big(\hat{\Sigma}_T^{-1} - \Sigma_T^{-1}\big) \Sigma_T^{1/2}\big\| \\ &= \big\|\Sigma_T^{-1/2} (\Sigma_T - \hat{\Sigma}_T) \hat{\Sigma}_T^{-1/2} \hat{\Sigma}_T^{-1/2} \Sigma_T^{1/2}\big\| \\ &= \big\|\Sigma_T^{-1/2} S_T (\Lambda_T - \hat{\Lambda}_T) S_T' \hat{\Sigma}_T^{-1/2} \hat{\Sigma}_T^{-1/2} \Sigma_T^{1/2}\big\| \\ &= \big\|\big(\Sigma_T^{-1/2} S_T L_T\big)\big(L_T' \hat{L}_T'^{-1} - L_T^{-1} \hat{L}_T\big)\big(\hat{L}_T S_T' \hat{\Sigma}_T^{-1/2}\big) \hat{\Sigma}_T^{-1/2} \Sigma_T^{1/2}\big\| \\ &\leq \big\|\Sigma_T^{-1/2} S_T L_T\big\| \big\|L_T' \hat{L}_T'^{-1} - L_T^{-1} \hat{L}_T\big\| \big\|\hat{L}_T S_T' \hat{\Sigma}_T^{-1/2}\big\| \big\|\hat{\Sigma}_T^{-1/2} \Sigma_T^{1/2}\big\| \\ &= O(1) o_p(1) O_p(1) O_p(1). \end{aligned}$$

The last line holds because

$$\begin{aligned}\|\Sigma_T^{-1/2} S_T L_T\|^2 &= \operatorname{tr}\big(L_T' S_T' \big(S_T L_T L_T' S_T'\big)^{-1} S_T L_T\big) = l,\\ \|\hat{L}_T S_T' \hat{\Sigma}_T^{-1/2}\| &= \operatorname{tr}\big(\hat{L}_T' S_T' \big(S_T \hat{L}_T \hat{L}_T' S_T'\big)^{-1} S_T \hat{L}_T\big) = l,\\ \|L_T' \hat{L}_T'^{-1} - L_T^{-1} \hat{L}_T\| &= o_p(1) \quad \text{under Assumption 1(vi).}\end{aligned}$$

Moreover,

$$\begin{aligned}\|\hat{\Sigma}_T^{-1/2} \Sigma_T^{1/2}\|^2 &= \|\hat{\Sigma}_T^{-1/2} (S_T L_T) [L_T' S_T' \big(S_T L_T L_T S_T'\big)^{-1}] \Sigma_T^{1/2}\|^2\\ &= \|\hat{\Sigma}_T^{-1/2} (S_T \hat{L}_T) \big(\hat{L}_T^{-1} L_T\big) \big(L_T' S_T' \Sigma_T^{-1/2}\big)\|^2\\ &\le \|\hat{\Sigma}_T^{-1/2} S_T \hat{L}_T\|^2 \|\hat{L}_T^{-1} L_T\|^2 \|L_T' S_T' \Sigma_T^{-1/2}\|^2\\ &= O_p(1) O_p(1) O(1) = O_p(1).\end{aligned}$$

This completes the proof for Case (ii). □

Lemma B2. *Suppose that a converging sequence* $\{\phi_T, q_T\}$ *satisfies the rank condition* $r(q_T) = r > 0$ *and* $V(q_T)$ *is a nonzero constant selection matrix for all* T*. Then there exists a subsequence* $\{T'\} \subset \{T\}$ *such that along the subsequence, we have*

(i)

$$D^{-1/2}(q_{T'}) S(q_{T'}) L(\phi_{T'}) \longrightarrow A,$$

where A *is a full rank matrix, and*

(ii)

$$\hat{D}^{-1/2}(q_{T'}) S(q_{T'}) \hat{L}(\hat{\phi}_{T'}) = D^{-1/2}(q_{T'}) S(q_{T'}) L(\phi_{T'}) + o_p(1).$$

Proof of Lemma B2. Part (i): Recall that $S_T = V_T \tilde{S}_T$. The rank reduction of $\tilde{S}_T$ is caused only by zero rows (see Section 2.3). Moreover, according to condition (ii) in the proof of Theorem 2(ii) the nonzero row selection matrix is V_T constant over T. Thus, we can construct an index set $\mathcal{J}$ of nonzero rows of $\tilde{S}_T$. By construction, the size of $\mathcal{J}$ is l and

$$S_T = [\tilde{S}_T^j]_{j \in \mathcal{J}}.$$

In turn, we obtain

$$D_T^{-1/2} S_T L_T = D_T^{-1/2} [\tilde{S}_T^j L_T]_{j \in \mathcal{J}}.$$

Recall from the definition of L and D that (omitting the T subscripts)

$$SLL'S' = D^{1/2} \Omega D^{1/2} \quad \text{and} \quad D^{-1/2} SLL'S' D^{-1/2} = \Omega,$$

where Ω is a correlation matrix with ones on its diagonal. Thus, $D_{ii}^{-1/2}$ normalizes the length of the i'th row of the matrix (SL) to one. Therefore,

$$D_{T'}^{-1/2} S_{T'} L_{T'} = \left[\frac{\tilde{S}_T^j L_T}{\|\tilde{S}_T^j L_T\|}\right]_{j \in \mathcal{J}} = \left[\frac{\tilde{S}_T^j}{\|\tilde{S}_T^j L_T\|}\right]_{j \in \mathcal{J}} L_T.$$

By construction, $\tilde{S}_T^j \neq 0$ for all T and $j \in \mathcal{J}$. Since $L_T > 0$, it follows that $\tilde{S}_T^j L_T \neq 0$ for all T and $j \in \mathcal{J}$. In turn, $\|\tilde{S}_T^j L_T\| > 0$ for all T and $j \in \mathcal{J}$ and $\tilde{S}_T^j L_T / \|\tilde{S}_T^j L_T\|$ is well-defined for all T and $j \in \mathcal{J}$. Notice that $\{\tilde{S}_T^j L_T / \|\tilde{S}_T^j L_T\|\}_T$ is a sequence on a unit sphere, which is compact. We can then choose a subsequence $\{T'\}$ such that $\tilde{S}_{T'}^j L_{T'} / \|\tilde{S}_{T'}^j L_{T'}\|$ converges for all $j \in \mathcal{J}$. Thus, we can write

$$D_{T'}^{-1/2} S_{T'} L_{T'} = \left[\frac{\tilde{S}_{T'}^j}{\|\tilde{S}_{T'}^j L_{T'}\|} \right]_{j \in \mathcal{J}} L_{T'} \longrightarrow A.$$

To obtain the desired result, it remains to be shown that A is full rank. Since $L_{T'}^{-1} \longrightarrow L^{-1} > 0$, it suffices to show that the limit

$$AL^{-1} = \lim_{T' \longrightarrow \infty} \left[\frac{\tilde{S}_{T'}^j}{\|\tilde{S}_{T'}^j L_{T'}\|} \right]_{j \in \mathcal{J}} \tag{A.10}$$

has full rank. Recall that $\tilde{S}(q) = (I \otimes q')\bar{S}\phi_q$. By construction, the nonzero rows of $\tilde{S}_{T'}^j$ are orthogonal to each other because $\{\tilde{S}_{T'}^j\}_{j \in \mathcal{J}}$ is composed of rows $(+/-)(I^j \otimes q)\bar{S}\phi_q$ that are orthogonal to each other. This implies that each row of the limit AL^{-1} is nonzero and orthogonal, which delivers the required result.

Part (ii): Consider the subsequence $\{T'\}$ in the proof of Part (i). Since $\hat{L}_{T'} > 0$ and $\tilde{S}_{T'}^j \neq 0$ for all T',

$$\|\tilde{S}_{T'}^j \hat{L}_{T'}\| > 0$$

for all T'. We will now show that

$$\frac{\tilde{S}_{T'}^j \hat{L}_{T'}}{\|\tilde{S}_{T'}^j \hat{L}_{T'}\|} = \frac{\tilde{S}_{T'}^j L_{T'}}{\|\tilde{S}_{T'}^j L_{T'}\|} + o_p(1)$$

for all $j \in \mathcal{J}$. Since it could be the case that $\|\tilde{S}_{T'}^j L_{T'}\| \longrightarrow 0$, we provide a detailed argument. Write

$$\begin{aligned}
&\frac{\tilde{S}_{T'}^j \hat{L}_{T'}}{\|\tilde{S}_{T'}^j \hat{L}_{T'}\|} - \frac{\tilde{S}_{T'}^j L_{T'}}{\|\tilde{S}_{T'}^j L_{T'}\|} \\
&= \frac{\tilde{S}_{T'}^j L_{T'}}{\|\tilde{S}_{T'}^j L_{T'}\|} \left(\frac{\|\tilde{S}_{T'}^j L_{T'}\|}{\|\tilde{S}_{T'}^j \hat{L}_{T'}\|} - 1 \right) + \frac{\tilde{S}_{T'}^j (\hat{L}_{T'} - L_{T'})}{\|\tilde{S}_{T'}^j \hat{L}_{T'}\|} \\
&= I + II, \quad \text{say.}
\end{aligned}$$

We begin with the following bound:

$$\frac{\|\tilde{S}_{T'}^j L_{T'}\|}{\|\tilde{S}_{T'}^j \hat{L}_{T'}\|} - 1 = \frac{\|\tilde{S}_{T'}^j L_{T'}\| - \|\tilde{S}_{T'}^j \hat{L}_{T'}\|}{\|\tilde{S}_{T'}^j \hat{L}_{T'}\|}$$

$$
\begin{aligned}
&= \frac{\|\tilde{S}^j_{T'}\hat{L}_{T'} - \tilde{S}^j_{T'}(\hat{L}_{T'} - L_{T'})\| - \|\tilde{S}^j_{T'}\hat{L}_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\
&\leq \frac{\|\tilde{S}^j_{T'}(\hat{L}_{T'} - L_{T'})\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\
&\leq \frac{\|\tilde{S}^j_{T'}\|\|\hat{L}_{T'} - L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\
&= \frac{\|\hat{L}_{T'} - L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|/\|\tilde{S}^j_{T'}\|}.
\end{aligned}
$$

The last equality holds because $\|\tilde{S}^j_{T'}\| > 0$ for all T'.

According to Assumption 1(vi) $\hat{L}_{T'} \xrightarrow{p} L$. Moreover, we deduce from (A.10) and $A^j L^{-1} \neq 0$ that

$$
\begin{aligned}
0 < \frac{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|}{\|\tilde{S}^j_{T'}\|} &\leq \frac{\|\tilde{S}^j_{T'}L_{T'}\| + \|\tilde{S}^j_{T'}(\hat{L}_{T'} - L_{T'})\|}{\|\tilde{S}^j_{T'}\|} \\
&\leq \frac{\|\tilde{S}^j_{T'}L_{T'}\|}{\|\tilde{S}^j_{T'}\|} + \|\hat{L}_{T'} - L_{T'}\| \\
&\xrightarrow{p} \frac{1}{\|A^j L^{-1}\|} > 0.
\end{aligned}
$$

Therefore,

$$
0 \leq \frac{\|\hat{L}_{T'} - L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|/\|\tilde{S}^j_{T'}\|} \leq o_p(1)\|A^j L^{-1}\| = o_p(1).
$$

Similarly, we obtain the bound

$$
\begin{aligned}
1 - \frac{\|\tilde{S}^j_{T'}L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} &= \frac{\|\tilde{S}^j_{T'}\hat{L}_{T'}\| - \|\tilde{S}^j_{T'}L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\
&= \frac{\|\tilde{S}^j_{T'}L_{T'} + \tilde{S}^j_{T'}(\hat{L}_{T'} - L_{T'})\| - \|\tilde{S}^j_{T'}L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\
&\leq \frac{\|\tilde{S}^j_{T'}(\hat{L}_{T'} - L_{T'})\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \xrightarrow{p} 0.
\end{aligned}
$$

Since $\tilde{S}^j_{T'}L_{T'}/\|\tilde{S}^j_{T'}L_{T'}\| = O(1)$, we have established that term I vanishes asymptotically:

$$
I = o_p(1).
$$

Term II can be bounded as follows:

$$\begin{aligned}\|II\| &= \frac{\|\tilde{S}^j_{T'}(\hat{L}_{T'} - L_{T'})\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\ &\leq \frac{\|\tilde{S}^j_{T'}\|\|\hat{L}_{T'} - L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|} \\ &\leq \frac{\|\hat{L}_{T'} - L_{T'}\|}{\|\tilde{S}^j_{T'}\hat{L}_{T'}\|/\|\tilde{S}^j_{T'}\|} \xrightarrow{p} 0,\end{aligned}$$

and so

$$II = o_p(1).$$

Combining the two $o_p(1)$ results completes the proof of Part (ii). □

LEMMA B3. *Suppose Assumption* 1 *is satisfied. Consider Case* (i) *in Step* 3 *of the proof of Theorem* 2(ii). *Along the* $\{T\}$ *sequence defined in Step* 1 *of the proof of Theorem* 2(ii),

$$c^\alpha_T\big(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}^*_T(q_T)\big) \to_p c^\alpha_T\big(A, B, \pi^*\big),$$

where the critical value function $c^\alpha_T(\cdot)$ *is defined in* (A.5) *and* (A.6).

PROOF OF LEMMA B3. The proof is very similar to that of Lemma 2(a) in Andrews and Soares (2010) and we provide a sketch. The proof proceeds in three steps. First, show

$$\big(\hat{\xi}_T, \hat{A}(q_T), \hat{B}(q_T)\big) \xrightarrow{p} (\pi, A, B) \quad \text{and} \quad \hat{\varphi}^*_T(q_T) \xrightarrow{p} \pi^*.$$

Second, show

$$\mathbb{P}\Big\{\min_{v \geq -\hat{\varphi}^*_T(q_T)} \big\|(\hat{A}(q_T)'Z_m - v\big\|^2_{\hat{B}(q_T)} \leq x\Big\} \xrightarrow{p} \mathbb{P}\Big\{\min_{v \geq -\pi^*}\{\big\|A'Z_m - v\big\|^2_B\} \leq x\Big\}.$$

Third, deduce $c^\alpha_T(\hat{A}(q_T), \hat{B}(q_T), \hat{\varphi}^*_T(q_T)) \xrightarrow{p} c^\alpha_T(\hat{A}(q_T), \hat{B}(q_T), \pi^*)$, as required for the lemma.

PROOF OF STEP 1. By the choice of the sequence $\{T\}$ and the limit result in Step 1 of the proof of Theorem 2(ii)

$$\big(\hat{\xi}_T, \hat{A}(q_T), \hat{B}(q_T)\big) \xrightarrow{p} (\pi, A, B).$$

Notice that if $\pi_j = 0$, then $\xi_T(q_T) < \kappa_T$ as $T \longrightarrow \infty$ and by an argument similar to the one used in the proof of Lemma B2(ii), we have $\hat{\xi}_T(q_T) < \kappa_T$ in probability as $T \longrightarrow \infty$. Therefore, plim $\hat{\varphi}^*_{j,T}(q_T) =$ plim $\hat{\varphi}_{j,T}(q_T) = 0 = \pi^*_j$ with probability one. On the other hand, if $\pi_j > 0$, then $\hat{\varphi}^*_{j,T}(q_T) = \infty = \pi^*_j$. Therefore, $\hat{\varphi}^*_T(q_T) \xrightarrow{p} \pi^*$. □

PROOF OF STEP 2. The desired result can be obtained by the same argument used in the proof of (S1.17) of Andrews and Soares (2010). □

Proof of Step 3. It is immediate from Step 2 and the fact that the distribution of

$$\min_{v \geq -\pi^*} \| A' Z_m - v \|_B^2$$

is continuous if $k \geq 1$, and continuous near the $(1-\alpha)'s$ quantile, where $\alpha < 1/2$, if $k = 0$. □

Appendix C: Description of Monte Carlo experiments

C.1 *Experiment 1: Bivariate* VAR(0)

This section presents the computations for the Monte Carlo experiment with the VAR(0) model, for example, Design 1 in Table 1 of the main article: $y_t = u_t$, $u_t \sim N(0, \Sigma)$. The population identified set is given by $F^\theta(\phi) = [0, \max\{\mathcal{I}\{\phi_2 \geq 0\}, \sqrt{\frac{\phi_3^2}{\phi_2^2+\phi_3^2}}\}]$ where $\phi = [\phi_1, \phi_2, \phi_3]' = [\Sigma_{11}^{\text{tr}}, \Sigma_{21}^{\text{tr}}, \Sigma_{22}^{\text{tr}}]'$ and Σ_{ij}^{tr} are the elements of Σ_{tr}, the lower triangular matrix from the Cholesky decomposition of Σ.

It is convenient to reparameterize q in spherical coordinates: $q = q(\varphi) = [\cos(\varphi) \sin(\varphi)]'$. However, for brevity we typically write q, omitting the φ argument. We generate a grid $\mathcal{Q}$ for q by dividing the domain of φ, $[-\frac{\pi}{2}, \frac{\pi}{2}]$, into equally sized partitions of length δ_φ. As discussed in the main text, since $\phi_1 = \Sigma_{11}^{\text{tr}} > 0$ the inequality restriction $\theta = q_1\phi_1 \geq 0$ implies that $q_1 \geq 0$. Thus, it suffices to conduct the grid search with respect to φ over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

The following steps are repeated n_{sim} times. The results reported in the main text are averages across these repetitions. We report the average length of the confidence intervals and compute the coverage probability as the fraction of times for which the upper bound of $F^\theta(\phi)$ is contained in the confidence interval. The upper bound of the identified set determines the lower bound of the coverage probability.

Generating Data: Generate a sample of length T of data from the VAR(0) using the parameters reported in Table 1.

Estimating the reduced-form parameters

- Compute the sample covariance $\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})'(y_t - \bar{y})$ where $\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t$. Denote by $\hat{\Sigma}_{\text{tr}}$ the lower triangular matrix from the Cholesky decomposition of $\hat{\Sigma}$. Then $\hat{\phi} = [\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3]' = [\hat{\Sigma}_{11}^{\text{tr}}, \hat{\Sigma}_{21}^{\text{tr}}, \hat{\Sigma}_{22}^{\text{tr}}]'$, where $\hat{\Sigma}_{ij}^{\text{tr}}$ are the elements of $\hat{\Sigma}_{\text{tr}}$.

- Estimate Λ, the asymptotic variance covariance matrix of $\hat{\phi}$, using a parametric bootstrap:

  – Generate bootstrap samples $b = 1, \dots, B$ of length T from $y_t^{(b)} = u_t^{(b)}$ where $u_t^{(b)} \sim N(0, \hat{\Sigma})$.

  – For each bootstrap sample, estimate $\hat{\Sigma}^{(b)}$ and compute $\hat{\phi}^{(b)} = [\hat{\phi}_1^{(b)}, \hat{\phi}_2^{(b)}, \hat{\phi}_3^{(b)}]'$.

  – Let $\hat{\Lambda} = \frac{1}{B}\sum_{b=1}^{B}[\sqrt{T}(\hat{\phi}^{(b)} - \hat{\phi})][\sqrt{T}(\hat{\phi}^{(b)} - \hat{\phi})]'$ with factorization $\hat{\Lambda} = \hat{L}\hat{L}'$.

Computing the confidence intervals

- *Step* 1*: Construct a* $(1-\alpha_1)$ *confidence set for* q. The following computations are executed for each $q \in \mathcal{Q}$. As before, it is convenient to express q in terms of the angle φ and generate $\mathcal{Q}$ by equally spaced grid points on the interval $[-\pi, \pi]$. Recall the definition of $\xi_{1,T}(q)$ and $\xi_{2,T}(q)$ in (24).

  – If $\varphi \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, the objective function is given by

$$G^q(q; \hat{\phi}, \hat{W}) = \min_{\mu \geq 0} \frac{T}{\hat{\Sigma}_{22}(q)} (q_1 \hat{\phi}_2 + q_2 \hat{\phi}_3 - \mu)^2.$$

* If $\xi_{2,T}(q) < \kappa_T$, the inequality condition is considered binding and the critical value $c^{\alpha_1}(q)$ is the $(1-\alpha_1)$ quantile of a squared truncated normal $Z^2\mathcal{I}\{Z \geq 0\}$.

* If $\xi_{2,T}(q) \geq \kappa_T$, the inequality condition is considered nonbinding and $c^{\alpha_1}(q) = 0$.

  – If $\varphi \notin \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, the objective function is given by

$$G^q(q; \hat{\phi}, \hat{W}) = \min_{\mu_1 \geq 0, \mu_2 \geq 0} T \left\| \hat{D}^{-1/2}(q) \begin{bmatrix} q_1 \hat{\phi}_1 - \mu_1 \\ q_1 \hat{\phi}_2 + q_2 \hat{\phi}_3 - \mu_2 \end{bmatrix} \right\|^2_{\hat{B}(q)}.$$

* If $\xi_{1,T}(q) < \kappa_T$ and $\xi_{2,T}(q) < \kappa_T$, both inequality conditions are considered binding. For $j = 1, \ldots, n_Z$ draw $Z_3^{(j)}$ from $N(0, I_3)$. The critical value is the $(1-\alpha_1)$ quantile of

$$\bar{\mathcal{G}}^{(j)}\big(q; \hat{B}(q)\big) = \min_{\nu \geq 0} \big\| \hat{D}^{-1/2}(q) S(q) \hat{L} Z_3^{(j)} - \nu \big\|^2_{\hat{B}(q)}.$$

The minimization can be executed with a numerical routine that solves quadratic programming problems.

* If $\xi_{1,T}(q) < \kappa_T$ and $\xi_{2,T}(q) \geq \kappa_T$ or if $\xi_{1,T}(q) \geq \kappa_T$ and $\xi_{2,T}(q) < \kappa_T$, i.e., only one inequality condition is considered binding, then $c^{\alpha_1}(q)$ is the $(1-\alpha_1)$th quantile of a squared truncated normal $Z^2\mathcal{I}\{Z \geq 0\}$.

* if $\xi_{1,T}(q) \geq \kappa_T$ and $\xi_{2,T}(q) \geq \kappa_T$, then no inequality condition is considered binding and $c^{\alpha_1}(q) = 0$.

  – Let $CS^q = \{q \in \mathcal{Q} \mid (G^q(q; \hat{\phi}, \hat{W}) - c^{\alpha_1}(q)) \leq 0\}$.

- *Step* 2*: Construct a* $(1-\alpha_2)$ *confidence set for* θ *conditional on* q:

$$CS^\theta_q = \left[\max\left\{0, q_1 \hat{\phi}_1 - z_{\alpha_2/2} \sqrt{q_1^2 \hat{\Lambda}_{11}/T}\right\},\ q_1 \hat{\phi}_1 + z_{\alpha_2/2} \sqrt{q_1^2 \hat{\Lambda}_{11}/T}\right],$$

where $z_{\alpha_2/2}$ is the $(1-\alpha_2/2)$ quantile of a $N(0,1)$ distribution and $\hat{\Lambda}_{11}$ is the $(1,1)$ element of the matrix $\hat{\Lambda}$.

- *Step* 3*: Construct the* $1-\alpha$ *Bonferroni set for* θ: Compute the minimum of the lower bounds of CS^θ_q and the maximum of the upper bounds of CS^θ_q for $q \in CS^q$.

C.2 *Experiment 2: Bivariate* VAR(1)

The computations are very similar to the computations for the VAR(0) experiment. Thus, we focus on highlighting the differences. The model takes the form (Designs 2 to 4 in Table 1 of the main article): $y_t = Ay_{t-1} + u_t$, where $u_t \sim N(0, \Sigma)$. Let Σ_{tr} denote the lower-triangular Cholesky factor of Σ. The reduced-form parameters are given by

$$\begin{aligned}\phi &= \text{vec}\big((A\Sigma_{\text{tr}})'\big) \\ &= [\phi_1, \phi_2, \phi_3, \phi_4]' \\ &= \big[A_{11}\Sigma_{11}^{\text{tr}} + A_{12}\Sigma_{21}^{\text{tr}}, A_{12}\Sigma_{22}^{\text{tr}}, A_{21}\Sigma_{11}^{\text{tr}} + A_{22}\Sigma_{21}^{\text{tr}}, A_{22}\Sigma_{22}^{\text{tr}}\big]',\end{aligned}$$

where Σ_{ij}^{tr} are the elements of Σ_{tr}. Under our three Monte Carlo designs, the identified set $F^q(\phi)$ has a geometry similar to that of the identified set for the VAR(0) design, depicted in Figure 1 of the main article. Roughly speaking, it is an arc located in the northeast section of the unit circle. Under the parameterization of the data-generating processes (DGPs), the top-left endpoint of $F^q(\phi)$ is given by the solution of

$$q_{1,l}^2 = \frac{1}{1 + (\phi_1/\phi_2)^2},$$

whereas the bottom-right endpoint of $F^q(\phi)$ is given by the solution of

$$q_{1,r}^2 = \frac{1}{1 + (\phi_3/\phi_4)^2}.$$

The structural parameter of interest is $\theta = q_1\phi_1 + q_2\phi_2$. For our Monte Carlo designs, the lower bound of the identified set $F^\theta(\phi)$ is determined by $\theta_l = q_{1,l}\phi_1 + q_{2,l}\phi_2$. The upper bound is $\theta_u = q_{1,r}\phi_1 + q_{2,r}\phi_2$ if $q_{2,r} > 0$; it is $\theta_u = q_{1,r}\phi_1 + q_{2,r}\phi_2$ otherwise.

As for the VAR(0) experiment, the minimization with respect to q is carried out using a grid $q \in \mathcal{Q}$, where $q = [\cos(\varphi)\ \sin(\varphi)]'$ and φ takes values on an equally spaced grid over $[-\pi, \pi]$ with spacing δ_φ.

Generating data: The DGP is now given by $y_t = Ay_{t-1} + u_t$.

Estimating the reduced-form parameters: Follow the same steps as in the VAR(0) experiment.

Bonferroni approach

- *Step* 1*: Construct a* $1 - \alpha_1$ *confidence set for q.*

– The objective function is

$$G^q(q; \hat{\phi}, \hat{W}) = \min_{\mu_1 \geq 0, \mu_2 \geq 0} T \left\| \hat{D}^{-1/2}(q) \begin{bmatrix} q_1\hat{\phi}_1 + q_2\hat{\phi}_2 - \mu_1 \\ q_1\hat{\phi}_2 + q_2\hat{\phi}_3 - \mu_2 \end{bmatrix} \right\|_{\hat{B}(q)}^2.$$

– If $\xi_{1,T}(q) < \kappa_T$ and $\xi_{2,T}(q) < \kappa_T$, both inequality conditions are considered binding. For $j = 1, \ldots, n_Z$ draw $Z_4^{(j)}$ from $N(0, I_4)$. The critical value is the $(1 - \alpha_1)$ quantile of

$$\bar{\mathcal{G}}^{(j)}\big(q; \hat{B}(q)\big) = \min_{\nu \geq 0} T \big\| \hat{D}^{-1/2}(q)S(q)\hat{L}Z_4^{(j)} - \nu \big\|_{\hat{B}(q)}^2.$$

– If $\xi_{1,T}(q) < \kappa_T$ and $\xi_{2,T}(q) \geq \kappa_T$ or if $\xi_{1,T}(q) \geq \kappa_T$ and $\xi_{2,T}(q) < \kappa_T$, that is, only one inequality condition is considered binding, then $c^{\alpha_1}(q)$ is the $(1-\alpha_1)$ quantile of a squared truncated normal $Z^2\mathcal{I}\{Z \geq 0\}$.

– If $\xi_{1,T}(q) \geq \kappa_T$ and $\xi_{2,T}(q) \geq \kappa_T$, then no inequality condition is considered binding and $c^{\alpha_1}(q) = 0$.

- *Step* 2*: Construct the* $(1-\alpha_2)$ *confidence set for* θ *conditional on* q. Follow the same steps as in Experiment 1.

- *Step* 3*: Construct the* $1-\alpha$ *Bonferroni set for* θ. Follow the same steps as in the Experiment 1.

C.3 *Experiment 3: Four-variable* VAR(2)

Design. The coefficient matrices for the DGP are given by

$$A_1' = \begin{bmatrix} 1.001 & -0.100 & 0.302 & -0.085 \\ 0.065 & 0.585 & 0.089 & -0.055 \\ 0.126 & 0.284 & 1.072 & -0.073 \\ 0.233 & 0.141 & 0.056 & 1.522 \end{bmatrix},$$

$$A_2' = \begin{bmatrix} -0.080 & 0.119 & -0.269 & 0.078 \\ -0.056 & 0.262 & 0.065 & 0.013 \\ -0.223 & -0.222 & -0.178 & 0.070 \\ -0.230 & -0.097 & -0.069 & -0.538 \end{bmatrix},$$

$$c = \begin{bmatrix} 0.626 \\ 0.175 \\ 0.064 \\ 0.204 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 0.542 & -0.124 & 0.199 & 0.095 \\ -0.124 & 1.164 & 0.129 & -0.369 \\ 0.199 & 0.129 & 0.912 & -0.263 \\ 0.095 & -0.369 & -0.263 & 0.549 \end{bmatrix}.$$

Appendix D: Further details on the empirical analysis

The construction of the data set follows Aruoba and Schorfheide (2011). Unless otherwise noted, the data are obtained from the FRED2 database maintained by the Federal Reserve Bank of St. Louis. Per capita output is defined as real GDP (GDPC96) divided by the civilian noninstitutionalized population (CNP16OV). The population series is provided at a monthly frequency and converted to quarterly frequency by simple averaging. We take the natural log of per capita output and extract a deterministic trend by OLS regression over the period 1959:I to 2006:IV. The deviations from the linear trend are scaled by 100 to convert them into percentages. Inflation is defined as the log difference of the GDP deflator (GDPDEF), scaled by 400 to obtain annualized percentage rates. Our measure of nominal interest rates corresponds to the federal funds rate (FEDFUNDS), which is provided at monthly frequency and converted to quarterly frequency by simple averaging. We use the sweep-adjusted M2S series provided by Cynamon, Dutkowsky, and Jones (2006). This series is recorded at monthly frequency without seasonal adjustments. The EVIEWS default version of the X12 filter is applied to remove seasonal variation. The M2S series is divided by quarterly nominal GDP to obtain inverse velocity. We then remove a linear trend from log inverse velocity and scale the deviations from trend by 100. Since our VAR is expressed in terms of real money balances, we take the sum of log inverse velocity and real GDP. Finally, we restrict our quarterly observations to the period from 1965:I to 2005:I.

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