Supplement to "Global identification of linearized DSGE models"

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A.1. SOME REMARKS ON APPLYING OUR IDENTIFICATION ANALYSIS TO THE SMETS–WOUTERS MODEL

In this supplementary material, we show how the widely cited medium-sized DSGE model of Smets and Wouters (2007) can be easily fitted into our identification analysis. The model has rich dynamics, its equilibrium conditions feature variables that show up both in conditional expectations and lags, and some shocks have an ARMA structure. These characteristics make this model a good object to demonstrate the generality of our approach.

We first present the full list of equations making up the Smets and Wouters (2007) model and show how to define the vector of states s_t and policy variables p_t. We next demonstrate how some of the model structural equations can be used to recover the policy function matrices F and G if one wants to speed up Algorithm 1 as discussed in Section 5.2.

A.1.1 *List of model equations and classification of variables*

The Smets–Wouters model is made up of the following 34 equations:

$$y_t = c_y^{ss} c_t + i_y^{ss} i_t + r_k^{ss} k_y^{ss} z_t + \varepsilon_t^g, \tag{A.1}$$

$$c_t = \frac{\lambda \gamma^{-1}}{1 + \lambda \gamma^{-1}} c_{t-1} + \frac{1}{1 + \lambda \gamma^{-1}} E_t c_{t+1} + \frac{w_{lc}^{ss}(\sigma_c - 1)}{\sigma_c(1 + \lambda \gamma^{-1})}(l_t - E_t l_{t+1})$$
$$- \frac{1 - \lambda \gamma^{-1}}{(1 + \lambda \gamma^{-1})\sigma_c}(r_t - E_t \pi_{t+1} + \varepsilon_t^b), \tag{A.2}$$

$$i_t = \frac{1}{1 + \beta \gamma^{1-\sigma_c}} i_{t-1} + \frac{\beta \gamma^{1-\sigma_c}}{1 + \beta \gamma^{1-\sigma_c}} E_t i_{t+1} + \frac{1}{(1 + \beta \gamma^{1-\sigma_c})\varphi \gamma^2} q_t + \varepsilon_t^i, \tag{A.3}$$

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$$q_t = \beta\gamma^{-\sigma_c}(1-\delta)E_t q_{t+1} + \left[1 - \beta\gamma^{-\sigma_c}(1-\delta)\right]E_t r_{t+1}^k - \left(r_t - E_t \pi_{t+1} + \varepsilon_t^b\right), \tag{A.4}$$

$$y_t = \phi_p\left[\alpha k_t^s + (1-\alpha)l_t + \varepsilon_t^a\right], \tag{A.5}$$

$$k_t^s = k_{t-1} + z_t, \tag{A.6}$$

$$z_t = \frac{1-\psi}{\psi}r_t^k, \tag{A.7}$$

$$r_t^k = -\left(k_t^s - l_t\right) + w_t, \tag{A.8}$$

$$k_t = (1-\delta)\gamma^{-1}k_{t-1} + \left[1 - (1-\delta)\gamma^{-1}\right]\left[i_t + (1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2\varepsilon_t^i\right], \tag{A.9}$$

$$\mu_t^p = \alpha\left(k_t^s - l_t\right) + \varepsilon_t^a - w_t, \tag{A.10}$$

$$\pi_t = \frac{\iota_p}{1+\beta\gamma^{1-\sigma_c}\iota_p}\pi_{t-1} + \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p}E_t\pi_{t+1}$$
$$- \frac{(1-\beta\gamma^{1-\sigma_c}\xi_p)(1-\xi_p)}{(1+\beta\gamma^{1-\sigma_c}\iota_p)\xi_p\left[(\phi_p-1)\varepsilon_p+1\right]}\mu_t^p + \varepsilon_t^p, \tag{A.11}$$

$$\mu_t^w = w_t - \sigma_l l_t - \frac{1}{1-\lambda\gamma^{-1}}\left(c_t - \lambda\gamma^{-1}c_{t-1}\right), \tag{A.12}$$

$$w_t = \frac{1}{1+\beta\gamma^{1-\sigma_c}}(w_{t-1} + \iota_w\pi_{t-1}) + \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}}E_t(w_{t+1} + \pi_{t+1})$$
$$- \frac{1+\beta\gamma^{1-\sigma_c}\iota_w}{1+\beta\gamma^{1-\sigma_c}}\pi_t - \frac{(1-\beta\gamma^{1-\sigma_c}\xi_w)(1-\xi_w)}{(1+\beta\gamma^{1-\sigma_c})\xi_w\left[(\phi_w-1)\varepsilon_w+1\right]}\mu_t^w + \varepsilon_t^w, \tag{A.13}$$

$$r_t = \rho r_{t-1} + (1-\rho)\left[r_\pi\pi_t + r_y(y_t - y_t^*)\right] + r_{\Delta y}\left[(y_t - y_t^*) - (y_{t-1} - y_{t-1}^*)\right] + \varepsilon_t^r, \tag{A.14}$$

$$y_t^* = c_y^{ss}c_t^* + i_y^{ss}i_t^* + r_k^{ss}k_y^{ss}z_t^* + \varepsilon_t^g, \tag{A.15}$$

$$c_t^* = \frac{\lambda\gamma^{-1}}{1+\lambda\gamma^{-1}}c_{t-1}^* + \frac{1}{1+\lambda\gamma^{-1}}E_t c_{t+1}^* + \frac{w_{lc}^{ss}(\sigma_c-1)}{\sigma_c(1+\lambda\gamma^{-1})}\left(l_t^* - E_t l_{t+1}^*\right)$$
$$- \frac{1-\lambda\gamma^{-1}}{(1+\lambda\gamma^{-1})\sigma_c}\left(r_t^* + \varepsilon_t^b\right), \tag{A.16}$$

$$i_t^* = \frac{1}{1+\beta\gamma^{1-\sigma_c}}i_{t-1}^* + \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}}E_t i_{t+1}^* + \frac{1}{(1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2}q_t^* + \varepsilon_t^i, \tag{A.17}$$

$$q_t^* = \beta\gamma^{-\sigma_c}(1-\delta)E_t q_{t+1}^* + \left[1 - \beta\gamma^{-\sigma_c}(1-\delta)\right]E_t r_{t+1}^{k*} - \left(r_t^* + \varepsilon_t^b\right), \tag{A.18}$$

$$y_t^* = \phi_p\left[\alpha k_t^{s*} + (1-\alpha)l_t^* + \varepsilon_t^a\right], \tag{A.19}$$

$$k_t^{s*} = k_{t-1}^* + z_t^*, \tag{A.20}$$

$$z_t^* = \frac{1-\psi}{\psi}r_t^{k*}, \tag{A.21}$$

$$r_t^{k*} = -\left(k_t^{s*} - l_t^*\right) + w_t^*, \tag{A.22}$$

$$k_t^* = (1-\delta)\gamma^{-1}k_{t-1}^* + \left[1 - (1-\delta)\gamma^{-1}\right]\left[i_t^* + (1+\beta\gamma^{1-\sigma_c})\varphi\gamma^2\varepsilon_t^i\right], \tag{A.23}$$

$$\mu_t^{p*} = \alpha\left(k_t^{s*} - l_t^*\right) + \varepsilon_t^a - w_t^*, \tag{A.24}$$

$$\mu_t^{p^*} = 0, \tag{A.25}$$

$$\mu_t^{w*} = w_t^* - \sigma_l l_t^* - \frac{1}{1 - \lambda\gamma^{-1}}\left(c_t^* - \lambda\gamma^{-1}c_{t-1}^*\right), \tag{A.26}$$

$$\mu_t^{w*} = 0, \tag{A.27}$$

$$\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \sigma_a \eta_t^a, \tag{A.28}$$

$$\varepsilon_t^b = \rho_b \varepsilon_{t-1}^b + \sigma_b \eta_t^b, \tag{A.29}$$

$$\varepsilon_t^g = \rho_g \varepsilon_{t-1}^g + \rho_{ga}\sigma_a \eta_t^a + \sigma_g \eta_t^g, \tag{A.30}$$

$$\varepsilon_t^i = \rho_i \varepsilon_{t-1}^i + \sigma_i \eta_t^i, \tag{A.31}$$

$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + \sigma_r \eta_t^r, \tag{A.32}$$

$$\varepsilon_t^p = \rho_p \varepsilon_{t-1}^p + \sigma_p \eta_t^p - \mu_p \sigma_p \eta_{t-1}^p, \tag{A.33}$$

$$\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \sigma_w \eta_t^w - \mu_w \sigma_w \eta_{t-1}^w. \tag{A.34}$$

Equations (A.1)–(A.14) are the equilibrium conditions summarizing the constraints and choices made by agents populating the model economy, equations (A.15)–(A.27) define a hypothetical economy that would prevail under flexible prices and wages in the absence of markup shocks,[1] while equations (A.28)–(A.34) describe the processes driving structural shocks. Symbols without a time subscript are the model structural parameters or their explicit functions, the latter indicated with a superscript "ss." See Smets and Wouters (2007) or Iskrev (2010) for more detail.

Since our representation of a DSGE model given by form (1) does not allow for lags in innovations to structural shocks, we need to rewrite the two ARMA processes (A.33) and (A.34) using their state space representations as follows:

$$\varepsilon_t^p = \tilde{\varepsilon}_{t-1}^p + \sigma_p \eta_t^p, \tag{A.35}$$

$$\tilde{\varepsilon}_t^p = \rho_p \tilde{\varepsilon}_{t-1}^p + (\rho_p - \mu_p)\sigma_p \eta_t^p, \tag{A.36}$$

$$\varepsilon_t^w = \tilde{\varepsilon}_{t-1}^w + \sigma_w \eta_t^w, \tag{A.37}$$

$$\tilde{\varepsilon}_t^w = \rho_w \tilde{\varepsilon}_{t-1}^w + (\rho_w - \mu_w)\sigma_w \eta_t^w. \tag{A.38}$$

Fitting this extended model, consisting of 36 equations (A.1)–(A.32) and (A.35)–(A.38), into form (1) is straightforward if we define the vectors of states s_t, policy variables p_t and structural shocks ε_t as follows:

$$s_t = \begin{bmatrix} y_t & c_t & i_t & k_t & \pi_t & w_t & r_t & y_t^* \\ c_t^* & i_t^* & k_t^* & \varepsilon_t^a & \varepsilon_t^b & \varepsilon_t^g & \varepsilon_t^i & \varepsilon_t^r & \tilde{\varepsilon}_t^p & \tilde{\varepsilon}_t^w \end{bmatrix}', \tag{A.39}$$

$$p_t = \begin{bmatrix} z_t & q_t & l_t & r_t^k & k_t^s & \mu_t^p & \mu_t^w \\ z_t^* & q_t^* & l_t^* & r_t^{k*} & k_t^{s*} & \mu_t^{p*} & \mu_t^{w*} & w_t^* & r_t^* & \varepsilon_t^p & \varepsilon_t^w \end{bmatrix}', \tag{A.40}$$

$$\varepsilon_t = \begin{bmatrix} \eta_t^a & \eta_t^b & \eta_t^g & \eta_t^i & \eta_t^r & \eta_t^p & \eta_t^w \end{bmatrix}' \sim i.i.d.\ N(0, \mathrm{I}_7). \tag{A.41}$$

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[1]This block is needed to describe the evolution of potential output y_t^* that enters the monetary policy rule (A.14).

Note that any variable that shows up in at least one of the model equilibrium conditions with a lag has to be a part of the state vector, even if in the underlying dynamic programming problem it is a control variable, that is, it is determined at any given point in time. For example, because the model features habit formation, the Euler equation (A.2) includes the lagged value of consumption c_{t-1}, and hence this variable must be included in vector s_t. It would be classified as a policy variable if we introduced an additional model equation defining the habit stock as $h_t = c_t$ and replaced c_{t-1} with h_{t-1} in equation (A.2), in which case h_t would enter the vector of states.

   This example shows that the classification of variables in our framework depends on whether some of them are substituted out from the original equilibrium conditions or not. However, the rule according to which we should assign the endogenous variables to either group is very simple: all variables showing up in lags enter vector s_t, the remaining ones form vector p_t.

   As discussed in the supplementary material to Komunjer and Ng (2011), the ABCD-representation corresponding to the states vector (A.39) violates Assumption 3. Therefore, to apply our global identification framework, we follow Komunjer and Ng (2011) and define an auxiliary variable

$$\tilde{r}_t = \rho r_t - r_{\Delta y}\big(y_t - y_t^*\big) \tag{A.42}$$

so that the monetary policy rule (A.14) can be rewritten as follows:

$$r_t = \tilde{r}_{t-1} + (1 - \rho)\big[r_\pi \pi_t + r_y\big(y_t - y_t^*\big)\big] + r_{\Delta y}\big(y_t - y_t^*\big) + \varepsilon_t^r. \tag{A.43}$$

In the system consisting of 37 equations (A.1)–(A.13), (A.15)–(A.32), (A.35)–(A.38), and (A.42)–(A.43), r_t, y_t, and y_t^* can now be classified as policy variables, and replaced in the state vector by a new variable $\tilde{r}_t$. As a result, vectors s_t and p_t become

$$s_t = \begin{bmatrix} c_t & i_t & k_t & \pi_t & w_t & c_t^* & i_t^* & k_t^* & \varepsilon_t^a & \varepsilon_t^b & \varepsilon_t^g & \varepsilon_t^i & \varepsilon_t^r & \tilde{\varepsilon}_t^p & \tilde{\varepsilon}_t^w & \tilde{r}_t \end{bmatrix}', \tag{A.44}$$

$$p_t = \begin{bmatrix} y_t & z_t & q_t & l_t & r_t^k & k_t^s & \mu_t^p & \mu_t^w & r_t \\ y_t^* & z_t^* & q_t^* & l_t^* & r_t^{k*} & k_t^{s*} & \mu_t^{p*} & \mu_t^{w*} & w_t^* & \varepsilon_t^p & \varepsilon_t^w & r_t^* \end{bmatrix}' \tag{A.45}$$

and the underlying ABCD-representation obeys Assumption 3; see the supplementary material to Komunjer and Ng (2011). Hence, our identification framework can be applied.

A.1.2 *Recovering policy functions F and G to speed up Algorithm 1*

As we discussed in Section 5.2, the time needed to execute Algorithm 1 can be reduced by using some of the structural model equations represented by form (1), together with matrices A and B that describe the law of motion for states s_t, to back out matrices F and G that determine equilibrium values of policy variables p_t. To this end, we have derived formulas that explicitly solve for these two policy matrices, or at least some of their rows. In this section, we show that this procedure allows to fully recover matrices F and G in the Smets and Wouters (2007) model.

The first step is to identify the model equilibrium conditions that fit the form given by (17). These are all of the 37 equations making up the model with state vector (A.44), except for (A.2), (A.4), (A.16), and (A.18). After some rearrangement in the sequence of these equations, the associated matrix Φ_0^p can be written as follows:

$$\Phi_0^p = \begin{bmatrix} Z & 0_{20 \times 1} \\ \times & 0_{13 \times 1} \end{bmatrix}, \tag{A.46}$$

where Z is a 20×20 lower block triangular matrix comprising six diagonal blocks placed in the following order:

$$Z_1 = \begin{bmatrix} 1 & -r_k^{ss}k_y^{ss} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{\left(1+\beta\gamma^{1-\sigma_c}\right)\varphi\gamma^2} & 0 & 0 & 0 \\ 1 & 0 & 0 & -\phi_p(1-\alpha) & 0 & -\phi_p\alpha \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\dfrac{1-\psi}{\psi} & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}, \tag{A.47}$$

$$Z_2 = I_2, \tag{A.48}$$

$$Z_3 = \begin{bmatrix} -\rho & -r_{\Delta y} \\ 1 & (1-\rho)r_y + r_{\Delta y} \end{bmatrix}, \tag{A.49}$$

$$Z_4 = \begin{bmatrix} -r_k^{ss}k_y^{ss} & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{1}{\left(1+\beta\gamma^{1-\sigma_c}\right)\varphi\gamma^2} & 0 & 0 & 0 \\ 0 & 0 & -\phi_p(1-\alpha) & 0 & -\phi_p\alpha \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -\dfrac{1-\psi}{\psi} & 0 \end{bmatrix}, \tag{A.50}$$

$$Z_5 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \tag{A.51}$$

$$Z_6 = I_2 \tag{A.52}$$

and the rows of which correspond to the following 20 model equations: Z_1—equations (A.1), (A.3), and (A.5)–(A.8); Z_2—equations (A.10) and (A.12); Z_3—equations (A.42)–(A.43); Z_4—equations (A.15), (A.17) and (A.19)–(A.21); Z_5—equations (A.24), (A.25), and (A.27); Z_6—equations (A.35) and (A.37). Note that Z_4 is the upper-right submatrix of Z_1.

Provided that each Z_i is nonsingular (which holds for almost all deep model parameters), the first 20 columns of Φ_0^p form a full column rank submatrix (since Z is nonsingular). It means that we can recover the first 20 out of 21 rows of policy matrices F and G as explained in Section 5.2. To obtain the last row, we need equations that can be written

in form (21). Bearing in mind the notation from Section 5.2, we have $\check{p}_t = r_t^*$. Since r_t^* shows up in equations (A.16) and (A.18) that do not involve conditional expectations of r_t^*, we get $\check{\Psi}_0^p = [\,\frac{1-\lambda\gamma^{-1}}{(1+\lambda\gamma^{-1})\sigma_c}\ 1]'$. Since the latter is of full column rank, we can recover the last row of F and G associated with $\check{p}_t = r_t^*$ as explained in Section 5.2.

To provide more intuition on what happens behind the scenes of these mechanical matrix operations, we now additionally show how matrices F and G can be fully retrieved by working directly with selected model equations. Note that, to convey our point, we just need to demonstrate how each of the policy variables included in vector (A.45) can be expressed as a function of (lagged, current, or expected future) states collected in vector (A.44), as well as shocks collected in vector (A.41). Having completed this step, the next one is trivial and boils down to using the law of motion for states (3) to eliminate current and expected future states so that, conformably with (4), each policy variable is expressed as a function of lagged states and current shocks.

Again, equations that do not involve taking expectations in policy variables (i.e., having form (17)) can be particularly useful in recovering F and G, and hence we will use them first. For a start, let us use equations (A.1), (A.5), (A.6), (A.7), and (A.8) to solve for z_t as a function of states

$$z_t = \left(\frac{(1-\alpha\psi)\phi_p}{1-\psi} - r_k^{ss}k_y^{ss}\right)^{-1}\left(c_y^{ss}c_t + i_y^{ss}i_t + \varepsilon_t^g + \phi_p\big[(1-\alpha)w_t - k_{t-1} - \varepsilon_t^a\big]\right). \quad \text{(A.53)}$$

A similar solution for z_t^* can be obtained by using equations (A.20), (A.21), (A.24), (A.25), (A.22), (A.26), and (A.27):

$$z_t^* = \frac{1-\psi}{\psi+(1-\alpha)(1-\psi)}\left((\alpha-1)k_{t-1}^* + \frac{\alpha-1}{\sigma_l\big(1-\lambda\gamma^{-1}\big)}\big(c_t^* - \lambda\gamma^{-1}c_{t-1}^*\big) + \varepsilon_t^a\right). \quad \text{(A.54)}$$

These two expressions allow us to obtain the formulas for all but one remaining variables collected in vector p_t by consecutive substitutions, so that we obtain y_t from (A.1), k_t^s from (A.6), l_t from (A.5), r_t^k from (A.7), q_t from (A.3), μ_t^p from (A.10), μ_t^w from (A.12), y_t^* from (A.15), k_t^{s*} from (A.20), l_t^* from (A.19), r_t^{k*} from (A.21), q_t^* from (A.17), w_t^* from (A.22), μ_t^{p*} from (A.24), μ_t^{w*} from (A.26), r_t from (A.43), ε_t^p from (A.35), and ε_t^w from (A.37).

The only policy variable that cannot be expressed as a function of states and shocks using equations in form (17) is r_t^*. This is because this variable shows up only in equations featuring conditional expectations of policy variables, which means that matrix Φ_0^p has a corresponding zero column (see (A.46)), and hence it is rank deficient. However, the missing policy functions can also be recovered using the obtained solutions for other policy variables and equilibrium conditions in form (21). In our case, we need just one equation of this type that has not been already used, and we can choose from equation (A.16) and (A.18). For example, if we pick the former, all we need to express r_t^* as a function of states and shocks is to plug into it the solution for l_t^* derived above.

The final step to obtain F and G is to get rid of the current period and expected future states. This is straightforward using the law of motion for states (3) with given matrices A and B, and the property of shocks $E_t \varepsilon_{t+1} = 0$ so that $E_t s_{t+1} = A s_t = A^2 s_{t-1} + AB \varepsilon_t$.

A.2. ANALYTICAL IDENTIFICATION ANALYSIS IN A SIMPLE A-THEORETICAL STATE-SPACE MODEL

Checking global identification of the simple state-space system discussed in Section 6 at some θ boils down to solving the five matrix restrictions included in Theorem 1 for $\bar{\theta}$, T, and U.

First note that, since $\Sigma = \bar{\Sigma} = 1$, restriction 5) of Theorem 1 implies that $U = 1$ or $U = -1$.[2] Then, from restriction 4) we get $\bar{\alpha}_2 = \alpha_2$ or $\bar{\alpha}_2 = 2 - \alpha_2$, respectively. The second solution does not lie within a unit interval, so it must be that $U = 1$ and $\bar{\alpha}_2 = \alpha_2$. Using this result, the remaining three restrictions included in Theorem 1 yield the following system of equations:

$$\bar{\alpha}_1^2 t_{11} = \alpha_1^2 t_{11} + (1 - \alpha_1^2 - \alpha_1^2 \alpha_2) t_{12}, \tag{A.55}$$

$$\bar{\alpha}_1^2 t_{12} = (1 - \alpha_1^2) t_{12}, \tag{A.56}$$

$$(1 - \bar{\alpha}_1^2 - \bar{\alpha}_1^2 \alpha_2) t_{11} + (1 - \bar{\alpha}_1^2) t_{21} = \alpha_1^2 t_{21} + (1 - \alpha_1^2 - \alpha_1^2 \alpha_2) t_{22}, \tag{A.57}$$

$$(1 - \bar{\alpha}_1^2 - \bar{\alpha}_1^2 \alpha_2) t_{12} + (1 - \bar{\alpha}_1^2) t_{22} = (1 - \alpha_1^2) t_{22}, \tag{A.58}$$

$$1 = t_{11} - \alpha_2 t_{12}, \tag{A.59}$$

$$-\alpha_2 = t_{21} - \alpha_2 t_{22}, \tag{A.60}$$

$$(1 - \bar{\alpha}_1^2 \alpha_2) t_{11} + (1 - \bar{\alpha}_1^2) t_{21} = 1 - \alpha_1^2 \alpha_2, \tag{A.61}$$

$$(1 - \bar{\alpha}_1^2 \alpha_2) t_{12} + (1 - \bar{\alpha}_1^2) t_{22} = 1 - \alpha_1^2, \tag{A.62}$$

where t_{ij} denotes the (i, j) element of T.

From equation (A.56), we can see that $t_{12} = 0$ or $\bar{\alpha}_1 = (1 - \alpha_1^2)^{\frac{1}{2}}$. Let us first deal with the case $t_{12} = 0$. From equation (A.59), we immediately have $t_{11} = 1$. Since equation (A.62) implies $t_{22} \neq 0$, solving equation (A.58) gives $\bar{\alpha}_1 = \alpha_1$. Then from equation (A.62), we have $t_{22} = 1$ and from equation (A.60) we get $t_{21} = 0$. It is easy to verify that this solution is also consistent with equations (A.55), (A.57), and (A.61), so that eventually we have $\bar{\theta} = \theta$, $T = I_2$, and $U = 1$, which is the point at which we are checking global identification.

If instead $\bar{\alpha}_1 = (1 - \alpha_1^2)^{\frac{1}{2}}$, then solving equations (A.58) and (A.62) result in

$$t_{12} = \frac{1 - 2\alpha_1^2}{(1 - \alpha_1^2)(1 - \alpha_2)}, \tag{A.63}$$

$$t_{22} = \frac{\alpha_1^2(1 + \alpha_2) - \alpha_2}{(1 - \alpha_1^2)(1 - \alpha_2)} \tag{A.64}$$

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[2]This also immediately follows from Corollary 1, according to which $U = V$, where V is orthogonal.

and plugging this solution into equations (A.59) and (A.60) gives

$$t_{11} = \frac{1 - \alpha_1^2(1 + \alpha_2)}{(1 - \alpha_1^2)(1 - \alpha_2)},$$ (A.65)

$$t_{21} = \frac{\alpha_2(2\alpha_1^2 - 1)}{(1 - \alpha_1^2)(1 - \alpha_2)}.$$ (A.66)

It is easy to verify that this solution also satisfies equations (A.55)–(A.57) and (A.61).

Overall, we can conclude that there are two solutions to the system of restrictions from Theorem 1 and there exist $\bar{\theta} = [(1 - \alpha_1^2)^{\frac{1}{2}} \ \alpha_2]'$ that is observationally equivalent to $\theta = [\alpha_1 \ \alpha_2]'$. These two parameterizations are linked to each other via the similarity transformation, with matrices T and U given by

$$T = \frac{1}{(1 - \alpha_1^2)(1 - \alpha_2)} \begin{bmatrix} 1 - \alpha_1^2(1 + \alpha_2) & 1 - 2\alpha_1^2 \\ \alpha_2(2\alpha_1^2 - 1) & \alpha_1^2(1 + \alpha_2) - \alpha_2 \end{bmatrix}, \qquad U = 1$$

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