Supplement to “Ambiguity and the historical equity premium”

(*Quantitative Economics*, Vol. 9, No. 2, July 2018, 945–993)

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1. Details of the numerical solution procedure

1.1 *Solution method*

This section describes the minimum weighted residuals method we use to obtain an approximate solution for the value function and the risky rate. We then explain how we assess the accuracy of the method.

Both the value function and the risky rate are approximated by a parametric function of the form

$$\Phi_y(X_t) = \exp\left(\sum_{i_c, i_h, i_\ell, i_\eta \in \mathcal{I}} \theta^y_{i_c, i_h, i_\ell, i_\eta} H_{i_c}\big(\varphi_c(C_t)\big) H_{i_h}\big(\varphi_h(\widehat{x}_{h,t})\big) H_{i_\ell}\big(\varphi_\ell(\widehat{x}_{\ell,t})\big) H_{i_\eta}\big(\varphi_\eta(\eta_t)\big)\right)$$

where $X_t \equiv (C_t, \widehat{x}_{h,t}, \widehat{x}_{\ell,t}, \eta_t)$ denotes the vector of state variables[1] and $y \in \{V, R\}$. The set of indices $\mathcal{I}$ is defined by

$$\mathcal{I} = \big\{i_z = 1, \dots, n_z; z \in \{C, h, \ell, \eta\} \mid i_c + i_h + i_\ell + i_\eta \le \max(n_c, n_h, n_\ell, n_\eta)\big\}.$$

Implicit in the definition of this set is that we are considering a complete basis of polynomials.[2] $H_\iota(\cdot)$ is a Hermite polynomial of order ι and $\varphi_z(\cdot)$ is a strictly increasing function that maps $\mathbb{R}$ into $\mathbb{R}$. This function is used to maps Hermitian nodes into values for

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[1] When persistence is known, the vector of state variables reduces to $X_t = (C_t, x_t)$ and the approximant takes the simpler form $\Phi_y(X_t) = \exp(\sum_{i_c, i_x \in \mathcal{I}} \theta^y_{i_c, i_x} H_{i_c}(\varphi_c(C_t)) H_{i_x}(\varphi_x(\widehat{x}_t)))$.

[2] See Judd (1998), Chapter 7.

Available at http://qeconomics.org. https://doi.org/10.3982/QE708

the vector of state variables, $X_t \equiv (C_t, \widehat{x}_{h,t}, \widehat{x}_{\ell,t}, \eta_t)$,[3] The parameters θ^y, $y \in \{V, R\}$, are then determined by a minimum weighted residuals method. More precisely, we define the residuals associated to both the direct Value function equation, $\mathcal{R}_V(\theta^V; X_t)$, and the Euler equations for risky assets (consumption claims and dividend claims), $\mathcal{R}_R(\theta^V; X_t)$, as

$$\mathcal{R}_V\big(\theta^V; X_t\big) \equiv \Phi_V\big(C_t, \widehat{x}_t^h, \widehat{x}_t^\ell, \eta_t\big) - (1-\beta)u(C_t) - \frac{\beta}{\alpha}\ln(\mathcal{V}_{t+1})$$

where

$$\begin{aligned}\mathcal{V}_{t+1} \equiv \eta_t &\int_{-\infty}^{\infty} \exp\biggl(-\alpha \iiint_{-\infty}^{\infty} \Phi_V\bigl(C_{t+1}^{(\ell)}, \widehat{x}_{h,t+1}^{(\ell)}, \widehat{x}_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}\bigr)\,\mathrm{d}F(\vec{\varepsilon}_{\ell,t+1})\biggr)\,\mathrm{d}F(x_{\ell,t}) \\ &+ (1-\eta_t) \\ &\times \int_{-\infty}^{\infty} \exp\biggl(-\alpha \iiint_{-\infty}^{\infty} \Phi_V\bigl(C_{t+1}^{(h)}, \widehat{x}_{h,t+1}^{(h)}, \widehat{x}_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}\bigr)\,\mathrm{d}F(\vec{\varepsilon}_{h,t+1})\biggr)\,\mathrm{d}F(x_{h,t})\end{aligned}$$

and

$$\mathcal{R}_R\big(\theta^R, \theta^V; X_t\big) \equiv u'(C_t) - \beta\mathcal{E}_{t+1}$$

where

$$\begin{aligned}\mathcal{E}_{t+1} \equiv \eta_t &\int_{-\infty}^{\infty} \biggl(\xi_{\ell,t} \iiint_{-\infty}^{\infty} u'\bigl(C_{t+1}^{(\ell)}\bigr)\Phi_R\bigl(C_{t+1}^{(\ell)}, \widehat{x}_{h,t+1}^{(\ell)}, \widehat{x}_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}\bigr) \\ &\times \underbrace{\frac{D_{t+1}^{(\ell)}}{D_t}}_{\text{(i)}}\,\mathrm{d}F(\vec{\varepsilon}_{\ell,t+1})\biggr)\,\mathrm{d}F(x_{\ell,t}) \\ &+ (1-\eta_t)\int_{-\infty}^{\infty} \biggl(\xi_{h,t} \iiint_{-\infty}^{\infty} u'\bigl(C_{t+1}^{(h)}\bigr)\Phi_R\bigl(C_{t+1}^{(h)}, \widehat{x}_{h,t+1}^{(h)}, \widehat{x}_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}\bigr) \\ &\times \underbrace{\frac{D_{t+1}^{(h)}}{D_t}}_{\text{(ii)}}\,\mathrm{d}F(\vec{\varepsilon}_{h,t+1})\biggr)\,\mathrm{d}F(x_{h,t})\end{aligned}$$

where $\vec{\varepsilon}_{\nu,t+1} = \{\varepsilon_{x_\nu,t+1}, \varepsilon_{d_\nu,t+1}, \varepsilon_{g_\nu,t+1}\}$, with $\nu \in \{h, \ell\}$ is a vector of standard normal shocks with distribution $F(\vec{\varepsilon}_{\nu,t+1})$. (i) and (ii) are only present in the dividend claim case. We also define

$$\begin{aligned}\Psi_t \equiv \eta_t &\int_{-\infty}^{\infty} \phi'\biggl(\iiint_{-\infty}^{\infty} \Phi_V\bigl(C_{t+1}^{(\ell)}, \widehat{x}_{h,t+1}^{(\ell)}, \widehat{x}_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}\bigr)\,\mathrm{d}F(\vec{\varepsilon}_{\ell,t+1})\biggr)\,\mathrm{d}F(x_{\ell,t}) \\ &+ (1-\eta_t)\int_{-\infty}^{\infty} \phi'\biggl(\iiint_{-\infty}^{\infty} \Phi_V\bigl(C_{t+1}^{(h)}, \widehat{x}_{h,t+1}^{(h)}, \widehat{x}_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}\bigr)\,\mathrm{d}F(\vec{\varepsilon}_{h,t+1})\biggr)\,\mathrm{d}F(x_{h,t}).\end{aligned}$$

In both cases, $C_{t+1}^{(\nu)}, \widehat{x}_{h,t+1}^{(\nu)}, \widehat{x}_{\ell,t+1}^{(\nu)}, \eta_{t+1}^{(h)}$, $\nu \in \{h, \ell\}$, are obtained using the dynamic equations described in Section B.1. These expressions are simplified when the agent is cer-

[3]We use this function in order to be able to narrow down the range of values taken by the state variables, such that the approximation performs better when evaluated on the data.

tain about the persistence. This case amounts to setting $\eta_t = 0$ for all t in the preceding expressions and consider only one process for $\widehat{x}_t$.

The vector of parameters θ^V and θ^R are then determined by projecting the residuals on Hermite polynomials. This then defines a system of orthogonality conditions which is solved for θ^V and θ^R. More precisely, we solve[4]

$$\langle \mathcal{R}_V(\theta^V; X_t) \mid \mathcal{H}(X_t) \rangle = \int \mathcal{R}_V(\theta^V; X_t) \mathcal{H}(X_t) \Omega(X_t) \, \mathrm{d}X_t = 0,$$

$$\langle \mathcal{R}_R(\theta^R, \theta^V; X_t) \mid \mathcal{H}(X_t) \rangle = \int \mathcal{R}_R(\theta^R, \theta^V; X_t) \mathcal{H}(X_t) \Omega(X_t) \, \mathrm{d}X_t = 0$$

where

$$\mathcal{H}(X_t) \equiv H_{i_c}(\varphi_h(C_t)) H_{i_h}(\varphi_h(\widehat{x}_t^h)) H_j(\varphi_\ell(\widehat{x}_t^\ell)) H_k(\varphi_\eta(\eta_t))$$
$$\text{with } i_c + i_h + i_\ell + i_\eta \leq \max(n_c, n_h, n_\ell, n_\eta)$$

and

$$\Omega(X_t) \equiv \omega(\varphi_h(C_t)) \omega(\varphi_h(x_t^h)) \omega(\varphi_\ell(x_t^\ell)) \omega(\varphi_\eta(\eta_t))$$

where $\omega(x) = \exp(-x^2)$ is the appropriate weighting function for Hermite polynomials. Note that since the knowledge of the risky interest rate is not needed to evaluate the direct value function in equilibrium, the system can be solved recursively. We therefore first solve the value function approximation problem, and use the result vector of parameters θ^V to solve for the risky rate problem.

Integrals are approximated using a monomial approach whenever we face a multidimensional integration problem (inner integrals in the computation of expectations and projections) and a Gauss–Hermitian quadrature approach when dealing with unidimensional integrals (outer integrals in the computation of expectations).[5]

The algorithm imposes that several important choices be made for the algorithm parameters. The first one corresponds to the degree of polynomials we use for the approximation. The results are obtained with polynomials of order:

- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (5, 2, 2, 2)$ for the value function when $\rho_h = 0.85$,
- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (4, 2, 2, 2)$ for the value function when $\rho_h = 0.90$,
- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (3, 3, 3, 3)$ for the interest rate,
- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (2, 4, 4, 1)$ for the asset prices.

The second choice pertains to the number of nodes. We use eight nodes in each dimension (4096 nodes). The transform functions $\varphi(\cdot)$ are assumed to be linear $\varphi_z(x) = \kappa_z x$ where κ_z, $z \in \{c, h, \ell, \eta\}$ is a constant chosen such that the focus of the approximation is put on values of state variables taken in the data. More precisely, we set $\kappa_c = 2.0817$, $\kappa_h = 40$, $\kappa_\ell = 350$, and $\kappa_\eta = 1$.

[4]It should be clear to the reader that the integral refers to a multidimensional integration problem, as we integrate over C, x^h, x^ℓ, and η.

[5]See Judd (1998), Chapter 7.

The number of nodes used in the unidimensional quadrature method used in the outer integral involved in the computation of expectations is set to 12. In the case of the multidimensional integrals, we use a degree 5 rule for an integrand on an unbounded range weighted by a standard normal.[6] Finally, the topping criterion is set to 1e−6.

Given these parameters, the algorithm associated to each problem works as follows:

1. Choose two candidate vectors of parameters θ^V and θ^R.

2. Find the nodes, r_{j_z}, $j_z = 1, \dots, m_z$, at which the residuals are evaluated. These nodes corresponds to the roots of the different Hermite polynomials involved in the approximation, then compute the values of the state variables as

$$C_{j_c} = \varphi_c^{-1}(r_{j_c}), x^h_{j_h} = \varphi_h^{-1}(r_{j_h}), x^\ell_{j_\ell} = \varphi_\ell^{-1}(r_{j_\ell}), \eta_{j_\eta} = \varphi_\eta^{-1}(r_{j_\eta}).$$

3. Evaluate the residuals $\mathcal{R}_V(\theta^V; X_t)$ and $\mathcal{R}_R(\theta^R, \theta^V; X_t)$ and compute the orthogonality conditions

$$\langle \mathcal{R}_V(\theta^V; X_t) \mid \mathcal{H}(X_t)\rangle \quad \text{and} \quad \langle \mathcal{R}_R(\theta^R, \theta^V; X_t) \mid \mathcal{H}(X_t)\rangle.$$

4. If the orthogonality conditions are satisfied, in the sense the residuals are lower than the stopping criterion ε, then the vector of parameters are given by θ^V and θ^R. Else update θ^V and θ^R using a Gauss–Newton algorithm and go back to step 1.

1.2 *Computation of returns*

Given an approximate solution for the value function and the risky return, and given a sequence $\{X_t\}_{t=t_1}^{t=t_2} = \{C_t, \widehat{x}_{h,t}, \widehat{x}_{\ell,t}, \eta_t\}_{t=t_1}^{t=t_N}$ of annual observations of aggregate per-capita consumption, beliefs and prior probabilities in the time periods $t = t_1$ through $t = t_N$ we compute the conditional nth order moment of the risky rate in period t as

$$E_t^n R_{t+1} = \iiiint_{-\infty}^{\infty} \Phi(X_{t+1})^n \, dF(\overrightarrow{\varepsilon}_{t+1}) \, dF(x_t). \tag{1}$$

The model average nth order moment is then computed as

$$ER^n = \frac{1}{t_2 - t_1}\left[\sum_{t=t_1}^{t=t_2} E_t^n R_{t+1} - \left(E_t^1 R_{t+1}\right)^n\right]. \tag{2}$$

[6] More precisely, we approximate

$$\int_{\mathbb{R}^k} F(x)\exp\left(\sum_{i=1}^k x_i^2\right) \mathrm{d}x \simeq a_0 F(0) + a_1 \sum_{i=1}^k \big(F(re_i) + F(-re_i)\big) +$$
$$+ a_2 \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \big(F(se_i + se_j) + F(se_i - se_j) + F(-se_i + se_j) + F(-se_i - se_j)\big)$$

where e_i denotes the ith column vector of the identity matrix of order k. $r = \sqrt{1 + \frac{k}{2}}$, $s = \frac{\sqrt{2r}}{2}$, $a_0 = \frac{2\pi^{\frac{k}{2}}}{k+2}$, $a_1 = \frac{4-k}{4(k+2)} a_0$ and $a_2 = \frac{a_0}{2(k+2)}$. See Judd (1998) for greater details.

Similarly, given a sequence $\{C_t, \widehat{x}_{h,t}, \widehat{x}_{\ell,t}, \eta_t\}_{t=t_1}^{t=t_N}$, the risk-free rate can be directly computed

$$R_t^f = \Bigg[\beta\eta_t \int_{-\infty}^{\infty} \xi_t^{(l)}(C_t, \widehat{x}_{l,t}, \widehat{x}_{h,t}, \eta_t)\bigg(\iiint_{-\infty}^{\infty}\big(U'\big(\exp(g_{l,t+1})\big)\big)\, dF(\vec{\varepsilon}_{l,t+1})\bigg)\, dF(x_{l,t})$$
$$+ \beta(1-\eta_t)\int_{-\infty}^{\infty} \xi_t^{(h)}(C_t, \widehat{x}_{l,t}, \widehat{x}_{h,t}, \eta_t)$$
$$\times \bigg(\iiint_{-\infty}^{\infty}\big(U'\big(\exp(g_{h,t+1})\big)\big)\, dF(\vec{\varepsilon}_{h,t+1})\bigg)\, dF(x_{h,t})\Bigg]^{-1}.$$

Just as in the preceding section, integrals are approximated using a monomial approach whenever we face a multidimensional integration problem (inner integrals in the computation of expectations and projections) and a Gauss–Hermitian quadrature approach when dealing with unidimensional integrals (outer integrals in the computation of expectations). The nth-order moments are then obtained in a similar fashion as for the risky rate.

The (conditional) equity premium at time t, is the random variable denoted $R_t^p \equiv E_t^1 R_{t+1} - R_t^f$. Therefore, the n-order moments of the equity premium can be computed as in equation (2).

References

Judd, K. (1998), *Numerical Methods in Economics*. MIT Press, Cambridge, MA. [1, 3, 4]

Co-editor Karl Schmedders handled this manuscript.

Manuscript received 22 April, 2016; final version accepted 23 November, 2017; available online 16 January, 2018.