

## Household debt and crises of confidence

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This paper develops a notion of consumer confidence within a dynamic competitive equilibrium framework. In any situation where multiple equilibrium prices on next-period spot markets are equally supported by the state of the economy, confidence is encoded in the subjective probabilities consumers attach to these multiple future outcomes. Our approach characterizes the set of all equilibrium-consistent subjective probabilities, and thereby endogenizes the extent of uncertainty faced by consumers. We use the structure of an economy with collateralized household debt and housing markets to develop and illustrate this concept. Our approach determines the specific range of debt levels at which this economy is vulnerable to crises of confidence, as well as the debt-level-specific extent of confidence-driven house price fluctuations.

**KEYWORDS.** Consumer confidence, asset price expectations, household debt, collateral constraints.

**JEL CLASSIFICATION.** D84, E32, E44, G01.

### 1. INTRODUCTION

Crises of confidence and household debt have been associated in the public debate with strong fluctuations in house prices and consumption in the 2000s, in particular during the 2007–2009 recession. According to Reinhart and Rogoff (2009, p. xxv) “*private sector borrowing binges can inflate housing [...] prices. [...] Such large-scale debt buildups pose risks because they make an economy vulnerable to crises of confidence [...]*” From a conceptual point of view, Kocherlakota (2010, p. 16) emphasized that “*phenomena like credit market crunches [...] rely on self-fulfilling beliefs about what others will do. [...] Macroeconomists need to do more to explore models that allow for the possibility of aggregate shocks to these kinds of self-fulfilling beliefs.*”

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We contribute to this debate by accommodating the notion of consumer confidence in a dynamic equilibrium framework. The paper develops this notion formally in a sequence of steps. The essence of our approach is intuitive: In specific situations (e.g., for a specific level of debt), an economy may support multiple market clearing prices in the future. Consumers form beliefs about which of the future market clearing prices will prevail, assigning subjective probabilities to any of the equilibrium outcomes. Our main contribution is to provide a method for constructing the set of those subjective probabilities, which are in line with equilibrium on current as well as on future spot markets. This construction determines the extent of exposure to variations in beliefs about future prices.

Given our introductory motivation, we use a model with collateralized debt and endogenous house prices for developing and illustrating this concept. Our analysis shows that the level of household debt determines an economy's vulnerability to a crisis of confidence. It turns out that such a crisis can spur strong fluctuations of house prices and consumption.

Most household debt in developed countries is secured by housing collateral. Fluctuations in the value of housing collateral thereby affect the borrowing opportunities and consumption choices of households. Figure 1 provides suggestive evidence that this matters at the aggregate level: both the relative price of homes and mortgage debt per GDP first increased in the 2000s and then fell substantially.<sup>1</sup> The 29% fall of relative house prices during the 2007–2009 recession has been accompanied by a 4% decrease of consumption per disposable income.<sup>2</sup> Models with endogenous loan-to-value ratios, such as Kubler and Schmedders (2003) or models of leverage cycles surveyed in Fostel and Geanakoplos (2013), would allow for further amplification of the effect of house prices on borrowing opportunities. In the data, however, the average loan-to-value ratio of households, as measured by mortgages per real-estate value in the bottom right panel of Figure 1, has increased temporarily only *after* relative house prices started to fall in the wake of the recession (Justiniano, Primiceri, and Tambalotti (2015)). The timing of the increase in leverage in Figure 1 suggests that higher leverage of households has been a consequence of, rather than a cause for, the 2007–2009 recession.

The empirical evidence in Figure 1 also motivates why we explore the role of changes in non-fundamentals such as consumer confidence to understand the large swings in house prices, debt and consumption in the 2000s. Models of price fluctuations, which rely on shocks to fundamentals like total factor productivity or on credit shocks to exogenous loan-to-value ratios, cannot readily explain the time series displayed in Figure 1. While shocks to fundamentals can generate a fall in asset prices that is three times the

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<sup>1</sup>The peak of the mortgage debt volume in billion US-\$ is in the first quarter of 2008. Since GDP fell during the recession, mortgage debt per GDP in Figure 1 only decreases after the end of the recession.

<sup>2</sup>A similar comovement of debt, house prices, and consumption has been documented across U.S. regions: Mian and Sufi (2011) show that, prior to the recession, household debt increased most in those U.S. regions in which house prices grew more because of inelastic housing supply; and Mian, Rao, and Sufi (2013) find that the slump in consumption in the aftermath of the last recession was more severe in U.S. regions with larger household debt.

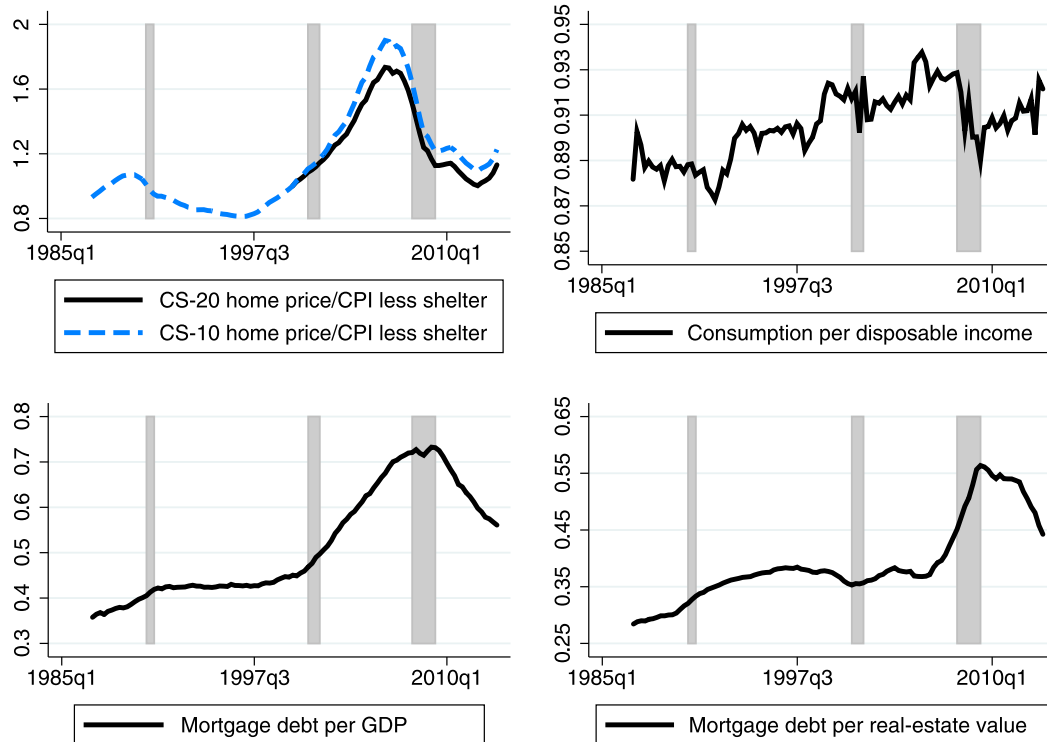


FIGURE 1. House prices, consumption, and mortgage debt in the U.S. Source: Federal Reserve Economic Database. Notes: Shaded areas are recessions dated by the NBER. The relative house price indexes are normalized to unity in the first quarter of 2000. The data series are (series abbreviation in brackets): the Case-Shiller 20-city and 10-city home price index (SPCS20RSA, SPCS10RSA) relative to the consumer price index for all urban consumers and all items less shelter (CUSR0000SA0L2); real consumption expenditures (PCECC96) per disposable income (DSPIC96); mortgage debt of households and nonprofit organizations (HHMSDODNS) per GDP (GDP); mortgage debt (HHMSDODNS) per real estate value held by households and nonprofit organizations (HNOREMQ027S).

size of the fall in output as in the 2007–2009 recession (Glover et al. (2014)), if the intertemporal elasticity of substitution is low enough, it is then challenging to explain that the relative price of homes has *not* fallen (and even increased) in the 2001 recession.

In this paper, we therefore propose changes of consumer confidence as an explanation for the observed fluctuations of house prices and consumption in the 2000s. Figure 2 displays a commonly used empirical measure of consumer confidence provided by the University of Michigan. We are going to argue that changes in consumer confidence, as illustrated by the changes of the index in Figure 2, are irrelevant for an economy with low levels of household debt. If household debt exceeds a critical level, however, changes in confidence cause changes in house prices and consumption. As shown in the lower-left panel of Figure 1, mortgage debt of households has increased since the 1980s. The increase of mortgage debt per GDP accelerated in the 2000s, resulting in a debt level 54%

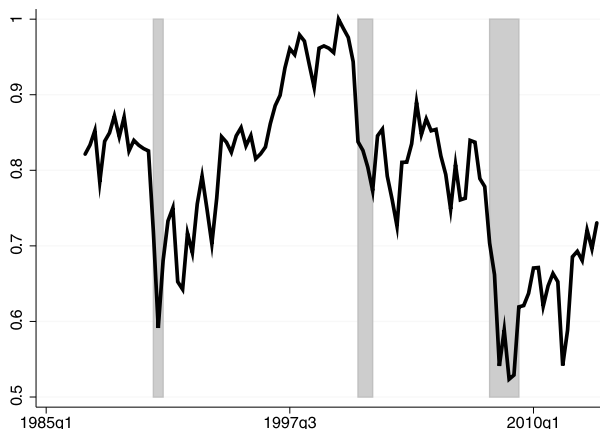


FIGURE 2. Fluctuations in consumer confidence. Source: Federal Reserve Economic Database. Notes: Shaded areas are recessions dated by the NBER. The consumer confidence index is normalized to unity in the first quarter of 2000. The data series is (series abbreviation in brackets) the University of Michigan Consumer Sentiment Index (UMCSENT).

higher at the beginning of the 2007–2009 recession than at the beginning of the 2001 recession.

The debt level is going to play an important role in our model, since it determines the strength of the liquidity feedback effect from house prices to borrowing opportunities. We show that multiple market clearing house prices coexist if this feedback effect is strong enough. Consumer confidence is going to matter in our model since agents form beliefs about these different house prices. We exploit equilibrium consistency requirements to delimit the set of equilibrium beliefs about future house prices, and show how variations within this set generate fluctuations in consumption and current house prices.

Which economic environments make crises of confidence more likely? We will show that temporarily low interest rates<sup>3</sup> and expected income growth provide a breeding ground for confidence-driven crises, extending the interval of debt levels for which such crises can occur, and increasing the range of possible fluctuations, as a larger set of beliefs becomes consistent with equilibrium. We will argue that this may explain why house prices and consumption have fluctuated much more in the 2000s than over previous business cycles, falling strongly during the 2007–2009 recession.

The application we use to illustrate our concept is related to the literature on financial frictions (surveyed by Bernanke, Gertler, and Gilchrist (1999) and Quadrini (2011) and to the seminal paper by Kiyotaki and Moore (1997)). Financial frictions have been combined with specific features, for example, large shocks to the stock of capital or to total factor productivity (Brunnermeier and Sannikov (2014)), or firms having to pledge collateral to finance current production costs (Mendoza (2010)), for achieving quanti-

<sup>3</sup>We consider interest rates exogenous to the mechanisms addressed by the model. This allows for an immediate focus on nontrivial amounts of debt, as relevant in the data, and as key in the concept for having an endogenously evolving state variable.

tatively strong *amplification* of shocks to fundamentals. In contrast, the collateral constraint in our paper justifies the role of variations in confidence as an *independent driving force* of fluctuations, which therefore does not rely on any fundamental shocks to the economy.

An important feature of our model is its liquidity feedback effect.<sup>4</sup> The price of homes depends on the funds available to households and the available funds depend on the liquidity provided by selling homes in the market. Using the terminology of Brunnermeier and Pedersen (2009), market liquidity and funding liquidity feed back into each other. Thus, as in Stein (1995), collateral requirements induce self-reinforcing effects so that multiple house prices may clear the market. Our work is not just about multiplicity, that is, our analysis does not stop at a point where multiple equilibria have emerged. Multiplicity is just an intermediate step for making our main contribution, namely the conceptualization of consumer confidence.

For our main contribution, we introduce multiplicity of prices in a dynamic setting, and at the same time we explicitly consider the state-dependence of equilibrium. The relevant endogenously evolving state in our economy is the level of household debt. Our approach allows us to conceptualize variations in confidence as an equilibrium phenomenon. More specifically, we identify confidence in terms of beliefs about future market prices. The explicit consideration of state-dependence provides for consistency requirements which allow us to determine the set of rationally entertainable beliefs according to the state of the economy. Variations of beliefs within this set generate fluctuations in house prices and consumption that are not related to changes in fundamentals.

This consideration of state-dependent variation of equilibrium beliefs about multiple future prices also distinguishes our approach from models of credit cycles (e.g., Gu et al. (2013) and He, Wright, and Zhu (2015)), models of boom episodes based on multiple steady states (e.g., Caballero, Farhi, and Hammour (2006)), and models with bubbly equilibria (e.g., Kocherlakota (2009) and Wang and Wen (2012)).

It is empirically plausible for the U.S. and across developed countries that macroeconomic fluctuations depend on the level of private debt. Schularick and Taylor (2012) and Jordà, Schularick, and Taylor (2013) showed for a sample of 14 developed economies between 1870 and 2008 that larger credit expansions in the private sector are associated with a higher probability of financial crises, deeper recessions and slower recoveries. Mian and Sufi (2010, 2014) found that the Great Recession was more severe in U.S. regions with larger household debt.

In our approach, the scope for beliefs to independently affect outcomes is state dependent, as the level of household debt determines the range of confidence-driven fluctuations. In comparison, in the conventional approach taken in the literature, surveyed by Benhabib and Farmer (1999), the case for an independent role of beliefs depends only

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<sup>4</sup>The presence of such a mechanism is related to the importance of balance-sheet feedback effects for crises that, for example, Schneider and Tornell (2004) have emphasized. They show how systemic bailout guarantees and borrowing constraints for firms may generate multiple equilibria and self-fulfilling crises in middle-income countries.

on the constant parameters of dynamic economies.<sup>5</sup> In our application, which features state-dependent multiple market clearing prices, such a method is no longer suitable.

The paper by [Heathcote and Perri \(2018\)](#) shares the finding of our paper that the volatility of an economy depends on its wealth level. However, in the equilibrium of their model there are no endogenously evolving state variables (there is no financial asset and housing clears within the type of household considered in their model). They thus construct a special type of sunspot equilibrium, in which they switch between two situations with constant probabilities. Moreover, they focus on constant asset prices and their model does not consider collateral constraints.

Quantitative work on macro-financial topics has created a natural need to deal with the possibility of multiple equilibria which depend on the (financial) state of an economy. For instance, this shows in some of the work on self-fulfilling sovereign debt crises (e.g., [Aguiar and Amador \(2014\)](#) and [Cole and Kehoe \(2000\)](#)). An approach to resolving such situations with multiple equilibria has been to make additional assumptions, which postulate specific—possibly probabilistic and state-dependent—processes for selecting or switching among equilibria.<sup>6</sup> Some prominent recent work in the literature applies particular versions of the previously mentioned approach. In a paper considering bank runs and financial positions, [Gertler and Kiyotaki \(2015\)](#) postulated a specific reduced form function to impose their assumption that the bank run probability depends in a particular way on the recovery rate of liquidation. In a paper considering collateral constraints in an international context, [Schmitt-Grohé and Uribe \(2016\)](#) postulated two versions of particular criteria for selecting a specific equilibrium, if multiple equilibria occur.

Instead of postulating specific probabilities assigned to multiple future equilibria, our approach determines the set of subjective probabilities assignable to multiple future outcomes as an equilibrium object. Our approach thereby endogenizes the extent of uncertainty households are exposed to.

The rest of the paper develops the link between confidence and house prices in the following sequence of steps: In Section 2, we present the model with debt collateralized by houses. In Section 3, we start the backward induction of competitive equilibria and show how it can be applied to characterize multiple equilibrium prices of houses and the corresponding levels of consumption. In Section 4, we continue backward induction from such a situation with multiple equilibria. This involves the key concept of our approach: The characterization of the set of equilibrium beliefs about future prices, as determined by the level of debt. This allows us to analyze the effects of changes in confidence, that is, variations within the set of equilibrium beliefs about future prices, on

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<sup>5</sup>More recently, this approach has been applied to (i) financial markets by [Farmer, Nourry, and Venditti \(2013\)](#) and [Liu and Wang \(2014\)](#), (ii) housing markets by [Kashiwagi \(2014\)](#) and [Mertens and Ravn \(2011\)](#), and (iii) international economics by [Perri and Quadrini \(2018\)](#). An important difference to [Perri and Quadrini \(2018\)](#) is that the collateral value is endogenously determined in our model by a pricing equation.

<sup>6</sup>Some earlier literature, pioneered by [Azariadis \(1981\)](#), relies on the special structure of overlapping-generations (OLG) models to consider a specific type of Markov switching among multiple equilibria. The approach we describe here does not presuppose a particular class of models. For instance, our way of constructing equilibrium beliefs is unaffected in the case of a fixed terminal date of the economy, for example, if truncating the infinite horizon.

consumption and on current house prices. We also discuss restrictions on the comovement of variables, obtained within this framework despite the dependence of outcomes on consumer confidence. Section 5 applies the approach defined in the previous sections to a setting which resembles the macroeconomic environment of the financial crisis along the dimensions relevant for our analysis. It turns out that this setting features the basic ingredients for the making of a confidence-driven house price boom and bust. Section 6 concludes the paper.

## 2. THE MODEL WITH COLLATERALIZED HOUSEHOLD DEBT

The model comprises a terminal stage of infinite duration (with time indices  $t = T, T + 1, T + 2, \dots, \infty$ ) preceded by two periods, namely the intermediate period  $T - 1$ , and the initial period  $T - 2$ . This schedule of the model serves two purposes: First, it reflects the approach of describing the development of equilibrium prices and consumption over time by backward induction, starting from the terminal stage. Second, it disentangles the concept of multiple equilibria, as generated by sufficiently strong liquidity effects in the intermediate period  $T - 1$ , and the concept of confidence, as the weighting of beliefs entertained in period  $T - 2$  over multiple equilibrium outcomes in  $T - 1$ . The debt level of the economy will determine whether multiplicity and confidence play a role. More generally, the analysis relies on the debt level to summarize the relevant state of the aggregate economy for the description of the connections between the periods considered.

In the following, we specify the assumptions made about goods, preferences, and markets. Further down we elaborate on the significance and motivation of particular assumptions. There are two types of goods in the economy, houses and a nondurable consumption good. Houses can be traded every period on a spot market at a price  $p_t$ . The price of nondurable consumption is normalized to one. Houses do not depreciate physically and are in fixed supply normalized to one.

We assume households to be identical. They receive endowments of earnings  $y_t$ , measured in units of nondurable consumption. Households optimize their portfolio holdings of two assets, housing, and a financial asset earning an interest rate  $r_t$ . Earnings  $y_t$  and the interest rate  $r_t$  are allowed to vary over time deterministically, that is, according to a pattern which is known with certainty from the initial period onwards. Households have preferences represented by the sum of present and discounted expected future utilities from nondurable consumption and housing. They maximize an objective taken to be of the form

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} E_t U(c_{\tau}, h_{\tau}), \quad (1)$$

where  $0 < \beta < 1$  is a discount factor, the utility function  $U$  is increasing, concave, and differentiable,  $c_t$  denotes nondurable consumption, and  $h_t$  denotes the quantity of housing owned by the household.



The choices of consumption and of portfolio investment made by households need to satisfy a sequence of budget constraints,

$$a_{t+1} + p_t h_{t+1} + c_t = (1 + r_t) a_t + p_t h_t + y_t, \quad (2)$$

where  $a_t$  denotes financial assets available at the beginning of period  $t$ , which represent a level of debt if they are negative, that is,  $a_t < 0$  is an amount of debt. The right-hand side of (2) collects the resources available to the household at the beginning of period  $t$ : interest-bearing financial assets, the value of the existing quantity of housing, and earnings. The left-hand side of (2) gathers spending on consumption and portfolio investment, in terms of financial assets and the value of housing accumulated for the next period.

In addition to (2), the portfolio choices made in the initial period and in the intermediate period, that is, the portfolio positions  $(a_{t+1}, h_{t+1})$  carried into the next period as chosen at times  $t = T - 2, T - 1$ , have to satisfy the collateral constraint

$$-a_{t+1}(1 + r_{t+1}) \leq \mu p_t h_{t+1}. \quad (3)$$

The collateral constraint (3) requires that debt (i.e., *negative* financial assets) with interest must not exceed the fraction  $0 \leq \mu \leq 1$  of the value of housing units held. Given this specification of the constraint, we refer to the parameter  $\mu$  as a loan-to-value ratio. In the terminal stage ( $t = T, T + 1, T + 2, \dots, \infty$ ), the sequence of budget constraints (2) is complemented just by the condition  $\lim_{t \rightarrow \infty} [a_t / (1 + r)^{t-T}] \geq 0$  to rule out Ponzi schemes. Appendix A.1 derives the Euler equations to characterize the portfolio investment decisions of the household, which maximize objective (1) subject to the constraints (2) and (3).

The following remarks discuss the assumptions made above and their motivation in terms of the essential features of the environment we want to capture. According to the setup specified above, housing conforms to the triple role played by owner-occupied housing: it serves as a durable consumption good in (1), as an asset in (2), and as a collateral item affecting borrowing opportunities in (3). This combination of features makes housing different from other goods.

The assumption of a fixed supply makes housing akin to land, as motivated by the empirical evidence that the price of land accounts for the largest part of house price fluctuations at low and business cycle frequencies (Davis and Heathcote (2007)). Furthermore, house price fluctuations are largest in cities in which zoning laws, geographic scarcity of land and constraints on the infrastructure limit supply (Burnside, Eichenbaum, and Rebelo (2016) and references therein). In particular, within the context of a confidence-induced house price bust, the assumption of a fixed-supply of houses accommodates the persistence of existing housing units and of the zoning development of residential areas these units are built on. This persistence follows from the physical irreversibility of investment in fixed structures for buildings combined with the lack (or ban) of a more profitable use of the corresponding plots of land. This makes the supply side of the housing market virtually fixed over the short run, during which the demand side may be hit by adverse liquidity effects from housing finance in a crisis of confidence.



We focus on owner-occupied housing as the determinant of household borrowing opportunities since data of the Survey of Consumer Finances in the 2000s reveal that the median working-age household in the U.S. owns a home and borrows against its collateral value. Mortgage debt of working-age households accounts for more than 90% of their debt (Hintermaier and Koeniger (2016)). The specification of the collateral constraint (3) in our model captures existing financial regulation and lending practices, which limit the extent of debt-financing by the valuation of collateral. In particular, limits on loan-to-value ratios from financial regulation are specified in terms of the value of collateral at the time of contracting, thus relying on the current price in the collateral constraint. This type of lending practices can just as well be based on an assumption of limited enforcement, when a lender can seize at most the house—losing a fraction  $1 - \mu$  in the process of appropriating it—but no other resources if the household defaults. As in Kiyotaki and Moore (1997), only one-period debt contracts need to be considered if borrowers can repudiate and renegotiate in each period. Lenders therefore ensure that their loan never exceeds the relevant liquidation value of the housing collateral.<sup>7</sup>

The assumption of risk-free interest rates  $r_t$  which are exogenously given—and thus are independent of liquidity effects from the endogenously priced collateral—means that we consider the spot market for the collateral good to be *more local* than some capital market for which market-clearing period-by-period would be plausible. In the context of the U.S. house price boom and bust, this captures the fact that *owner-occupied* houses needed to be bought (or held) by residents of the local U.S. economy. At the same time, the interest rate for financing of home-ownership in the local economy was in line with a world capital market—thus clearing beyond U.S. borders—under conditions which at times were associated with terms such as *global imbalances* or a *global saving glut*. Considering the U.S. economy as a whole, during the buildup of the crisis there is clear empirical support of persistent capital (in)flows. The U.S. current account was in a deficit every year from 1992 until the start of the 2007–2009 recession (and beyond). From 2002 until 2007, the current account deficit constantly exceeded 4% of GDP.<sup>8</sup> The appropriateness of accommodating outside capital flows is naturally reinforced if considering a subgroup of borrowers, and the relevant debt positions, within the U.S. economy.

More generally, our framework is meant to address any situation where the market for a specific durable collateral good needs to clear among a given collection of households who face a common financial friction, involving the price of the collateral good. If the relevant collateral good is identified by a specific regional stratification, a perfectly closed market for land ownership among villagers represents an example of

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<sup>7</sup>The assumption that the current value of collateral appears in the collateral constraint is frequently made in the literature on financial frictions (e.g., Mendoza (2010), Jeanne and Korinek (2010)). It can be rationalized by a setting where the borrower takes actions which imply future default already during the period of loan origination. This might be labeled a *time-to-cheat* assumption. Such a setting is complemented by the assumption that the creditor can observe these actions immediately, appropriate the collateral item and resell it at the current price by the end of the period.

<sup>8</sup>Data reference: OECD, Main Economic Indicators, as provided by Federal Reserve Economic Data (FRED), Series ID: BPBLT01USA188S.

this structure. In this respect, empirical evidence by Mian and Sufi (2010, 2011, 2014) and Mian, Rao, and Sufi (2013) suggests that local housing markets, for U.S. counties, MSAs or ZIP code areas, imply different comovements of house prices, household debt, and consumption. *Even within* a given regional entity, further stratification according to demographic or social features may be necessary to identify those households who participate in the market for the relevant collateral good. For instance, during a housing boom the specific type of new housing units developed and built may be targeted toward, say, young families who are typically collateral constrained when demanding these units. To the extent that older and financially unconstrained households do not share any demand for these specific units (because of their location, layout, size, etc.), such a situation equally fits the assumptions of our model.

Like the above-mentioned sequence of interest rates  $r_t$  also the sequence of earnings endowments  $y_t$  is assumed to be known with certainty from the initial period onwards. This means that there is no uncertainty about the fundamental exogenous variables featured in our assumptions. The only source of uncertainty relevant for the households when forming their expectations is uncertainty about future prices. This uncertainty about prices, which are endogenous variables in our model, is a consequence of multiple equilibria. Our analysis shows how liquidity feedback effects from the collateral constraint give rise to equilibrium multiplicity over some range of debt levels. The belief-weightings of future prices which are *rationally entertainable*—in the sense that they are in line with an equilibrium law-of-motion—will be dependent on the aggregate state of the economy, as described by its financial asset position. We denote the financial asset position in the economy<sup>9</sup> by  $A_t$ . The evolution of this aggregate state variable is governed by the aggregate constraint on motion

$$A_{t+1} = (1 + r_t)A_t + y_t - c_t. \quad (4)$$

This constraint is based on the individual budget constraint (2) and uses the following two implications of the assumptions made in our model: First, all—existing as well as chosen—quantities pertaining to a single type of identical households coincide with aggregate quantities. Second, the aggregate constraint takes into account that the aggregate housing stock must correspond to its fixed supply, which is constant across periods. In Section 3, when we deploy the backward-induction analysis of our model at its terminal stage, we will formally introduce the dependence of equilibrium prices on the aggregate state variable  $A_t$ .

For the quantitative implementation of the approach described in this paper, we assume that the utility function contained in the forward-looking objective (1) takes the standard constant-relative-risk-aversion (CRRA) form

$$U(c_t, h_t) = \frac{\psi(c_t, h_t)^{1-\sigma} - 1}{1-\sigma}, \quad (5)$$

<sup>9</sup>Note that in line with the discussion provided above for the assumptions made about agents and assets in this model, the aggregate financial state  $A_t$  can just as well be interpreted as the financial asset position of a specific type of household: A group who shares demand for a specific type (local or layout-specific, etc.) of housing, all under financial conditions which make collateral constraints equally relevant for them.

TABLE 1. Parameter values for the benchmark case of the model solution.

	Parameters	
Discount factor (annual)	$\beta$	$1/(1+r)$
Weight of $c$ in consumption basket	$\theta$	0.7
Intertemporal elasticity of substitution	$1/\sigma$	1/10
Loan-to-value ratio	$\mu$	0.80
Interest rate (annual)	$r$	0.04
Earnings endowment, GDP (annual)	$y$	1

with a consumption basket

$$\psi(c_t, h_t) = c_t^\theta h_t^{1-\theta}, \quad (6)$$

as composed by the two goods, nondurable consumption and housing. Note that the specification combining (5) and (6) nests the case of separability between the goods, since logarithmic and separable utility is obtained for  $\sigma = 1$ .

For the quantitative illustration of our findings, we will also need to assign values to the parameters of our model. Table 1 displays the parameter values for which we present the solution. The discount factor is in line with a long-run (terminal stage) interest rate of 4%. In our benchmark parameterization, we apply this interest rate constantly in all periods of our model. The weight of nondurable consumption of 0.7 falls in the range of commonly used values, as guided by long-run averages of expenditure shares.

The relatively low elasticity of intertemporal substitution is not required for the main patterns of our findings, concerning multiplicity and confidence-driven outcomes. The correspondingly high degree of risk aversion is familiar from other work on asset pricing with simple (separable) preferences. Like in other work, this parameter is ultimately going to affect the relative magnitudes of fluctuations in asset (house) prices and consumption. Assuming that a fraction of 20% of the collateral is wasted when appropriated by the lender, the credit market operates with a maximum loan-to-value ratio  $\mu = 0.8$ .

In the benchmark case, we assume for simplicity that the earnings endowment is constant. In Section 5, we discuss how expected income changes affect the solution. The annual endowment is normalized at unity here, thus providing for the unit in which the remaining quantities (consumption and asset levels) in the budget constraint are measured. In a small-open-economy interpretation of the model, this normalized endowment unit corresponds to GDP. In other interpretations of the model, which are just as valid, the unit is to be read as the annual endowment of participants in the market for a specific type of dwelling.

In the quantitative solution of our model, we consider the length of a period in the model to be 10 years. This is meant to approximate a realistic time line for investment in owner-occupied housing, for its adjustability as a component of total wealth, and for the maturity of housing debt, while fitting the schedule of our model. This schedule precisely allows for a clear focus on our conceptualization of confidence, for a description in the most transparent environment, where one period is used for sparking multiplicity

of prices, and an earlier period is used for pinning down rational belief-weightings of those future prices.

### 3. MULTIPLICITY OF STATE-DEPENDENT PRICES

#### 3.1 Prices in the terminal stage

The terminal stage serves as our point of departure in a backward-induction approach to characterize house prices and consumption in all periods. During the terminal stage, households are assumed to receive constant earnings  $y_t = y$ , in every period  $t = T, T + 1, \dots, \infty$ . The interest rate on financial asset positions chosen during the terminal stage is assumed to be constant, that is,  $r_{t+1} = r$  for  $t = T, T + 1, \dots, \infty$ . This interest rate and the discount factor  $\beta$  are assumed to satisfy the condition  $\beta = 1/(1 + r)$ . Imposing this condition in the terminal stage with constant endowments allows for the interpretation of  $r$  as the long-run equilibrium (natural) interest rate of this economy.

In Appendix A.2, we show how to derive consumption and asset price functions for the terminal stage. Equilibrium ensures market clearing for housing and consistency of individual financial asset positions and choices with the corresponding aggregate positions. Both consumption and house prices are equilibrium functions of the aggregate financial state  $A_t$ . At the beginning of the terminal stage, that is, in period  $T$ , the consumption function is

$$c_T(A_T) = \frac{r(1 + r_T)}{1 + r} A_T + y, \quad (7)$$

and the house price function is

$$p_T(A_T) = \frac{1 - \theta}{\theta} \frac{c_T(A_T)}{r}. \quad (8)$$

Equation (7) says that consumption is determined by the annuity value of existing financial resources and by the current period earnings endowment. The price function (8) reveals the roles of the relative weight of housing in preferences, as captured by  $1 - \theta$ , and of the real interest rate, which rules the discounting of future (utility) dividends of the infinitely-lived asset housing. Our previously stated assumption, that choices in the terminal period are not subject to a collateral constraint, ensures that the terminal (long-run) price is purged of any influence from financial frictions. Furthermore, our assumptions allowed us to derive closed-form solutions for consumption and prices as functions of the aggregate financial state  $A_T$  of the economy. These closed-form expressions for the terminal stage provide a basis for obtaining results about multiplicity and the role of confidence in an analytic form, as pursued in later sections of this paper, when considering earlier periods of our model.

In the next step, we will rely on these functions for performing backward induction to derive equilibrium house prices and consumption in the preceding period, that is, the intermediate period  $T - 1$ . Having characterized equilibrium in period  $T$  as a function of the aggregate state  $A_T$  is key to determine equilibrium in the preceding period  $T - 1$ : An aggregate equilibrium law of motion connecting from some position of the aggregate

state  $A_{T-1}$  to a specific level of the successor state  $A_T$  needs to be in line with prices (and the corresponding future consumption choices) prevailing at this level of the successor state  $A_T$ .

### 3.2 Backward induction with state-dependent constraint patterns

In the periods preceding the previously discussed terminal stage, the presence of collateral constraints implies feedback effects of conditions in housing finance on the house price. The collateral constraint (3) involves the housing value and the debt level chosen for financing. Since these are endogenous variables of our model, we need to accommodate the determination of the relevant pattern of bindingness of this constraint, that is, identify situations where the collateral constraint is binding and those where it is slack, in every step of our analysis.

In the following, we present the analysis for the intermediate period  $T - 1$  in two parts. First, we will consider optimal choices (of consumption, debt, and housing quantities) at the level of the individual household, taking the relevant sequence of house prices as given. Next, we consider the determination of equilibrium house prices, which needs to ensure that the housing market clears and that expectations of future prices are in line with the equilibrium law-of-motion of aggregate financial assets in the economy.

**3.2.1 Household portfolio choices** The solution of the individual household problem for the intermediate period  $T - 1$  relies on the Euler equations derived in Appendix A.1. These Euler equations characterize the optimization of portfolio choices, including the determination of Lagrange multipliers on the collateral constraint, thus pinning down those states for which the constraint is binding. Household policy functions at time  $T - 1$  are obtained by solving the system consisting of the two Euler equations (29) and (30), and the household budget constraint (2). This system is solved for the unknowns (i.e., consumption, portfolio choices, and the multiplier on the collateral constraint), taking combinations of the household's state variables  $a_{T-1}$  and  $h_{T-1}$  as given, and assuming a given sequence of house prices  $(p_{T-1}, p_T, p_{T+1}, \dots)$ .<sup>10</sup>

Since the optimal household decision is conditional on individual asset holdings and on prices, there are two main ways of illustrating the solution of the household problem: First, we may consider the response of household decisions to variations in its asset position, taking specific prices as given. This is the perspective taken by the corresponding policy functions of the household. Second, we may focus on the relationship between household choices and prices, conditioning on specific asset levels of the household. This is illustrated by (excess) demand functions.

Figure 3 displays the policy functions of the household, considering variations in the financial asset position  $a_{T-1}$  available to the household at the time of decision making. The figure collects functions for the following choice variables (top to bottom): the chosen financial asset position  $a_T$ , consumption  $c_{T-1}$ , and the chosen quantity of housing

<sup>10</sup>The computation of solutions to illustrate household demand policies is based on the method described by Hintermaier and Koeniger (2010), for solving portfolio choice problems with occasionally binding constraints.

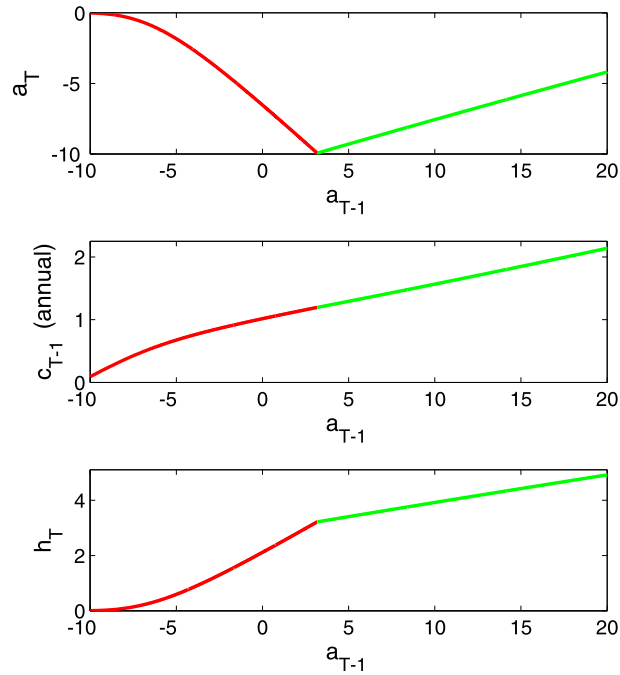


FIGURE 3. Policy functions of the household for a given house price,  $p_{T-1} = 5.72$ , assuming a constant house price of 7.42 in all later periods, and a given housing position of  $h_{T-1} = 1$ . Notes: Constrained household if dark gray (red color), unconstrained household if light gray (green color). Financial assets and consumption are measured in units of annual earnings. The underlying parameter values are specified in Table 1.

$h_T$ . The assumptions of our model ensure regularity conditions of the recursive optimization problem, which imply that the household's decision variables are unique and continuous functions of the household's state variables. The figure reveals that the collateral constraint is binding for low levels of financial assets, as depicted by the dark gray (red) parts of the functions, and that the collateral constraint is slack for higher levels of financial assets, as depicted by the light gray (green) parts of the functions.<sup>11</sup> At the asset level where the bindingness pattern of the constraint changes, the functions for portfolio choices of financial assets and housing have a kink.

Figure 4 illustrates the excess demand function for housing. Under our assumption of homogeneity, this coincides with the difference between housing demand of the representative household and (fixed) housing supply.<sup>12</sup> A root of this excess demand func-

<sup>11</sup>To facilitate reading of a black and white printout of our paper, we use the term *dark gray* while also mentioning the color (*red*) in brackets, and we use the term *light gray* while also mentioning the color (*green*) in brackets.

<sup>12</sup>This illustration of the problem implicitly features housing demand and supply. This provides for a link to alternative settings with price-elastic housing supply, as these—depending on a particular way of modeling and parameterizing such a setting—may equally display the relevant nonmonotonicities present in this illustration. We would like to thank a referee for pointing us to such alternatives, which made us spot this link.

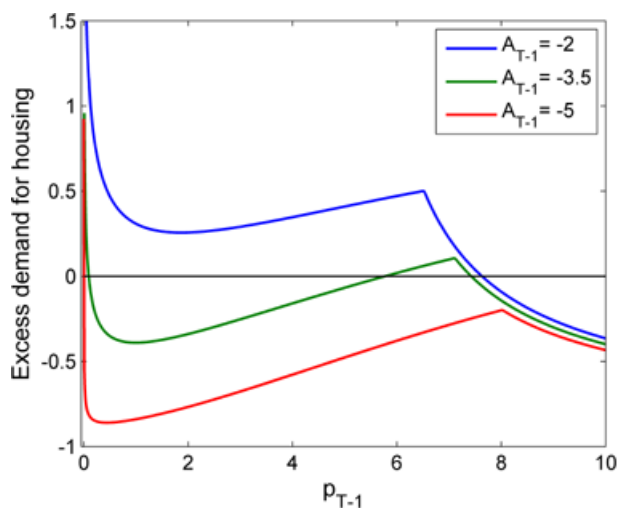


FIGURE 4. Excess demand function for housing in period  $T - 1$ , plotted for different levels of debt, assuming a constant house price of 7.42 in all later periods. Notes: The level of financial assets conditioned on is measured in units of annual earnings. The underlying parameter values are specified in Table 1.

tion thus corresponds to housing market clearing, that is,  $h_T = 1$ . The figure shows excess demands conditional on various levels of the aggregate financial asset position, which likewise coincides with the financial asset position of the representative household. The figure reveals that there may be multiple market clearing prices for housing. The multiplicity of equilibrium prices is a consequence of a liquidity feedback effect of price changes. This effect operates through the collateral constraint and may make the demand function upward-sloping over a specific range. The figure also shows that the possibility for multiple equilibria depends on the financial asset position. The financial asset position is crucial for pinning down the position of the excess demand curve. It thereby determines whether the upward-sloping parts of the demand curve have a bite on equilibrium intersections. In Figure 4, only the excess demand function which is conditional on the depicted middle level of debt (corresponding to a financial asset position of  $-3.5$ , as measured in units of annual earnings) has three market clearing house prices. The two other excess demand functions (conditioning on financial asset positions of  $-2$  and  $-5$ ) have just one market clearing house price each.

**3.2.2 State-dependent competitive equilibrium prices** The characterization of equilibrium prices and the corresponding levels of consumption relies on a set of conditions required to hold in a competitive equilibrium. The following definition specifies these conditions.

**DEFINITION 1.** A competitive equilibrium for a given aggregate state of the financial asset position  $A_t$  involves household choices, an equilibrium level of the house price  $p_t$ , and an equilibrium law-of-motion, which connects  $A_t$  to a successor-state  $A_{t+1}$ . These need to satisfy the following requirements:



(a) Households optimize portfolio choices of financial assets  $a_{t+1}$  and housing  $h_{t+1}$ . The optimization is based on a given household financial asset position  $a_t$ , which corresponds to the aggregate position  $A_t$  under consideration, and on a given housing position  $h_t$ , corresponding to the fixed aggregate housing quantity.

(b) Households take the current house price  $p_t$  as given.

(c) Households form expectations about the next-period house price  $p_{t+1}$ .

Household choices at an equilibrium level of the house price  $p_t$ , using their expectations formed about the next-period house price, have to satisfy the following two conditions:

(d) The housing market clears, such that  $h_{t+1} = 1$ .

(e) Household choices imply an aggregate successor-state  $A_{t+1}$  for which the expectations formed about the next-period price are justified by the next-period state-dependent equilibrium. This requirement of consistency between the price expectations formed and the implied successor state, which needs to support these price expectations, identifies an equilibrium law-of-motion, connecting the current aggregate state  $A_t$  to a successor-state  $A_{t+1}$ .

Some remarks on this definition are meant to highlight features which are key for the application to the present model. The definition includes the standard features of decentralized optimization, price taking, and market clearing. It is based on the consideration of a specific aggregate (financial) state  $A_t$ , to build a requirement of consistency with an aggregate successor-state  $A_{t+1}$ . This link between successive states of the economy makes the concept amenable to backward induction, which is going to be key for the analysis of our model.

The formation of price expectations is in line with the market arrangements of our decentralized economy. It is relevant for the households since they participate in asset markets. Their investment activities take into consideration the price prevailing at their next (period) occasion to adjust portfolios, that is, by trading on the next-period spot market, on which they are price takers.

The definition requires an equilibrium law-of-motion to be *consistent with* expectations of prices prevailing at a successor-state. The concept thus can accommodate situations when there is more than one equilibrium law-of-motion from a given current aggregate (financial) state  $A_t$  to some successor-state  $A_{t+1}$ . This feature plays a role in characterizing multiple state-dependent recursive equilibria of our economy below. By considering the formation of price expectations, and requiring them to be supported by a consistent successor-state, the definition can equally accommodate situations where multiple prices are supported by a successor-state. In such a context, the formation of expectations captures nonfundamental uncertainty about multiple competitive equilibrium outcomes. This feature will play a role in Section 4, when we continue backward induction from a situation which already features multiplicity.

This approach allows for the characterization of equilibria in dynamic economies, without excluding the possibility of multiplicity of equilibria, which in turn may depend on the state of the economy. Put differently, it allows for the recursive characterization

of competitive equilibria by backward induction, *without* being confined to situations where prices and the equilibrium law-of-motion are unique *functions* of the aggregate state. When we apply the approach to the model below, it will turn out that over *specific* ranges of the current aggregate state  $A_t$  of the economy there is *just one* equilibrium law-of-motion, which is *consistent with* price expectations supported by some successor-state  $A_{t+1}$ . The approach therefore results in *verifying* uniqueness of state-dependent equilibrium, rather than *codifying* uniqueness as part of the concept.

We complement Definition 1 by the following Definition 2, which introduces a fixed terminology to refer to types of equilibria, as distinguished by the pattern of bindingness of the collateral constraint.

**DEFINITION 2.** The term *unconstrained equilibrium* refers to an equilibrium which satisfies the requirements of Definition 1 for a house price and portfolio choices which imply that the collateral constraint is slack. The term *constrained equilibrium* refers to an equilibrium which satisfies the requirements of Definition 1 for a house price and portfolio choices which imply that the collateral constraint is binding.

This distinction of equilibria by their constraint pattern will be convenient for describing how to characterize all equilibria. The pattern of bindingness of the collateral constraint is an endogenous object in our model, since the price  $p_t$  and the portfolio choices  $(a_{t+1}, h_{t+1})$  involved in the collateral constraint (3) are endogenous to the equilibrium of the economy.

We implement the optimality of household choices required in Definition 1 by relying on the Euler equations (29) and (30), derived in Appendix A.1. The requirement of housing market clearing is taken into account by imposing the fixed aggregate housing quantity in all places where housing appears in these Euler equations. Next-period consumption in the Euler equations is required to be in line with price expectations, which in turn have to be supported by the next-period financial state. Imposing these requirements and the functional form of utility according to equations (5) and (6), equilibrium in the intermediate period  $T - 1$  is characterized by the following collection of relationships:

$$\theta c_{T-1}^{\theta(1-\sigma)-1} = \beta(1+r_T)\theta c_T^{\theta(1-\sigma)-1} + \kappa_{T-1}(1+r_T), \quad (9)$$

$$\theta c_{T-1}^{\theta(1-\sigma)-1} p_{T-1} = \beta((1-\theta)c_T^{\theta(1-\sigma)} + p_T \theta c_T^{\theta(1-\sigma)-1}) + \kappa_{T-1} \mu p_{T-1}, \quad (10)$$

$$A_T = (1+r_{T-1})A_{T-1} + y_{T-1} - c_{T-1}, \quad (11)$$

$$-A_T(1+r_T) \leq \mu p_{T-1}, \quad (12)$$

$$\kappa_{T-1} \geq 0, \quad (13)$$

$$\kappa_{T-1} \cdot (-A_T(1+r_T) - \mu p_{T-1}) = 0. \quad (14)$$

Equation (9) corresponds to the Euler equation for financial investment and equation (10) to the Euler equation for housing investment. In these equations, two endogenous next-period variables appear, namely the next-period price  $p_T$  and the corresponding

next-period level of consumption  $c_T$ . According to the definition of competitive equilibrium, these next-period variables need to be supported by the next-period aggregate financial state  $A_T$ . At this stage of performing backward-induction of state-dependent competitive equilibria in period  $T - 1$ , we can rely on the equilibrium price *function* obtained for the terminal stage. Any aggregate successor-state  $A_T$ —which is a result of choices by homogeneous households, given any price-expectation they form—supports just one price  $p_T(A_T)$ , as expressed in equation (8), and one corresponding level of consumption  $c_T(A_T)$ , as expressed in equation (7). Since in the intermediate period  $T - 1$  an equilibrium law-of-motion to  $A_T$  pins down the relevant next-period price, rational price-expectations in this period are fully taken care of by the successor-state  $A_T$ . The definition of equilibrium does involve this successor-state. Therefore, in  $T - 1$  no further consideration of expectations in the Euler equations is required, once equilibrium has been imposed.

Equation (11) applies the aggregate constraint on motion (4) to the variables relevant in the  $T - 1$  equilibrium problem. Inequality (12) imposes housing market clearing,  $h_{t+1} = 1$ , and the fact that the homogeneous household financial position coincides with the aggregate financial position, on the collateral constraint (3). The sign restriction on the multiplier of the inequality constraint is specified in (13). Equation (14) states the complementary slackness condition of the inequality-constrained portfolio choice problem, with equilibrium restrictions imposed.

In our economy, and in particular depending on the financial state  $A_{T-1}$  prevailing in this economy, two different types of equilibria may exist, as distinguished in Definition 2 by their pattern of bindingness of the collateral constraint. The equilibrium conditions described above, which involve a complementary slackness condition, accommodate both types of equilibria. *Unconstrained equilibria* are obtained by considering the equilibrium conditions for the case where the complementary slackness condition (14) is satisfied by the combination of

$$\kappa_{T-1} = 0 \quad \text{and} \quad -A_T(1 + r_T) \leq \mu p_{T-1}, \quad (15)$$

whereas<sup>13</sup> *constrained equilibria* are obtained by considering the equilibrium conditions for the case where (14) is satisfied by the combination of

$$\kappa_{T-1} \geq 0 \quad \text{and} \quad -A_T(1 + r_T) = \mu p_{T-1}. \quad (16)$$

Constructing all equilibria boils down to finding all combinations of a current aggregate financial state  $A_{T-1}$ , consumption  $c_{T-1}$ , a house price  $p_{T-1}$ , the value of a multiplier  $\kappa_{T-1}$  (for constrained equilibria), and an aggregate successor-state  $A_T$ , such that any equilibrium combination of these variables satisfies the required conditions, as characterized above. The successor-state  $A_T$  pins down the next-period variables which show up in the Euler equations for the characterization of both types of equilibria. The next-period price  $p_T(A_T)$  is pinned down according to function (8), and the corresponding level of consumption  $c_T(A_T)$  is pinned down according to function (7).

<sup>13</sup>The limiting case which combines a multiplier of just zero and a collateral constraint holding just with equality is covered by both branches of this equilibrium characterization.

The following Proposition 1, derived in Appendix A.3, lists the elements of a prescription to construct all equilibria, which depend on the financial state  $A_{T-1}$  of our economy.

**PROPOSITION 1.** (i) *Existence of either type of equilibrium, unconstrained or constrained, can be identified in terms of the successor-state  $A_T$  to be reached by an equilibrium law-of-motion.*

(ii) *Equilibrium combinations of variables  $(A_{T-1}, c_{T-1}, p_{T-1}, \kappa_{T-1}, A_T)$  for both types of equilibria, unconstrained and constrained, can be constructed by using a successor-state  $A_T$  as the starting point of such a construction, and then using an appropriate composition of functions to determine the corresponding current state  $A_{T-1}$ , as well as the other variables involved in some equilibrium combination.*

*For unconstrained equilibria, with  $\kappa_{T-1} = 0$ , these functions are composed as follows:*

$$\begin{aligned} c_{T-1} &= [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} c_T(A_T), \\ A_{T-1} &= \frac{A_T + c_{T-1} - y_{T-1}}{1+r_{T-1}}, \\ p_{T-1} &= \frac{(1-\theta)c_T(A_T)^{\theta(1-\sigma)} + p_T(A_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}}{(1+r_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}}. \end{aligned}$$

*For constrained equilibria, these functions are composed as follows:*

$$\begin{aligned} p_{T-1} &= \frac{-A_T(1+r_T)}{\mu}, \\ \kappa_{T-1} &= \frac{\beta}{1+r_T-\mu} \left\{ \frac{(1-\theta)}{p_{T-1}} c_T(A_T)^{\theta(1-\sigma)} + \theta c_T(A_T)^{\theta(1-\sigma)-1} \left[ \frac{p_T(A_T)}{p_{T-1}} - (1+r_T) \right] \right\}, \\ c_{T-1} &= \left\{ \frac{1+r_T}{\theta} (\beta\theta c_T(A_T)^{\theta(1-\sigma)-1} + \kappa_{T-1}) \right\}^{\frac{1}{\theta(1-\sigma)-1}}, \\ A_{T-1} &= \frac{A_T + c_{T-1} - y_{T-1}}{1+r_{T-1}}. \end{aligned}$$

(iii) *There is a lower bound  $\underline{A}_T$  for the successor-state, such that all  $A_T \geq \underline{A}_T$  are part of an unconstrained equilibrium combination and all negative  $A_T \geq \underline{A}_T$  are also part of a constrained equilibrium combination.*

The main significance of Proposition 1 is that it delivers a prescription for constructing all equilibria of this economy. It is fit for application to the relevant dynamic setting, where equilibria depend on the current financial state of the economy, and where future prices need to be supported by future financial states which are in line with current household behavior. The construction employs the distinction of types of equilibria, according to the bindingness of the collateral constraint. The two types of equilibria, unconstrained and constrained, share common principles in their construction. The following principles turn out to be particularly useful in our analysis.

First, the construction is based on successor-states,<sup>14</sup>  $A_T$ , that is, financial states in the next period. The existence of either type of equilibrium can be verified by relying on successor-states only. This is particularly suitable for a backward-induction approach, since all successor-state-dependent properties—as decisive for whether or not a specific successor-state can be part of an equilibrium—are naturally available as outcomes of a previous stage of backward induction.

Second, all values of variables in an equilibrium can be constructed by relying only on a composition of *functions*. This will be particularly relevant in our application, where it is possible that multiple equilibria exist for some current financial states  $A_{T-1}$ . The construction specified in Proposition 1 is unaffected by—and hence robust to—this possibility. Departing from the equilibrium range of successor-states  $A_T$ , it traces out all equilibria by just applying functions. When viewing the outcome traced by this construction from the perspective of a fixed current financial state  $A_{T-1}$ , it may well be possible that instances sampled from this outcome at a specific  $A_{T-1}$  correspond to multiple equilibria.

Therefore, our approach handles multiplicity of state-dependent equilibria as a perfectly legitimate *result*, to be detected by switching the axes of the current state and of the successor-state of the economy. This *backward-construction* based on *successor-states*  $A_T$  avoids any need for amendments of the analysis—such as launching a *precautionary* pursuit (typically nontrivial) of multiple equilibrium solutions at *given* levels of the *current* state  $A_{T-1}$ —when uniqueness of equilibrium is not warranted, as might be the case in situations with financial frictions. Our specific way of implementing *backward-induction* of state-dependent equilibria distinguishes itself by applying functions in a *backward-construction* based on the value of a *successor-state* involved in an equilibrium combination of variables. This makes our construction immune to the possibility of nonuniqueness.

Relying on compositions of known functions to characterize all equilibria is also key for obtaining analytic results. For instance, further down, the condition for multiplicity of equilibria in Proposition 2 exploits the derivatives of these functions, calculated separately for the unconstrained and constrained branches of equilibria.

Figure 5 displays the results for the state-dependent equilibria in the intermediate period  $T - 1$ , using the parameter values specified for our economy in Table 1. It shows the *equilibrium curves* obtained by applying the composition of functions according to Proposition 1. Branches of these equilibrium curves which are obtained for an unconstrained case (i.e., where the collateral constraint is not binding in equilibrium) are drawn in light gray (green), branches obtained for a constrained case (i.e., where the collateral constraint turns out to be binding in equilibrium) are drawn in dark gray (red). The figure considers financial states  $A_{T-1}$  of the economy ranging from *debt* (i.e., *negative* financial assets) at a level of six annual earnings up to a financial asset position of one annual endowment of earnings. The top panel of Figure 5 shows the combinations

<sup>14</sup>Gains in efficiency from addressing dynamic optimization problems by solutions based on the next-period state have been pointed out by Carroll (2006). In our competitive equilibrium problem, the construction of equilibria based on next-period states becomes key for identifying state-dependent multiple equilibrium prices and, even more, for characterizing the set of equilibrium price expectations.

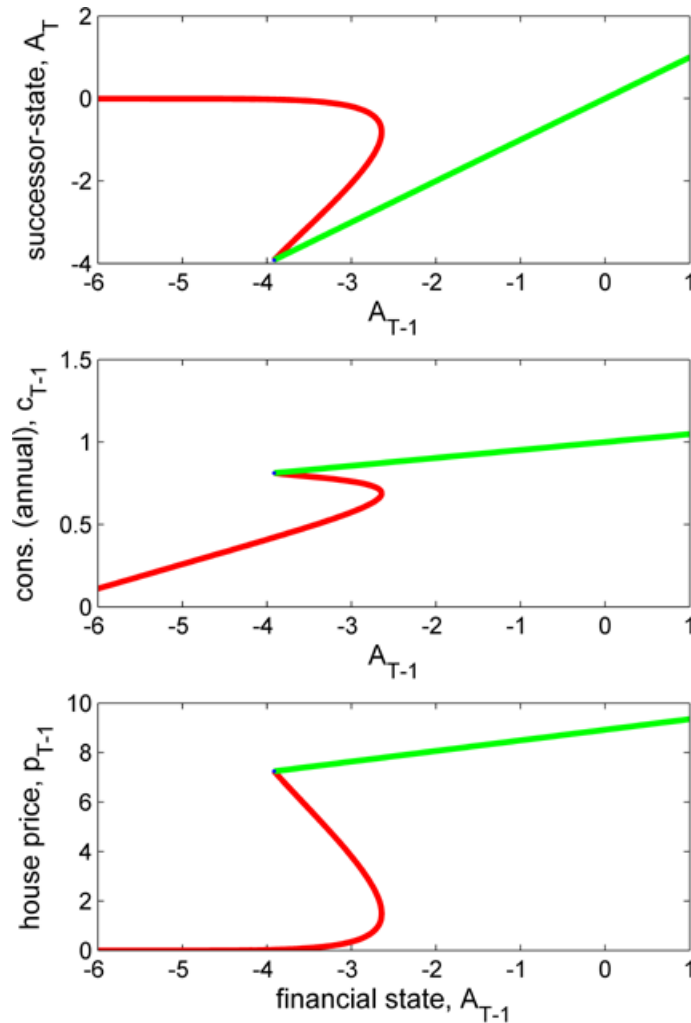


FIGURE 5. Equilibrium curves in period  $T - 1$ . Constrained equilibrium in dark gray (red color); unconstrained equilibrium in light gray (green color). Notes: Financial assets and consumption are measured in units of annual earnings. The underlying parameter values are specified in Table 1.

of  $A_{T-1}$  and the corresponding financial successor-state  $A_T$ , which are in line with an equilibrium law-of-motion for some type of equilibrium, unconstrained or constrained. This panel illustrates the fact that—once equilibria are distinguished by whether or not they imply a binding collateral constraint—equilibrium combinations of variables are traced out by *functions* of the *successor-state*  $A_T$ . Viewing these combinations from the angle of  $A_{T-1}$ , as is done naturally when assigning the horizontal axis in Figure 5 to the current state of the economy, reveals that financial states  $A_{T-1}$  in a specific intermediate range are combined with multiple equilibrium laws-of-motion to successor-states  $A_T$ .

This pattern of state-dependent equilibrium multiplicity carries over to the level of consumption in the economy and to the house price, as displayed in the two bottom

panels of Figure 5. For strongly negative financial asset positions  $A_{T-1}$ , there is a unique equilibrium which turns out to involve a binding collateral constraint. For sufficiently high financial asset positions, there is also a unique equilibrium, which implies that the collateral constraint is slack there. Over an intermediate range of negative financial asset positions multiple equilibria exist. For those intermediate levels of debt, there are three equilibria in our economy: There is an unconstrained equilibrium with high aggregate consumption demand and high house prices. The same debt levels in our economy also support two constrained equilibria, which are combined with lower consumption levels and lower prices in the housing market.

To gain further intuition for this coexistence of equilibria, consider the link between state-dependent equilibria in Figure 5 and the underlying excess demand function for housing in Figure 4. For instance, for a fixed aggregate financial state  $A_{T-1}$  of  $-3.5$  annual endowments the excess demand function in Figure 4 has three roots, which correspond to multiple market-clearing house prices. This pattern of equilibrium multiplicity is reflected in Figure 5, where that fixed financial state  $A_{T-1}$  of  $-3.5$  supports three intersections with the equilibrium price curve, combined with three corresponding intersections with the equilibrium consumption curve.

In the unconstrained equilibrium combination, a high house price  $p_{T-1}$  is in line with a high level of consumption  $c_{T-1}$ . In order to finance this level of consumption, households use high debt (i.e., a low level of financial assets  $A_T$  in the portfolio chosen at  $T-1$ ). In the unconstrained equilibrium, the price is high enough to leave slack in the collateral constraint at this high debt level. When the collateral constraint is binding, a liquidity effect comes into play: In any constrained equilibrium, the price of the housing asset is low. Such a low house price enters the collateral constraint, which transmits a low collateral asset price to low liquidity for funding consumption  $c_{T-1}$ . The strength of this liquidity effect is reflected by the value of the multiplier  $\kappa_{T-1}$  in the conditions for equilibrium portfolio choices of both assets. In equation (9), the multiplier matches the distortion of the intertemporal allocation of consumption, stemming from a liquidity constrained choice of  $c_{T-1}$ . Equations (9) and (10) can be combined to solve for

$$p_{T-1} = \frac{\beta((1-\theta)c_T^{\theta(1-\sigma)} + p_T\theta c_T^{\theta(1-\sigma)-1})}{\beta(1+r_T)\theta c_T^{\theta(1-\sigma)-1} + \kappa_{T-1}(1+r_T-\mu)}. \quad (17)$$

Equation (17) closes the feedback loop, rationalizing the collateral asset price which introduced the liquidity effect: A low house price is justified by an adequate value of the multiplier  $\kappa_{T-1}$  in a constrained equilibrium.

The construction of equilibria according to Proposition 1 traces out equilibrium curves entirely and, therefore, already allows for the identification of all—possibly multiple—equilibria for any given current financial state  $A_{T-1}$  of the economy. Nevertheless, it may be convenient to complement such an outright construction of all equilibria with a criterion to compactly check the parameters of the economy for their potential to entail equilibrium multiplicity. The following Proposition 2, derived in Appendix A.4, supplies a sufficient condition for the equilibrium curves to become backward bending over some range of the financial state variable  $A_{T-1}$ .



PROPOSITION 2. *There is a range of financial states  $A_{T-1}$  of the economy for which unconstrained and constrained equilibria with distinct prices coexist, if the parameters of the economy satisfy the following condition:*

$$\frac{r[\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}}}{1+r_T-\mu} \left[ \frac{\theta(1+r_T)^2}{(\theta(1-\sigma)-1)(1+r)} \left( 1-\sigma + \frac{1+r_T}{\mu(1-\theta)} \right) - \mu \frac{1+r_T}{1+r} \right] > -1. \quad (18)$$

The criterion from Proposition 2 implies that the coexistence of unconstrained and constrained equilibria does not hinge on assumptions<sup>15</sup> about the separability of preferences. In particular, our specification of preferences by the functions (5) and (6) nests the case of utility being separable in housing and nonhousing consumption. This separable case is obtained by setting the curvature parameter  $\sigma = 1$ , corresponding to logarithmic utility. For this special case, the sufficient condition from Proposition 2 simplifies to

$$\frac{r}{1+r_T-\mu} \left[ -\frac{\theta(1+r_T)^2}{\mu(1-\theta)} - \mu \right] > -1.$$

The inequality in (18) is qualified as a *sufficient* condition for two reasons. First, it ensures multiplicity of a specific type, namely an overlap—as viewed from the axis of the current financial state  $A_{T-1}$ —of the unconstrained and unconstrained branches of equilibrium curves. Second, it performs a check based on derivatives evaluated at a specific point, namely the so-called *kink*, where the unconstrained and constrained branches connect.

Figure 6 illustrates how the scope for equilibrium multiplicity depends on the loan-to-value ratio  $\mu$ . It shows the region of parameter combinations for  $\mu$  and  $\sigma$ , for which the sufficient condition in (18) holds, as delimited by the dashed line. Figure 6 also displays the region for which *any type* of multiplicity obtains for some financial states of the economy. This broader region captures more general patterns of coexistence than condition (18), for example, with backward-bendingness *not* directly at the point where unconstrained and constrained branches connect, and is delimited by a solid line. On the one hand, Figure 6 shows that a loan-to-value ratio sufficiently close to zero rules out multiplicity. This is quite intuitive, since  $\mu = 0$  corresponds to an exogenous debt limit, breaking the liquidity feedback mechanism between the endogenous house price and the critical financial constraint. On the other hand, loan-to-value ratios sufficiently close to one entail multiple equilibria for a broad range of preferences. Figure 6 also confirms that our benchmark case with  $\mu = 0.8$  and  $\sigma = 10$  is comfortably located in the interior of the multiplicity region. This means that the relatively low intertemporal elasticity of substitution would not necessarily be required for having multiplicity over *some* range of the financial state  $A_{T-1}$ . This parameterization is constructive for having multiplicity over a *specific* reasonable range of  $A_{T-1}$  in our stylized economy.

<sup>15</sup>We also checked that the qualitative properties of our model, such as backward-bendingness of equilibrium curves, are unchanged if the terminal price function is constantly zero for all financial states  $A_T$ . This would be the case in an alternative setup, which just considers three periods, letting the economy literally terminate in period  $T$ , instead of letting it enter a terminal stage of infinite duration. The analysis for such an alternative setup proceeds by a specific modification of the proofs we provide here, replacing the period  $T$  equilibrium price and consumption functions with those applicable in such an alternative setup.

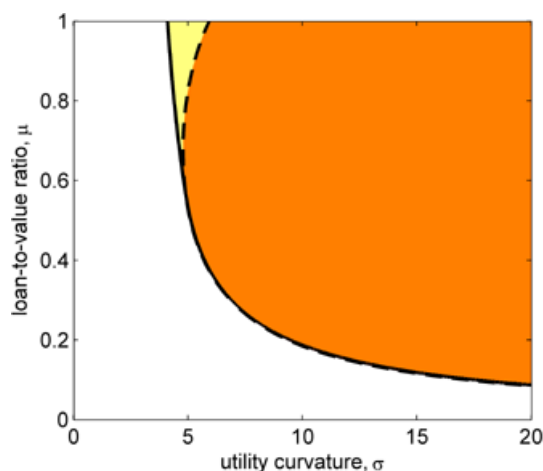


FIGURE 6. Combinations of the loan-to-value ratio  $\mu$  and utility curvature  $\sigma$ , leading to multiplicity in the following sense: In this region of the parameter space, unconstrained and constrained equilibria coexist over some range of  $A_{T-1}$ . The dark (orange) subregion delimited by the dashed line depicts those parameters for which this goes along with backward-bendingness at the kink, as analytically characterized by condition (18) in Proposition 2. Note: The remaining parameters values are as specified in Table 1.

#### 4. CONFIDENCE AND STATE-DEPENDENT PRICES

This section continues our backward-induction approach, relying on the state-dependent outcomes from the intermediate period  $T - 1$  to infer market equilibria in the initial period  $T - 2$ . At this stage, our approach needs to address the possibility of multiple equilibria in  $T - 1$ . Definition 1 accommodates this situation. Applied to the initial period  $T - 2$ , it requires an equilibrium law-of-motion from a given current state  $A_{T-2}$  of the economy to a successor-state  $A_{T-1}$  to be consistent with price *expectations*. Since multiple next-period prices may be equally in line with equilibrium of this economy, we refer to these expectations by the notion of *confidence* in future outcomes. More specifically, we conceptualize confidence by belief-weightings of potential equilibrium outcomes in the future.

The goal of this section is to fully characterize equilibria for all financial states  $A_{T-2}$  of the economy in  $T - 2$ . Such a full characterization now needs to take into account the combination of two objects: the aggregate state of the economy and confidence of households living in this economy. The *backward-construction* approach, already employed above in the intermediate period  $T - 1$ , turns out to be key for handling this *combination* of equilibrium objects. It allows us to effectively *separate* the construction of equilibrium belief-weightings from the construction of all remaining equilibrium variables. The crucial link of this construction is provided by the fact that price expectations—even if they may reflect confidence unrelated to fundamentals—need to be supported by an equilibrium successor-state. Therefore, in the characterization of equilibria in  $T - 2$  the corresponding successor-state  $A_{T-1}$  serves as the point of departure. It first allows us to identify all equilibrium belief-weightings, and then—equipped

with any equilibrium belief-weighting—lets us infer all other variables involved in an equilibrium, including the corresponding current financial state  $A_{T-2}$  of the economy.

#### 4.1 Equilibrium confidence

The elements of Definition 1 provide the framework for constructing all equilibria in period  $T - 2$ . The construction implements the optimization of household portfolio choices and housing market clearing, to identify equilibrium laws-of-motion for the aggregate financial state. The required element of optimized choices of financial assets and housing is captured by the corresponding two Euler equations, derived in Appendix A.1. With housing market clearing imposed, and using the specification of preferences in (5) and (6), these Euler equations are:

$$\theta c_{T-2}^{\theta(1-\sigma)-1} = \beta(1 + r_{T-1})E_{T-2}[\theta c_{T-1}^{\theta(1-\sigma)-1}] + \kappa_{T-2}(1 + r_{T-1}), \quad (19)$$

$$\theta c_{T-2}^{\theta(1-\sigma)-1} p_{T-2} = \beta E_{T-2}[(1 - \theta)c_{T-1}^{\theta(1-\sigma)} + p_{T-1}\theta c_{T-1}^{\theta(1-\sigma)-1}] + \kappa_{T-2}\mu p_{T-2}. \quad (20)$$

Applying the constraint on the motion of the aggregate financial state from (4) to the variables relevant for equilibrium in period  $T - 2$ , we have

$$A_{T-1} = (1 + r_{T-2})A_{T-2} + y_{T-2} - c_{T-2}. \quad (21)$$

Equation (21) links the current financial state  $A_{T-2}$  to the choices of current consumption and of asset accumulation, as embodied in equations (19) and (20).

The previous conditions need to be complemented by the following:

$$-A_{T-1}(1 + r_{T-1}) \leq \mu p_{T-2}, \quad (22)$$

$$\kappa_{T-2} \geq 0, \quad (23)$$

$$\kappa_{T-2} \cdot (-A_{T-1}(1 + r_{T-1}) - \mu p_{T-2}) = 0. \quad (24)$$

Inequality (22) states the collateral constraint (3), having imposed housing market clearing and the equilibrium congruence of individual and aggregate financial assets. Inequality (23) states the sign restriction on the multiplier of this constraint. Equation (24) represents the complementary slackness condition, allowing for an endogenous pattern of bindingness of the collateral constraint in equilibrium.

There is a key difference between the equilibrium conditions presented earlier for the intermediate period  $T - 1$ , and those just listed now for the initial period  $T - 2$ . In period  $T - 1$ , once equilibrium had been imposed on the economy, expectations about the next-period price and the corresponding optimal choice of next-period consumption were *degenerate* and, therefore, redundant. That was explained by the property that any next-period aggregate financial state, brought about by some equilibrium law-of-motion, could only give rise to a *unique* market price in the next period. That property, in turn, was a consequence of two features: First, in our economy there is no uncertainty about any of the exogenous variables (i.e., *no fundamental uncertainty*) which could cause any further variation of next-period prices, on top of the variation explained by

the endogenous next-period aggregate financial state. Second, in relation to that state, the next-period price  $p_T$  relevant for equilibrium choices in period  $T - 1$  is *uniquely* determined as a function  $p_T(A_T)$  by (8), and so is consumption by the function (7).

However, when pursuing another step of backward induction to characterize equilibrium in period  $T - 2$ , such uniqueness of state-dependent next-period prices  $p_{T-1}$  in terms of successor-states  $A_{T-1}$  is not assured anymore. In fact, we already know from the previous step that financial states  $A_{T-1}$  in some range do support *multiple* market-clearing prices in  $T - 1$ . The construction of all equilibria in this situation therefore needs to consider the formation of expectations about multiple future outcomes.

These expectations become part of the requirements for equilibrium consistency by the following principle: The corresponding weighting of beliefs about future prices and the corresponding levels of consumption need to be in line with all variables involved in an equilibrium combination. Therefore, the equilibrium conditions imply restrictions to pin down the set of admissible belief-weightings. The extent to which such a set allows for variations in beliefs determines the scope for *nonfundamental uncertainty*, in terms of *nondegenerate* expectations justified by multiple future equilibria.

Backward induction suggests a way of taking advantage of the state-dependent setting to delimit the scope for equilibrium-consistent variations in beliefs: In order to satisfy the consistency requirements of Definition 1, the specific next-period consequences which are subject to some variation of belief-weightings must be supported by the successor-state reached by an equilibrium law-of-motion. The following Proposition 3, derived in Appendix A.5, shows that a prescription to construct *all*—possibly belief-driven—equilibria in period  $T - 2$  can be *built on* financial successor-states  $A_{T-1}$ . The crucial feature of this construction is its separation into parts. Using the next-period state of the economy as an anchor permits to separate the identification of equilibrium belief-weightings from the determination of all remaining variables involved in an equilibrium combination.

**PROPOSITION 3.** (i) *Belief-weightings of next-period prices, to form all those expectations which are consistent with equilibrium conditions for period  $T - 2$ , can be identified by relying just on the successor-state  $A_{T-1}$  involved in an equilibrium law-of-motion.*

*For an unconstrained equilibrium, these beliefs can be identified by the condition*

$$-A_{T-1}(1 + r_{T-1}) \leq \mu \frac{E_{T-2}[(1 - \theta)c_{T-1}^{\theta(1-\sigma)} + p_{T-1}\theta c_{T-1}^{\theta(1-\sigma)-1}]}{(1 + r_{T-1})E_{T-2}[\theta c_{T-1}^{\theta(1-\sigma)-1}]},$$

*and for a constrained equilibrium these beliefs can be identified by the condition*

$$\frac{\beta}{1 + r_{T-1} - \mu} E_{T-2} \left\{ \frac{-\mu(1 - \theta)}{A_{T-1}(1 + r_{T-1})} c_{T-1}^{\theta(1-\sigma)} + \theta c_{T-1}^{\theta(1-\sigma)-1} \left[ -\frac{\mu p_{T-1}}{A_{T-1}(1 + r_{T-1})} - (1 + r_{T-1}) \right] \right\} \geq 0.$$

*The previous two conditions are equivalent, if evaluated for some  $A_{T-1} < 0$ . Expectations denoted by  $E_{T-2}$  are formed by weightings of beliefs entertained in period  $T - 2$ . These*

belief-weightings are nonnegative and sum to unity. They assign weights to those next-period prices  $p_{T-1}$ , and to the corresponding next-period choices induced by those prices, which are contained in the set of—possibly multiple—equilibrium outcomes supported by a successor-state  $A_{T-1}$ .

(ii) Equilibrium combinations of variables  $(A_{T-2}, c_{T-2}, p_{T-2}, \kappa_{T-2}, A_{T-1})$  for both types of equilibria, unconstrained and constrained, can be constructed for any equilibrium belief-weighting, as identified in part (i) above. The construction uses a successor-state  $A_{T-1}$  and an associated equilibrium belief-weighting as a starting point. It then determines the corresponding current state  $A_{T-2}$ , as well as the other variables involved in some equilibrium combination, by an appropriate composition of functions.

For unconstrained equilibria, with  $\kappa_{T-2} = 0$ , these functions are composed as follows:

$$\begin{aligned} c_{T-2} &= [\beta(1+r_{T-1})E_{T-2}c_{T-1}^{\theta(1-\sigma)-1}]^{\frac{1}{\theta(1-\sigma)-1}}, \\ A_{T-2} &= \frac{A_{T-1} + c_{T-2} - y_{T-2}}{1+r_{T-2}}, \\ p_{T-2} &= \frac{E_{T-2}[(1-\theta)c_{T-1}^{\theta(1-\sigma)} + p_{T-1}\theta c_{T-1}^{\theta(1-\sigma)-1}]}{(1+r_{T-1})E_{T-2}[\theta c_{T-1}^{\theta(1-\sigma)-1}]}. \end{aligned}$$

For constrained equilibria these functions are composed as follows:

$$\begin{aligned} p_{T-2} &= \frac{-A_{T-1}(1+r_{T-1})}{\mu}, \\ \kappa_{T-2} &= \frac{\beta}{1+r_{T-1}-\mu} E_{T-2} \left\{ \frac{(1-\theta)}{p_{T-2}} c_{T-1}^{\theta(1-\sigma)} + \theta c_{T-1}^{\theta(1-\sigma)-1} \left[ \frac{p_{T-1}}{p_{T-2}} - (1+r_{T-1}) \right] \right\}, \\ c_{T-2} &= \left\{ \frac{1+r_{T-1}}{\theta} (\beta E_{T-2} [\theta c_{T-1}^{\theta(1-\sigma)-1}] + \kappa_{T-2}) \right\}^{\frac{1}{\theta(1-\sigma)-1}}, \\ A_{T-2} &= \frac{A_{T-1} + c_{T-2} - y_{T-2}}{1+r_{T-2}}. \end{aligned}$$

The following discussion of the parts of this proposition prepares their quantitative use for determining the shapes of—possibly belief-driven—state-dependent equilibrium outcomes in the initial period  $T-2$ . The first part of the proposition delimits equilibrium belief-weightings assigned to next-period market outcomes, which need to be supported by  $A_{T-1}$ , that is, by the next-period aggregate state of the economy.

Therefore, this part builds on the equilibrium curves obtained in the previous stage of backward induction for period  $T-1$ . Those  $A_{T-1}$  which support a unique equilibrium in  $T-1$  can only be combined with degenerate expectations  $E_{T-2}$ , having full weight on that unique next-period equilibrium. However, for those financial successor-states  $A_{T-1}$  which support multiple equilibria in  $T-1$ , weightings of these multiple future outcomes may form a nondegenerate range of expectations  $E_{T-2}$ , all equally consistent with dynamic equilibrium. Part (i) of Proposition 3 forms all these equilibrium belief-weightings. Figure 7 illustrates this formation. Figure 7 represents a situation where a

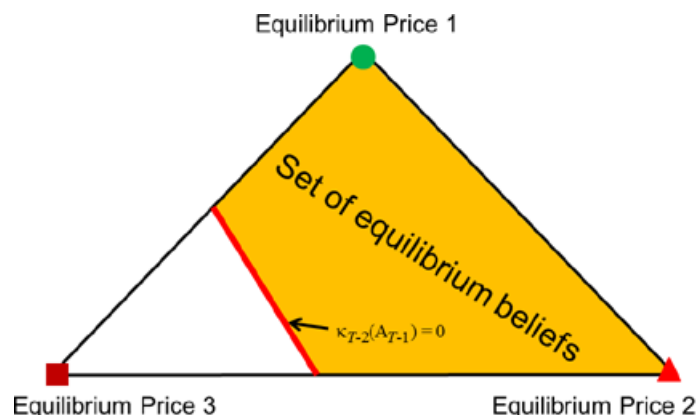


FIGURE 7. Simplex with set of equilibrium beliefs.

specific financial successor-state  $A_{T-1}$  supports three different next-period equilibrium prices  $p_{T-1}$ . Beliefs about these next-period prices are captured by probability weights assigned to each of them. Any such belief-weighting of prices carries over to the next-period choices of households, who behave as price takers. The equilibrium conditions involve expectations formed according to these belief-weightings. Part (i) of Proposition 3 determines the set of all belief-weightings which are consistent with the equilibrium conditions. Figure 7 illustrates this set of equilibrium beliefs, depicting it as a subset of all probability weights assignable to prices at the vertices of the corresponding simplex. The delimitation of the set of equilibrium beliefs is identified by the relevant condition in part (i) of Proposition 3, for each type of equilibrium, unconstrained or constrained.

This step is pivotal for our main contribution. It implies that the set of equilibrium belief-weightings, and thus the degree of uncertainty faced by market participants, is endogenously determined. This distinguishes our approach from conventional approaches to accommodate multiplicity in a dynamic setting with price takers. These other approaches postulate specific ways of assigning probabilities to multiple outcomes, for example, Cole and Kehoe (2000), Gertler and Kiyotaki (2015), or postulate specific criteria for selecting one of the outcomes, for example, Schmitt-Grohé and Uribe (2016).

Having identified all equilibrium weightings of future outcomes for all successor-states  $A_{T-1}$ , we can rely on part (ii) of Proposition 3 to complete the construction of state-dependent equilibria in period  $T - 2$ . This part provides a prescription for constructing all remaining variables involved in some equilibrium combination. It therefore, finally, pins down the value of a current financial state  $A_{T-2}$  which connects to a successor-state  $A_{T-1}$  for some equilibrium belief-weighting of next-period house prices. Put differently, the value of the current state  $A_{T-2}$  connected to a particular successor-state  $A_{T-1}$  by an equilibrium law-of-motion depends on the specific weighting considered from the set of equilibrium beliefs. This is illustrated in Figure 8.

Figure 8 shows two horizontal axes, and equilibrium connections between them. The top axis refers to the financial successor-state  $A_{T-1}$ , and the bottom axis refers to

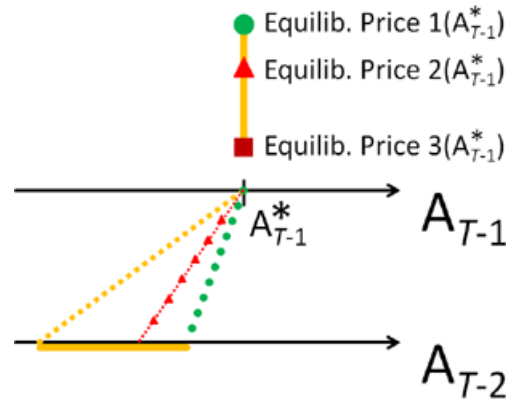


FIGURE 8. Consistency of beliefs and aggregate law-of-motion.

the current financial state  $A_{T-2}$ . The illustration relies on a particular successor-state  $A_{T-1}^*$ , as well as three next-period equilibrium prices supported by  $A_{T-1}^*$ . The figure depicts different equilibrium laws-of-motion connecting to  $A_{T-1}^*$ , if different beliefs about next-period prices are entertained in period  $T - 2$ . For instance, the connection with green circles is applicable if the highest price, also depicted by a green circle, is expected to prevail. Accordingly, the red triangles describe the equilibrium connection if expectations put full weight on the medium price level. We know from part (i) of Proposition 3 that a broad range of weightings of next-period prices may be consistent with equilibrium. The yellow range on the bottom horizontal axis is meant to trace out the variation in current states  $A_{T-2}$  obtained by applying the construction according to part (ii) of Proposition 3 for all possible equilibrium belief-weightings. Figure 8 therefore illustrates how variation within the set of equilibrium beliefs translates into variation of equilibrium laws-of-motion. The quantitative application of this principle will ultimately determine the extent of belief-driven equilibrium variations for our economy.

Parts (i) and (ii) of Proposition 3 provide a prescription for constructing all equilibria in period  $T - 2$ . Like in the earlier construction of equilibria for the intermediate period  $T - 1$ , this is again an instance of a *backward-construction* approach to backward induction: Starting at a *successor-state*  $A_{T-1}$ , we subsequently construct a current state  $A_{T-2}$ , or even a range of current states, if  $A_{T-1}$  supports a range of equilibrium belief-weightings of next-period equilibrium outcomes.

At this stage of constructing equilibria for period  $T - 2$ , such a backward-construction approach is key for a clear-cut characterization of equilibrium belief-weightings, as provided by part (i) of Proposition 3 for any particular successor-state  $A_{T-1}$ . The consideration of all successor-states, which can be associated with at least some equilibrium belief-weighting, traces out all equilibria. Viewing the result of this construction from the perspective of current states  $A_{T-2}$  reveals the shapes of state-dependent equilibrium combinations in period  $T - 2$ . Figure 9 singles out two successor-states,  $A_{T-1}^*$  and  $A_{T-1}^{**}$ , to illustrate this principle.

The successor-state  $A_{T-1}^*$  is reached by various equilibrium laws-of-motion, departing from current financial states  $A_{T-2}$  in a specific range. Likewise,  $A_{T-1}^{**}$  is reached by



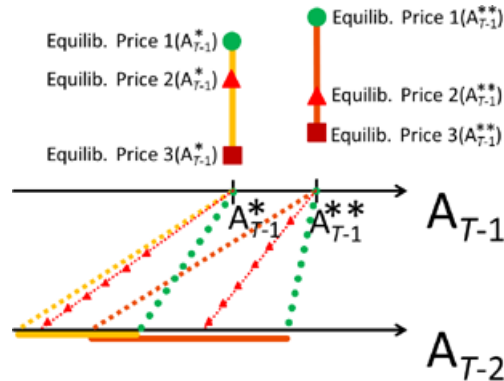


FIGURE 9. Consistent equilibrium beliefs for different successor-states  $A_{T-1}$ .

equilibrium laws-of-motion from another specific range of current states  $A_{T-2}$ . If considered in isolation, these are just two instances of the principle already presented earlier in Figure 8, depicting the influence of variations within the set of equilibrium belief-weightings. In addition, the combined consideration of these instances, facilitated by Figure 9, reveals an overlap of the corresponding ranges of current states  $A_{T-2}$ . Let us now view the result of this backward-construction from the perspective of the current financial state: For current states  $A_{T-2}$  in the overlap an equilibrium law-of-motion may connect to  $A_{T-1}^*$  just as well as to  $A_{T-1}^{**}$ . The path ultimately taken in such a situation depends on the particular weighting of beliefs about next-period prices entertained in this economy. The precise leeway—as *exactly* determined by part (i) of Proposition 3—admitted by nondegenerate weightings of next-period outcomes therefore determines the *shapes* of all equilibrium combinations which may occur for given current financial states  $A_{T-2}$ .

Such a view on the construction of the specific equilibrium shapes for period  $T - 2$  highlights the importance of having considered *state-dependence* explicitly during all stages of backward induction: It has allowed us to rely on successor-states  $A_{T-1}$  for pinning down the specific state-dependent next-period outcomes to be weighted by beliefs. Having clearly delimited the range of equilibrium belief-weightings, these again needed to be combined with their supporting successor-state to determine the specific range of all current states  $A_{T-2}$  involved in an equilibrium combination of variables.

For a given current level of the financial state  $A_{T-2}$ , the previously described backward-construction implies a specific set of belief-weightings of next-period prices. With regard to their foundation in terms of dynamic equilibrium consistency requirements, these belief-weightings constitute the set of all *rationaly entertainable* beliefs about next-period prices. Variations within this set may thus be regarded as variations in *confidence* about future outcomes. Our approach therefore conceptualizes the role which confidence may play in equilibrium. By the same principle, this endogenizes the extent of uncertainty which households may be subject to in equilibrium.

The following Proposition 4, derived in Appendix A.6, provides a condition to ensure that the market equilibrium leaves scope for confidence-driven outcomes.

PROPOSITION 4. *State-dependent competitive equilibria combine specific successor-states  $A_{T-1}$  with a nondegenerate range of equilibrium belief-weightings of future prices, if condition (18) and the following condition (25) hold:*

$$\begin{aligned} & \frac{r_T - r_{T-1}}{1 + r_{T-1}} \left[ \frac{1 + r}{r(1 + r_T)} + \left( \frac{1 + r_T}{1 + r} \right)^{\frac{1}{\theta(1-\sigma)-1}} \right] \frac{\mu(1 - \theta)}{\theta(1 + r_T) + \mu(1 - \theta)} \\ & + \left( \frac{1 + r_T}{1 + r} \right)^{\frac{1}{\theta(1-\sigma)-1}} - \frac{y_{T-1}}{y} \geq 0. \end{aligned} \quad (25)$$

Proposition 4 is complementary to the earlier Proposition 2. If condition (25) holds in addition to the earlier condition (18), then variations in confidence, as conceptualized by belief-weightings of future prices, are guaranteed to play a role at least for some financial states of the economy. In terms of the backward-induction approach, Proposition 4 assures that some of those states which give rise to multiplicity of market prices in the intermediate period—thus providing the support for nondegenerate belief-weightings of future prices—can actually be reached by a corresponding equilibrium law-of-motion in the initial period.

Proposition 4 inherits some of its characteristics from Proposition 2: It is tailored for situations where multiplicity of equilibrium in  $T - 1$  occurs right next to the point where unconstrained and constrained equilibrium curves connect. Beyond that, it specifically considers those belief-weightings which are sufficiently close to a full weight on an unconstrained equilibrium. It therefore identifies a sufficient condition.

Condition (25) highlights the importance of the patterns of interest rates and of income anticipated by the households. Both a temporarily low interest rate ( $r_{T-1} < r_T$ ) and the anticipation of income growth ( $y > y_{T-1}$ ) facilitate satisfying the condition, thereby boosting the scope for confidence-driven fluctuations. We will further investigate these risk factors for a crisis of confidence in Section 5 below.

#### 4.2 Confidence-driven prices and consumption

The previously explained construction of all state-dependent equilibria, when confidence is a potential driver of outcomes, is essential for the quantitative analysis of our economy. The results obtained earlier for the intermediate period  $T - 1$ , as depicted in Figure 5, show that a specific range of financial states  $A_{T-1}$  supports multiple equilibria. Our construction of equilibria according to Proposition 3 is fit for accommodating such a situation: It enables the implementation of the next backward induction step in period  $T - 2$ , despite the nonuniqueness of outcomes in period  $T - 1$ .

Figure 10 shows the results of applying Proposition 3 for the construction of equilibria in our economy in the initial period  $T - 2$ . The underlying parameters are specified in Table 1. Figure 10 depicts the *shapes* formed by equilibrium combinations of variables. Parts of these equilibrium shapes obtained for an unconstrained case (i.e., where the collateral constraint is not binding in equilibrium) are painted in light gray (green). Parts obtained for a constrained case (i.e., where the collateral constraint turns out to be binding in equilibrium) are rendered in dark gray (red). The dark (blue) parts depict the

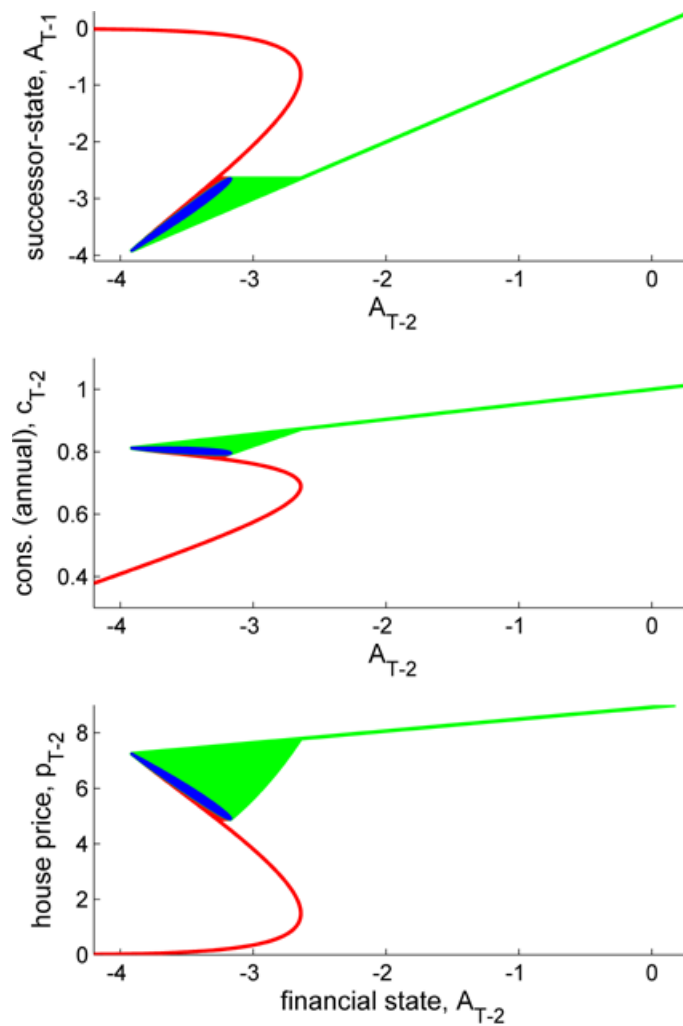


FIGURE 10. Equilibrium shapes in period  $T - 2$ . Constrained equilibrium in dark gray (red color); unconstrained equilibrium in light gray (green color). The dark (blue color) area depicts the overlap of constrained and unconstrained areas. Notes: Financial assets and consumption are measured in units of annual earnings. The underlying parameter values are specified in Table 1.

overlap of constrained and unconstrained parts. Along the horizontal axis, the financial state  $A_{T-2}$  of the economy is *negative* for the corresponding amounts of *debt*, and it is measured in units of annual endowments of earnings.

The top panel of Figure 10 depicts the shape formed by combinations of  $A_{T-2}$  and the corresponding financial successor-state  $A_{T-1}$ , which are in line with an equilibrium law-of-motion for *some* equilibrium weighting of next-period prices supported by  $A_{T-1}$ . Note that Figure 10 presents a quantitative version of the principle illustrated in Figures 8 and 9. The range of variation in current states  $A_{T-2}$  consistent with some level of  $A_{T-1}$  shows the influence of the underlying variation in equilibrium weightings

of next-period prices. The dark (blue) overlap of unconstrained and constrained equilibrium parts is explained by the same principle: Laws-of-motion for the two different types of equilibrium, unconstrained and constrained, may connect the same current states  $A_{T-2}$  to the same successor-state  $A_{T-1}$ . However, this observed identity of laws-of-motion hides the underlying differences in belief-weightings of next-period prices. Overlaps from the same source reappear in the panels for other variables.

The two bottom panels of Figure 10 show the *equilibrium shapes* for state-dependent consumption and the house price. These variables mirror the confidence-driven variation explained above in terms of the corresponding laws-of-motion. For given levels of the current financial state  $A_{T-2}$  in a specific range, variations in belief-weightings assigned to future outcomes continuously influence consumption and the house price. Our economy becomes vulnerable to crises of confidence<sup>16</sup> if household debt surpasses a threshold, which for this parameterization is located at a debt level of approximately 2.7 annual earnings. Very high levels of debt would make the economy immune to variations in confidence again, while supporting only considerably lower consumption and lower house prices. Figure 10 also reveals the relative magnitudes of equilibrium responses of variables: Collateralized borrowing provides a basis for strong reactions of the house price to changes in confidence of consumers.

#### 4.3 *Equilibrium restrictions despite dependence on confidence*

While confidence may drive asset prices and consumption in our economy, it is certainly not the case that *anything goes*. The consistency requirements imposed by dynamic equilibrium imply specific restrictions on outcomes, despite their possible dependence on confidence.

Some of these restrictions are readily revealed by the *shapes* of equilibrium combinations of variables presented in Figure 10. Along the horizontal axis, equilibrium in our economy pins down the range of current financial states  $A_{T-2}$  for which variations in beliefs about next-period prices matters. Along the vertical axis, for  $A_{T-2}$  in the relevant range, the theory delivers state-dependent bounds on the extent to which confidence may drive the house price and consumption. This delimitation of *equilibrium shapes*, both in the direction of the current-state and in the direction of endogenous variables, is a consequence of the clear-cut characterization of equilibrium belief-weightings from part (i) of Proposition 3.

The theory also implies restrictions on the *comovement* of the variables. Figure 11 develops an illustration of this type of restriction. The top of Figure 11 presents confidence-driven shapes for consumption and for the house price in separate panels, both in terms of their dependence on the current financial state  $A_{T-2}$ . The bottom panel of Figure 11 reveals the predictions about the comovement of these variables, by depicting the joint outcomes for a fixed level of the current state of the economy. The comovement of variables is explained by considering how a specific weighting from the set of equilibrium beliefs enters the determination of variables.

<sup>16</sup>Interestingly, Fisher's (1933, p. 343) stylized chain of events for great depressions starts with a shock to confidence, also assigning a key role to the corresponding debt dynamics.

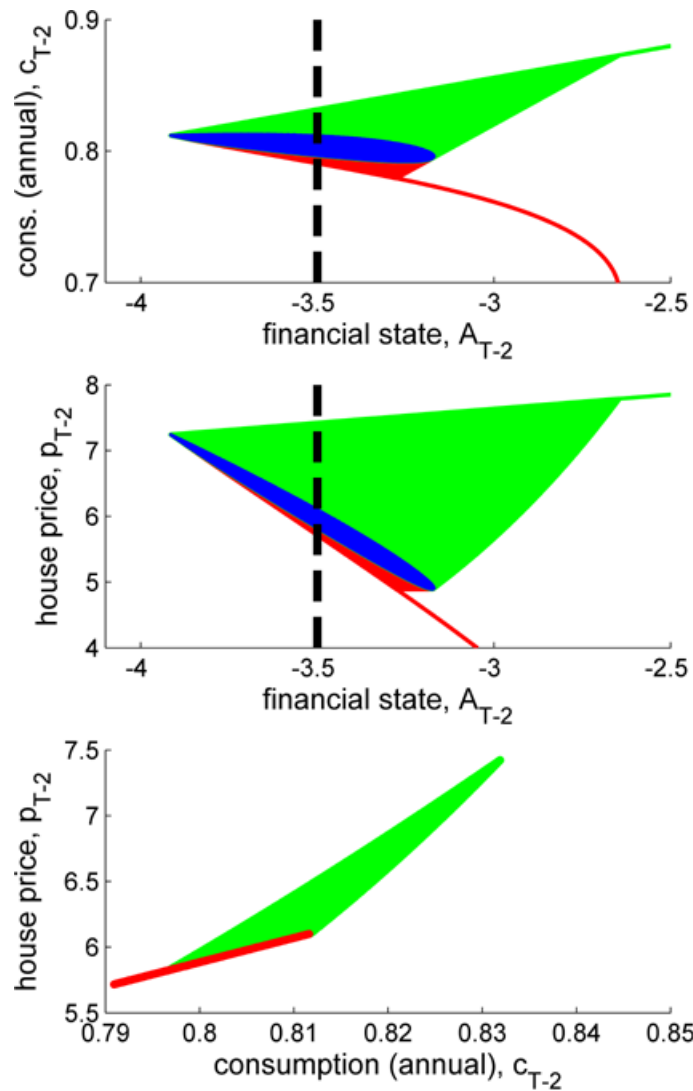


FIGURE 11. Tight restrictions on equilibrium combinations of consumption and the house price, for given  $A_{T-2} = -3.5$ . Notes: Constrained equilibrium in dark gray (red color); unconstrained equilibrium in light gray (green color). The dark (blue color) area depicts the overlap of constrained and unconstrained areas. Financial assets and consumption are measured in units of annual earnings. The underlying parameter values are specified in Table 1.

Current consumption  $c_{T-2}$  is obtained from equation (19),

$$c_{T-2} = \left\{ \frac{1 + r_{T-1}}{\theta} (\beta E_{T-2} [\theta c_{T-1}^{\theta(1-\sigma)-1}] + \kappa_{T-2}) \right\}^{\frac{1}{\theta(1-\sigma)-1}}.$$

This level of consumption and, in particular, the *same belief-weightings* underlying  $E_{T-2}$  also enter the asset pricing equation for houses in our economy. This asset-pricing

equation is obtained by solving equation (20) for the house price,

$$p_{T-2} = \frac{\beta E_{T-2}[(1-\theta)c_{T-1}^{\theta(1-\sigma)} + p_{T-1}\theta c_{T-1}^{\theta(1-\sigma)-1}]}{\theta c_{T-2}^{\theta(1-\sigma)-1} - \kappa_{T-2}\mu}.$$

The key driver of the predicted comovement of variables<sup>17</sup> is therefore the following: All contemporaneously determined variables are influenced by the same realization of an equilibrium belief-weighting of future outcomes.

## 5. THE MAKING OF A CONFIDENCE-DRIVEN CRISIS

This section uses the approach developed in the previous sections to investigate key ingredients for the making of a confidence-driven crisis. To this end, we analyze a scenario with specific patterns for interest rates and for anticipated income growth rates. These parameters are specified in Table 2. All other parameters are kept constant at their values in the previous benchmark scenario, as specified in Table 1. In the present *making-of-a-crisis scenario* short-run and medium-run interest rates, that is, those in the initial period  $T - 2$  and in the intermediate period  $T - 1$ , are temporarily lower than the long-run natural interest rate  $r = 1/\beta - 1$ , which was assumed for all periods of the previously discussed benchmark scenario. We combine this pattern with expectations of higher long-run income growth. Such a combination may be thought of as reproducing—along the dimensions captured by our model—salient features of the U.S. economy during the buildup of the 2008 financial crisis.

Figures 12 and 13 present the *equilibrium shapes* for consumption and house prices in this economy in period  $T - 2$ . Comparisons of Figures 12 and 13 with the benchmark

TABLE 2. Interest rates and growth rates of earnings for the making-of-a-crisis scenario. Notes: The entries in each column combine the growth rate of earnings prevailing in that period with the interest rate applicable to financial investments chosen in that period. The interest rate on pre-existing financial assets in period  $T - 2$  is set equal to the interest rate applicable to the first period choice of investment. The values of all other parameters are as specified in Table 1.

	Initial Period $T - 2$	Intermediate Period $T - 1$	Terminal Stage $T, T + 1, \dots, \infty$
Making-of-a-crisis scenario			
Interest rate (annual)	0.01	0.01	0.04
Growth rate of earnings (annual)	0	0	0.03
Benchmark scenario			
Interest rate (annual)	0.04	0.04	0.04
Growth rate of earnings (annual)	0	0	0

<sup>17</sup>This type of comovement may have implications for an empirical analysis of these variables: If changes in confidence are the common cause driving changes of both housing values and consumption, then elasticities obtained from regressions of consumption on housing wealth have to be interpreted with caution, as acknowledged by Case, Quigley, and Shiller (2013, pp. 125–126).

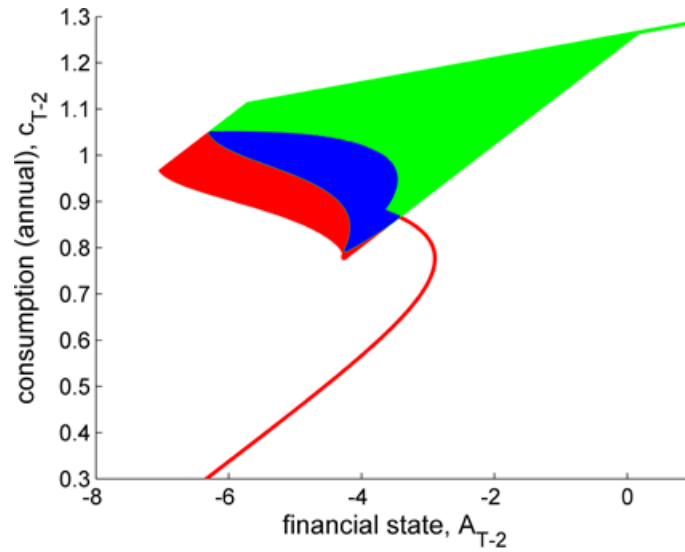


FIGURE 12. Equilibrium shape of consumption in period  $T - 2$ . Constrained equilibrium in dark gray (red color); unconstrained equilibrium in light gray (green color). The dark (blue color) area depicts the overlap of constrained and unconstrained areas. Notes: Financial assets and consumption are measured in units of annual earnings. The underlying parameter values are specified in Table 2.

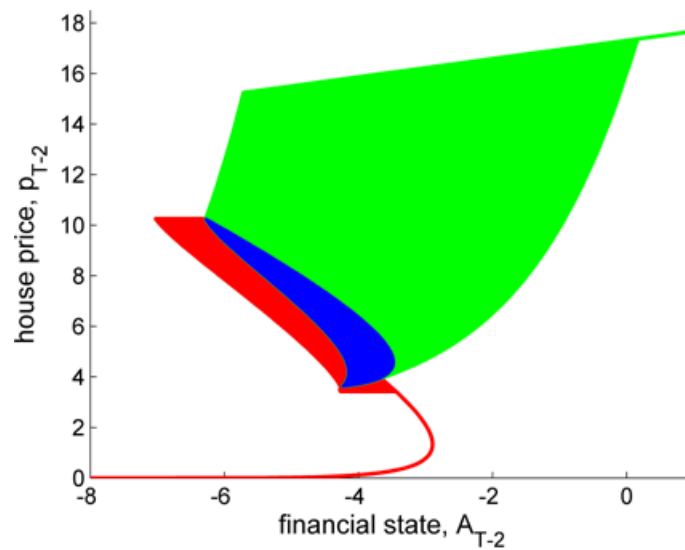


FIGURE 13. Equilibrium shape of the house price in period  $T - 2$ . Constrained equilibrium in dark gray (red color); unconstrained equilibrium in light gray (green color). The dark (blue color) area depicts the overlap of constrained and unconstrained areas. Notes: Financial assets and consumption are measured in units of annual earnings. The underlying parameter values are specified in Table 2.



results in Figure 10 show the impact of the modifications considered in the making-of-a-crisis scenario. The present scenario considerably stretches equilibrium shapes in both directions. First, observe that the domain of current financial states  $A_{T-2}$  for which confidence-driven crises can occur is extended. Second, for given financial states within that domain, the range of confidence-driven fluctuations increases for both consumption and the house price. The combination of temporarily low interest rates and higher growth expectations therefore makes the economy *more vulnerable* to crises of confidence: It broadens the spectrum of relevant risk factors, and it adds to the potential severity of a crisis, once the looming risk has materialized. This is a consequence of a corresponding extension of the set of equilibrium beliefs about future prices.

The intuition behind this result is that low interest rates and positive income growth make the representative household more willing to borrow, thus making the liquidity feedback relevant for a broader range of debt levels. In the intermediate period  $T - 1$ , this implies an extension of debt levels which support multiple house prices, and a corresponding increase in the variation across these house prices. From the perspective of the initial period  $T - 2$ , the broader interval of successor-states, which support multiple next-period prices, implies that a larger set of current beliefs about future prices can be sustained in equilibrium.

Viewing the economy represented by the making-of-a-crisis scenario of our model as a stylized characterization of the U.S. economy before the Great Recession, Figures 12 and 13 show that shifts in equilibrium confidence provide a potential rationalization of the large fluctuations in consumption and house prices observed in the U.S., which we have documented in the [Introduction](#).

The results show that the extent of exposure to confidence-driven fluctuations depends on the level of debt. For example, regarding price ranges for the 54% increase in the level of debt documented earlier in Figure 1, from  $A_{T-2} = -0.47$  at the beginning of the 2001 recession to  $A_{T-2} = -0.72$  at the beginning of the 2007–2009 recession, exposure changes as follows: The maximum extent of a purely confidence-driven house price drop (as covered by the shape in Figure 13) is 25.7% at  $A_{T-2} = -0.47$ , and it is 33.8% at  $A_{T-2} = -0.72$ .<sup>18</sup>

In particular, the results for this scenario imply that the empirically observed variations in consumption and in house prices during the Great Recession are fully covered by the extent of confidence-driven equilibria for this parameterization. At  $A_{T-2} = -0.72$ , which represents the debt level at the beginning of the 2007–2009 recession, the model-consistent extent of 6% of variation in consumption covers the 4% drop mentioned in the [Introduction](#). Likewise, at this pre-recession debt level, the model-consistent extent of 33.8% of house price variation covers the 29% house price drop, which was illustrated in Figure 1 to motivate our paper.

We would like to supplement these findings with some qualifying remarks. Besides the debt level, which has been central in the previous paragraphs, other parameters of

<sup>18</sup>For comparability, we report the extent of *relative* changes within the set of belief-driven outcomes here. For example, the 33.8% reported for a house price drop at  $A_{T-2} = -0.72$  are obtained as the proportional deviation between the maximum (17.05) and the minimum (11.29) house prices for the given debt level, expressed as a percentage of the maximum amount.

the economy are equally important. The comparison of the model to the 2007–2009 recession relies on the quantity of debt mentioned above *in combination with* the other parameters (e.g., low interest rates) we have used to address this specific episode. The empirical links have been pursued in a setting which was stylized for a clear presentation of our main contribution—the development of a concept for the quantification of state-dependent equilibrium exposure to belief-driven uncertainty. Despite the stylized nature of our model, these findings suggest that persistently low interest rates and substantial household debt in developed countries make the vulnerability to confidence-driven crises a relevant issue.

## 6. CONCLUSIONS

Motivated by the challenge of rationalizing the large swings of house prices and consumption which accompanied the Great Recession in the U.S., we have conceptualized the role assignable to consumer confidence in a dynamic equilibrium framework. Our analysis is built around housing collateral and exploits equilibrium consistency requirements to determine the scope for confidence-driven variations in prices and consumption.

Collateralized borrowing introduces a liquidity feedback effect, whose strength depends on an endogenously evolving aggregate state, that is, on the level of household debt in the economy. For sufficiently high loan-to-value ratios, this liquidity effect can give rise to multiple market-clearing house prices. We conceptualize confidence by belief-weightings which households assign to multiple future prices. The explicit consideration of the state-dependence of equilibria turns out to be crucial for determining the range of rationally entertainable beliefs about future prices: The next-period state reached by an aggregate law-of-motion needs to support these prices. This principle identifies equilibrium confidence and the domain of states, that is, debt levels, for which it can play a role, as well as the range of confidence-driven outcomes. Our approach therefore endogenously determines the amount of uncertainty households may be subject to in equilibrium.

Our work shows how the widespread recursive paradigm can be enhanced to accommodate situations when market prices are not necessarily guaranteed to be unique functions of the state of the economy. Our approach constructs all equilibrium combinations of laws of motion and the corresponding beliefs about future prices. It nests the special case where allocations and prices turn out to be unique functions of the state of the economy. It detects—a potential continuum of—multiple equilibria if they are existent. Therefore, we trust that the concept developed for the application in this paper will prove to be valuable for researchers across a diverse range of specializations.

When applied to the model with housing collateral, our approach predicts that high levels of household debt make the economy vulnerable to a crisis of confidence. Moreover, we find that an environment with low interest rates and high growth expectations aggravates this vulnerability, making an economy even more prone to confidence-driven booms and busts of house prices. These results may speak to the debate about

factors affecting the relationship between household debt and macroeconomic fluctuations. Our approach may inform economic policy makers how to accommodate endogenously evolving states, for example, debt levels, in the management of systemic risk.

## APPENDIX

### A.1 Characterizing the portfolio investment decision of the household

The maximization of the discounted sum of expected utilities, according to the objective function (1), for all periods  $t = T - 2, T - 1, T, \dots, \infty$  of the model can be expressed in recursive form as follows:

$$v_t(a_t, h_t) = \max_{a_{t+1}, h_{t+1}} [U(c_t, h_t) + \beta E_t v_{t+1}(a_{t+1}, h_{t+1})] \quad (26)$$

subject to the budget constraint (2) and the collateral constraint (3), restated here for clarity:

$$a_{t+1} + p_t h_{t+1} + c_t = (1 + r_t) a_t + p_t h_t + y_t, \quad (27)$$

$$-a_{t+1}(1 + r_{t+1}) \leq \mu p_t h_{t+1}. \quad (28)$$

The assumptions made for our analysis imply that we abstract from all possible sources of *fundamental risk*, which would be characterized by objectively known (conditional) probability distributions for features like next-period endowments  $y_{t+1}$ . Nevertheless, households in our economy may face *nonfundamental*<sup>19</sup> *uncertainty* about the consequences of their investment decisions. For a specific range of the aggregate state of the economy liquidity effects operating through the collateral constraint (28) may be strong enough to give rise to *multiple future equilibrium prices* of the (housing) asset.

This possibility of nonunique prices is precluded during the terminal stage  $t = T, \dots, \infty$  with perfect capital markets, that is, without the collateral constraint. Therefore, the explicit consideration of expectations  $E_t$  is redundant in (26) for investment decisions made in the intermediate period  $T - 1$  and in all later periods, in which households have perfect foresight of future equilibrium prices. In the initial period  $T - 2$ , the expectations-based weighting  $E_{T-2}$  of continuation values in the Bellman equation (26) becomes key, and is induced by weightings of those equilibrium prices which may arise in  $T - 1$ .

The first-order and envelope conditions for the forward-looking decisions taken according to (26), subject to (27) and (28), imply the following Euler equations:

$$\frac{\partial U(c_t, h_t)}{\partial c_t} = \beta(1 + r_{t+1}) E_t \left[ \frac{\partial U(c_{t+1}, h_{t+1})}{\partial c_{t+1}} \right] + \kappa_t(1 + r_{t+1}), \quad (29)$$

and

$$\frac{\partial U(c_t, h_t)}{\partial c_t} p_t = \beta E_t \left[ \frac{\partial U(c_{t+1}, h_{t+1})}{\partial h_{t+1}} + p_{t+1} \frac{\partial U(c_{t+1}, h_{t+1})}{\partial c_{t+1}} \right] + \kappa_t \mu p_t, \quad (30)$$

<sup>19</sup>Tantamount to *extrinsic uncertainty*, as defined by Cass and Shell (1983).

where  $\kappa_t \geq 0$  denotes the Kuhn–Tucker–Lagrangian multiplier on the inequality constraint (28), that is,  $\kappa_t > 0$  if the collateral constraint is binding,  $\kappa_t = 0$  if it is slack. While the intuition for Euler equation (29) is familiar from the large literature on consumption and borrowing constraints, Euler equation (30) deserves further explanation. In an optimum, marginal utility of consumption forgone due to housing investment (on the left-hand side) equals the sum of marginal benefits (on the right-hand side): the discounted dividend in terms of the marginal utility of housing, the resale value of the house in terms of the marginal utility of consumption, and the collateral value of the house, if the collateral constraint is binding, such that  $\kappa_t > 0$ .

Note that portfolio choices, consumption, and the multiplier are characterized recursively as functions of the household's state variables  $a_t$  and  $h_t$ . In the initial period  $T - 2$  choices and the multiplier additionally depend on the household's belief-weighting—as captured by the formation of expectations  $E_{T-2}$ —about which of multiple equilibrium prices will prevail in the intermediate period  $T - 1$ .

### A.2 Deriving the equilibrium house price in the terminal stage as a function of the financial asset position

In the periods of the infinite terminal stage with perfect capital markets collateral constraints do not enter the characterization of the economy. Given the implied omission of multipliers from the Euler equations (as derived for the relevant portfolio choice situation in Appendix A.1), optimal sequences of consumption  $c_t$  for all periods of the terminal stage  $t \geq T$  need to satisfy

$$\frac{\partial U(c_t, h_t)}{\partial c_t} = \beta(1 + r_{t+1}) \frac{\partial U(c_{t+1}, h_{t+1})}{\partial c_{t+1}}, \quad (31)$$

and

$$\frac{\partial U(c_t, h_t)}{\partial c_t} p_t = \beta \left[ \frac{\partial U(c_{t+1}, h_{t+1})}{\partial h_{t+1}} + p_{t+1} \frac{\partial U(c_{t+1}, h_{t+1})}{\partial c_{t+1}} \right]. \quad (32)$$

Note that expectations are dropped from these Euler equations: With perfect capital markets and given the lack of fundamental sources of risk, the future equilibrium price relevant for items on the right-hand side of these equations is a uniquely determined function of the future financial asset position, which is consistent with an aggregate equilibrium law-of-motion. This is verified in equilibrium below.

Substituting for the basket (6) in the utility function (5), we have

$$U(c_t, h_t) = \frac{c_t^{\theta(1-\sigma)} h_t^{(1-\theta)(1-\sigma)} - 1}{1 - \sigma}, \quad (33)$$

and marginal utilities are

$$\frac{\partial U(c_t, h_t)}{\partial c_t} = \theta c_t^{\theta(1-\sigma)-1} h_t^{(1-\theta)(1-\sigma)}, \quad (34)$$

$$\frac{\partial U(c_t, h_t)}{\partial h_t} = (1 - \theta) c_t^{\theta(1-\sigma)} h_t^{(1-\theta)(1-\sigma)-1}. \quad (35)$$

Imposing market-clearing for housing, we have  $h_t = 1$  (the normalized constant housing supply), and using the assumption that interest rates applicable during the terminal stage are constant  $r_{t+1} = r$ ,  $t \geq T$ , (31) implies

$$c_t = [\beta(1+r)]^{\frac{k}{\theta(1-\sigma)-1}} c_{t+k}, \quad \text{for } k = 1, 2, \dots, \quad (36)$$

or

$$c_{t+k} = [\beta(1+r)]^{\frac{k}{1-\theta(1-\sigma)}} c_t, \quad \text{for } k = 1, 2, \dots. \quad (37)$$

In order to determine consumption in the terminal stage as a function of the aggregate financial state, we need to combine the dynamics of consumption across all periods from (37) with the corresponding intertemporal constraint involving all these amounts of consumption. Iterating forward the aggregate constraint on motion (4) from period  $T$  onwards, taking into account the no-Ponzi-game condition and the assumption that utility is strictly increasing in consumption, we obtain the following equation for the aggregate intertemporal financial constraint:

$$\sum_{s=T}^{\infty} \left(\frac{1}{1+r}\right)^{s-T} c_s = (1+r_T)A_T + \sum_{s=T}^{\infty} \left(\frac{1}{1+r}\right)^{s-T} y_s.$$

Substituting for future amounts of consumption from (37), we get

$$c_T \sum_{s=T}^{\infty} \left(\frac{1}{1+r}\right)^{s-T} [\beta(1+r)]^{\frac{s-T}{1-\theta(1-\sigma)}} = (1+r_T)A_T + \sum_{s=T}^{\infty} \left(\frac{1}{1+r}\right)^{s-T} y_s.$$

Using the assumption that the (long-run) terminal stage interest rate is in line with the discount factor, that is,  $\beta = 1/(1+r)$ , and the assumption of constant endowments in all periods of the terminal stage,  $y_t = y$ ,  $t \geq T$ , we have

$$\begin{aligned} c_T \sum_{s=T}^{\infty} \left(\frac{1}{1+r}\right)^{s-T} &= (1+r_T)A_T + y \sum_{s=T}^{\infty} \left(\frac{1}{1+r}\right)^{s-T}, \\ c_T \frac{1+r}{r} &= (1+r_T)A_T + y \frac{1+r}{r} \end{aligned}$$

and thus the equilibrium consumption function

$$c_T(A_T) = \frac{r(1+r_T)}{1+r} A_T + y.$$

Note that equation (37) with  $\beta = 1/(1+r)$  implies that the amounts of consumption stay constant during all periods  $t \geq T$  of the terminal stage.

The equilibrium price of housing is determined using the equilibrium consumption sequence to provide the relevant discount factor for the asset pricing equation as follows. Using the Euler equation for financial assets (31) to substitute for marginal utility of current period consumption in the Euler equation for housing investment (32), imposing the functional forms of marginal utilities from (34) and (35), and evaluating these

marginal utilities at the fixed level of housing supply, such that  $h_t = 1$ , we have

$$p_t = \frac{(1 - \theta)c_{t+1}^{\theta(1-\sigma)} + \theta c_{t+1}^{\theta(1-\sigma)-1} p_{t+1}}{(1 + r)\theta c_{t+1}^{\theta(1-\sigma)-1}}.$$

Substituting for

$$p_{t+1} = \frac{(1 - \theta)c_{t+2}^{\theta(1-\sigma)} + \theta c_{t+2}^{\theta(1-\sigma)-1} p_{t+2}}{(1 + r)\theta c_{t+2}^{\theta(1-\sigma)-1}},$$

we get

$$\begin{aligned} p_t &= \frac{(1 - \theta)c_{t+1}^{\theta(1-\sigma)} + \theta c_{t+1}^{\theta(1-\sigma)-1} \frac{(1 - \theta)c_{t+2}^{\theta(1-\sigma)} + \theta c_{t+2}^{\theta(1-\sigma)-1} p_{t+2}}{(1 + r)\theta c_{t+2}^{\theta(1-\sigma)-1}}}{(1 + r)\theta c_{t+1}^{\theta(1-\sigma)-1}} \\ &= \frac{(1 - \theta)c_t^{\theta(1-\sigma)} + \frac{(1 - \theta)c_t^{\theta(1-\sigma)}}{1 + r} + \frac{\theta c_t^{\theta(1-\sigma)-1} p_{t+2}}{1 + r}}{(1 + r)\theta c_t^{\theta(1-\sigma)-1}}, \end{aligned}$$

where the second equality follows from consumption being constant under our assumption  $\beta = 1/(1 + r)$  during all periods of the terminal stage.

Successive substitutions for  $p_{t+2}$ ,  $p_{t+3}$ , etc., yield

$$p_t = \frac{(1 - \theta)c_t^{\theta(1-\sigma)} \sum_{s=t}^{\infty} \left(\frac{1}{1 + r}\right)^{s-t} + \theta c_t^{\theta(1-\sigma)-1} \lim_{S \rightarrow \infty} \frac{p_{t+S+1}}{(1 + r)^S}}{(1 + r)\theta c_t^{\theta(1-\sigma)-1}}.$$

Requiring that  $\lim_{S \rightarrow \infty} \frac{p_{t+S+1}}{(1 + r)^S} = 0$  for bubble-free asset prices we get the house-pricing function

$$p_t(A_t) = \frac{1 - \theta}{\theta} \frac{c_t(A_t)}{r}, \quad \text{for all periods } t \geq T \text{ of the terminal stage.}$$

### A.3 Deriving Proposition 1: Constructing all state-dependent equilibria in $T - 1$

The derivations are organized by type of equilibrium, unconstrained and constrained. This layout is meant to maintain the focus on a constant structure of type-specific relationships, when taking steps to derive the results according to the items (i), (ii), and (iii) of the proposition. These steps draw on the equilibrium conditions stated by equations (9), (10), (11), (12), (13), and (14). Irrespective of the type of equilibrium, we base the construction of equilibrium combinations of variables on a successor-state  $A_T$ . We can thus rely on the next-period price  $p_T(A_T)$ , as determined by function (8), and on the corresponding next-period level of consumption  $c_T(A_T)$ , as determined by function (7).

**A.3.1 Unconstrained equilibrium** For an unconstrained equilibrium, the derivation imposes  $\kappa_{T-1} = 0$  in the equations for the construction in part (ii), and verifies the inequality  $-A_T(1+r_T) \leq \mu p_{T-1}$  for the existence of this type of equilibrium in part (i).

Part (ii): Starting from a successor-state  $A_T$ , we obtain current-period consumption  $c_{T-1}$  by setting  $\kappa_{T-1} = 0$  in equation (9), and solving for

$$c_{T-1} = [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} c_T(A_T).$$

Equipped with  $A_T$  and  $c_{T-1}$ , the constraint on aggregate motion (11) determines the corresponding current-period state by

$$A_{T-1} = \frac{A_T + c_{T-1} - y_{T-1}}{1+r_{T-1}}.$$

The current house price is obtained from equation (10) by setting  $\kappa_{T-1} = 0$  and substituting for  $c_{T-1}$ ,

$$p_{T-1} = \frac{(1-\theta)c_T(A_T)^{\theta(1-\sigma)} + p_T(A_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}}{(1+r_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}}.$$

Part (i): Substituting for  $p_{T-1}$  in the condition  $-A_T(1+r_T) \leq \mu p_{T-1}$ , we get

$$-A_T(1+r_T) \leq \mu \frac{(1-\theta)c_T(A_T)^{\theta(1-\sigma)} + p_T(A_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}}{(1+r_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}},$$

which therefore is expressed by relying only on the successor-state  $A_T$  involved in the corresponding equilibrium combination.

Part (iii): Using function (8) to substitute for  $p_T(A_T)$ , the previous inequality from the proof of part (i) delivers

$$-A_T(1+r_T) \leq \mu \frac{(1-\theta)c_T(A_T)^{\theta(1-\sigma)} + \frac{1-\theta}{r}c_T(A_T)^{\theta(1-\sigma)}}{(1+r_T)\theta c_T(A_T)^{\theta(1-\sigma)-1}} = \mu \frac{(1-\theta)(1+r)}{\theta r(1+r_T)} c_T(A_T).$$

Using function (7) to substitute for  $c_T(A_T)$ , we get

$$-A_T(1+r_T) \leq \mu \frac{(1-\theta)(1+r)}{\theta r(1+r_T)} \left[ \frac{r(1+r_T)}{1+r} A_T + y \right],$$

which is satisfied by all  $A_T \geq \underline{A}_T$ , where the lower bound  $\underline{A}_T$  stands for

$$\underline{A}_T \equiv - \left[ \frac{\mu(1-\theta)}{\theta(1+r_T) + \mu(1-\theta)} \right] \left[ \frac{1+r}{r(1+r_T)} \right] y.$$

**A.3.2 Constrained equilibrium** For a constrained equilibrium, the derivation imposes  $-A_T(1+r_T) = \mu p_{T-1}$  in the equations for the construction in part (ii), and verifies the inequality  $\kappa_{T-1} \geq 0$  for the existence of this type of equilibrium in part (i).



First of all, note that, under our assumptions, the equilibrium conditions imply a positive house price: Rearrange equation (10) to separate the house price

$$p_{T-1} = \frac{\beta((1-\theta)c_T^{\theta(1-\sigma)} + p_T \theta c_T^{\theta(1-\sigma)-1})}{\theta c_{T-1}^{\theta(1-\sigma)-1} - \kappa_{T-1} \mu}.$$

Use (9) to substitute for  $\theta c_{T-1}^{\theta(1-\sigma)-1}$ , to obtain

$$p_{T-1} = \frac{\beta((1-\theta)c_T^{\theta(1-\sigma)} + p_T \theta c_T^{\theta(1-\sigma)-1})}{\beta(1+r_T)\theta c_T^{\theta(1-\sigma)-1} + \kappa_{T-1}(1+r_T) - \kappa_{T-1}\mu},$$

which, for any  $\kappa_{T-1} \geq 0$ , is positive if  $\mu < (1+r_T)$ , as guaranteed by our assumptions for any positive interest rate. Considering any *negative* position of the aggregate financial successor-state  $A_T$ , while imposing  $-A_T(1+r_T) = \mu p_{T-1}$ , is therefore sufficient for covering all potential constrained equilibrium prices.

Part (ii): The condition  $-A_T(1+r_T) = \mu p_{T-1}$  implies a house price of

$$p_{T-1} = \frac{-A_T(1+r_T)}{\mu}.$$

Use (9) to substitute for  $\theta c_{T-1}^{\theta(1-\sigma)-1}$  in (10) and solve for

$$\begin{aligned} \kappa_{T-1} = \frac{\beta}{1+r_T-\mu} & \left\{ \frac{1}{p_{T-1}} (1-\theta)c_T(A_T)^{\theta(1-\sigma)} \right. \\ & \left. + \theta c_T(A_T)^{\theta(1-\sigma)-1} \left[ \frac{p_T(A_T)}{p_{T-1}} - (1+r_T) \right] \right\}. \end{aligned}$$

Solve (9) for

$$c_{T-1} = \left\{ \frac{1+r_T}{\theta} (\beta \theta c_T(A_T)^{\theta(1-\sigma)-1} + \kappa_{T-1}) \right\}^{\frac{1}{\theta(1-\sigma)-1}}.$$

From  $A_T$ , as used from the beginning of this construction, and  $c_{T-1}$  the constraint on aggregate motion (11) determines the corresponding current-period state by

$$A_{T-1} = \frac{A_T + c_{T-1} - y_{T-1}}{1+r_{T-1}}.$$

Part (i): Substituting for  $p_{T-1}$ , the multiplier  $\kappa_{T-1}$  can be written as

$$\begin{aligned} \kappa_{T-1} = \frac{\beta}{1+r_T-\mu} & \left\{ \frac{-\mu(1-\theta)}{A_T(1+r_T)} c_T(A_T)^{\theta(1-\sigma)} \right. \\ & \left. + \theta c_T(A_T)^{\theta(1-\sigma)-1} \left[ -\frac{\mu p_T(A_T)}{A_T(1+r_T)} - (1+r_T) \right] \right\}, \end{aligned}$$

so the requirement  $\kappa_{T-1} \geq 0$  can be verified by relying only on the successor-state  $A_T$  involved in the corresponding equilibrium combination.

Part (iii): Using the function (8) to substitute for  $p_T(A_T)$ , the requirement  $\kappa_{T-1} \geq 0$  amounts to

$$\frac{\beta}{1+r_T-\mu} \left\{ \frac{-\mu(1-\theta)}{A_T(1+r_T)} c_T(A_T)^{\theta(1-\sigma)} + \theta c_T(A_T)^{\theta(1-\sigma)-1} \left[ -\frac{1-\theta}{\theta} \frac{\mu c_T(A_T)}{A_T r(1+r_T)} - (1+r_T) \right] \right\} \geq 0.$$

Since our assumptions ensure that  $1+r_T-\mu > 0$ , this can be simplified to

$$\frac{\mu(1-\theta)}{-A_T(1+r_T)} + \frac{\mu(1-\theta)}{-A_T r(1+r_T)} - \frac{\theta(1+r_T)}{c_T(A_T)} \geq 0.$$

For the  $A_T < 0$ , which need to be considered in the case of a constrained equilibrium, this can be stated as

$$-A_T(1+r_T) \leq \mu \frac{(1-\theta)(1+r)}{\theta r(1+r_T)} c_T(A_T).$$

This is the same expression as above in the derivation of item (iii) for the unconstrained case. Therefore, using (7) to substitute for  $c_T(A_T)$ , we find again that this is satisfied by all  $A_T \geq \underline{A}_T$ , where the lower bound  $\underline{A}_T$  stands for

$$\underline{A}_T \equiv - \left[ \frac{\mu(1-\theta)}{\theta(1+r_T) + \mu(1-\theta)} \right] \left[ \frac{1+r}{r(1+r_T)} \right]^y.$$

#### A.4 Deriving Proposition 2: Coexistence of unconstrained and constrained equilibria

First, observe that the composition of functions to construct equilibria according to part (ii) of Proposition 1 produces exactly the same combination of variables for the unconstrained case as for the constrained case, if this construction is based *exactly on the lower bound  $\underline{A}_T$  for the successor-state*. This holds because the price  $p_{T-1}$  in the corresponding unconstrained equilibrium implies that the collateral constraint is satisfied just with equality, such that  $-\underline{A}_T(1+r_T) = \mu p_{T-1}$ , while a constrained equilibrium connecting to  $\underline{A}_T$  comes along with the limiting multiplier of just zero,  $\kappa_{T-1} = 0$ .

Second, pursue the *unconstrained branch* of equilibrium curves, by considering unconstrained equilibrium combinations connecting to the interior of the relevant range of successor-states for which unconstrained equilibria exist, for any  $A_T > \underline{A}_T$ . The changes in equilibrium combinations of variables on that branch of equilibrium curves can be inferred from the total differential of functions involved in the construction of an unconstrained equilibrium. Using the composition of functions stated for unconstrained equilibria in part (ii) of Proposition 1 and substituting for  $c_T(A_T)$  from function (7) we get

$$dc_{T-1} = [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} \frac{r(1+r_T)}{1+r} dA_T,$$

$$dA_{T-1} = \frac{dA_T + dc_{T-1}}{1+r_{T-1}}.$$

Thus any  $dA_T > 0$  goes along with an equilibrium combination of variables tied to  $dA_{T-1} > 0$ . From the corresponding construction of the price we get, after substituting for  $p_T(A_T)$  from function (8) and for  $c_T(A_T)$  from function (7)

$$dp_{T-1} = \frac{1-\theta}{\theta} dA_T.$$

Moving into the interior of the range of successor-states which are reached by an unconstrained equilibrium, by considering a variation  $dA_T > 0$ , thus produces  $dp_{T-1} > 0$ .

Finally, we pursue the *constrained branch* of equilibrium combinations, departing from the same point which involves the same lower bound of the successor-state  $\underline{A}_T$ , as in the previous analysis for the unconstrained branch. We check whether the movement along the constrained branch combines lower prices with the same direction of variation  $dA_{T-1} > 0$  in the current financial state of the economy. More specifically, we show that the price variation is unambiguously negative,  $dp_{T-1} < 0$ , and provide a condition to support  $dA_{T-1} > 0$ .

The following derivation of total differentials is based on the composition of functions for a constrained equilibrium, according to part (ii) of Proposition 1. For the price, we obtain

$$dp_{T-1} = \frac{-(1+r_T)}{\mu} dA_T,$$

and, therefore,  $dp_{T-1} < 0$ , when having  $dA_T > 0$ , for moving beyond the lower bound  $\underline{A}_T$  of the successor-state. For the financial state, the aggregate constraint on motion implies

$$dA_{T-1}(1+r_{T-1}) = dA_T + dc_{T-1} = dA_T \left( 1 + \frac{dc_{T-1}}{dA_T} \right).$$

Therefore,  $dA_T > 0$  combines with  $dA_{T-1} > 0$ , if  $\frac{dc_{T-1}}{dA_T} > -1$ . The remainder of the derivation determines  $\frac{dc_{T-1}}{dA_T}$ .

Defining  $\underline{c}_T \equiv c_T(\underline{A}_T)$ , substituting for  $\underline{A}_T$  from part (iii) of Proposition 1, and using the terminal stage consumption function (7) we obtain

$$\underline{c}_T = \frac{\theta(1+r_T)}{\theta(1+r_T) + \mu(1-\theta)} y,$$

and

$$\frac{\underline{c}_T}{\underline{A}_T} = -\frac{\theta r(1+r_T)^2}{\mu(1-\theta)(1+r)}.$$

Substituting for the terminal price from function (8) the multiplier becomes

$$\kappa_{T-1} = \frac{\beta}{1+r_T-\mu} \left[ \frac{1+r}{rp_{T-1}} (1-\theta)c_T^{\theta(1-\sigma)} - (1+r_T)\theta c_T^{\theta(1-\sigma)-1} \right].$$

Using this expression for the multiplier, we get

$$c_{T-1}^{\theta(1-\sigma)-1} = \frac{1+r_T}{\theta} \left( \beta [\theta c_T^{\theta(1-\sigma)-1}] \right. \\ \left. + \frac{\beta}{1+r_T-\mu} \left[ \frac{1+r}{rp_{T-1}} (1-\theta) c_T^{\theta(1-\sigma)} - (1+r_T) \theta c_T^{\theta(1-\sigma)-1} \right] \right).$$

Substituting for the price in a constrained equilibrium, this becomes

$$c_{T-1}^{\theta(1-\sigma)-1} = -\frac{\beta\mu}{\theta(1+r_T-\mu)} \frac{1+r}{rA_T} (1-\theta) c_T^{\theta(1-\sigma)} - \frac{\mu(1+r_T)\beta}{(1+r_T-\mu)} c_T^{\theta(1-\sigma)-1}.$$

Calculating the total differential:

$$[\theta(1-\sigma)-1] c_{T-1}^{\theta(1-\sigma)-2} dc_{T-1} \\ = -\frac{\beta\mu}{\theta(1+r_T-\mu)} \frac{1+r}{rA_T} (1-\theta) \theta(1-\sigma) c_T^{\theta(1-\sigma)-1} dc_T \\ + \frac{\beta\mu}{\theta(1+r_T-\mu)} \frac{1+r}{r} (1-\theta) c_T^{\theta(1-\sigma)} \frac{1}{(A_T)^2} dA_T \\ - \frac{\mu(1+r_T)\beta}{(1+r_T-\mu)} [\theta(1-\sigma)-1] c_T^{\theta(1-\sigma)-2} dc_T.$$

Since all the derivatives are evaluated at the specific equilibrium combination which includes the lower bound of the successor-state  $\underline{A}_T$ , meaning that the multiplier is just zero at this point, we can rely on the limiting—effectively unconstrained—relationship,

$$\underline{c}_{T-1} \equiv [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} \underline{c}_T.$$

Hence, substituting for

$$\underline{c}_{T-1}^{\theta(1-\sigma)-2} \equiv [\beta(1+r_T)]^{1-\frac{1}{\theta(1-\sigma)-1}} \underline{c}_T^{\theta(1-\sigma)-2}$$

and dividing by  $\underline{c}_T^{\theta(1-\sigma)-2}$  the total differential evaluated at this point is

$$[\theta(1-\sigma)-1][\beta(1+r_T)]^{1-\frac{1}{\theta(1-\sigma)-1}} dc_{T-1} \\ = -\frac{\beta\mu}{\theta(1+r_T-\mu)} \frac{1+r}{r\underline{A}_T} (1-\theta) \theta(1-\sigma) \underline{c}_T dc_T \\ + \frac{\beta\mu}{\theta(1+r_T-\mu)} \frac{1+r}{r} (1-\theta) \frac{1}{(\underline{A}_T)^2} \underline{c}_T^2 dA_T \\ - \frac{\mu(1+r_T)\beta}{(1+r_T-\mu)} [\theta(1-\sigma)-1] dc_T$$

Obtaining  $dc_T$  according to the terminal stage consumption function (7) and substituting for the ratio  $\underline{c}_T/\underline{A}_T$ , we get

$$\begin{aligned} & [\theta(1-\sigma)-1][\beta(1+r_T)]^{1-\frac{1}{\theta(1-\sigma)-1}} dc_{T-1} \\ &= \frac{\beta r(1+r_T)^3}{(1+r_T-\mu)(1+r)} \theta(1-\sigma) dA_T \\ &+ \frac{\beta}{(1+r_T-\mu)} \frac{\theta r(1+r_T)^4}{\mu(1-\theta)(1+r)} dA_T \\ &- \frac{\mu r(1+r_T)^2 \beta}{(1+r_T-\mu)(1+r)} [\theta(1-\sigma)-1] dA_T \end{aligned}$$

This can be rearranged as

$$\frac{dc_{T-1}}{dA_T} = \frac{r[\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}}}{1+r_T-\mu} \left[ \frac{\theta(1+r_T)^2}{(\theta(1-\sigma)-1)(1+r)} \left( 1-\sigma + \frac{1+r_T}{\mu(1-\theta)} \right) - \mu \frac{1+r_T}{1+r} \right].$$

The three steps of the derivation can be summarized as follows: If  $\frac{dc_{T-1}}{dA_T} > -1$ , a variation  $dA_T > 0$ —departing from  $\underline{A}_T$  to pursue the constrained branch of equilibria—produces a variation  $dA_{T-1} > 0$  in the current financial state involved in a constrained equilibrium combination, while lowering the price with respect to the point of departure. Unconstrained equilibrium combinations, at the specific point of departure coinciding with the constrained type, also exist for the same direction of variation  $dA_{T-1} > 0$ . The price response combined with pursuing the unconstrained branch is distinct from the constrained case.

#### A.5 Deriving Proposition 3: Constructing equilibrium belief-weightings and all equilibria in $T-2$

The derivations are organized by type of equilibrium, unconstrained and constrained, to maintain the focus on a constant structure of type-specific relationships. The steps exploit the equilibrium conditions stated by equations (19), (20), (21), (22), (23), and (24). Irrespective of the type of equilibrium, we base the construction of equilibrium combinations of variables on a successor-state  $A_{T-1}$ , in a first stage determine the set of all equilibrium belief-weightings of outcomes supported by  $A_{T-1}$ , and in a second stage construct equilibrium combinations of the remaining variables.

**A.5.1 Unconstrained equilibrium** For an unconstrained equilibrium, the derivation imposes  $\kappa_{T-2} = 0$  and verifies that  $-A_{T-1}(1+r_{T-1}) \leq \mu p_{T-2}$ .

(i) Setting  $\kappa_{T-2} = 0$  in equations (19) and (20), these two equations can be solved for the house price

$$p_{T-2} = \frac{E_{T-2}[(1-\theta)c_{T-1}^{\theta(1-\sigma)} + p_{T-1}\theta c_{T-1}^{\theta(1-\sigma)-1}]}{(1+r_{T-1})E_{T-2}[\theta c_{T-1}^{\theta(1-\sigma)-1}]} \quad (38)$$

Using this expression for  $p_{T-2}$  in the inequality  $-A_{T-1}(1+r_{T-1}) \leq \mu p_{T-2}$ , to be verified, we obtain the condition

$$-A_{T-1}(1+r_{T-1}) \leq \mu \frac{E_{T-2}[(1-\theta)c_{T-1}^{\theta(1-\sigma)} + p_{T-1}\theta c_{T-1}^{\theta(1-\sigma)-1}]}{(1+r_{T-1})E_{T-2}[\theta c_{T-1}^{\theta(1-\sigma)-1}]} \quad (39)$$

(ii) Setting  $\kappa_{T-2} = 0$ , equation (19) can be solved for  $c_{T-2}$ . Relying on the successor-state  $A_{T-1}$ —which our construction was based on—and on  $c_{T-2}$ , we obtain  $A_{T-2}$  from the aggregate constraint on motion (21). The house price  $p_{T-2}$  was already derived above in (38).

**A.5.2 Constrained equilibrium** For a constrained equilibrium, the derivation imposes  $-A_{T-1}(1+r_{T-1}) = \mu p_{T-2}$  and verifies that  $\kappa_{T-2} \geq 0$ .

(i) Using (19) to substitute for  $\theta c_{T-2}^{\theta(1-\sigma)-1}$ , equation (20) can be solved for

$$\kappa_{T-2} = \frac{\beta}{1+r_{T-1}-\mu} E_{T-2} \left\{ \frac{(1-\theta)}{p_{T-2}} c_{T-1}^{\theta(1-\sigma)} + \theta c_{T-1}^{\theta(1-\sigma)-1} \left[ \frac{p_{T-1}}{p_{T-2}} - (1+r_{T-1}) \right] \right\} \quad (40)$$

Using  $-A_{T-1}(1+r_{T-1}) = \mu p_{T-2}$  to substitute for  $p_{T-2}$ , the requirement  $\kappa_{T-2} \geq 0$  can therefore be expressed as

$$\begin{aligned} & \frac{\beta}{1+r_{T-1}-\mu} E_{T-2} \left\{ \frac{-\mu(1-\theta)}{A_{T-1}(1+r_{T-1})} c_{T-1}^{\theta(1-\sigma)} \right. \\ & \left. + \theta c_{T-1}^{\theta(1-\sigma)-1} \left[ \frac{-\mu p_{T-1}}{A_{T-1}(1+r_{T-1})} - (1+r_{T-1}) \right] \right\} \geq 0. \end{aligned} \quad (41)$$

Note that under our assumption  $\mu < 1+r_{T-1}$  and considering the case  $A_{T-1} < 0$ , as relevant for constrained equilibria, inequality (41) is equivalent to inequality (39).

(ii) The house price  $p_{T-2}$  is obtained from the constraint  $-A_{T-1}(1+r_{T-1}) = \mu p_{T-2}$ , which is binding in this equilibrium. This enables the use of (40), as already derived above, to determine  $\kappa_{T-2}$ . Using such an equilibrium value for the multiplier, we can then solve (19) for  $c_{T-2}$ . Finally, relying on this  $c_{T-2}$  and on the successor-state  $A_{T-1}$ —which our construction was based on—we obtain  $A_{T-2}$  from the aggregate constraint on motion (21).

#### A.6 Deriving Proposition 4: Nondegenerate range of equilibrium belief-weightings

We analyze the potential of specific financial states  $A_{T-1}$  to be reached as successor-states by equilibrium laws-of-motion in the initial period  $T-2$ . More specifically, we consider  $A_{T-1}$  in a range, which was already shown to support coexisting unconstrained and constrained equilibria under condition (18). Under the additional condition in Proposition 4, we show that such  $A_{T-1}$  satisfy existence requirements *strictly*, if beliefs in  $T-2$  have *full* weight on the unconstrained equilibrium price supported by  $A_{T-1}$ . It then follows that such an  $A_{T-1}$  is *still* a valid successor-state of an equilibrium law-of-motion, if beliefs vary in a *range* with some positive weights on other equilibrium prices

supported by  $A_{T-1}$ . We pursue unconstrained equilibria in  $T - 2$ , and perform the corresponding check of existence according to part (i) of Proposition 3, therefore aiming for the price  $p_{T-2}$  to be expressed in terms of  $A_{T-1}$ .<sup>20</sup>

The prices and the corresponding levels of consumption on the unconstrained branch of equilibrium curves in  $T - 1$ , according to the construction of unconstrained equilibria in Proposition 1, satisfy

$$p_{T-1} = \frac{1 - \theta}{\theta} \frac{c_T}{1 + r_T} + \frac{p_T}{1 + r_T},$$

$$c_{T-1} = [\beta(1 + r_T)]^{\frac{1}{\theta(1-\sigma)-1}} c_T.$$

Putting full weight on variables on this branch, and substituting for  $c_T$  (from equation (7)) and for  $p_T$  (from equation (8)), the equation for the price in an unconstrained equilibrium in  $T - 2$  becomes

$$p_{T-2} = \frac{1 - \theta}{\theta(1 + r_{T-1})} \left( [\beta(1 + r_T)]^{\frac{1}{\theta(1-\sigma)-1}} + \frac{1}{1 + r_T} \right) \left( \frac{r(1 + r_T)}{1 + r} A_T + y \right) + \frac{\frac{1 - \theta}{\theta} \left( \frac{1 + r_T}{1 + r} A_T + \frac{y}{r} \right)}{(1 + r_T)(1 + r_{T-1})}. \quad (42)$$

The constraint on the motion of the financial state

$$A_T = (1 + r_{T-1})A_{T-1} + y_{T-1} - c_{T-1},$$

and unconstrained equilibrium consumption

$$c_{T-1} = [\beta(1 + r_T)]^{\frac{1}{\theta(1-\sigma)-1}} \left( \frac{r(1 + r_T)}{1 + r} A_T + y \right),$$

in  $T - 1$  imply a positive linear relationship for the unconstrained branch of the equilibrium law-of-motion  $A_T(A_{T-1})$ ,

$$A_T(A_{T-1}) = \frac{(1 + r_{T-1})A_{T-1} + y_{T-1} - [\beta(1 + r_T)]^{\frac{1}{\theta(1-\sigma)-1}} y}{1 + [\beta(1 + r_T)]^{\frac{1}{\theta(1-\sigma)-1}} \frac{r(1 + r_T)}{1 + r}}. \quad (43)$$

The combination of (42) and (43) establishes a positive linear (and thus monotonic) relationship  $p_{T-2}(A_{T-1})$ . Let  $\underline{A}_{T-1}$  denote the level of the financial state in  $T - 1$  which, in line with equation (43), connects to the lower bound  $\underline{A}_T$  of the successor-state, such that

$$A_T(\underline{A}_{T-1}) = \underline{A}_T.$$

By monotonicity of (43),  $\underline{A}_{T-1}$  is the lower bound of those financial states which support an unconstrained equilibrium in  $T - 1$ . Therefore, if at  $\underline{A}_{T-1}$  the condition

<sup>20</sup>Constrained equilibria in  $T - 2$  and their existence, according to part (i) of Proposition 3, could be pursued equivalently, by aiming to express the multiplier  $\kappa_{T-2} \geq 0$  in terms of  $A_{T-1}$ .



$p_{T-2}(\underline{A}_{T-1}) \geq -\frac{(1+r_{T-1})}{\mu} \underline{A}_{T-1}$  is satisfied with weak inequality, as required for the existence of an unconstrained equilibrium in  $T-2$ , this condition is satisfied with *strict* inequality for all  $A_{T-1} > \underline{A}_{T-1}$ . This includes the range for  $A_{T-1}$  on which coexistence of unconstrained and constrained equilibria is ensured by Proposition 2. In the remainder of this derivation, we show under which condition  $p_{T-2}(\underline{A}_{T-1}) \geq -\frac{(1+r_{T-1})}{\mu} \underline{A}_{T-1}$  is satisfied.

Use  $A_T(\underline{A}_{T-1}) = \underline{A}_T$ , when substituting from (42) on the LHS of the previous inequality, and substituting from (43) on the RHS of this inequality:

$$\begin{aligned} & \frac{1-\theta}{\theta(1+r_{T-1})} \left( [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} + \frac{1}{1+r_T} \right) \left( \frac{r(1+r_T)}{1+r} \underline{A}_T + y \right) \\ & + \frac{1-\theta}{\theta} \left( \frac{1+r_T}{1+r} \underline{A}_T + \frac{y}{r} \right) \\ & + \frac{1-\theta}{(1+r_T)(1+r_{T-1})} \\ & \geq -\frac{1}{\mu} \left[ \underline{A}_T \left( 1 + [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} \frac{r(1+r_T)}{1+r} \right) + [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} y - y_{T-1} \right]. \end{aligned}$$

Substituting for the lower bound  $\underline{A}_T$  from part (iii) of Proposition 1, this can be simplified as

$$\begin{aligned} & \frac{r_T - r_{T-1}}{1+r_{T-1}} \left( \frac{1+r}{r(1+r_T)} + [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} \right) \frac{\mu(1-\theta)}{\theta(1+r_T) + \mu(1-\theta)} \\ & + [\beta(1+r_T)]^{\frac{1}{\theta(1-\sigma)-1}} - \frac{y_{T-1}}{y} \geq 0. \end{aligned}$$

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