Supplement to "Estimating nonseparable models with mismeasured endogenous variables"

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Suyong Song
Department of Economics, University of Iowa

Susanne M. Schennach
Department of Economics, Brown University

Halbert White
Department of Economics, University of California, San Diego

This supplementary material contains discussions on the optimal bandwidths selection, technical proofs, and additional asymptotic results for weighted averages of effects. All assumptions, notations, and references are as given in the main body of the paper.

Appendix A: Selection of bandwidths

In this appendix, we provide a useful guideline on how to use leave-one-out cross-validation (Wahba (1990)) in estimating bandwidths. Because formal treatment of the methods described here is beyond the scope of this paper, our discussion is purely heuristic.

Suppose for the moment that X is observable, as the local linear approach assumes. Let $Z \equiv (X, W)$, and let $\hat{\mu}_h(z)$ be the nonparametric regression estimate at $z \equiv (x, w)$ with bandwidth h. For observation k and smoothing parameter α, let $h^{[k,\alpha]}$ be the minimizer of

$$Q_\alpha^k(h) = \frac{1}{n}\sum_{i=1, i\neq k}^{n} \bigl(Y_i - \hat{\mu}_h(Z_i)\bigr)^2 + \alpha \frac{1}{n}\sum_{i=1, i\neq k}^{n} \bigl\| D^2\hat{\mu}_h(Z_i)\bigr\|^2,$$

where $D^2\hat{\mu}_h(Z_i)$ is the second derivative matrix of $\hat{\mu}_h$ evaluated at Z_i, and $\|\cdot\|$ denotes a suitable matrix norm. Here, it is convenient to choose $\|\cdot\|$ to be the Frobenius norm,

Suyong Song: suyong-song@uiowa.edu
Susanne M. Schennach: smschenn@brown.edu

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that is, the square root of the sum of squares of the matrix elements. Let $\hat{\mu}_{h^{[k,\alpha]}}$ denote $\hat{\mu}_h$ evaluated at $h^{[k,\alpha]}$. Then the cross-validation function $V_0(\alpha)$ is

$$V_0(\alpha) = \frac{1}{n}\sum_{k=1}^{n}\big(Y_k - \hat{\mu}_{h^{[k,\alpha]}}(Z_k)\big)^2.$$

We obtain a cross-validated estimate of the smoothing parameter α by minimizing the cross-validation function $V_0(\alpha)$. Using this estimated α, we then obtain a cross-validated estimate of the bandwidths $h \equiv (h_1, h_2)$ by minimizing $Q_\alpha^0(h)$, which leaves out no observations. This procedure is similar to standard cross-validation, as in Wahba (1990), except that here h_1 and h_2 are additional smoothing parameters to be estimated, and instead of an integral, a sample average is used in the second term of $Q_\alpha^k(h)$. Here $Q_\alpha^k(h)$ embodies a trade-off between fidelity to the data and smoothness of the regression function. The first is represented by the mean square of residuals and the second is represented by the generalized mean square of the second derivative. As the smoothing parameter α controls the trade-off between fidelity and smoothness, one can choose useful bandwidths even with quite noisy data.

Although the above method is feasible when X is observable, a more elaborate approach is required when X is mismeasured. A straightforward analog of the method for observable X follows by suitably replacing the functions Q_α^k and $V_0(\alpha)$. Specifically, we replace Q_α^k with an estimate of

$$\begin{aligned}\tilde{Q}_\alpha(h) &= \int \big(y - \hat{\mu}_h(x, w)\big)^2 f_{Y|X,W}(y \mid x, w) f_{X|W}(x \mid w) f_W(w)\, dy\, dx\, dw \\ &\quad + \alpha \int \big\|D^2 \hat{\mu}_h(x, w)\big\|^2 f_{X|W}(x \mid w) f_W(w)\, dx\, dw\end{aligned}$$

that does not require X to be observable, where now $\hat{\mu}_h(x, w) = \hat{g}_{Y,0}(x, w; h)/\hat{g}_{1,0}(x, w; h)$. Inspecting this expression, we see that a new quantity appears: $f_{Y|X,W}$, the density of Y given (X, W). Using the fact that

$$f_{Y|X,W} = \frac{f_{X|Y,W} f_{Y|W}}{f_{X|W}}$$

and substituting this into $\tilde{Q}_\alpha(h)$ gives

$$\begin{aligned}\tilde{Q}_\alpha(h) &= \int \big(y - \hat{\mu}_h(x, w)\big)^2 f_{X|Y,W}(x \mid y, w) f_{Y,W}(y, w)\, dy\, dx\, dw \\ &\quad + \alpha \int \big\|D^2 \hat{\mu}_h(x, w)\big\|^2 f_{X|W}(x \mid w; h) f_W(w)\, dx\, dw.\end{aligned}$$

Replacing $f_{Y,W}$ and f_W with their empirical counterparts yields the analog of Q_α^k:

$$\begin{aligned}\hat{Q}_\alpha^k(h) &= \frac{1}{n}\sum_{i=1, i\neq k}^{n} \int \big(Y_i - \hat{\mu}_h(x, W_i)\big)^2 \hat{f}_{X|Y,W}(x \mid Y_i, W_i; h)\, dx \\ &\quad + \alpha \frac{1}{n}\sum_{i=1, i\neq k}^{n} \int \big\|D^2 \hat{\mu}_h(x, W_i)\big\|^2 \hat{f}_{X|W}(x \mid W_i; h)\, dx.\end{aligned}$$

With $h^{[k,\alpha]}$ now denoting the minimizer of $\hat{Q}^k_\alpha(h)$, we define a similar analog of V_0 as

$$\hat{V}_0(\alpha) = \frac{1}{n}\sum_{k=1}^{n}\int \big(Y_k - \hat{\mu}_{h^{[k,\alpha]}}(x, W_k)\big)^2 \hat{f}_{X|Y,W}\big(x \mid Y_k, W_k; h^{[k,\alpha]}\big)\, dx.$$

To implement this method, we require an estimator $\hat{f}_{X|Y,W}$ for $f_{X|Y,W}$. But this can be obtained in a manner precisely parallel to the way we obtain $f_{X|W}$, simply by replacing W in Lemma 3.1 and the following development[1] with (Y, W) for the case $(V = 1, \lambda = 0)$. The estimates for α, h_1, and h_2 are then obtained using the method described above for observable X.[2]

Appendix B: Mathematical proofs of main results

This appendix presents mathematical proofs of results in the main text.

Proof of Lemma 3.1. By Assumption 3.1, all expectations below exist and are finite. We first observe that $U_2 \perp (X, W)$ implies $U_2 \perp X$ and $U_2 \perp W$. By Assumptions 2.3, 3.2, and 3.4 and the law of iterated expectation, we get

$$\begin{aligned}
\frac{iE[X_1 e^{i\xi X_2}]}{E[e^{i\xi X_2}]} &= \frac{iE[Xe^{i\xi(X+U_2)}] + iE[E(U_1 e^{i\xi(X+U_2)} \mid X, U_2)]}{E[e^{i\xi(X+U_2)}]} \\
&= \frac{iE[Xe^{i\xi(X+U_2)}] + iE[E(U_1 \mid X, U_2)e^{i\xi(X+U_2)}]}{E[e^{i\xi(X+U_2)}]} \\
&= \frac{iE[Xe^{i\xi(X+U_2)}]}{E[e^{i\xi(X+U_2)}]} \\
&= \frac{iE[Xe^{i\xi X}]}{E[e^{i\xi X}]} \\
&= D_\xi \ln\big(E\big[e^{i\xi X}\big]\big),
\end{aligned} \tag{S1}$$

as considered by SWC. We use $E[U_1 \mid X, U_2] = 0$ in the step from the second to the third equality and use $U_2 \perp X$ in the step from the third to the fourth equality.

We note that $U_2 \perp (X, W)$ implies $U_2 \perp X \mid W$, and we also note that $U_2 \perp X \mid W$ implies $U_2 \perp (X, W) \mid W$ by Lemma 4.1 and Lemma 4.2(ii) in Dawid (1979). Then for each real ζ, we have

$$\begin{aligned}
\phi_V(\zeta, W) &\equiv E\big[Ve^{i\zeta X} \mid W\big] \\
&= \frac{E[Ve^{i\zeta X} \mid W]E[e^{i\zeta U_2}]}{E[e^{i\zeta X}]E[e^{i\zeta U_2}]} E\big[e^{i\zeta X}\big]
\end{aligned}$$

[1] This includes the stated regularity conditions involving W, except for Assumption 3.2(iii).

[2] Although we use the same h for notational simplicity, the bandwidths for $\hat{f}_{X|Y,W}$ are different from those for $\hat{f}_{X|W}$. In estimating the bandwidth and smoothing parameters, we allow for different bandwidths for $\hat{f}_{X|Y,W}$ and $\hat{f}_{X|W}$. We use the same bandwidths for Y and W for computational simplicity; this is not a strong restriction.

$$
\begin{aligned}
&= \frac{E[Ve^{i\zeta X} \mid W]E[e^{i\zeta U_2} \mid W]}{E[e^{i\zeta X}]E[e^{i\zeta U_2}]} E\big[e^{i\zeta X}\big] \\
&= \frac{E[E[Ve^{i\zeta X} \mid X, W] \mid W]E[e^{i\zeta U_2} \mid W]}{E[e^{i\zeta X}]E[e^{i\zeta U_2}]} E\big[e^{i\zeta X}\big] \\
&= \frac{E[E[V \mid X, W]e^{i\zeta X}e^{i\zeta U_2} \mid W]}{E[e^{i\zeta X}]E[e^{i\zeta U_2}]} E\big[e^{i\zeta X}\big] \\
&= \frac{E[E[V \mid X, U_2, W]e^{i\zeta X}e^{i\zeta U_2} \mid W]}{E[e^{i\zeta X}]E[e^{i\zeta U_2}]} E\big[e^{i\zeta X}\big] \\
&= \frac{E[E[Ve^{i\zeta X_2} \mid X, U_2, W] \mid W]}{E[e^{i\zeta X}]E[e^{i\zeta U_2}]} E\big[e^{i\zeta X}\big] \\
&= \frac{E[Ve^{i\zeta X_2} \mid W]}{E[e^{i\zeta X_2}]} \exp\left(\int_0^\zeta D_\xi \ln\big(E\big[e^{i\zeta X}\big]\big)\, d\xi\right) \\
&= \frac{E[Ve^{i\zeta X_2} \mid W]}{E[e^{i\zeta X_2}]} \exp\left(\int_0^\zeta \frac{iE[X_1 e^{i\xi X_2}]}{E[e^{i\xi X_2}]}\, d\xi\right),
\end{aligned}
$$

where $U_2 \perp W$, $U_2 \perp (X, W) \mid W$, and $E[V \mid X, U_2, W] = E[V \mid X, W]$ are used in the steps from the second to the third line, from the fifth to the sixth line, and from the sixth to the seventh line, respectively.

Given Assumptions 3.3–3.5, we get

$$
\begin{aligned}
&(-i\zeta)^\lambda E\big[Ve^{i\zeta X} \mid W = w\big] \\
&\quad = (-i\zeta)^\lambda \int E[V \mid W = w, X = x] f_{X|W}(x \mid w) e^{i\zeta x}\, dx \\
&\quad = (-1)^\lambda \int E[V \mid W = w, X = x] f_{X|W}(x \mid w) D_x^\lambda e^{i\zeta x}\, dx \\
&\quad = \int D_x^\lambda\big(E[V \mid W = w, X = x] f_{X|W}(x \mid w)\big) e^{i\zeta x}\, dx \\
&\quad = \int g_{V,\lambda}(x, w) e^{i\zeta x}\, dx,
\end{aligned}
$$

where the third equality follows by integration by parts. The last expression is the Fourier transform of $g_{V,\lambda}(x, w)$. For each $\lambda \in \{0, \dots, \Lambda\}$ and $(x, w) \in \operatorname{supp}(X, W)$, we have

$$
\begin{aligned}
&\frac{1}{2\pi} \int (-i\zeta)^\lambda \phi_V(\zeta, w) \exp(-i\zeta x)\, d\zeta \\
&\quad = \frac{1}{2\pi} \int (-i\zeta)^\lambda E\big[Ve^{i\zeta X} \mid W = w\big] \exp(-i\zeta x)\, d\zeta.
\end{aligned}
$$

Since the right hand side is the inverse Fourier transform of $(-i\zeta)^\lambda E[Ve^{i\zeta X} \mid W = w]$, the result follows. □

Proof of Lemma 3.2. Assumptions 3.1 and 3.3–3.6 ensure the existence of

$$\begin{aligned} g_{V,\lambda}(x, w, h_1) &\equiv \int \frac{1}{h_1} k\left(\frac{\tilde{x} - x}{h_1}\right) g_{V,\lambda}(\tilde{x}, w)\, d\tilde{x} \\ &= \int \frac{1}{h_1} k\left(\frac{\tilde{x} - x}{h_1}\right) D_{\tilde{x}}^{\lambda}\big(E[V \mid X = \tilde{x}, W = w] f_{X|W}(\tilde{x} \mid w)\big)\, d\tilde{x}. \end{aligned}$$

By the convolution theorem, the inverse Fourier transform of the product of $\kappa(h_1\zeta)$ and $(-i\zeta)^{\lambda} E[V e^{i\zeta X} \mid W = w]$ is the convolution between the inverse Fourier transform of $\kappa(h_1\zeta)$ and the inverse Fourier transform of $(-i\zeta)^{\lambda} E[V e^{i\zeta X} \mid W = w]$. The inverse Fourier transform of $\kappa(h_1\zeta)$ is $h_1^{-1} k(x/h_1)$, and the inverse Fourier transform of $(-i\zeta)^{\lambda} E[V e^{i\zeta X} \mid W = w]$ is $D_x^{\lambda}(E[V \mid X = x, W = w] f_{X|W}(x \mid w))$. It follows that

$$\begin{aligned} g_{V,\lambda}(x, w, h_1) &= \frac{1}{2\pi} \int \kappa(h_1\zeta)\big((-i\zeta)^{\lambda} E\big[V e^{i\zeta X} \mid W = w\big]\big) \exp(-i\zeta x)\, d\zeta \\ &= \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \phi_V(\zeta, w) \exp(-i\zeta x)\, d\zeta. \end{aligned}$$ □

Proof of Lemma 4.1. For $A \in \{1, X_1\}$, we let $\theta_A(\zeta) \equiv E[A e^{i\zeta X_2}]$ and for $V \in \{1, Y\}$,

$$\theta_V(\zeta, w) \equiv E\big[V e^{i\zeta X_2} \mid W = w\big] = \frac{E[V e^{i\zeta X_2} \mid W = w] f_W(w)}{f_W(w)} = \frac{\chi_V(\zeta, w)}{f_W(w)},$$

where $\chi_V(\zeta, w) \equiv E[V e^{i\zeta X_2} \mid W = w] f_W(w)$. Also we let $\hat{\theta}_A(\zeta) \equiv \hat{E}[A e^{i\zeta X_2}]$ and $\delta\hat{\theta}_A(\zeta) \equiv \hat{\theta}_A(\zeta) - \theta_A(\zeta)$. Similarly, $\hat{\theta}_V(\zeta, w) \equiv \hat{E}[V e^{i\zeta X_2} \mid W = w] \equiv \hat{\chi}_V(\zeta, w)/\hat{f}_W(w)$, where

$$\hat{\chi}_V(\zeta, w) = \frac{1}{n} \sum_{j=1}^{n} k_{h_2}(W_j - w) V_j e^{i\zeta X_{2j}} = \hat{E}\big[V e^{i\zeta X_2} k_{h_2}(W - w)\big],$$

$$\hat{f}_W(w) = \frac{1}{n} \sum_{j=1}^{n} k_{h_2}(W_j - w) = \hat{E}\big[k_{h_2}(W - w)\big],$$

so that $\delta\hat{\chi}_V(\zeta, w) \equiv \hat{\chi}_V(\zeta, w) - \chi_V(\zeta, w)$ and $\delta\hat{f}_W(w) \equiv \hat{f}_W(w) - f_W(w)$. We state a useful representation for $\hat{\theta}_{X_1}(\zeta)/\hat{\theta}_1(\zeta)$,

$$\frac{\hat{\theta}_{X_1}(\zeta)}{\hat{\theta}_1(\zeta)} = \frac{\theta_{X_1}(\zeta) + \delta\hat{\theta}_{X_1}(\zeta)}{\theta_1(\zeta) + \delta\hat{\theta}_1(\zeta)} = q_{X_1}(\zeta) + \delta\hat{q}_{X_1}(\zeta), \tag{S2}$$

where $q_{X_1}(\zeta) = \theta_{X_1}(\zeta)/\theta_1(\zeta)$ and where $\delta\hat{q}_{X_1}(\zeta)$ can be written as either

$$\delta\hat{q}_{X_1}(\zeta) = \left(\frac{\delta\hat{\theta}_{X_1}(\zeta)}{\theta_1(\zeta)} - \frac{\theta_{X_1}(\zeta)\delta\hat{\theta}_1(\zeta)}{(\theta_1(\zeta))^2}\right)\left(1 + \frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^{-1}$$

or $\delta\hat{q}_{X_1}(\zeta) = \delta_1\hat{q}_{X_1}(\zeta) + \delta_2\hat{q}_{X_1}(\zeta)$ with

$$\delta_1\hat{q}_{X_1}(\zeta) \equiv \frac{\delta\hat{\theta}_{X_1}(\zeta)}{\theta_1(\zeta)} - \frac{\theta_{X_1}(\zeta)\delta\hat{\theta}_1(\zeta)}{(\theta_1(\zeta))^2},$$

$$\delta_2\hat{q}_{X_1}(\zeta) \equiv \frac{\theta_{X_1}(\zeta)}{\theta_1(\zeta)}\left(\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^2\left(1+\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^{-1} - \frac{\delta\hat{\theta}_{X_1}(\zeta)}{\theta_1(\zeta)}\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\left(1+\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^{-1}.$$

For $\hat{\chi}_V(\zeta,w)/\hat{\theta}_1(\zeta)$,

$$\frac{\hat{\chi}_V(\zeta,w)}{\hat{\theta}_1(\zeta)} = \frac{\chi_V(\zeta,w)+\delta\hat{\chi}_V(\zeta,w)}{\theta_1(\zeta)+\delta\hat{\theta}_1(\zeta)} = q_V(\zeta,w)+\delta\hat{q}_V(\zeta,w), \tag{S3}$$

where $q_V(\zeta,w) \equiv \chi_V(\zeta,w)/\theta_1(\zeta)$ and where $\delta\hat{q}_V(\zeta,w)$ can be written as either

$$\delta\hat{q}_V(\zeta,w) = \left(\frac{\delta\hat{\chi}_V(\zeta,w)}{\theta_1(\zeta)} - \frac{\chi_V(\zeta,w)\delta\hat{\theta}_1(\zeta)}{(\theta_1(\zeta))^2}\right)\left(1+\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^{-1}$$

or $\delta\hat{q}_V(\zeta,w) = \delta_1\hat{q}_V(\zeta,w)+\delta_2\hat{q}_V(\zeta,w)$ with

$$\delta_1\hat{q}_V(\zeta,w) \equiv \frac{\delta\hat{\chi}_V(\zeta,w)}{\theta_1(\zeta)} - \frac{\chi_V(\zeta,w)\delta\hat{\theta}_1(\zeta)}{(\theta_1(\zeta))^2},$$

$$\delta_2\hat{q}_V(\zeta,w) \equiv \frac{\chi_V(\zeta,w)}{\theta_1(\zeta)}\left(\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^2\left(1+\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^{-1} - \frac{\delta\hat{\chi}_V(\zeta,w)}{\theta_1(\zeta)}\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\left(1+\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)^{-1}.$$

Similarly, for $1/\hat{f}_W(w)$,

$$\frac{1}{\hat{f}_W(w)} = \frac{1}{f_W(w)+\delta\hat{f}_W(w)} = q_1(w)+\delta\hat{q}_1(w), \tag{S4}$$

where $q_1(w) \equiv 1/f_W(w)$ and where $\delta\hat{q}_1(w)$ can be written as either

$$\delta\hat{q}_1(w) = \left(-\frac{\delta\hat{f}_W(w)}{(f_W(w))^2}\right)\left(1+\frac{\delta\hat{f}_W(w)}{f_W(w)}\right)^{-1}$$

or $\delta\hat{q}_1(w) = \delta_1\hat{q}_1(w)+\delta_2\hat{q}_1(w)$ with

$$\delta_1\hat{q}_1(w) \equiv -\frac{\delta\hat{f}_W(w)}{(f_W(w))^2},$$

$$\delta_2\hat{q}_1(w) \equiv \frac{1}{f_W(w)}\left(\frac{\delta\hat{f}_W(w)}{f_W(w)}\right)^2\left(1+\frac{\delta\hat{f}_W(w)}{f_W(w)}\right)^{-1}.$$

For $Q_{X_1}(\zeta) \equiv \int_0^\zeta (i\theta_{X_1}(\xi)/\theta_1(\xi))\,d\xi$, $\delta\hat{Q}_{X_1}(\zeta) \equiv \int_0^\zeta (i\hat{\theta}_{X_1}(\xi)/\hat{\theta}_1(\xi))\,d\xi - Q_{X_1}(\zeta)$ and some random function $\delta\bar{Q}_{X_1}(\zeta)$ such that $|\delta\bar{Q}_{X_1}(\zeta)| \le |\delta\hat{Q}_{X_1}(\zeta)|$ for all ζ,

$$\begin{aligned}&\exp\bigl(Q_{X_1}(\zeta)+\delta\hat{Q}_{X_1}(\zeta)\bigr)\\ &\quad= \exp\bigl(Q_{X_1}(\zeta)\bigr)\left(1+\delta\hat{Q}_{X_1}(\zeta)+\frac{1}{2}\bigl[\exp\bigl(\delta\bar{Q}_{X_1}(\zeta)\bigr)\bigr]\bigl(\delta\hat{Q}_{X_1}(\zeta)\bigr)^2\right).\end{aligned} \tag{S5}$$

By substituting eqs. (S2)–(S5) into

$$\hat{g}_{V,\lambda}(x,w,h)-g_{V,\lambda}(x,w,h_1)$$
$$=\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\biggl[\frac{\hat{\theta}_V(\zeta,w)}{\hat{\theta}_1(\zeta)}\exp\biggl(\int_0^{\zeta}\frac{i\hat{\theta}_{X_1}(\xi)}{\hat{\theta}_1(\xi)}\,d\xi\biggr)$$
$$-\frac{\theta_V(\zeta,w)}{\theta_1(\zeta)}\exp\biggl(\int_0^{\zeta}\frac{i\theta_{X_1}(\xi)}{\theta_1(\xi)}\,d\xi\biggr)\biggr]\,d\zeta,$$

we have

$$\hat{g}_{V,\lambda}(x,w,h)-g_{V,\lambda}(x,w,h_1)$$
$$=\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\biggl[-\frac{\theta_V(\zeta,w)}{\theta_1(\zeta)}\exp\biggl(\int_0^{\zeta}\frac{i\theta_{X_1}(\xi)}{\theta_1(\xi)}\,d\xi\biggr)$$
$$+\biggl\{\frac{\chi_V(\zeta,w)}{\theta_1(\zeta)}+\frac{\delta\hat{\chi}_V(\zeta,w)}{\theta_1(\zeta)}-\frac{\chi_V(\zeta,w)\delta\hat{\theta}_1(\zeta)}{(\theta_1(\zeta))^2}+\delta_2\hat{q}_V(\zeta,w)\biggr\}$$
$$\times\biggl\{\frac{1}{f_W(w)}-\frac{\delta\hat{f}_W(w)}{(f_W(w))^2}+\delta_2\hat{q}_1(w)\biggr\}\exp\bigl(Q_{X_1}(\zeta)\bigr)$$
$$\times\biggl\{1+\int_0^{\zeta}i\delta_1\hat{q}_{X_1}(\xi)\,d\xi+\int_0^{\zeta}i\delta_2\hat{q}_{X_1}(\xi)\,d\xi$$
$$+\frac{1}{2}\exp\bigl(\delta\bar{Q}_{X_1}(\zeta)\bigr)\biggl(\int_0^{\zeta}i\delta\hat{q}_{X_1}(\xi)\,d\xi\biggr)^2\biggr\}\biggr]\,d\zeta.$$

Keeping the terms linear in $\delta\hat{\theta}_1(\zeta)$, $\delta\hat{\theta}_{X_1}(\zeta)$, $\delta\hat{\chi}_V(\zeta,w)$, and $\delta\hat{f}_W(w)$ gives the linearization of $\hat{g}_{V,\lambda}(x,w,h)$, denoted by $\bar{g}_{V,\lambda}(x,w,h)$:

$$\bar{g}_{V,\lambda}(x,w,h)-g_{V,\lambda}(x,w,h_1)$$
$$=\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\biggl[\frac{\theta_V(\zeta,w)}{\theta_1(\zeta)}\exp(Q_{X_1})$$
$$\times\int_0^{\zeta}\biggl(\frac{i\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)}-\frac{i\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\biggr)\,d\xi-\exp\bigl(Q_{X_1}(\zeta)\bigr)\frac{\chi_V(\zeta,w)}{\theta_1(\zeta)}\frac{\delta\hat{f}_W(w)}{(f_W(w))^2}$$
$$+\exp\bigl(Q_{X_1}(\zeta)\bigr)\frac{1}{f_W(w)}\biggl(\frac{\delta\hat{\chi}_V(\zeta,w)}{\theta_1(\zeta)}-\frac{\chi_V(\zeta,w)\delta\hat{\theta}_1(\zeta)}{(\theta_1(\zeta))^2}\biggr)\biggr]\,d\zeta$$
$$=\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\phi_V(\zeta,w)$$
$$\times\int_0^{\zeta}\biggl(\frac{i\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)}-\frac{i\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\biggr)\,d\xi\,d\zeta$$
$$+\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\phi_V(\zeta,w)$$
$$\times\biggl(-\frac{\delta\hat{f}_W(w)}{f_W(w)}+\frac{\delta\hat{\chi}_V(\zeta,w)}{\chi_V(\zeta,w)}-\frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\biggr)\,d\zeta.$$

Using the identity

$$\int_{-\infty}^{\infty}\int_{0}^{\zeta} f(\zeta,\xi)\,d\xi\,d\zeta = \int_{0}^{\infty}\int_{\xi}^{\infty} f(\zeta,\xi)\,d\zeta\,d\xi + \int_{-\infty}^{0}\int_{\xi}^{-\infty} f(\zeta,\xi)\,d\zeta\,d\xi$$
$$\equiv \int\int_{\xi}^{\pm\infty} f(\zeta,\xi)\,d\zeta\,d\xi$$

for any absolutely integrable function f, we get

$$\begin{aligned}
&L_{V,\lambda}(x,w,h)\\
&\quad\equiv \bar{g}_{V,\lambda}(x,w,h) - g_{V,\lambda}(x,w,h_1)\\
&\quad= \frac{1}{2\pi}\int\int_{\xi}^{\pm\infty}(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\phi_V(\zeta,w)\,d\zeta\\
&\qquad\times\left(\frac{i\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)} - \frac{i\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\right)d\xi\\
&\qquad+\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\phi_V(\zeta,w)\\
&\qquad\times\left(\frac{\delta\hat{\chi}_V(\zeta,w)}{\chi_V(\zeta,w)} - \frac{\delta\hat{f}_W(w)}{f_W(w)} - \frac{\delta\hat{\theta}_1(\zeta)}{\theta_1(\zeta)}\right)d\zeta\\
&\quad= \int\Bigg[\Bigg\{-\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2}\int_{\zeta}^{\pm\infty}(-i\xi)^{\lambda}\kappa(h_1\xi)\exp(-i\xi x)\phi_V(\xi,w)\,d\xi\\
&\qquad-\frac{1}{2\pi}(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{\theta_1(\zeta)}\Bigg\}\delta\hat{\theta}_1(\zeta)\\
&\qquad+\left\{\frac{1}{2\pi}\frac{i}{\theta_1(\zeta)}\int_{\zeta}^{\pm\infty}(-i\xi)^{\lambda}\kappa(h_1\xi)\exp(-i\xi x)\phi_V(\xi,w)\,d\xi\right\}\delta\hat{\theta}_{X_1}(\zeta)\\
&\qquad+\left\{\frac{1}{2\pi}(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{\chi_V(\zeta,w)}\right\}\delta\hat{\chi}_V(\zeta,w)\\
&\qquad+\left\{-\frac{1}{2\pi}(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{f_W(w)}\right\}\delta\hat{f}_W(w)\Bigg]d\zeta\\
&\quad= \int\Big[\Psi_{V,\lambda,1}(\zeta,x,w,h_1)\big(\hat{E}[e^{i\zeta X_2}] - E[e^{i\zeta X_2}]\big)\\
&\qquad+\Psi_{V,\lambda,X_1}(\zeta,x,w,h_1)\big(\hat{E}[X_1e^{i\zeta X_2}] - E[X_1e^{i\zeta X_2}]\big)\\
&\qquad+\Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)\big(\hat{E}[Ve^{i\zeta X_2}k_{h_2}(W-w)] - E[Ve^{i\zeta X_2}k_{h_2}(W-w)]\big)\\
&\qquad+\Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)\big(\hat{E}[k_{h_2}(W-w)] - E[k_{h_2}(W-w)]\big)\Big]d\zeta\\
&\quad= \hat{E}[\ell_{V,\lambda}(x,w,h;V,X_1,X_2,W)],
\end{aligned}$$

where $\Psi_{V,\lambda,A}(\zeta,x,w,h_1)$ and $\ell_{V,\lambda}(x,w,h;V,X_1,X_2,W)$ are defined in the statement of the Lemma 4.1. □

We define the following convenient notation as employed in SWC.

Definition B.1. We write $f(\zeta) \preceq g(\zeta)$ for $f, g : \mathbb{R} \mapsto \mathbb{R}$ when there exists a constant $C > 0$, independent of ζ, such that $f(\zeta) \leq Cg(\zeta)$ for all $\zeta \in \mathbb{R}$ (and similarly for $\succeq$). Analogously, we write $a_n \preceq b_n$ for two sequences a_n, b_n when there exists a constant C independent of n such that $a_n \leq Cb_n$ for all $n \in \mathbb{N}$.

Lemma B.1. *Let the conditions of Lemma* 4.1 *hold, and suppose in addition that Assumption* 4.1(ii) *holds. Then for* $V \in \{1, Y\}$, *each* $\lambda \in \{0, \ldots, \Lambda\}$ *and* $h_1 > 0$,

$$\sup_{(x,w)\in \operatorname{supp}(X,W)} \left|B_{V,\lambda}(x, w; h_1)\right| = O\left(\left(h_1^{-1}\right)^{\gamma_{\lambda,B}} \exp\left(\alpha_B \left(h_1^{-1}\right)^{\nu_B}\right)\right),$$

where $\alpha_B \equiv \alpha_\phi \bar{\xi}^{\nu_\phi}$, $\nu_B \equiv \nu_\phi$, *and* $\gamma_{\lambda,B} \equiv \gamma_\phi + \lambda + 1$.

Proof. Using Parseval's identity, we have

$$\begin{aligned}
\left|B_{V,\lambda}(x, w, h_1)\right| &= \left|g_{V,\lambda}(x, w, h_1) - g_{V,\lambda}(x, w)\right| \\
&= \left|\frac{1}{2\pi} \int \kappa(h_1\zeta)(-i\zeta)^\lambda \phi_V(\zeta, w) \exp(-i\zeta x)\, d\zeta \right. \\
&\quad \left. - \frac{1}{2\pi} \int (-i\zeta)^\lambda \phi_V(\zeta, w) \exp(-i\zeta x)\, d\zeta \right| \\
&\leq \frac{1}{2\pi} \int \left|\left(\kappa(h_1\zeta) - 1\right)\right| |\zeta|^\lambda \left|\phi_V(\zeta, w)\right| d\zeta \\
&\preceq \int_{\bar{\xi}/h_1}^{\infty} |\zeta|^\lambda \left|\phi_V(\zeta, w)\right| d\zeta,
\end{aligned}$$

since Assumption 3.6 ensures $\kappa(\zeta) = 1$ for $|\zeta| \leq \bar{\xi}$ and $\sup_\zeta |\kappa(h_1\zeta)| < \infty$. Thus, by Assumption 4.1(ii), we have

$$\begin{aligned}
\sup_{(x,w)\in \operatorname{supp}(X,W)} \left|B_{V,\lambda}(x, w, h_1)\right| &\preceq \int_{\bar{\xi}/h_1}^{\infty} |\zeta|^\lambda C_\phi \left(1 + |\zeta|\right)^{\gamma_\phi} \exp\left(\alpha_\phi |\zeta|^{\nu_\phi}\right) d\zeta \\
&= O\left((\bar{\xi}/h_1)^{\gamma_\phi+\lambda+1} \exp\left(\alpha_\phi (\bar{\xi}/h_1)^{\nu_\phi}\right)\right) \\
&= O\left(\left(h_1^{-1}\right)^{\gamma_{\lambda,B}} \exp\left(\alpha_B \left(h_1^{-1}\right)^{\nu_B}\right)\right),
\end{aligned}$$

where the second line follows by Lemma 7 of Schennach (2004a). □

Lemma B.2. *Suppose the conditions of Lemma* 4.1 *hold. For each* ζ *and* $h \equiv (h_1, h_2)$, *and for* $A \in \{1, X_1, \chi_V, f_W\}$, *let* $\Psi^+_{V,\lambda,A}(\zeta, h_1) \equiv \sup_{(x,w)\in \operatorname{supp}(X,W)} |\Psi_{V,\lambda,A}(\zeta, x, w, h_1)|$, *and define*

$$\Psi^+_{V,\lambda}(h) \equiv \sum_{A=\{1,X_1\}} \int \Psi^+_{V,\lambda,A}(\zeta, h_1)\, d\zeta + h_2^{-1} \sum_{A=\{\chi_V, f_W\}} \int \Psi^+_{V,\lambda,A}(\zeta, h_1)\, d\zeta.$$

If Assumption 4.1 *also holds, then for* $h>0$,

$$\Psi^{+}_{V,\lambda}(h)=O\Big(\max\big\{\big(1+h_1^{-1}\big)^{\gamma_1+1},h_2^{-1}\big\}\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1}\times\exp\big((\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}}-\alpha_\theta)\big(h_1^{-1}\big)^{\nu_\theta}\big)\Big).$$

PROOF. We obtain rates for each term of $\Psi^{+}_{V,\lambda}(h)$. First,

$$\begin{aligned}
&\Psi^{+}_{V,\lambda,1}(\zeta,h_1)\\
&\quad\equiv\sup_{(x,w)\in\operatorname{supp}(X,W)}\big|\Psi_{V,\lambda,1}(\zeta,x,w,h_1)\big|\\
&\quad=\sup_{(x,w)\in\operatorname{supp}(X,W)}\bigg|-\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2}\\
&\qquad\times\int_\zeta^{\pm\infty}(-i\xi)^\lambda\kappa(h_1\xi)\exp(-i\xi x)\phi_V(\xi,w)\,d\xi\\
&\qquad-\frac{1}{2\pi}(-i\zeta)^\lambda\kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{\theta_1(\zeta)}\bigg|\\
&\quad\preceq\frac{|\theta_{X_1}(\zeta)|}{|\theta_1(\zeta)|^2}\int_\zeta^{\pm\infty}|\xi|^\lambda\big|\kappa(h_1\xi)\big|\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi\\
&\qquad+|\zeta|^\lambda\big|\kappa(h_1\zeta)\big|\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\zeta,w)\big|\Big)\frac{1}{|\theta_1(\zeta)|}\\
&\quad=\frac{1}{|\theta_1(\zeta)|}\bigg[\big|D_\zeta\ln\phi_1(\zeta)\big|\int_\zeta^{\pm\infty}|\xi|^\lambda\big|\kappa(h_1\xi)\big|\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi\\
&\qquad+|\zeta|^\lambda\big|\kappa(h_1\zeta)\big|\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\zeta,w)\big|\Big)\bigg]
\end{aligned}$$

because we have $\theta_{X_1}(\zeta)/\theta_1(\zeta)=-iD_\zeta\ln\phi_1(\zeta)$ by eq. (S4) in the proof of Lemma 2.1. Then

$$\begin{aligned}
&\Psi^{+}_{V,\lambda,1}(\zeta,h_1)\\
&\quad\preceq\frac{1}{|\theta_1(\zeta)|}\bigg[\big|D_\zeta\ln\phi_1(\zeta)\big|\int_\zeta^{\pm\infty}|\xi|^\lambda 1_{\{|\xi|\le\bar\xi h_1^{-1}\}}\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi\\
&\qquad+|\zeta|^\lambda 1_{\{|\zeta|\le\bar\zeta h_1^{-1}\}}\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\zeta,w)\big|\Big)\bigg]\\
&\quad\preceq\frac{1}{|\theta_1(\zeta)|}1_{\{|\zeta|\le h_1^{-1}\}}\bigg[\big|D_\zeta\ln\phi_1(\zeta)\big|\int_\zeta^{h_1^{-1}}|\xi|^\lambda\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi\\
&\qquad+|\zeta|^\lambda\Big(\sup_{w\in\operatorname{supp}(W)}\big|\phi_V(\zeta,w)\big|\Big)\bigg].
\end{aligned}$$

By using Assumption 4.1 and integrating $\Psi^{+}_{V,\lambda,1}(\zeta,h_1)$ with respect to ζ, we obtain

$$\begin{aligned}
&\int \Psi^{+}_{V,\lambda,1}(\zeta,h_1)\,d\zeta \\
&\quad\preceq \int \frac{1}{|\theta_1(\zeta)|}1_{\{|\zeta|\le h_1^{-1}\}}\Big[\big|D_\zeta \ln\phi_1(\zeta)\big|\int_\zeta^{h_1^{-1}}|\xi|^\lambda\Big(\sup_{w\in\mathrm{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi \\
&\qquad + |\zeta|^\lambda\Big(\sup_{w\in\mathrm{supp}(W)}\big|\phi_V(\zeta,w)\big|\Big)\Big]\,d\zeta \\
&\quad\preceq \int_0^{h_1^{-1}}\big(1+|\zeta|\big)^{-\gamma_\theta}\exp\big(-\alpha_\theta|\zeta|^{\nu_\theta}\big)\Big[\big(1+|\zeta|\big)^{\gamma_1} \\
&\qquad\times\int_0^{h_1^{-1}}|\xi|^\lambda\big(1+|\xi|\big)^{\gamma_\phi}\exp\big(\alpha_\phi|\xi|^{\nu_\phi}\big)\,d\xi + |\zeta|^\lambda\big(1+|\zeta|\big)^{\gamma_\phi}\exp\big(\alpha_\phi|\zeta|^{\nu_\phi}\big)\Big]\,d\zeta \\
&\quad\preceq \big(1+h_1^{-1}\big)^{1-\gamma_\theta}\exp\big(-\alpha_\theta\big(h_1^{-1}\big)^{\nu_\theta}\big)\big[\big(1+h_1^{-1}\big)^{\gamma_1}\big(1+h_1^{-1}\big)^{\lambda+\gamma_\phi+1}\exp\big(\alpha_\phi\big(h_1^{-1}\big)^{\nu_\phi}\big) \\
&\qquad + \big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda}\exp\big(\alpha_\phi\big(h_1^{-1}\big)^{\nu_\phi}\big)\big] \\
&\quad\preceq \big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+2}\exp\big(-\alpha_\theta\big(h_1^{-1}\big)^{\nu_\theta}\big)\exp\big(\alpha_\phi\big(h_1^{-1}\big)^{\nu_\phi}\big).
\end{aligned}$$

Second,

$$\begin{aligned}
&\Psi^{+}_{V,\lambda,X_1}(\zeta,h_1) \\
&\quad\equiv \sup_{(x,w)\in\mathrm{supp}(X,W)}\big|\Psi_{V,\lambda,X_1}(\zeta,x,w,h_1)\big| \\
&\quad= \sup_{(x,w)\in\mathrm{supp}(X,W)}\left|\frac{1}{2\pi}\frac{i}{\theta_1(\zeta)}\int_\zeta^{\pm\infty}(-i\xi)^\lambda\kappa(h_1\xi)\exp(-i\xi x)\phi_V(\xi,w)\,d\xi\right| \\
&\quad\preceq \frac{1}{|\theta_1(\zeta)|}\int_\zeta^{\pm\infty}|\xi|^\lambda\big|\kappa(h_1\xi)\big|\Big(\sup_{w\in\mathrm{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi \\
&\quad\preceq \frac{1}{|\theta_1(\zeta)|}1_{\{|\zeta|\le h_1^{-1}\}}\int_\zeta^{h_1^{-1}}|\xi|^\lambda\Big(\sup_{w\in\mathrm{supp}(W)}\big|\phi_V(\xi,w)\big|\Big)\,d\xi,
\end{aligned}$$

so that

$$\begin{aligned}
&\int \Psi^{+}_{V,\lambda,X_1}(\zeta,h_1)\,d\zeta \\
&\quad\preceq \int_0^{h_1^{-1}}\big(1+|\zeta|\big)^{-\gamma_\theta}\exp\big(-\alpha_\theta|\zeta|^{\nu_\theta}\big) \\
&\qquad\times\left(\int_0^{h_1^{-1}}|\xi|^\lambda\big(1+|\xi|\big)^{\gamma_\phi}\exp\big(\alpha_\phi|\xi|^{\nu_\phi}\big)\,d\xi\right)d\zeta \\
&\quad\preceq \big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda-\gamma_\theta+2}\exp\big(-\alpha_\theta\big(h_1^{-1}\big)^{\nu_\theta}\big)\exp\big(\alpha_\phi\big(h_1^{-1}\big)^{\nu_\phi}\big).
\end{aligned}$$

Third,

$$\begin{aligned}\Psi^+_{V,\lambda,\chi_V}(\zeta,h_1) &\equiv \sup_{(x,w)\in\operatorname{supp}(X,W)} \left|\Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)\right| \\ &= \sup_{(x,w)\in\operatorname{supp}(X,W)} \left|\frac{1}{2\pi}(-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{\chi_V(\zeta,w)}\right| \\ &\preceq |\zeta|^\lambda 1_{\{|\zeta|\le h_1^{-1}\}}\left(\sup_{w\in\operatorname{supp}(W)}\left|\frac{\phi_V(\zeta,w)}{\chi_V(\zeta,w)}\right|\right),\end{aligned}$$

so that

$$\begin{aligned}&h_2^{-1}\int \Psi^+_{V,\lambda,\chi_V}(\zeta,h_1)\,d\zeta \\ &\quad\preceq h_2^{-1}\int_0^{h_1^{-1}} |\zeta|^\lambda\big(1+|\zeta|\big)^{-\gamma_\theta}\exp\big(-\alpha_\theta|\zeta|^{\nu_\theta}\big)\big(1+|\zeta|\big)^{\gamma_\phi}\exp\big(\alpha_\phi|\zeta|^{\nu_\phi}\big)\,d\zeta \\ &\quad\preceq h_2^{-1}\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1}\exp\big(-\alpha_\theta\big(h_1^{-1}\big)^{\nu_\theta}\big)\exp\big(\alpha_\phi\big(h_1^{-1}\big)^{\nu_\phi}\big).\end{aligned}$$

Because $\inf_{w\in\operatorname{supp}(W)} f_W(w) > 0$ by Assumption 3.3(i), we finally obtain

$$\begin{aligned}\Psi^+_{V,\lambda,f_W}(\zeta,h_1) &\equiv \sup_{(x,w)\in\operatorname{supp}(X,W)} \left|\Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)\right| \\ &= \sup_{(x,w)\in\operatorname{supp}(X,W)} \left|-\frac{1}{2\pi}(-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{f_W(w)}\right| \\ &\preceq |\zeta|^\lambda 1_{\{|\zeta|\le h_1^{-1}\}}\left(\sup_{w\in\operatorname{supp}(W)}\left|\phi_V(\zeta,w)\right|\right),\end{aligned}$$

so that

$$\begin{aligned}h_2^{-1}\int \Psi^+_{V,\lambda,f_W}(\zeta,h)\,d\zeta &\preceq h_2^{-1}\int_0^{h_1^{-1}} |\zeta|^\lambda\big(1+|\zeta|\big)^{\gamma_\phi}\exp\big(\alpha_\phi|\zeta|^{\nu_\phi}\big)\,d\zeta \\ &\preceq h_2^{-1}\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda+1}\exp\big(\alpha_\phi\big(h_1^{-1}\big)^{\nu_\phi}\big).\end{aligned}$$

Collecting these rates for each term of $\Psi^+_{V,\lambda}(h)$ gives the desired result. □

LEMMA B.3. *For a finite integer J and K, let $P_{n,j}(x_2)$ define a sequence of nonrandom real-valued continuously differentiable functions of a real variable x_2, $j=1,\ldots,J$, and let $Q_{n,k}(w)$ define a sequence of nonrandom real-valued continuously differentiable functions of a real variable w, $k=1,\ldots,K$. For some C_1, C_2 and $\delta>0$, let A_j and X_2 be random variables satisfying $E[A_j^{2+\delta}\mid X_2=x_2]\le C_1$ for all $x_2\in\operatorname{supp}(X_2)$, $j=1,\ldots,J$, and let B_k and W be random variables satisfying $E[B_k^{2+\delta}\mid W=w]\le C_2$ for all $w\in\operatorname{supp}(W)$, $k=1,\ldots,K$, such that $\sup_{n\ge N}\sigma_n<\infty$ and $\inf_{n\ge N}\sigma_n>0$ for some $N\in\mathbb{N}^+$, where*

$$\sigma_n \equiv \left(\operatorname{var}\left[\sum_{j=1}^J A_jP_{n,j}(X_2)+\sum_{k=1}^K B_kQ_{n,k}(W)\right]\right)^{1/2}.$$

If there exists some $\eta > 0$ *such that*

$$\max\Big\{\sup_{x_2\in\text{supp}(X_2)}\big|D_{x_2}P_{n,j}(x_2)\big|, \sup_{w\in\text{supp}(W)}\big|D_wQ_{n,k}(w)\big|\Big\} = O\big(n^{(3/2)-\eta}\big)$$

for $j = 1, \dots, J$, *and* $k = 1, \dots, K$, *then*

$$\sigma_n^{-1}n^{1/2}\Bigg(\hat{E}\Bigg[\sum_{j=1}^{J}A_jP_{n,j}(X_2) + \sum_{k=1}^{K}B_kQ_{n,k}(W)\Bigg] - E\Bigg[\sum_{j=1}^{J}A_jP_{n,j}(X_2) + \sum_{k=1}^{K}B_kQ_{n,k}(W)\Bigg]\Bigg) \xrightarrow{d} N(0,1).$$

Proof. Apply Lemma 9 in Schennach (2004a) and the Lindeberg–Feller central limit theorem. □

Lemma B.4. *Let the conditions of Lemma* 4.1 *hold.* (i) *Then for* $V \in \{1, Y\}$ *and for each* $\lambda \in \{0, \dots, \Lambda\}$, $(x, w) \in \text{supp}(X, W)$, *and* $h > 0$, $E[L_{V,\lambda}(x, w; h)] = 0$, *and if Assumption* 4.2 *also holds, then*

$$E\big[\big(L_{V,\lambda}(x, w; h_n)\big)^2\big] = n^{-1}\Omega_{V,\lambda}(x, w; h_n),$$

where

$$\Omega_{V,\lambda}(x, w; h_n) \equiv E\big[\big(\ell_{V,\lambda}(x, w, h_n; V, X_1, X_2, W)\big)^2\big]$$

is finite. Furthermore, if Assumption 4.1 *holds, then*

$$\sqrt{\sup_{(x,w)\in\text{supp}(X,W)}\Omega_{V,\lambda}(x, w; h_n)} = O\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_L}, h_{2n}^{-1}\big\}\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,L}}\exp\big(\alpha_L\big(h_{1n}^{-1}\big)^{\nu_L}\big)\big),$$

with $\alpha_L \equiv \alpha_\phi 1_{\{\nu_\phi=\nu_\theta\}} - \alpha_\theta$, $\nu_L \equiv \nu_\theta$, $\gamma_{\lambda,L} \equiv 1 + \gamma_\phi - \gamma_\theta + \lambda$, *and* $\delta_L \equiv 1 + \gamma_1$. *We also have*

$$\sup_{(x,w)\in\text{supp}(X,W)}\big|L_{V,\lambda}(x, w; h_n)\big| = O_p\big(n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_L}, h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,L}}\exp\big(\alpha_L\big(h_{1n}^{-1}\big)^{\nu_L}\big)\big).$$

(ii) *If Assumptions* 4.3 *and* 4.4 *also hold, and if for* $V \in \{1, Y\}$ *and for each* $\lambda \in \{0, \dots, \Lambda\}$, $(x, w) \in \text{supp}(X, W)$, $\Omega_{V,\lambda}(x, w; h_n) > 0$ *for all n sufficiently large, then*

$$n^{1/2}\big(\Omega_{V,\lambda}(x, w; h_n)\big)^{-1/2}L_{V,\lambda}(x, w; h_n) \xrightarrow{d} N(0,1).$$

Proof. (i) It follows that $E[L_{V,\lambda}(x, w, h)] = 0$ by the definition of $L_{V,\lambda}(x, w, h)$. Assumption 4.2 guarantees that $L_{V,\lambda}(x, w, h)$ has a finite variance so that

$$E\big[\big(L_{V,\lambda}(x, w, h)\big)^2\big] = E\big[\big(\hat{E}\big[\ell_{V,\lambda}(x, w, h; V, X_1, X_2, W)\big]\big)^2\big] = n^{-1}\Omega_{V,\lambda}(x, w, h).$$

Because $L_{V,\lambda}(x,w,h) \equiv \bar{g}_{V,\lambda}(x,w,h) - g_{V,\lambda}(x,w,h_1)$, we have by Minkowski inequality that

$$\begin{aligned}
&\Omega_{V,\lambda}(x,w,h)\\
&\quad = nE\big[\big(\bar{g}_{V,\lambda}(x,w,h) - g_{V,\lambda}(x,w,h_1)\big)^2\big]\\
&\quad = E\bigg[\bigg(\int \Psi_{V,\lambda,1}(\zeta,x,w,h_1) n^{1/2}\delta\hat{\theta}_1(\zeta)\,d\zeta\\
&\qquad + \int \Psi_{V,\lambda,X_1}(\zeta,x,w,h_1) n^{1/2}\delta\hat{\theta}_{X_1}(\zeta)\,d\zeta\\
&\qquad + \int \Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1) n^{1/2}\delta\hat{\chi}_V(\zeta,w)\,d\zeta\\
&\qquad + \int \Psi_{V,\lambda,f_W}(\zeta,x,w,h_1) n^{1/2}\delta\hat{f}_W(w)\,d\zeta\bigg)^2\bigg]\\
&\quad \le \bigg[\bigg\{\int\int \Psi_{V,\lambda,1}(\zeta,x,w,h_1) E\big[n\delta\hat{\theta}_1(\zeta)\delta\hat{\theta}_1^\dagger(\xi)\big]\big(\Psi_{V,\lambda,1}(\xi,x,w,h_1)\big)^\dagger\,d\zeta\,d\xi\bigg\}^{1/2}\\
&\qquad + \bigg\{\int\int \Psi_{V,\lambda,X_1}(\zeta,x,w,h_1)\\
&\qquad \times E\big[n\delta\hat{\theta}_{X_1}(\zeta)\delta\hat{\theta}_{X_1}^\dagger(\xi)\big]\big(\Psi_{V,\lambda,X_1}(\xi,x,w,h_1)\big)^\dagger\,d\zeta\,d\xi\bigg\}^{1/2}\\
&\qquad + \bigg\{h_2^{-2}\int\int \Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)\\
&\qquad \times E\Big[n\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{\chi}_V(\zeta,w)\Big)\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{\chi}_V^\dagger(\xi,w)\Big)\Big]\\
&\qquad \times \big(\Psi_{V,\lambda,\chi_V}(\xi,x,w,h_1)\big)^\dagger\,d\zeta\,d\xi\bigg\}^{1/2}\\
&\qquad + \bigg\{h_2^{-2}\int\int \Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)\\
&\qquad \times E\Big[n\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{f}_W(w)\Big)\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{f}_W(w)\Big)\Big]\\
&\qquad \times \big(\Psi_{V,\lambda,f_W}(\xi,x,w,h_1)\big)^\dagger\,d\zeta\,d\xi\bigg\}^{1/2}\bigg]^2.
\end{aligned}$$

Note that by Assumption 4.2,

$$\begin{aligned}
E\big[n\delta\hat{\theta}_1(\zeta)\delta\hat{\theta}_1^\dagger(\xi)\big] &= E\big[n\big(\hat{\theta}_1(\zeta)-\theta_1(\zeta)\big)\big(\hat{\theta}_1^\dagger(\xi)-\theta_1^\dagger(\xi)\big)\big]\\
&= E\big[\big(e^{i\zeta X_2}-\theta_1(\zeta)\big)\big(e^{-i\xi X_2}-\theta_1^\dagger(\zeta)\big)\big]\\
&= E\big[e^{i(\zeta-\xi)X_2}\big] - \theta_1(\zeta)\theta_1^\dagger(\xi) - \theta_1(\zeta)\theta_1^\dagger(\xi) + \theta_1(\zeta)\theta_1^\dagger(\xi)\\
&= \theta_1(\zeta-\xi) - \theta_1(\zeta)\theta_1(-\xi),
\end{aligned}$$

so that

$$\begin{aligned}\left|E\left[n\delta\hat{\theta}_1(\zeta)\delta\hat{\theta}_1^\dagger(\xi)\right]\right| &= \left|\theta_1(\zeta-\xi)-\theta_1(\zeta)\theta_1(-\xi)\right| \\ &\le E\left[\left|e^{i(\zeta-\xi)X_2}\right|\right]+E\left[\left|e^{i\zeta X_2}\right|\right]E\left[\left|e^{-i\xi X_2}\right|\right]\preceq 1; \\ E\left[n\delta\hat{\theta}_{X_1}(\zeta)\delta\hat{\theta}_{X_1}^\dagger(\xi)\right] &= E\left[n\left(\hat{\theta}_{X_1}(\zeta)-\theta_{X_1}(\zeta)\right)\left(\hat{\theta}_{X_1}^\dagger(\xi)-\theta_{X_1}^\dagger(\xi)\right)\right] \\ &= E\left[\left(X_1 e^{i\zeta X_2}-\theta_{X_1}(\zeta)\right)\left(X_1 e^{-i\xi X_2}-\theta_{X_1}^\dagger(\xi)\right)\right] \\ &= E\left[X_1X_1 e^{i(\zeta-\xi)X_2}\right]-\theta_{X_1}(\zeta)\theta_{X_1}^\dagger(\xi),\end{aligned}$$

so that

$$\begin{aligned}\left|E\left[n\delta\hat{\theta}_{X_1}(\zeta)\delta\hat{\theta}_{X_1}^\dagger(\xi)\right]\right| &= \left|E\left[X_1X_1e^{i(\zeta-\xi)X_2}\right]-\theta_{X_1}(\zeta)\theta_{X_1}^\dagger(\xi)\right| \\ &\le E\left[|X_1X_1|\left|e^{i(\zeta-\xi)X_2}\right|\right]+E\left[|X_1|\left|e^{i\zeta X_2}\right|\right]E\left[|X_1|\left|e^{-i\xi X_2}\right|\right] \\ &\le E\left[|X_1X_1|\right]+E\left[|X_1|\right]E\left[|X_1|\right]\preceq 1;\end{aligned}$$

$$\begin{aligned}&E\left[n\left(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{\chi}_V(\zeta,w)\right)\left(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{\chi}_V^\dagger(\zeta,w)\right)\right] \\ &\quad= E\left[n\left(\sup_{w\in\operatorname{supp}(W)} h_2\left(\hat{\chi}_V(\zeta,w)-\chi_V(\zeta,w)\right)\right)\right. \\ &\qquad\left.\times\left(\sup_{w\in\operatorname{supp}(W)} h_2\left(\delta\hat{\chi}_V^\dagger(\zeta,w)-\chi_V^\dagger(\zeta,w)\right)\right)\right] \\ &\quad= E\left[\sup_{w\in\operatorname{supp}(W)} h_2\left(Ve^{i\zeta X_2}k_{h_2}(W-w)-\chi_V(\zeta,w)\right)\right. \\ &\qquad\left.\times\sup_{w\in\operatorname{supp}(W)} h_2\left(Ve^{-i\xi X_2}k_{h_2}(W-w)-\chi_V^\dagger(\xi,w)\right)\right],\end{aligned}$$

so that

$$\begin{aligned}&\left|E\left[n\left(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{\chi}_V(\zeta,w)\right)\left(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{\chi}_V^\dagger(\zeta,w)\right)\right]\right| \\ &\quad\le E\left[|V|^2\left|e^{i(\zeta-\xi)X_2}\right|\left|\sup_{w\in\operatorname{supp}(W)} h_2k_{h_2}(W-w)\right|^2\right] \\ &\qquad+3E\left[|V|\left|e^{i\zeta X_2}\right|\left|\sup_{w\in\operatorname{supp}(W)} h_2k_{h_2}(W-w)\right|\right] \\ &\qquad\times E\left[|V|\left|e^{-i\xi X_2}\right|\left|\sup_{w\in\operatorname{supp}(W)} h_2k_{h_2}(W-w)\right|\right] \\ &\quad\preceq 1,\end{aligned}$$

where the last line is obtained by Assumption 4.2 and the fact that

$$\sup_{w\in\operatorname{supp}(W)}\left|h_2k_{h_2}(w)\right| = \sup_{w\in\operatorname{supp}(W)}\left|\frac{h_2}{2\pi}\int\kappa(h_2\zeta)e^{-i\zeta w}\,d\zeta\right|\le\frac{1}{2\pi}\int\left|\kappa(\bar{\zeta})\right|d\bar{\zeta}\preceq 1.$$

Finally,

$$\begin{aligned}
&E\Big[n\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{f}_W(w)\Big)\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{f}_W(w)\Big)\Big]\\
&\quad= E\Big[n\Big(\sup_{w\in\operatorname{supp}(W)} h_2\big(\hat{f}_W(w)-f_W(w)\big)\Big)\Big(\sup_{w\in\operatorname{supp}(W)} h_2\big(\hat{f}_W(w)-f_W(w)\big)\Big)\Big]\\
&\quad= E\Big[\Big(\sup_{w\in\operatorname{supp}(W)} h_2\big(k_{h_2}(W-w)-E\big[k_{h_2}(W-w)\big]\big)\Big)\\
&\qquad\times\Big(\sup_{w\in\operatorname{supp}(W)} h_2\big(k_{h_2}(W-w)-E\big[k_{h_2}(W-w)\big]\big)\Big)\Big],
\end{aligned}$$

so that

$$\begin{aligned}
&\Big|E\Big[n\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{f}_W(w)\Big)\Big(\sup_{w\in\operatorname{supp}(W)} h_2\delta\hat{f}_W(w)\Big)\Big]\Big|\\
&\quad\le E\Big[\Big|\sup_{w\in\operatorname{supp}(W)} h_2k_{h_2}(W-w)\Big|^2\Big]\\
&\qquad+E\Big[\Big|\sup_{w\in\operatorname{supp}(W)} h_2k_{h_2}(W-w)\Big|\Big]E\Big[\Big|\sup_{w\in\operatorname{supp}(W)} h_2k_{h_2}(W-w)\Big|\Big]\\
&\quad\preceq 1.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\Omega_{V,\lambda}(x,w,h)\\
&\quad\preceq\Big(\sum_{A=1,X_1}\int\big|\Psi_{V,\lambda,A}(\zeta,x,w,h_1)\big|\,d\zeta\\
&\qquad+h_2^{-1}\sum_{B=\chi_V,f_W}\int\big|\Psi_{V,\lambda,B}(\zeta,x,w,h_1)\big|\,d\zeta\Big)^2\\
&\quad\le\Big(\sum_{A=1,X_1}\int\Psi^+_{V,\lambda,A}(\zeta,h_1)\,d\zeta+h_2^{-1}\sum_{B=\chi_V,f_W}\int\Psi^+_{V,\lambda,B}(\zeta,h_1)\,d\zeta\Big)^2\\
&\quad=\big(\Psi^+_{V,\lambda}(h)\big)^2,
\end{aligned}$$

where, for $A\in\{1,X_1,\chi_V,f_W\}$,

$$\begin{aligned}
\Psi^+_{V,\lambda,A}(\zeta,h_1)&\equiv\sup_{(x,w)\in\operatorname{supp}(X,W)}\big|\Psi_{V,\lambda,A}(\zeta,x,w,h_1)\big|,\\
\Psi^+_{V,\lambda}(h)&\equiv\sum_{A=\{1,X_1\}}\int\Psi^+_{V,\lambda,A}(\zeta,h_1)\,d\zeta+h_2^{-1}\sum_{A=\{\chi_V,f_W\}}\int\Psi^+_{V,\lambda,A}(\zeta,h_1)\,d\zeta\\
&=O\Big(\max\big\{\big(1+h_1^{-1}\big)^{\gamma_1+1},h_2^{-1}\big\}\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1}\\
&\qquad\times\exp\big((\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}}-\alpha_\theta)\big(h_1^{-1}\big)^{\nu_\theta}\big)\Big).
\end{aligned}$$

It follows that

$$\sqrt{\sup_{(x,w)\in\text{supp}(X,W)} \Omega_{V,\lambda}(x,w,h)}$$
$$= O\big(\max\{(h_1^{-1})^{\delta_L}, h_2^{-1}\}(h_1^{-1})^{\gamma_{\lambda,L}} \exp\big(\alpha_L (h_1^{-1})^{\nu_L}\big)\big),$$

with $\alpha_L \equiv \alpha_\phi 1_{\{\nu_\phi=\nu_\theta\}} - \alpha_\theta$, $\nu_L \equiv \nu_\theta$, $\gamma_{\lambda,L} \equiv 1+\gamma_\phi-\gamma_\theta+\lambda$, and $\delta_L \equiv \gamma_1+1$.

To show uniform convergence, we write

$$\sup_{(x,w)\in\text{supp}(X,W)} \big|\bar{g}_{V,\lambda}(x,w,h) - g_{V,\lambda}(x,w,h_1)\big|$$
$$= \sup_{(x,w)\in\text{supp}(X,W)} \bigg| \int \big[\Psi_{V,\lambda,1}(\zeta,x,w,h_1)\big(\hat{E}\big[e^{i\zeta X_2}\big] - E\big[e^{i\zeta X_2}\big]\big)$$
$$+ \Psi_{V,\lambda,X_1}(\zeta,x,w,h_1)\big(\hat{E}\big[X_1 e^{i\zeta X_2}\big] - E\big[X_1 e^{i\zeta X_2}\big]\big)$$
$$+ \Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)\big(\hat{E}\big[V e^{i\zeta X_2} k_{h_2}(W-w)\big] - E\big[V e^{i\zeta X_2} k_{h_2}(W-w)\big]\big)$$
$$+ \Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)\big(\hat{E}\big[k_{h_2}(W-w)\big] - E\big[k_{h_2}(W-w)\big]\big)\big]\, d\zeta \bigg|$$
$$= \int \Big[\Psi^+_{V,\lambda,1}(\zeta,h_1)\big|\hat{E}\big[e^{i\zeta X_2}\big] - E\big[e^{i\zeta X_2}\big]\big|$$
$$+ \Psi^+_{V,\lambda,X_1}(\zeta,h_1)\big|\hat{E}\big[X_1 e^{i\zeta X_2}\big] - E\big[X_1 e^{i\zeta X_2}\big]\big|$$
$$+ h_2^{-1}\Psi^+_{V,\lambda,\chi_V}(\zeta,h_1)\Big(\sup_{w\in\text{supp}(W)} \big|\hat{E}\big[V e^{i\zeta X_2} h_2 k_{h_2}(W-w)\big]$$
$$- E\big[V e^{i\zeta X_2} h_2 k_{h_2}(W-w)\big]\big|\Big)$$
$$+ h_2^{-1}\Psi^+_{V,\lambda,f_W}(\zeta,h_1)\Big(\sup_{w\in\text{supp}(W)} \big|\hat{E}\big[h_2 k_{h_2}(W-w)\big]$$
$$- E\big[h_2 k_{h_2}(W-w)\big]\big|\Big)\Big]\, d\zeta,$$

where the integrals are finite since $|\hat{E}[e^{i\zeta X_2}] - E[e^{i\zeta X_2}]| \preceq 1$, $|\hat{E}[X_1 e^{i\zeta X_2}] - E[X_1 e^{i\zeta X_2}]| \preceq 1$, $\sup_{w\in\text{supp}(W)} |\hat{E}[V e^{i\zeta X_2} h_2 k_{h_2}(W-w)] - E[V e^{i\zeta X_2} h_2 k_{h_2}(W-w)]| \preceq 1$, and $\sup_{w\in\text{supp}(W)} |\hat{E}[h_2 k_{h_2}(W-w)] - E[h_2 k_{h_2}(W-w)]| \preceq 1$, and since Lemma B.2 implies that $\Psi^+_{V,\lambda}(h) < \infty$. Then we have

$$E\Big[\sup_{(x,w)\in\text{supp}(X,W)} \big|\bar{g}_{V,\lambda}(x,w,h) - g_{V,\lambda}(x,w,h_1)\big|\Big]$$
$$\le \int \Big[\Psi^+_{V,\lambda,1}(\zeta,h_1) E\big\{\big(\big|\hat{E}\big[e^{i\zeta X_2} - E\big[e^{i\zeta X_2}\big]\big]\big|^2\big)^{1/2}\big\}$$
$$+ \Psi^+_{V,\lambda,X_1}(\zeta,h_1) E\big\{\big(\big|\hat{E}\big[X_1 e^{i\zeta X_2} - E\big[X_1 e^{i\zeta X_2}\big]\big]\big|^2\big)^{1/2}\big\}$$
$$+ h_2^{-1}\Psi^+_{V,\lambda,\chi_V}(\zeta,h_1) E\Big\{\Big(\Big|\sup_{w\in\text{supp}(W)} \big(\hat{E}\big[V e^{i\zeta X_2} h_2 k_{h_2}(W-w)$$

$$
\begin{aligned}
&\quad - E\big[Ve^{i\zeta X_2} h_2 k_{h_2}(W-w)\big]\big]\big)\Big|^2\Big)^{1/2}\Big\}\\
&\quad + h_2^{-1}\Psi^+_{V,\lambda,f_W}(\zeta,h_1)E\Big\{\Big(\Big|\sup_{w\in\operatorname{supp}(W)}\big(\hat{E}\big[h_2 k_{h_2}(W-w)\\
&\quad - E\big[h_2 k_{h_2}(W-w)\big]\big]\big)\Big|^2\Big)^{1/2}\Big\}\Big]\,d\zeta\\
&\le \int \Big[\Psi^+_{V,\lambda,1}(\zeta,h_1)\big\{n^{-1}E\big(\big|e^{i\zeta X_2}-E\big[e^{i\zeta X_2}\big]\big|^2\big)\big\}^{1/2}\\
&\quad + \Psi^+_{V,\lambda,X_1}(\zeta,h_1)\big\{n^{-1}E\big(\big|X_1e^{i\zeta X_2}-E\big[X_1e^{i\zeta X_2}\big]\big|^2\big)\big\}^{1/2}\\
&\quad + h_2^{-1}\Psi^+_{V,\lambda,\chi_V}(\zeta,h_1)\Big\{n^{-1}E\Big(\Big|\sup_{w\in\operatorname{supp}(W)}\big(Ve^{i\zeta X_2}h_2k_{h_2}(W-w)\\
&\quad - E\big[Ve^{i\zeta X_2}h_2k_{h_2}(W-w)\big]\big)\Big|^2\Big)\Big\}^{1/2}\\
&\quad + h_2^{-1}\Psi^+_{V,\lambda,f_W}(\zeta,h_1)\Big\{n^{-1}E\Big(\Big|\sup_{w\in\operatorname{supp}(W)}\big(h_2k_{h_2}(W-w)\\
&\quad - E\big[h_2k_{h_2}(W-w)\big]\big)\Big|^2\Big)\Big\}^{1/2}\Big]\,d\zeta\\
&\le n^{-1/2}\Big(\sum_{A=\{1,X_1\}}\int \Psi^+_{V,\lambda,A}(\zeta,h_1)\,d\zeta + h_2^{-1}\sum_{A=\{\chi_V,f_W\}}\int \Psi^+_{V,\lambda,A}(\zeta,h_1)\,d\zeta\Big)\\
&= n^{-1/2}\Psi^+_{V,\lambda}(h),
\end{aligned}
$$

where $\Psi^+_{V,\lambda}(h) = O(\max\{(1+h_1^{-1})^{\gamma_1+1}, h_2^{-1}\}(1+h_1^{-1})^{\gamma_\phi+\lambda-\gamma_\theta+1}\exp((\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}}-\alpha_\theta)\times (h_1^{-1})^{\nu_\theta}))$. Markov's inequality then gives

$$
\begin{aligned}
&\sup_{(x,w)\in\operatorname{supp}(X,W)}\big|L_{V,\lambda}(x,w,h)\big|\\
&= O_p\big(n^{-1/2}\big(\max\big\{\big(1+h_1^{-1}\big)^{\gamma_1+1}, h_2^{-1}\big\}\big)\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1}\\
&\quad \times \exp\big(\big(\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}}-\alpha_\theta\big)\big(h_1^{-1}\big)^{\nu_\theta}\big)\big).
\end{aligned}
$$

(ii) To show asymptotic normality, for fixed x and w, we apply Lemma B.3 to

$$
\begin{aligned}
&\sum_{j=1}^{2} A_j P_{n,j}(X_2) + \sum_{k=1}^{2} B_k Q_{n,k}(W)\\
&\equiv \int \Psi_{V,\lambda,1}(\zeta,x,w,h_1)\big(e^{i\zeta X_2}\big)\,d\zeta + \int \Psi_{V,\lambda,X_1}(\zeta,x,w,h_1)\big(X_1e^{i\zeta X_2}\big)\,d\zeta\\
&\quad + \int \Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)\big(Ve^{i\zeta X_2}k_{h_2}(W-w)\big)\,d\zeta\\
&\quad + \int \Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)\big(k_{h_2}(W-w)\big)\,d\zeta,
\end{aligned}
$$

with

$$P_{n,1}(x_2) = \int \Psi_{V,\lambda,1}(\zeta, x, w, h_1)e^{i\zeta x_2}\,d\zeta,$$

$$P_{n,2}(x_2) = \int \Psi_{V,\lambda,X_1}(\zeta, x, w, h_1)e^{i\zeta x_2}\,d\zeta,$$

$$Q_{n,1}(\tilde{w}) = \int \Psi_{V,\lambda,\chi_V}(\zeta, x, w, h_1)e^{i\zeta X_2}k_{h_2}(\tilde{w} - w)\,d\zeta,$$

$$Q_{n,2}(\tilde{w}) = \int \Psi_{V,\lambda,f_W}(\zeta, x, w, h_1)k_{h_2}(\tilde{w} - w)\,d\zeta$$

corresponding to $A_1 = 1$, $A_2 = X_1$, $B_1 = V$, and $B_2 = 1$, respectively. We assume that $\inf_{n>N} \Omega_{V,\lambda}(x, w, h) > 0$, and the imposed conditions ensure that for some finite N, $\sup_{n>N} \Omega_{V,\lambda}(x, w, h) = \sup_{n>N} \mathrm{var}[\ell_{V,\lambda}(x, w, h_n; V, X_1, X_2)] < \infty$. We need to verify that $\max\{\sup_{x_2\in\mathrm{supp}(X_2)} |D_{x_2}P_{n,j}(x_2)|, \sup_{\tilde{w}\in\mathrm{supp}(W)} |D_{\tilde{w}}Q_{n,k}(\tilde{w})|\} = O(n^{(3/2)-\eta})$ for $j = 1, 2$ and $k = 1, 2$. To do this, we use Lemma B.2. For $j = 1, 2$,

$$\begin{aligned}\sup_{x_2\in\mathrm{supp}(X_2)} \left|D_{x_2}P_{n,j}(x_2)\right| &= \sup_{x_2\in\mathrm{supp}(X_2)} \left|\int i\zeta\Psi_{V,\lambda,j}(\zeta, x, w, h_1)e^{i\zeta x_2}\,d\zeta\right| \\ &\preceq \sup_{(x,w)\in\mathrm{supp}(X,W)} \int_0^{h_{1n}^{-1}} |\zeta|\left|\Psi_{V,\lambda,j}(\zeta, x, w, h_1)\right|d\zeta \\ &\leq h_{1n}^{-1} \int_0^{h_{1n}^{-1}} \Psi^+_{V,\lambda,j}(\zeta, h_1)\,d\zeta \\ &\preceq \left(1 + h_{1n}^{-1}\right)^{\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3} \exp\left(\left(\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}} - \alpha_\theta\right)\left(h_{1n}^{-1}\right)^{\nu_\theta}\right).\end{aligned}$$

By Assumption 4.4, if $\nu_\theta \neq 0$, we have $h_{1n}^{-1} = O((\ln n)^{1/\nu_\theta-\eta})$ for some $\eta > 0$. Thus we have for $j = 1, 2$,

$$\begin{aligned}&\sup_{x_2\in\mathrm{supp}(X_2)} \left|D_{x_2}P_{n,j}(x_2)\right| \\ &\quad\preceq \left(1 + (\ln n)^{1/\nu_\theta-\eta}\right)^{\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3} \exp\left(\left(\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}} - \alpha_\theta\right)\left((\ln n)^{1/\nu_\theta-\eta}\right)^{\nu_\theta}\right).\end{aligned}$$

Because the right-hand side grows more slowly than any power of n, we certainly have

$$\sup_{x_2\in\mathrm{supp}(X_2)} \left|D_{x_2}P_{n,j}(x_2)\right| = O\left(n^{(3/2)-\eta}\right)$$

for $j = 1, 2$. If $\nu_\theta = 0$, we have $h_{1n}^{-1} = O(n^{-\eta}n^{(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)})$ for some $\eta > 0$. Thus we have

$$\begin{aligned}\sup_{x_2\in\mathrm{supp}(X_2)} \left|D_{x_2}P_{n,j}(x_2)\right| &\preceq \left(1 + n^{-\eta}n^{(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)}\right)^{\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3} \\ &= O\left(n^{(3/2)-\eta}\right).\end{aligned}$$

Because the Fourier transform of $D_x^{\lambda} k_{h_1}(x)$ is $(-i\zeta)^{\lambda}\kappa(h_1\zeta)$, we have

$$\left|h_1^{\lambda+1} D_x^{\lambda} k_{h_1}(x)\right| = \left|\frac{h_1^{\lambda+1}}{2\pi}\int(-i\zeta)^{\lambda}\kappa(h_1\zeta)e^{-i\zeta x}\,d\zeta\right| \le \frac{1}{2\pi}\int_{-1}^{1}|\bar{\zeta}|^{\lambda}\left|\kappa(\bar{\zeta})\right|d\bar{\zeta} < \infty.$$

Therefore, we get

$$\begin{aligned}
&\sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}Q_{n,1}(\tilde{w})\right| \\
&= \sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}\int \Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)e^{i\zeta x_2}k_{h_2}(\tilde{w}-w)\,d\zeta\right| \\
&= h_2^{-2}\int \Psi^{+}_{V,\lambda,\chi_V}(\zeta,h_1)\,d\zeta \\
&= O\Big(\big(1+h_2^{-1}\big)^2\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1}\exp\Big(\big(\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}}-\alpha_\theta\big)\big(h_1^{-1}\big)^{\nu_\theta}\Big)\Big).
\end{aligned}$$

By Assumption 4.4, if $\nu_\theta \neq 0$, we have $h_{1n}^{-1} = O((\ln n)^{1/\nu_\theta-\eta})$ and $h_{2n}^{-1} = O(n^{1/4-\eta})$ for some $\eta > 0$. Then

$$\begin{aligned}
\sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}Q_{n,1}(\tilde{w})\right| &\preceq \big(1+n^{1/4-\eta}\big)^2\big(1+(\ln n)^{1/\nu_\theta-\eta}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1} \\
&\quad\times\exp\Big(\big(\alpha_\phi 1_{\{\nu_\theta=\nu_\phi\}}-\alpha_\theta\big)\big((\ln n)^{1/\nu_\theta-\eta}\big)^{\nu_\theta}\Big) \\
&\preceq n^{1/2-\eta}n^{1-\eta} < n^{(3/2)-\eta}.
\end{aligned}$$

If $\nu_\theta = 0$, we have

$$h_{1n}^{-1} = O\big(n^{-\eta}n^{(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)}\big)$$

and

$$h_{2n}^{-1} = O\big(n^{((\gamma_1+2)/2)(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)}\big)$$

for some $\eta > 0$. Then

$$\begin{aligned}
&\sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}Q_{n,1}(\tilde{w})\right| \\
&\preceq \big(1+n^{((\gamma_1+2)/2)(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)}\big)^2 \\
&\quad\times\big(1+n^{-\eta}n^{(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)}\big)^{\gamma_\phi+\lambda-\gamma_\theta+1} \\
&\preceq n^{(\gamma_1+2)(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)}n^{-\eta}n^{(\gamma_\phi+\lambda-\gamma_\theta+1)(3/2)/(\gamma_\phi+\lambda+\gamma_1-\gamma_\theta+3)} = n^{(3/2)-\eta}.
\end{aligned}$$

Thus, the bandwidth sequences in Assumption 4.4 guarantee that

$$\sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}Q_{n,1}(\tilde{w})\right| = O\big(n^{(3/2)-\eta}\big).$$

Similarly,

$$\sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}Q_{n,2}(\tilde{w})\right| = \sup_{\tilde{w}\in\operatorname{supp}(W)}\left|D_{\tilde{w}}\int \Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)k_{h_2}(\tilde{w}-w)\,d\zeta\right|$$
$$\preceq h_2^{-2}\int \Psi^{+}_{V,\lambda,f_W}(\zeta,h_1)\,d\zeta$$
$$= O\big(\big(1+h_2^{-1}\big)^2\big(1+h_1^{-1}\big)^{\gamma_\phi+\lambda+1}\exp\big(\big(\alpha_\phi\big(h^{-1}\big)^{\nu_\phi}\big)\big).$$

Because $\sup_{\tilde{w}\in\operatorname{supp}(W)}|D_{\tilde{w}}Q_{n,2}(\tilde{w})| \preceq \sup_{\tilde{w}\in\operatorname{supp}(W)}|D_{\tilde{w}}Q_{n,1}(\tilde{w})|$, the result follows. □

Lemma B.5. *Let A and X_2 be random variables satisfying $E[|A|^2]<\infty$ and $E[|A||X_2|]<\infty$, and let $\{A_i,X_{2,i}\}_{i=1,\dots,n}$ be a corresponding IID sample. Then for any u, $U\geq 0$, and $\epsilon>0$,*

$$\sup_{\zeta\in[-Un^u,Un^u]}\left|\hat{E}\big[A\exp(i\zeta X_2)\big]-E\big[A\exp(i\zeta X_2)\big]\right| = O_p\big(n^{-1/2+\epsilon}\big).$$

Proof. The result immediately follows by Lemma 6 in Schennach (2004b). □

Lemma B.6. (i) *Suppose the conditions of Lemma B.4 hold, together with Assumptions 4.5 and 4.6. Then for $V\in\{1,Y\}$, each $\lambda\in\{0,\dots,\Lambda\}$, and some $\epsilon>0$,*

$$\sup_{(x,w)\in\operatorname{supp}(X,W)}\left|R_{V,\lambda}(x,w;h_n)\right|$$
$$= O_p\big(n^{-1/2+\epsilon}\big(h_{2n}^{-1}\big)^3\big(h_{1n}^{-1}\big)^{\gamma_1-\gamma_\theta}\exp\big(-\alpha_\theta\big(h_{1n}^{-1}\big)^{\nu_\theta}\big)\big)$$
$$\times O_p\big(n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_L},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,L}}\exp\big(\alpha_L\big(h_{1n}^{-1}\big)^{\nu_L}\big)\big).$$

(ii) *If Assumption 4.7 holds in place of Assumption 4.4, then for $V\in\{1,Y\}$ and each $\lambda\in\{0,\dots,\Lambda\}$,*

$$\sup_{(x,w)\in\operatorname{supp}(X,W)}\left|R_{V,\lambda}(x,w;h_n)\right|$$
$$= o_p\big(n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_L},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,L}}\exp\big(\alpha_L\big(h_{1n}^{-1}\big)^{\nu_L}\big)\big).$$

Proof. By substituting eqs. (S2)–(S5) into

$$\hat{g}_{V,\lambda}(x,w,h)-g_{V,\lambda}(x,w,h_1)$$
$$=\frac{1}{2\pi}\int(-i\zeta)^\lambda\kappa(h_1\zeta)\exp(-i\zeta x)\left[\frac{\hat{\theta}_V(\zeta,w)}{\hat{\theta}_1(\zeta)}\exp\left(\int_0^\zeta\frac{i\hat{\theta}_{X_1}(\xi)}{\hat{\theta}_1(\zeta)}\,d\xi\right)\right.$$
$$\left.-\frac{\theta_V(\zeta,w)}{\theta_1(\zeta)}\exp\left(\int_0^\zeta\frac{i\theta_{X_1}(\xi)}{\theta_1(\zeta)}\,d\xi\right)\right]d\zeta,$$

and removing the terms linear in $\delta\hat{\theta}_1(\zeta)$, $\delta\hat{\theta}_{X_1}(\zeta)$, $\delta\hat{\chi}_V(\zeta,w)$, and $\delta\hat{f}_W(w)$, we obtain the nonlinear remainder term such that $R_{V,\lambda}(x,w,h)\equiv\hat{g}_{V,\lambda}(x,w,h)-\bar{g}_{V,\lambda}(x,w,h)=$

$\sum_{i=1}^{22} R_i$, where

$$R_1 = \frac{1}{4\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) q_V(\zeta,w) q_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\exp\big(\delta\bar{Q}_{X_1}(\zeta)\big)\left(\int_0^\zeta i\delta\hat{q}_{X_1}(\xi)\,d\xi\right)^2 d\zeta,$$

$$R_2 = \frac{1}{4\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) \delta\hat{q}_V(\zeta,w) q_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\exp\big(\delta\bar{Q}_{X_1}(\zeta)\big)\left(\int_0^\zeta i\delta\hat{q}_{X_1}(\xi)\,d\xi\right)^2 d\zeta,$$

$$R_3 = \frac{1}{4\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) q_V(\zeta,w) \delta\hat{q}_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\exp\big(\delta\bar{Q}_{X_1}(\zeta)\big)\left(\int_0^\zeta i\delta\hat{q}_{X_1}(\xi)\,d\xi\right)^2 d\zeta,$$

$$R_4 = \frac{1}{4\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) \delta\hat{q}_V(\zeta,w) \delta\hat{q}_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\exp\big(\delta\bar{Q}_{X_1}(\zeta)\big)\left(\int_0^\zeta i\delta\hat{q}_{X_1}(\xi)\,d\xi\right)^2 d\zeta,$$

$$R_5 = \frac{1}{2\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) q_V(\zeta,w) q_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\int_0^\zeta i\delta_2\hat{q}_{X_1}(\xi)\,d\xi\,d\zeta,$$

$$R_6 = \frac{1}{2\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) \delta\hat{q}_V(\zeta,w) q_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\int_0^\zeta i\delta_2\hat{q}_{X_1}(\xi)\,d\xi\,d\zeta,$$

$$R_7 = \frac{1}{2\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) q_V(\zeta,w) \delta\hat{q}_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\int_0^\zeta i\delta_2\hat{q}_{X_1}(\xi)\,d\xi\,d\zeta,$$

$$R_8 = \frac{1}{2\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) \delta\hat{q}_V(\zeta,w) \delta\hat{q}_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\int_0^\zeta i\delta_2\hat{q}_{X_1}(\xi)\,d\xi\,d\zeta,$$

$$R_9 = \frac{1}{2\pi}\int (-i\zeta)^\lambda \kappa(h_1\zeta)\exp(-i\zeta x) \delta_1\hat{q}_V(\zeta,w) q_1(w) \times \exp\big(Q_{X_1}(\zeta)\big)\int_0^\zeta i\delta_1\hat{q}_{X_1}(\xi)\,d\xi\,d\zeta,$$

$$R_{10} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_2 \hat{q}_V(\zeta, w) q_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_{X_1}(\xi)\, d\xi\, d\zeta,$$

$$R_{11} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) q_V(\zeta, w) \delta_1 \hat{q}_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_1(\xi)\, d\xi\, d\zeta,$$

$$R_{12} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_1 \hat{q}_V(\zeta, w) \delta_1 \hat{q}_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_{X_1}(\xi)\, d\xi\, d\zeta,$$

$$R_{13} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_2 \hat{q}_V(\zeta, w) \delta_1 \hat{q}_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_{X_1}(\xi)\, d\xi\, d\zeta,$$

$$R_{14} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) q_V(\zeta, w) \delta_2 \hat{q}_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_{X_1}(\xi)\, d\xi\, d\zeta,$$

$$R_{15} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_1 \hat{q}_V(\zeta, w) \delta_2 \hat{q}_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_{X_1}(\xi)\, d\xi\, d\zeta,$$

$$R_{16} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_2 \hat{q}_V(\zeta, w) \delta_2 \hat{q}_1(w) \times \exp\bigl(Q_{X_1}(\zeta)\bigr) \int_0^{\zeta} i\delta_1 \hat{q}_{X_1}(\xi)\, d\xi\, d\zeta,$$

$$R_{17} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_2 \hat{q}_V(\zeta, w) q_1(w) \exp\bigl(Q_{X_1}(\zeta)\bigr)\, d\zeta,$$

$$R_{18} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_1 \hat{q}_V(\zeta, w) \delta_1 \hat{q}_1(w) \exp\bigl(Q_{X_1}(\zeta)\bigr)\, d\zeta,$$

$$R_{19} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_2 \hat{q}_V(\zeta, w) \delta_1 \hat{q}_1(w) \exp\bigl(Q_{X_1}(\zeta)\bigr)\, d\zeta,$$

$$R_{20} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) q_V(\zeta, w) \delta_2 \hat{q}_1(w) \exp\bigl(Q_{X_1}(\zeta)\bigr)\, d\zeta,$$

$$R_{21} = \frac{1}{2\pi} \int (-i\zeta)^{\lambda} \kappa(h_1\zeta) \exp(-i\zeta x) \delta_1 \hat{q}_V(\zeta, w) \delta_2 \hat{q}_1(w) \exp\bigl(Q_{X_1}(\zeta)\bigr)\, d\zeta,$$

$$R_{22} = \frac{1}{2\pi}\int (-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\delta_2\hat{q}_V(\zeta,w)\delta_2\hat{q}_1(w)\exp\bigl(Q_{X_1}(\zeta)\bigr)\,d\zeta.$$

Because $E[Y^2] < \infty$ by Assumption 4.2 and $E[|YX_2|] < \infty$ by Assumption 4.5, Lemma B.5 gives that for any $\epsilon > 0$,

$$\begin{aligned}
&\sup_{w\in\operatorname{supp}(W)}\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\hat{\chi}_V(\zeta,w)-\chi_V(\zeta,w)\bigr| \\
&\quad=\sup_{w\in\operatorname{supp}(W)}\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\hat{E}\bigl[Vk_{h_{2n}}(W-w)\exp(i\zeta X_2)\bigr] \\
&\qquad-E\bigl[Vk_{h_{2n}}(W-w)\exp(i\zeta X_2)\bigr]\bigr| \\
&\quad=h_{2n}^{-1}\sup_{w\in\operatorname{supp}(W)}\bigl|h_{2n}k_{h_{2n}}(W-w)\bigr| \\
&\qquad\times\sup_{\zeta\in[-Un^u,Un^u]}\bigl|\hat{E}\bigl[V\exp(i\zeta X_2)\bigr]-E\bigl[V\exp(i\zeta X_2)\bigr]\bigr| \\
&\quad=O_p\bigl(h_{2n}^{-1}n^{-1/2+\epsilon}\bigr).
\end{aligned}$$

We define $Y(h_n)$ and $\hat{\Phi}_n$ as

$$\begin{aligned}
Y(h_n)&\equiv\bigl(1+h_{2n}^{-1}\bigr)\Bigl(\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|D_\zeta\ln\phi_1(\zeta)\bigr|\Bigr) \\
&\qquad\times\Bigl(\max\Bigl\{\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\sup_{w\in\operatorname{supp}(W)}\bigl|\chi_V(\zeta,w)\bigr|^{-1},\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\theta_1(\zeta)\bigr|^{-1}\Bigr\}\Bigr) \\
&\qquad=O\bigl(\bigl(1+h_{2n}^{-1}\bigr)\bigl(1+h_{1n}^{-1}\bigr)^{\gamma_1-\gamma_\theta}\exp\bigl(-\alpha_\theta\bigl(h_{1n}^{-1}\bigr)^{\nu_\theta}\bigr)\bigr), \\
\hat{\Phi}_n&\equiv\max\Bigl\{\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\hat{\theta}_1(\zeta)-\theta_1(\zeta)\bigr|,\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\hat{\theta}_{X_1}(\zeta)-\theta_{X_1}(\zeta)\bigr|, \\
&\qquad\sup_{w\in\operatorname{supp}(W)}\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\hat{\chi}_V(\zeta,w)-\chi_V(\zeta,w)\bigr|,\sup_{w\in\operatorname{supp}(W)}\bigl|\hat{f}_W(w)-f_W(w)\bigr|\Bigr\} \\
&\quad=O_p\bigl(h_{2n}^{-1}n^{-1/2+\epsilon}\bigr)
\end{aligned}$$

for any $\epsilon > 0$. Note that the suprema associated with ζ can be taken over $[-h_{1n}^{-1}, h_{1n}^{-1}]$, since $\kappa(h_{1n}\zeta)$ vanishes outside the interval by Assumption 3.6. The second order of magnitude follows from Lemma B.5 and Assumption 4.6, since $h_{2n}^{-1}n^{-1/2+\epsilon} = h_{2n}^{-1/2}n^{-1/2}(n^{\epsilon}\times h_{2n}^{-1/2}) > h_{2n}^{-1/2}n^{-1/2}(\ln n)^{1/2} + h_{2n}^2$ for any choices of h_{2n} from Assumption 4.4 and 4.7. Then the terms in the nonlinear remainder can be bounded in terms of $\Psi_{V,\lambda}^{+}(h_n)$, $Y(h_n)$, and $\hat{\Phi}_n$. We note that

$$\begin{aligned}
&\hat{\Phi}_n\times\Bigl(\max\Bigl\{\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\sup_{w\in\operatorname{supp}(W)}\bigl|\chi_V(\zeta,w)\bigr|^{-1},\sup_{\zeta\in[-h_{1n}^{-1},h_{1n}^{-1}]}\bigl|\theta_1(\zeta)\bigr|^{-1}\Bigr\}\Bigr) \\
&\quad\preceq\hat{\Phi}_nY(h_n)
\end{aligned}$$

$$
\begin{aligned}
&= O_p\big(h_{2n}^{-1} n^{-1/2+\varepsilon}\big) O\big((1+h_{2n}^{-1})(1+h_{1n}^{-1})^{\gamma_1-\gamma_\theta} \exp\big(-\alpha_\theta (h_{1n}^{-1})^{\nu_\theta}\big)\big) \\
&= o_p(1).
\end{aligned}
$$

We now find upper bounds for each term, R_i, $i = 1, \dots, 22$. Because all other terms are also bounded by the upper bound for R_1, we focus on the bound for R_1.

$$
\begin{aligned}
&\sup_{(x,w)\in \operatorname{supp}(X,W)} |R_1| \\
&\preceq \sup_{w\in\operatorname{supp}(W)} \int_0^\infty |\zeta|^\lambda \big|\kappa(h_1\zeta)\big| \big|q_V(\zeta,w)\big| \big|q_1(w)\big| \\
&\quad \times \exp\big(Q_{X_1}(\zeta)\big) \exp\big(\big|\delta \bar{Q}_{X_1}(\zeta)\big|\big) \bigg(\int_0^\zeta \big|\delta \hat{q}_{X_1}(\xi)\big|\, d\xi\bigg)^2 d\zeta \\
&\preceq \exp\big(o_p(1)\big) \int_0^\infty |\zeta|^\lambda \big|\kappa(h_1\zeta)\big| \Big(\sup_{w\in\operatorname{supp}(W)} \big|\phi_V(\zeta,w)\big|\Big) \\
&\quad \times \int_0^\zeta \bigg|\bigg(\frac{\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)} - \frac{\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\bigg)\bigg(1+\frac{\delta\hat{\theta}_1(\xi)}{\theta_1(\xi)}\bigg)^{-1}\bigg|\, d\xi \\
&\quad \times \int_0^\zeta \bigg|\bigg(\frac{\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)} - \frac{\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\bigg)\bigg(1+\frac{\delta\hat{\theta}_1(\xi)}{\theta_1(\xi)}\bigg)^{-1}\bigg|\, d\xi\, d\zeta \\
&\preceq \exp\big(o_p(1)\big) \Upsilon(h) \hat{\Phi}_n^2 \big|1+o_p(1)\big|^{-2} \int_0^\infty |\zeta|^\lambda \big|\kappa(h_1\zeta)\big| \\
&\quad \times \Big(\sup_{w\in\operatorname{supp}(W)} \big|\phi_V(\zeta,w)\big|\Big) \int_0^\zeta \bigg(\frac{1}{|\theta_1(\xi)|} + \frac{|\theta_{X_1}(\xi)|}{|\theta_1(\xi)|^2}\bigg) d\xi\, d\zeta \\
&= \exp\big(o_p(1)\big) \Upsilon(h) \hat{\Phi}_n^2 \big|1+o_p(1)\big|^{-2} \int_0^\infty \bigg(\int_\xi^\infty |\zeta|^\lambda \big|\kappa(h_1\zeta)\big| \\
&\quad \times \Big(\sup_{w\in\operatorname{supp}(W)} \big|\phi_V(\zeta,w)\big|\Big) d\zeta\bigg) \bigg(\frac{1}{|\theta_1(\xi)|} + \frac{|\theta_{X_1}(\xi)|}{|\theta_1(\xi)|^2}\bigg) d\xi \\
&= O_p(1) \Upsilon(h) \hat{\Phi}_n^2 \Psi_{V,\lambda}^+(h) \\
&\preceq \Upsilon(h) \hat{\Phi}_n^2 \Psi_{V,\lambda}^+(h).
\end{aligned}
$$

When the conditions of Lemma B.4 hold, we get the bound needed for part (i):

$$
\begin{aligned}
\sup_{(x,w)\in\operatorname{supp}(X,W)} |R_1| &= O_p\big(\big(h_2^{-1}\big)\big(h_1^{-1}\big)^{\gamma_1-\gamma_\theta} \exp\big(-\alpha_\theta \big(h_1^{-1}\big)^{\nu_\theta}\big)\big(h_2^{-1}\big)^2 n^{-1+2\epsilon} \\
&\quad \times \big(\max\big\{\big(h_1^{-1}\big)^{\delta_L}, h_2^{-1}\big\}\big)\big(h_1^{-1}\big)^{\gamma_{\lambda,L}} \exp\big(\alpha_L \big(h_1^{-1}\big)^{\nu_L}\big)\big).
\end{aligned}
$$

To get the bound for $R_{V,\lambda}(x,w,h_n)$ when Assumption 4.7 holds in place of Assumption 4.4 in the conditions of Lemma B.4, we note that

$$
\begin{aligned}
&\Upsilon(h)\hat{\Phi}_n^2 \Psi_{V,\lambda}^+(h) = \big(\Upsilon(h)\hat{\Phi}_n^2 n^{1/2}\big) n^{-1/2} \Psi_{V,\lambda}^+(h), \\
&n^{-1/2}\Psi_{V,\lambda}^+(h) = O_p\big(n^{-1/2}\big(\max\big\{\big(h_1^{-1}\big)^{\delta_L}, h_2^{-1}\big\}\big)\big(h_1^{-1}\big)^{\gamma_{\lambda,L}} \exp\big(\alpha_L \big(h_1^{-1}\big)^{\nu_L}\big)\big),
\end{aligned}
$$

where the second equality is obtained using Lemma B.2. Now we show that $\Upsilon(h_n)\hat{\Phi}_n^2 \times n^{1/2} = o_p(1)$. When $\nu_\theta \neq 0$, we have $h_{1n}^{-1} = O((\ln n)^{1/\nu_\theta - \eta})$ and $h_{2n}^{-1} = O(n^{1/6-\eta})$ by Assumption 4.7, so that

$$
\begin{aligned}
&\Upsilon(h_n)\hat{\Phi}_n^2 n^{1/2} \\
&\quad = \Upsilon(h_n) O_p\bigl(h_{2n}^{-2} n^{-1+2\epsilon}\bigr) n^{1/2} \\
&\quad = O_p\bigl(\bigl(1 + h_{2n}^{-1}\bigr)^3 \bigl(1 + h_{1n}^{-1}\bigr)^{\gamma_1 - \gamma_\theta} \exp\bigl(-\alpha_\theta \bigl(h_{1n}^{-1}\bigr)^{\nu_\theta}\bigr) n^{-1/2+2\epsilon}\bigr) \\
&\quad = O_p\bigl(\bigl(1 + n^{1/6-\eta}\bigr)^3 \bigl(1 + (\ln n)^{1/\nu_\theta - \eta}\bigr)^{\gamma_1 - \gamma_\theta} \exp\bigl(-\alpha_\theta (\ln n)^{1-\eta\nu_\theta}\bigr) n^{-1/2+2\epsilon}\bigr) \\
&\quad = O_p\bigl(n^{1/2-3\eta} (\ln n)^{(1/\nu_\theta - \eta)(\gamma_1 - \gamma_\theta)} \exp\bigl(-\alpha_\theta (\ln n)^{1-\eta\nu_\theta}\bigr) n^{-1/2+2\epsilon}\bigr) \\
&\quad = O_p\bigl(\exp\bigl[(1/2 - 3\eta - 1/2 + 2\epsilon)\ln n \\
&\qquad + (1/\nu_\theta - \eta)(\gamma_1 - \gamma_\theta)\ln(\ln n) - \alpha_\theta (\ln n)^{1-\eta\nu_\theta}\bigr]\bigr) \\
&\quad = o_p(1),
\end{aligned}
$$

where the last equality follows because $\ln n$ dominates $(\ln n)^{1-\eta\nu_\theta}$ and $\ln(\ln n)$, and because $1/2 - 3\eta - 1/2 + 2\epsilon < 0$ by selecting $\eta > 2\epsilon/3$. When $\nu_\theta = 0$, we have $h_{1n}^{-1} = O(n^{-\eta} n^{1/(14\gamma_1 - 14\gamma_\theta)})$ and $h_{2n}^{-1} = O(n^{1/7-\eta})$, so that

$$
\begin{aligned}
\Upsilon(h_n)\hat{\Phi}_n^2 n^{1/2} &= \Upsilon(h_n) O_p\bigl(h_{2n}^{-2} n^{-1+2\epsilon}\bigr) n^{1/2} \\
&= O_p\bigl(\bigl(1 + h_{2n}^{-1}\bigr)^3 \bigl(1 + h_{1n}^{-1}\bigr)^{\gamma_1 - \gamma_\theta} n^{-1/2+2\epsilon}\bigr) \\
&= O_p\bigl(\bigl(n^{1/7-\eta}\bigr)^3 \bigl(n^{-\eta} n^{1/(14\gamma_1 - 14\gamma_\theta)}\bigr)^{\gamma_1 - \gamma_\theta} n^{-1/2+2\epsilon}\bigr) \\
&\preceq O_p\bigl(n^{-4\eta+2\epsilon}\bigr) \\
&= o_p(1)
\end{aligned}
$$

by selecting $\eta > \epsilon/2$. One can show that the bounds for the remaining terms contain the same leading term, $\Upsilon(h)\hat{\Phi}_n^2 \Psi_{V,\lambda}^+(h)$. We thus omit them for brevity. □

Proof of Theorem 4.2. Combining Lemma B.1, Lemma B.4 and Lemma B.6(ii) immediately yields the result. □

Proof of Theorem 4.3. Because the bias and the remainder term will never dominate the variance term by Assumption 4.8, the result immediately follows by Lemma B.4, Lemma B.6(i) and the fact that $\hat{g}_{V,\lambda}(x, w, h_n) - g_{V,\lambda}(x, w) = B_{V,\lambda}(x, w, h_{1n}) + L_{V,\lambda}(x, w, h_n) + R_{V,\lambda}(x, w, h_n)$. □

Proof of Theorem 4.4. From a first-order Taylor expansion of $\hat{\beta}(x, w, h_n) - \beta(x, w)$ in $\hat{g}_{V,\lambda}(x, w, h_n) - g_{V,\lambda}(x, w)$, we get

$$
\begin{aligned}
&\hat{\beta}(x, w, h_n) - \beta(x, w) \\
&\quad = \sum_{V \in \{1, Y\}} \sum_{\lambda = 0,1} s_{V,\lambda}(x, w)\bigl(\hat{g}_{V,\lambda}(x, w, h_n) - g_{V,\lambda}(x, w)\bigr) \qquad \text{(S6)} \\
&\qquad + R_{V,\lambda}\bigl(\bar{g}_{V,\lambda}(x, w, h_n), \bigl(\hat{g}_{V,\lambda}(x, w, h_n) - g_{V,\lambda}(x, w)\bigr)\bigr),
\end{aligned}
$$

where $R_{V,\lambda}(\bar{g}_{V,\lambda}(x,w,h_n), (\hat{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w)))$ is a remainder term in which $\bar{g}_{V,\lambda}(x,w,h_n)$ lies between $\hat{g}_{V,\lambda}(x,w,h_n)$ and $g_{V,\lambda}(x,w)$ for each (x,w,h_n), and $s_{V,\lambda}(x,w)$ is given in front of the theorem.

We note that by Theorem 4.2,

$$\max_{V\in\{1,Y\}} \max_{\lambda=0,1} \sup_{(x,w)\in(X,W)} \left|\hat{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w)\right| = O_p(\varepsilon_n),$$

$$\begin{aligned}\varepsilon_n &\equiv \left(h_{1n}^{-1}\right)^{\gamma_{1,B}} \exp\left(\alpha_B\left(h_{1n}^{-1}\right)^{\nu_B}\right) \\ &\quad + n^{-1/2}\left(\max\left\{\left(h_{1n}^{-1}\right)^{\delta_L}, h_{2n}^{-1}\right\}\right)\left(h_{1n}^{-1}\right)^{\gamma_{1,L}} \exp\left(\alpha_L\left(h_{1n}^{-1}\right)^{\nu_L}\right) \\ &\to 0.\end{aligned}$$

The first terms in the Taylor expansion of $\hat{\beta}(x,w,h_n) - \beta(x,w)$ can be shown to be $O_p(\varepsilon_n/\tau_n^3)$ uniformly for $(x,w) \in \Gamma_\tau$. Each term of $s_{V,\lambda}(x,w)$ consists of products of functions of the form $g_{V,\lambda}(x,w)$ divided by products of at most three functions of the form $g_{1,0}(x,w)$. Because $g_{V,\lambda}(x,w)$ are uniformly bounded over $\mathbb{R}$ by assumption and $g_{1,0}(x,w)$ are bounded below by τ_n uniformly for $(x,w) \in \Gamma_\tau$ by construction, we have that $\sup_{(x,w)\in\Gamma_\tau} |s_{V,\lambda}(x,w)(\hat{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w))| = O(1)O_p(\tau_n^{-3})O_p(\varepsilon_n) = O_p(\varepsilon_n/\tau_n^3)$.

The remainder terms in the Taylor expansion of $\hat{\beta}(x,w,h_n) - \beta(x,w)$ can be shown to be $o_p(\varepsilon_n/\tau_n^3)$ uniformly for $(x,w) \in \Gamma_\tau$. These terms involve a finite sum of (i) finite products of the functions $\bar{g}_{V,\lambda}(x,w,h_n)$ for $V \in \{1,Y\}$ and $\lambda = 0,1$, (ii) division by a product of at most four functions of the form $\bar{g}_{1,0}(x,w,h_n)$, and (iii) pairwise products of functions of the form $\hat{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w)$ for $V \in \{1,Y\}$ and $\lambda = 0,1$. First, the contribution of (i) is bounded in probability uniformly for $(x,w) \in \Gamma_\tau$ because

$$\begin{aligned}\left|\bar{g}_{V,\lambda}(x,w,h_n)\right| &\le \left|g_{V,\lambda}(x,w)\right| + \left|\bar{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w)\right| \\ &\le \left|g_{V,\lambda}(x,w)\right| + \left|\hat{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w)\right| \\ &\le \left|g_{V,\lambda}(x,w)\right| \\ &\quad + \max_{V\in\{1,Y\}} \max_{\lambda=0,1} \sup_{(x,w)\in(X,W)} \left|\hat{g}_{V,\lambda}(x,w,h_n) - g_{V,\lambda}(x,w)\right| \\ &= O_p(1).\end{aligned}$$

Second, the contribution of (ii) is bounded as well. We note that for $(x,w) \in \Gamma_\tau$,

$$\begin{aligned}\bar{g}_{1,0}(x,w,h_n) &= g_{1,0}(x,w)\left(1 + \frac{\bar{g}_{1,0}(x,w,h_n) - g_{1,0}(x,w)}{g_{1,0}(x,w)}\right) \\ &= f_{X|W}(x \mid w)\left(1 + O_p\left(\frac{\varepsilon_n}{\tau_n}\right)\right).\end{aligned}$$

By selecting $\{\tau_n\}$ such that $\tau_n > 0$, $\tau_n \to 0$ as $n \to \infty$, and $\varepsilon_n/\tau_n^3 \to 0$, we also have $\varepsilon_n/\tau_n \to 0$. Thus, we get

$$\bar{g}_{1,0}(x,w,h_n) = f_{X|W}(x \mid w)\left(1 + o_p(1)\right) \ge \tau_n/2$$

with probability approaching 1 since $f_{X|W}(x \mid w) \geq \tau_n$ for $(x, w) \in \Gamma_\tau$ by construction. Therefore, we have that the contribution of (ii) is $\bar{g}_{1,0}^{-4}(x, w, h_n) = O_p(\tau_n^{-4})$. Finally, the contribution of (iii) is $O_p(\varepsilon_n^2)$. Putting all together gives

$$R_{V,\lambda}\big(\bar{g}_{V,\lambda}(x, w, h_n), \big(\hat{g}_{V,\lambda}(x, w, h_n) - g_{V,\lambda}(x, w)\big)\big)$$
$$= O_p(1)O_p\big(\tau_n^{-4}\big)O_p\big(\varepsilon_n^2\big) = O_p\left(\frac{\varepsilon_n}{\tau_n^3}\right)O_p\left(\frac{\varepsilon_n}{\tau_n}\right) = o_p\left(\frac{\varepsilon_n}{\tau_n^3}\right),$$

so that

$$\sup_{(x,w)\in\Gamma_\tau} \big|\hat{\beta}(x, w, h_n) - \beta(x, w)\big| = O_p\left(\frac{\varepsilon_n}{\tau_n^3}\right) + o_p\left(\frac{\varepsilon_n}{\tau_n^3}\right) = o_p(1). \qquad \square$$

Proof of Theorem 4.5. We have established the asymptotic normality of $\hat{g}_{V,\lambda}(x, w, h_n) - g_{V,\lambda}(x, w)$ in Theorem 4.3 and we have the Taylor expansion in eq. (S6). Thus, the result is immediate from the delta method. $\square$

Appendix C: Asymptotic properties of weighted averages effects

This appendix provides asymptotic properties of functionals of the general form $\hat{g}_{V,\lambda}(x, w; h)$ in Section C.1 and of functionals of covariate-conditioned average marginal effects $\hat{\beta}(x, w; h)$ in Section C.2, which covers various forms of treatment effects. Section C.3 establishes their mathematical proofs.

C.1 *Asymptotics for functionals of the general form*

In addition to structurally identified $\beta(x, w)$, we are interested in weighted averages of $\beta(x, w)$ such as

$$\beta_m(x) \equiv \int \beta(x, w)m(w)\, dw,$$

$$\beta_{\tilde{m}} \equiv \int\int \beta(x, w)\tilde{m}(x, w)\, dw\, dx,$$

$$\beta_{mf_W}(x) \equiv \int \beta(x, w)m(w)f_W(w)\, dw,$$

$$\beta_{\tilde{m}f_{W|X}} \equiv \int\int \beta(x, w)\tilde{m}(x, w)f_{W|X}(w \mid x)\, dw\, dx,$$

$$\beta_{mf_{W|X}}(x) \equiv \int \beta(x, w)m(w)f_{W|X}(w \mid x)\, dw,$$

$$\beta_{\tilde{m}f_{W,X}} \equiv \int\int \beta(x, w)\tilde{m}(x, w)f_{W,X}(w, x)\, dw\, dx,$$

where $m(\cdot)$ and $\tilde{m}(\cdot, \cdot)$ are user-supplied weight functions, and where f_W, $f_{W|X}$, and $f_{W,X}$ are the marginal density of W, conditional density of W given X, and joint density of W and X, respectively. When $m(w) = 1$, for instance, $\beta_{mf_W}(x)$ is analogous to the derivative of the average structural function of Blundell and Powell (2004) and the average treatment effect of Florens, Heckman, Meghir, and Vytlacil (2008). When $m(w) = 1$, $\beta_{mf_{W|X}}(x)$ corresponds to the local average response of Altonji and Matzkin (2005) and the effect of treatment on the treated (Florens, Heckman, Meghir, and Vytlacil (2008)). When $\tilde{m}(x, w) = m(w)$, $\beta_{\tilde{m}f_{W|X}}$ is the weighted average of the local average response (Altonji and Matzkin (2005)).

To cover the above cases in a common framework, we consider general functionals b of J-vectors $g_x \equiv (g_{V_1,\lambda_1}(x, \cdot), \dots, g_{V_J,\lambda_J}(x, \cdot))$ and $g \equiv (g_{V_1,\lambda_1}, \dots, g_{V_J,\lambda_J})$ with finite J, and we establish the asymptotic properties of $b(\hat{g}_x(h)) - b(g_x) \equiv b((\hat{g}_{V_1,\lambda_1}(x, \cdot, h), \dots, \hat{g}_{V_J,\lambda_J}(x, \cdot, h))) - b((g_{V_1,\lambda_1}(x, \cdot), \dots, g_{V_J,\lambda_J}(x, \cdot)))$ and $b(\hat{g}(h)) - b(g) \equiv b((\hat{g}_{V_1,\lambda_1}(\cdot, h), \dots, \hat{g}_{V_J,\lambda_J}(\cdot, h))) - b((g_{V_1,\lambda_1}, \dots, g_{V_J,\lambda_J}))$. The first of the following theorems is relevant to estimating $\beta_m(x)$, $\beta_{mf_W}(x)$, and $\beta_{mf_{W|X}}(x)$. Because the weighted average of coordinates of g_x is taken only over w, functionals of g_x obtain a rate between $\sqrt{n}$ and that obtained in Theorem 4.2. It is not easy to use a functional delta method to obtain asymptotic normality of the functional, because we need to show tightness of integrands by introducing trimming of the tails of characteristic functions in the theorem. We therefore leave formal treatment of asymptotic normality results to future research. The second theorem is useful for estimating $\beta_{\tilde{m}}$, $\beta_{\tilde{m}f_{W,X}}$, and $\beta_{\tilde{m}f_{W|X}}$, and delivers $\sqrt{n}$ consistency and asymptotic normality results for the weighted averages of interest. Because it involves a weighted average over both x and w, it achieves the standard parametric rate of convergence. Each theorem relies on the existence of an asymptotically linear representation of the functional b.

Assumption C.1. *Suppose that for each $x \in \operatorname{supp}(X)$, the real-valued functional $b(\cdot)$ satisfies, for any $\tilde{g}_x \equiv (\tilde{g}_{V_1,\lambda_1}(x, \cdot), \dots, \tilde{g}_{V_J,\lambda_J}(x, \cdot))$ in an L_∞ neighborhood of the J-vector $g_x \equiv (g_{V_1,\lambda_1}(x, \cdot), \dots, g_{V_J,\lambda_J}(x, \cdot))$,*

$$\begin{aligned} b(\tilde{g}_x) - b(g_x) = {} & \sum_{j=1}^{J} \int \big(\tilde{g}_{V_j,\lambda_j}(x, w) - g_{V_j,\lambda_j}(x, w)\big) s_j(x, w)\, dw \\ & + \sum_{j=1}^{J} O\big(\|\tilde{g}_{V_j,\lambda_j}(x, \cdot) - g_{V_j,\lambda_j}(x, \cdot)\|_\infty^2\big) \end{aligned} \tag{S7}$$

for some real-valued functions s_j, $j = 1, \dots, J$.

This assumption states Fréchet differentiability of $b(\tilde{g}_x)$ with respect to $\tilde{g}_x$ in the norm $\|\tilde{g}_{V_j,\lambda_j}(x, \cdot)\|_\infty^2$, where the derivative is $s_j(x, w)$. To obtain a faster rate for functionals of g_x than that for $g_{V,\lambda}(x, w)$, we first impose a bound on the tail behavior of the Fourier transforms involved, as in Assumption 4.1.

Assumption C.2. *For each $x \in \operatorname{supp}(X)$, the functional derivatives s_j, defined in Assumption C.1, are such that $\sup_{x \in \operatorname{supp}(X)} \int |s_j(x, w)|\, dw < \infty$. Also, for $V \in \{1, Y\}$, there* *exist constants* $C_{\phi s} > 0$, $\alpha_{\phi s} \le 0$, $\nu_{\phi s} \ge \nu_\phi \ge 0$, *and* $\gamma_{\phi s} \in \mathbb{R}$ *such that* $\nu_{\phi s}\gamma_{\phi s} \ge 0$ *and if* $\nu_{\phi s} = \nu_\phi = 0$, $\gamma_\phi \ge \gamma_{\phi s}$, *and*

$$\sup_{x\in \mathrm{supp}(X)} \left|\int \phi_V(\zeta, w) s_j(x, w)\, dw\right| \le C_{\phi s}\big(1+|\zeta|\big)^{\gamma_{\phi s}} \exp\big(\alpha_{\phi s}|\zeta|^{\nu_{\phi s}}\big),$$

and in addition if $\alpha_{\phi s} = 0$, *then* $\gamma_{\phi s} < -\lambda - 1$ *for given* $\lambda \in \{0, \dots, \Lambda\}$.

The assumption above formalizes the intuition that averaging a quantity typically improves the convergence rate. It is natural to assume $\nu_{\phi s} \ge \nu_\phi$ and if $\nu_{\phi s} = \nu_\phi = 0$, $\gamma_\phi \ge \gamma_{\phi s}$, because, for some nonzero constant C, we have

$$\begin{aligned}\sup_{w\in \mathrm{supp}(W)} \big|\phi_V(\zeta, w)\big| &\ge C\Big(\sup_{w\in \mathrm{supp}(W)} \big|\phi_V(\zeta, w)\big|\Big)\Big(\sup_{x\in \mathrm{supp}(X)} \left|\int s(x, w)\, dw\right|\Big)\\ &\ge \sup_{x\in \mathrm{supp}(X)} \left|\int \phi_V(\zeta, w) s(x, w)\, dw\right|.\end{aligned}$$

Observe, however, that the inequality above can hold even when $\nu_{\phi s} < \nu_\phi$ or $\gamma_\phi < \gamma_{\phi s}$, because both bounds on $\sup_{w\in \mathrm{supp}(W)} |\phi_V(\zeta, w)|$ and on $\sup_{x\in \mathrm{supp}(X)} |\int \phi_V(\zeta, w) \times s(x, w)\, dw|$ given in Assumptions 4.1(ii) and C.2, respectively, are upper bounds. Thus, a faster convergence rate due to averaging over W is not a necessary result.

We next impose minimum convergence rates in a high-level form for conciseness, since primitive conditions can be obtained via Lemmas B.1, B.4, and B.6.

Assumption C.3. *The bandwidth sequence* $h_n \to 0$ *as* $n \to \infty$ *such that for all* $\lambda \in \{0, \dots, \Lambda\}$, *we have* (i) *if* $\nu_{\phi s} = \nu_\phi > 0$ *or* $\gamma_\phi = \gamma_{\phi s}$ *for* $\nu_{\phi s} = \nu_\phi = 0$,

$$\sup_{(x,w)\in \mathrm{supp}(X,W)} \big|B_{V,\lambda}(x, w; h_{1n})\big| = o(\alpha_{1n}),$$

$$\sup_{(x,w)\in \mathrm{supp}(X,W)} \big|L_{V,\lambda}(x, w; h_n)\big| = o_p\big(\alpha_{1n}^{1/2}\big),$$

and

$$\sup_{(x,w)\in \mathrm{supp}(X,W)} \big|R_{V,\lambda}(x, w; h_n)\big| = o_p(\alpha_{1n}),$$

where $\alpha_{1n} \equiv (h_{1n}^{-1})^{\gamma_{\lambda,B}} \exp(\alpha_B (h_{1n}^{-1})^{\nu_B}) + n^{-1/2}(\max\{(h_{1n}^{-1})^{\delta_L}, h_{2n}^{-1}\})(h_{1n}^{-1})^{\gamma_{\lambda,L}} \exp(\alpha_L \times (h_{1n}^{-1})^{\nu_L})$ *and where* α_B, ν_B, $\gamma_{\lambda,B}$, α_L, ν_L, $\gamma_{\lambda,L}$, *and* δ_L *are as defined in Lemmas* B.1 *and* B.4.

(ii) *If* $\nu_{\phi s} > \nu_\phi > 0$ *or* $\gamma_\phi > \gamma_{\phi s}$ *for* $\nu_{\phi s} = \nu_\phi = 0$, $\sup_{(x,w)\in \mathrm{supp}(X,W)} |B_{V,\lambda}(x, w; h_{1n})| = o(\alpha_{2n})$, $\sup_{(x,w)\in \mathrm{supp}(X,W)} |L_{V,\lambda}(x, w; h_n)| = o_p(\alpha_{2n}^{1/2})$, *and* $\sup_{(x,w)\in \mathrm{supp}(X,W)} |R_{V,\lambda}(x, w; h_n)| = o_p(\alpha_{2n})$, *where* $\alpha_{2n} \equiv (h_{1n}^{-1})^{\gamma_{\lambda,B,s}} \exp(\alpha_{B,s}(h_{1n}^{-1})^{\nu_{B,s}}) + n^{-1/2}(\max\{(h_{1n}^{-1})^{\delta_{L,s}}, h_{2n}^{-1}\})(h_{1n}^{-1})^{\gamma_{\lambda,L,s}} \exp(\alpha_{L,s}(h_{1n}^{-1})^{\nu_{L,s}})$, *and where* $\alpha_{B,s} \equiv \alpha_{\phi s}\bar{\xi}^{\nu_{\phi s}}$, $\nu_{B,s} \equiv \nu_{\phi s}$, $\gamma_{\lambda,B,s} \equiv \gamma_{\phi s} + \lambda + 1$, $\alpha_{L,s} \equiv \alpha_{\phi s} 1_{\{\nu_{\phi s} \ge \nu_\theta\}} - \alpha_\theta 1_{\{\nu_{\phi s} \le \nu_\theta\}}$, $\nu_{L,s} \equiv \max\{\nu_\theta, \nu_{\phi s}\}$, $\gamma_{\lambda,L,s} \equiv 1 + \gamma_{\phi s} - \gamma_\theta + \lambda$, *and* $\delta_{L,s} \equiv 1 + \gamma_1$.

We now establish a faster convergence rate for functionals of g_x than that for $g_{V,\lambda}(x,w)$, which is useful for analyzing $\beta_m(x)$, $\beta_{mf_W}(x)$, and $\beta_{mf_{W|X}}(x)$.

Theorem C.1. *For given $\Lambda, J \in \mathbb{N}$, let $\lambda_1, \ldots, \lambda_J$ belong to $\{0, \ldots, \Lambda\}$, let $V_1, \ldots, V_J$ belong to $\{1, Y\}$, and suppose that the conditions of Theorem 4.2 and Assumption C.2 hold. In addition, suppose that Assumption C.1 holds with s_j such that $\sup_{x \in \mathrm{supp}(X)} \int |s_j(x,w)|\,dw < \infty$, and let $\hat{g}_x(h_n) \equiv (\hat{g}_{V_1,\lambda_1}(x,\cdot;h_n), \ldots, \hat{g}_{V_J,\lambda_J}(x,\cdot;h_n))$.*

(i) *If Assumption C.3(i) holds, then*

$$\begin{aligned}\sup_{x\in\mathrm{supp}(X)} &\big|b\big(\hat{g}_x(h_n)\big) - b(g_x)\big| \\ &= O\big(\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,B}} \exp\big(\alpha_B\big(h_{1n}^{-1}\big)^{\nu_B}\big)\big) \\ &\quad + O_p\big(n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_L}, h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,L}} \exp\big(\alpha_L\big(h_{1n}^{-1}\big)^{\nu_L}\big)\big).\end{aligned}$$

(ii) *If Assumption C.3(ii) holds, then*

$$\begin{aligned}\sup_{x\in\mathrm{supp}(X)} &\big|b\big(\hat{g}_x(h_n)\big) - b(g_x)\big| \\ &= O\big(\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,B,s}} \exp\big(\alpha_{B,s}\big(h_{1n}^{-1}\big)^{\nu_{B,s}}\big)\big) \\ &\quad + O_p\big(n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_{L,s}}, h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{\lambda,L,s}} \exp\big(\alpha_{L,s}\big(h_{1n}^{-1}\big)^{\nu_{L,s}}\big)\big).\end{aligned}$$

We impose conditions on the minimum convergence rates for the next theorem in a high-level form, which can be readily verified in terms of more primitive conditions using Lemmas B.1, B.4, and B.6.

Assumption C.4. *The bandwidth sequence $h_n \to 0$ as $n \to \infty$ such that for all $\lambda \in \{0, \ldots, \Lambda\}$, we have*

$$\sup_{(x,w)\in\mathrm{supp}(X,W)} \big|B_{V,\lambda}(x,w;h_{1n})\big| = o\big(n^{-1/2}\big),$$

$$\sup_{(x,w)\in\mathrm{supp}(X,W)} \big|L_{V,\lambda}(x,w;h_n)\big| = o_p\big(n^{-1/4}\big),$$

$$\sup_{(x,w)\in\mathrm{supp}(X,W)} \big|R_{V,\lambda}(x,w;h_n)\big| = o_p\big(n^{-1/2}\big),$$

and

$$\sup_{w\in\mathrm{supp}(W)} \big|\hat{f}_W(w) - f_W(w)\big| = o_p\big(n^{-1/4}\big).$$

This assumption is required to ensure that weighted average derivatives converge at the parametric rate, $\sqrt{n}$. Note that because the rate of divergence of $L_{V,\lambda}(x,w;h_n)$ depends on the smoothness of various quantities as given in Lemma B.4, its $n^{-1/4}$ rate may not always be possible. Nevertheless, we can achieve the $n^{-1/4}$ rate when the functional form of the regression model is sufficiently smooth (e.g., polynomial or exponential functions in Taupin (2001) or, more generally, functions with rapidly decaying Fourier transforms) or when the distribution of the measurement error exhibits a slowly decaying Fourier transforms (e.g., range-restricted distributions in Hu and Ridder (2010)).

Define

$$\bar{\Psi}_{V,\lambda,s} \equiv \sum_{j=1}^{J} \int \Psi_{V_j,\lambda_j,s_j}(\zeta)\,d\zeta + \left|\sigma_{f_W,s}(\tilde{w})\right|,$$

where

$$\begin{aligned}
\Psi_{V,\lambda,s}(\zeta) &\equiv \frac{1}{|\theta_1(\zeta)|}\left(1+\frac{|\theta_{X_1}(\zeta)|}{|\theta_1(\zeta)|}\right)\int_{|\zeta|}^{\infty}\left|\sigma_{V,1,s}(\xi)\right||\xi|^{\lambda}\,d\xi \\
&\quad + |\zeta|^{\lambda}\left(\frac{|\sigma_{V,1,s}(\zeta)|}{|\theta_1(\zeta)|} + \left|\sigma_{V,\chi_V,s}(\zeta; v, x_2, \tilde{w})\right| + \left|\sigma_{V,f_W,s}(\zeta; \tilde{w})\right|\right), \\
\sigma_{f_W,s}(\tilde{w}) &\equiv \int \lim_{h_2\to 0}\int s_{J+1}(x,w)k_{h_2}(\tilde{w}-w)\,dw\,dx,
\end{aligned}$$

with

$$\begin{aligned}
\sigma_{V,1,s}(\zeta) &\equiv \int \exp(i\zeta x)\int s(x,w)\phi_V(\zeta,w)\,dw\,dx, \\
&\sigma_{V,\chi_V,s}(\zeta; v, x_2, \tilde{w}) \\
&\quad \equiv \int \exp(i\zeta x)\lim_{h_2\to 0}\int \frac{1}{\chi_V(\zeta,w)}s(x,w)\phi_V(\zeta,w)ve^{i\zeta x_2}k_{h_2}(\tilde{w}-w)\,dw\,dx, \\
\sigma_{V,f_W,s}(\zeta;\tilde{w}) &\equiv \int \exp(i\zeta x)\lim_{h_2\to 0}\int \frac{1}{f_W(w)}s(x,w)\phi_V(\zeta,w)k_{h_2}(\tilde{w}-w)\,dw\,dx.
\end{aligned}$$

Also define $\psi_s(v, x_1, x_2, \tilde{w}) \equiv \sum_{j=1}^{J}\psi_{V_j,\lambda_j}(s_j; v_j, x_1, x_2, \tilde{w}) + \psi_f(s_{J+1}; \tilde{w})$, where

$$\begin{aligned}
&\psi_{V,\lambda}(s; v, x_1, x_2, \tilde{w}) \\
&\quad \equiv \int \Big\{\Psi_{V,\lambda,1,s}(\zeta)\big(e^{i\zeta x_2} - E\big[e^{i\zeta X_2}\big]\big) + \Psi_{V,\lambda,X_1,s}(\zeta)\big(x_1 e^{i\zeta x_2} - E\big[X_1 e^{i\zeta X_2}\big]\big) \\
&\qquad + \big(\Xi_{V,\lambda,\chi_V}(s,\zeta; v, x_2, \tilde{w}) - E\big[\Xi_{V,\lambda,\chi_V}(s,\zeta; V, X_2, W)\big]\big) \\
&\qquad + \big(\Xi_{V,\lambda,f_W}(s,\zeta;\tilde{w}) - E\big[\Xi_{V,\lambda,f_W}(s,\zeta; W)\big]\big)\Big\}\,d\zeta, \\
&\psi_f(s_{J+1};\tilde{w}) \equiv \int \lim_{h_2\to 0}\int s_{J+1}(x,w)\big(k_{h_2}(\tilde{w}-w) - E\big[k_{h_2}(W-w)\big]\big)\,dw\,dx,
\end{aligned}$$

with

$$\begin{aligned}
&\Psi_{V,\lambda,1,s}(\zeta) \\
&\quad \equiv -\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2}\int_{\zeta}^{\pm\infty}\left(\int \exp(-i\xi x)\int s(x,w)\phi_V(\xi,w)\,dw\,dx\right)(-i\xi)^{\lambda}\,d\xi
\end{aligned}$$

$$-\frac{1}{2\pi}\frac{(-i\zeta)^{\lambda}}{\theta_1(\zeta)}\left(\int \exp(-i\zeta x)\int s(x,w)\phi_V(\zeta,w)\,dw\,dx\right),$$

$$\begin{aligned}
&\Psi_{V,\lambda,X_1,s}(\zeta)\\
&\quad\equiv \frac{1}{2\pi}\frac{i}{\theta_1(\zeta)}\int_{\zeta}^{\pm\infty}\left(\int \exp(-i\xi x)\int s(x,w)\phi_V(\xi,w)\,dw\,dx\right)(-i\xi)^{\lambda}\,d\xi,\\
&\mathcal{Z}_{V,\lambda,\chi_V}(s,\zeta;v,x_2,\tilde{w})\\
&\quad\equiv \frac{1}{2\pi}(-i\zeta)^{\lambda}\int \exp(-i\zeta x)\lim_{h_2\to 0}\int \frac{1}{\chi_V(\zeta,w)}s(x,w)\phi_V(\zeta,w)\\
&\qquad\times ve^{i\zeta x_2}k_{h_2}(\tilde{w}-w)\,dw\,dx,\\
&\mathcal{Z}_{V,\lambda,f_W}(s,\zeta;\tilde{w})\\
&\quad\equiv -\frac{1}{2\pi}(-i\zeta)^{\lambda}\int \exp(-i\zeta x)\lim_{h_2\to 0}\int \frac{1}{f_W(w)}s(x,w)\phi_V(\zeta,w)k_{h_2}(\tilde{w}-w)\,dw\,dx.
\end{aligned}$$

The following theorem gives a convenient asymptotic normality and $\sqrt{n}$ consistency result useful for analyzing $\beta_{\tilde{m}}$, $\beta_{\tilde{m}f_{W,X}}$, and $\beta_{\tilde{m}f_{W|X}}$.

Theorem C.2. *For given $\Lambda, J\in\mathbb{N}$, let $\lambda_1,\ldots,\lambda_J$ belong to $\{0,\ldots,\Lambda\}$, let $V_1,\ldots,V_J$ belong to $\{1,Y\}$, and suppose that the conditions of Theorem 4.3 and Assumption C.4 hold. Define $b(\cdot,\cdot)$ as a real-valued functional satisfying, for any $\tilde{g}\equiv(\tilde{g}_{V_1,\lambda_1},\ldots,\tilde{g}_{V_J,\lambda_J})$ in an L_∞ neighborhood of the J-vector $g\equiv(g_{V_1,\lambda_1},\ldots,g_{V_J,\lambda_J})$ and for any $\tilde{f}\equiv\tilde{f}_W$ in a neighborhood of $f\equiv f_W$,*

$$\begin{aligned}
b(\tilde{g},\tilde{f})-b(g,f) &= \sum_{j=1}^{J}\int\int\big(\tilde{g}_{V_j,\lambda_j}(x,w)-g_{V_j,\lambda_j}(x,w)\big)s_j(x,w)\,dw\,dx\\
&\quad+\int\int\big(\tilde{f}_W(w)-f_W(w)\big)s_{J+1}(x,w)\,dw\,dx\\
&\quad+\sum_{j=1}^{J}O\big(\|\tilde{g}_{V_j,\lambda_j}-g_{V_j,\lambda_j}\|_\infty^2\big)+O\big(\|\tilde{f}_W-f_W\|_\infty^2\big)
\end{aligned}\tag{S8}$$

for some real-valued functions s_j, $j=1,\ldots,J+1$. If s_j is such that $\int\int|s_j(x,w)|\,dw\,dx<\infty$ and $\bar{\Psi}_{V,\lambda,s}<\infty$, then for $\hat{g}(h_n)\equiv(\hat{g}_{V_1,\lambda_1}(\cdot;h_n),\ldots,\hat{g}_{V_J,\lambda_J}(\cdot;h_n))$ and $\hat{f}(h_{2n})\equiv\hat{E}[k_{h_{2n}}(\cdot)]$, we get

$$b\big(\hat{g}(h_n),\hat{f}(h_{2n})\big)-b(g,f)=\hat{E}\big[\psi_s(V,X_1,X_2,W)\big]+o_p\big(n^{-1/2}\big).$$

Moreover,

$$n^{1/2}\big(b\big(\hat{g}(h_n),\hat{f}(h_{2n})\big)-b(g,f)\big)\xrightarrow{d}N(0,\Omega_b),$$

where

$$\Omega_b\equiv E\big[\big(\psi_s(V,X_1,X_2,W)\big)^2\big]<\infty.$$

C.2 *Asymptotics for weighted averages of effects*

We now consider the asymptotic properties of the estimators of the weighted averages defined in Section C.1:

$$\hat{\beta}_m(x, h_n) = \int_{S^w_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) m(w)\, dw, \tag{S9}$$

$$\hat{\beta}_{mf_W}(x, h_n) = \int_{S^w_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) m(w) \hat{f}_W(w)\, dw, \tag{S10}$$

$$\begin{aligned} \hat{\beta}_{mf_{W|X}}(x, h_n) &= \int_{S^w_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) m(w) \hat{f}_{W|X}(w \mid x)\, dw \\ &= \int_{S^w_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) m(w) \frac{\hat{g}_{1,0}(x, w; h_n)\hat{f}_W(w)}{\int_{S^w_{\hat{\beta}(\cdot,h_n)}} \hat{g}_{1,0}(x, w; h_n)\hat{f}_W(w)\, dw}\, dw, \end{aligned} \tag{S11}$$

$$\hat{\beta}_{\tilde{m}}(h_n) = \int_{S^{x,w}_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) \tilde{m}(x, w)\, dw\, dx, \tag{S12}$$

$$\begin{aligned} \hat{\beta}_{\tilde{m}f_{W|X}}(h_n) &= \int_{S^{x,w}_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) \tilde{m}(x, w) \hat{f}_{W|X}(w \mid x)\, dw\, dx \\ &= \int_{S^{x,w}_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) \tilde{m}(x, w) \frac{\hat{g}_{1,0}(x, w; h_n)\hat{f}_W(w)}{\int_{S^w_{\hat{\beta}(\cdot,h_n)}} \hat{g}_{1,0}(x, w; h_n)\hat{f}_W(w)\, dw}\, dw\, dx, \end{aligned} \tag{S13}$$

$$\begin{aligned} \hat{\beta}_{\tilde{m}f_{W,X}}(h_n) &= \int_{S^{x,w}_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) \tilde{m}(x, w) \hat{f}_{W,X}(w, x)\, dw\, dx \\ &= \int_{S^{x,w}_{\hat{\beta}(\cdot,h_n)}} \hat{\beta}(x, w; h_n) \tilde{m}(x, w) \hat{g}_{1,0}(x, w; h_n) \hat{f}_W(w)\, dw\, dx, \end{aligned} \tag{S14}$$

where $S^w_{\hat{\beta}(\cdot,h_n)} \equiv \{w \in \mathbb{R} : \hat{g}_{1,0}(x, w; h_n) > 0\}$ and $S^{x,w}_{\hat{\beta}(\cdot,h_n)} \equiv \{(x, w) \in \mathbb{R}^2 : \hat{g}_{1,0}(x, w; h_n) > 0\}$, and where $\hat{f}_W(w)$ is a nonparametric estimator of the density of W. The next assumption restricts the weight functions, m and $\tilde{m}$.

Assumption C.5. *Let* $\mathbb{M}$ *and* $\tilde{\mathbb{M}}$ *be bounded measurable subsets of* $\mathbb{R}$ *and* $\mathbb{R}^2$, *respectively.* (i) *The weight functions* $m : \mathbb{R} \to \mathbb{R}$ *and* $\tilde{m} : \mathbb{R}^2 \to \mathbb{R}$ *are measurable and supported on* $\mathbb{M}$ *and* $\tilde{\mathbb{M}}$, *respectively. Additionally,* (ii) $\inf_{(x,w)\in\tilde{\mathbb{M}}} f_{X|W}(x \mid w) > 0$ *and* (iii) $\max_{V\in\{1,Y\}} \max_{\lambda=0,1} \sup_{(x,w)\in\tilde{\mathbb{M}}} |g_{V,\lambda}(x, w)| < \infty$.

The next two theorems establish asymptotic properties for these estimators by applying Theorems C.1 and C.2. We first establish asymptotic results for the semiparametric functionals taking the forms of eqs. (S9)–(S11) by applying Theorem C.1.

Theorem C.3. *Suppose the conditions of Theorem C.1 hold for* $\Lambda = 1$ *and that Assumption C.5 holds. Then*

(i)

$$\begin{aligned}&\sup_{x\in\mathbb{M}}\big|\hat{\beta}_m(x,h_n)-\beta_m(x)\big|\\&\quad=O\big(\tau^{-3}\big(h_{1n}^{-1}\big)^{\gamma_{1,B,s}}\exp\big(\alpha_{B,s}\big(h_{1n}^{-1}\big)^{\nu_{B,s}}\big)\big)\\&\qquad+O_p\big(\tau^{-3}n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_{L,s}},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{1,L,s}}\exp\big(\alpha_{L,s}\big(h_{1n}^{-1}\big)^{\nu_{L,s}}\big)\big),\end{aligned}$$

(ii)

$$\begin{aligned}&\sup_{x\in\mathbb{M}}\big|\hat{\beta}_{mf_W}(x,h_n)-\beta_{mf_W}(x)\big|\\&\quad=O\big(\tau^{-3}\big(h_{1n}^{-1}\big)^{\gamma_{1,B,s}}\exp\big(\alpha_{B,s}\big(h_{1n}^{-1}\big)^{\nu_{B,s}}\big)\big)\\&\qquad+O_p\big(\tau^{-3}n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_{L,s}},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{1,L,s}}\exp\big(\alpha_{L,s}\big(h_{1n}^{-1}\big)^{\nu_{L,s}}\big)\big),\end{aligned}$$

and

(iii)

$$\begin{aligned}&\sup_{x\in\mathbb{M}}\big|\hat{\beta}_{mf_{W|X}}(x,h_n)-\beta_{mf_{W|X}}(x)\big|\\&\quad=O\big(\tau^{-3}\big(h_{1n}^{-1}\big)^{\gamma_{1,B,s}}\exp\big(\alpha_{B,s}\big(h_{1n}^{-1}\big)^{\nu_{B,s}}\big)\big)\\&\qquad+O_p\big(\tau^{-3}n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_{L,s}},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{1,L,s}}\exp\big(\alpha_{L,s}\big(h_{1n}^{-1}\big)^{\nu_{L,s}}\big)\big),\end{aligned}$$

where $\alpha_{B,s}$, $\nu_{B,s}$, $\gamma_{\lambda,B,s}$, $\alpha_{L,s}$, $\nu_{L,s}$, $\gamma_{\lambda,L,s}$, *and* $\delta_{L,s}$ *are as defined in Assumption* C.3.

We now define useful notations for asymptotic normality results. Recall that $\psi_{V,\lambda}$ is defined in front of Theorem C.2 in Section C.1. Let

$$\psi_{\beta_{\tilde{m}}}(v,x_1,x_2,\tilde{w})\equiv\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_1;v,x_1,x_2,\tilde{w}),$$

where P_1 denotes the function mapping (x,w) to $\tilde{m}(x,w)s_{V,\lambda}(x,w)$. Let

$$\begin{aligned}\psi_{\beta_{\tilde{m}f_{W|X}}}(v,x_1,x_2,\tilde{w})\equiv&\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_2;v,x_1,x_2,\tilde{w})+\psi_{1,0}(P_3;1,x_1,x_2,\tilde{w})\\&-\psi_{1,0}(P_4;1,x_1,x_2,\tilde{w})+\psi_f(P_5;\tilde{w}),\end{aligned}$$

where P_2, P_3, P_4, and P_5 denote the functions mapping (x,w) to $\tilde{m}(x,w)f_{W|X}(w\mid x)s_{V,\lambda}(x,w)$, $\beta(x,w)\tilde{m}(x,w)f_W(w)/f_X(x)$, $\int_{S^w_{\hat{\beta}(\cdot,h_n)}}\beta(x,w)\tilde{m}(x,w)f_{W|X}(w\mid x)\,dw/f_X(x)$, and $\beta(x,w)\tilde{m}(x,w)f_{X|W}(x\mid w)/f_X(x)$, respectively. Also let

$$\begin{aligned}\psi_{\beta_{\tilde{m}f_{W,X}}}(v,x_1,x_2,\tilde{w})\equiv&\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_6;v,x_1,x_2,\tilde{w})\\&+\psi_{1,0}(P_7;1,x_1,x_2,\tilde{w})+\psi_f(P_8;\tilde{w}),\end{aligned}$$

where P_6, P_7, and P_8 denote the functions mapping (x, w) to $\tilde{m}(x, w) f_{W,X}(w, x) \times s_{V,\lambda}(x, w)$, $\beta(x, w)\tilde{m}(x, w) f_W(w)$, and $\beta(x, w)\tilde{m}(x, w) f_{X|W}(x \mid w)$, respectively.

The following theorem establishes asymptotic results for the semiparametric functionals taking the forms of eqs. (S12)–(S14) by straightforward application of Theorem C.2.

Theorem C.4. *Suppose the conditions of Theorem C.2 hold for $\Lambda = 1$ and that Assumption C.5 holds. Then*

(i)

$$n^{1/2}(\Omega_{\tilde{m}})^{-1/2}\big(\hat{\beta}_{\tilde{m}}(h_n) - \beta_{\tilde{m}}\big) \xrightarrow{d} N(0, 1),$$

provided that

$$\Omega_{\tilde{m}} \equiv E\big[\big(\psi_{\beta_{\tilde{m}}}(V, X_1, X_2, W)\big)^2\big] < \infty;$$

(ii)

$$n^{1/2}(\Omega_{\tilde{m}f_{W|X}})^{-1/2}\big(\hat{\beta}_{\tilde{m}f_{W|X}}(h_n) - \beta_{\tilde{m}f_{W|X}}\big) \xrightarrow{d} N(0, 1),$$

provided that

$$\Omega_{\tilde{m}f_{W|X}} \equiv E\big[\big(\psi_{\beta_{\tilde{m}f_{W|X}}}(V, X_1, X_2, W)\big)^2\big] < \infty;$$

(iii)

$$n^{1/2}(\Omega_{\tilde{m}f_{W,X}})^{-1/2}\big(\hat{\beta}_{\tilde{m}f_{W,X}}(h_n) - \beta_{\tilde{m}f_{W,X}}\big) \xrightarrow{d} N(0, 1),$$

provided that

$$\Omega_{\tilde{m}f_{W,X}} \equiv E\big[\big(\psi_{\beta_{\tilde{m}f_{W,X}}}(V, X_1, X_2, W)\big)^2\big] < \infty.$$

C.3 *Mathematical proofs*

Lemma C.5. *Suppose the conditions of Lemma 4.1 hold. For each ζ and $h \equiv (h_1, h_2)$, and for $A \in \{1, X_1, \chi_V, f_W\}$, let*

$$\Psi^+_{V,\lambda,A,s}(\zeta, h_1) \equiv \sup_{x \in \text{supp}(X)} \left| \int \Psi_{V,\lambda,A}(\zeta, x, w, h_1) s(x, w)\, dw \right|$$

and define

$$\Psi^+_{V,\lambda,s}(h) \equiv \sum_{A=\{1,X_1\}} \int \Psi^+_{V,\lambda,A,s}(\zeta, h_1)\, d\zeta + h_2^{-1} \sum_{A=\{\chi_V,f_W\}} \int \Psi^+_{V,\lambda,A,s}(\zeta, h_1)\, d\zeta.$$

If Assumption C.2 also holds, then for $h > 0$,

$$\Psi^{+}_{V,\lambda,s}(h) = O\Big(\max\big\{\big(1+h_1^{-1}\big)^{\gamma_1+1}, h_2^{-1}\big\}\big(1+h_1^{-1}\big)^{\gamma_{\phi s}+\lambda-\gamma_\theta+1} \times \exp\big((\alpha_{\phi s}1_{\{\nu_{\phi s}\geq\nu_\theta\}} - \alpha_\theta 1_{\{\nu_{\phi s}\leq\nu_\theta\}})\big(h_1^{-1}\big)^{\max\{\nu_\theta,\nu_{\phi s}\}}\big)\Big).$$

Proof. We obtain rates for each term of $\Psi^{+}_{V,\lambda,s}(h)$. First,

$$\begin{aligned}
&\Psi^{+}_{V,\lambda,1,s}(\zeta, h_1)\\
&\equiv \sup_{x\in\mathrm{supp}(X)}\left|\int \Psi_{V,\lambda,1}(\zeta,x,w,h_1)s(x,w)\,dw\right|\\
&= \sup_{x\in\mathrm{supp}(X)}\left|\int\left(-\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2}\int_\zeta^{\pm\infty}(-i\xi)^\lambda\kappa(h_1\xi)\exp(-i\xi x)\phi_V(\xi,w)\,d\xi\right.\right.\\
&\qquad\left.\left.-\frac{1}{2\pi}(-i\zeta)^\lambda\kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{\theta_1(\zeta)}\right)s(x,w)\,dw\right|\\
&\preceq \frac{|\theta_{X_1}(\zeta)|}{|\theta_1(\zeta)|^2}\int_\zeta^{\pm\infty}|\xi|^\lambda\big|\kappa(h_1\xi)\big|\left(\sup_{x\in\mathrm{supp}(X)}\left|\int\phi_V(\xi,w)s(x,w)\,dw\right|\right)d\xi\\
&\qquad+|\zeta|^\lambda\big|\kappa(h_1\zeta)\big|\frac{1}{|\theta_1(\zeta)|}\left(\sup_{x\in\mathrm{supp}(X)}\left|\int\phi_V(\zeta,w)s(x,w)\,dw\right|\right)\\
&\preceq \frac{1}{|\theta_1(\zeta)|}1_{\{|\zeta|\leq h_1^{-1}\}}\Bigg[\big|D_\zeta\ln\phi_1(\zeta)\big|\\
&\qquad\times\int_\zeta^{h_1^{-1}}|\xi|^\lambda\left(\sup_{x\in\mathrm{supp}(X)}\left|\int\phi_V(\xi,w)s(x,w)\,dw\right|\right)d\xi\\
&\qquad+|\zeta|^\lambda\left(\sup_{x\in\mathrm{supp}(X)}\left|\int\phi_V(\zeta,w)s(x,w)\,dw\right|\right)\Bigg].
\end{aligned}$$

By Assumptions 4.1 and C.2, we obtain

$$\begin{aligned}
&\int\Psi^{+}_{V,\lambda,1,s}(\zeta,h_1)\,d\zeta\\
&\preceq\int\frac{1}{|\theta_1(\zeta)|}1_{\{|\zeta|\leq h_1^{-1}\}}\\
&\qquad\times\Bigg[\big|D_\zeta\ln\phi_1(\zeta)\big|\int_\zeta^{h_1^{-1}}|\xi|^\lambda\left(\sup_{x\in\mathrm{supp}(X)}\left|\int\phi_V(\xi,w)s(x,w)\,dw\right|\right)d\xi\\
&\qquad+|\zeta|^\lambda\left(\sup_{x\in\mathrm{supp}(X)}\left|\int\phi_V(\zeta,w)s(x,w)\,dw\right|\right)\Bigg]d\zeta\\
&\preceq\int\big(1+|\zeta|\big)^{-\gamma_\theta}\exp\big(-\alpha_\theta|\zeta|^{\nu_\theta}\big)1_{\{|\zeta|\leq h_1^{-1}\}}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \Big[(1+|\zeta|)^{\gamma_1} \int_0^{h_1^{-1}} |\xi|^{\lambda} (1+|\xi|)^{\gamma_{\phi s}} \exp(\alpha_{\phi s}|\xi|^{\nu_{\phi s}})\, d\xi \\
&\quad + |\zeta|^{\lambda}(1+|\zeta|)^{\gamma_{\phi s}} \exp(\alpha_{\phi s}|\zeta|^{\nu_{\phi s}}) \Big]\, d\zeta \\
&\preceq (1+h_1^{-1})^{1-\gamma_\theta} \exp(-\alpha_\theta (h_1^{-1})^{\nu_\theta}) \\
&\quad \times \big[(1+h_1^{-1})^{\gamma_1} (1+h_1^{-1})^{\lambda+\gamma_{\phi s}+1} \exp(\alpha_{\phi s}(h_1^{-1})^{\nu_{\phi s}}) \\
&\quad + (1+h_1^{-1})^{\lambda+\gamma_{\phi s}} \exp(\alpha_{\phi s}(h_1^{-1})^{\nu_{\phi s}}) \big] \\
&\preceq (1+h_1^{-1})^{\gamma_{\phi s}+\lambda+\gamma_1-\gamma_\theta+2} \exp(-\alpha_\theta (h_1^{-1})^{\nu_\theta}) \exp(\alpha_{\phi s}(h_1^{-1})^{\nu_{\phi s}}).
\end{aligned}
$$

Second, similarly,

$$
\begin{aligned}
&\Psi^+_{V,\lambda,X_1,s}(\zeta, h_1) \\
&\quad \equiv \sup_{x\in \operatorname{supp}(X)} \left| \int \Psi_{V,\lambda,X_1}(\zeta, x, w, h_1) s(x,w)\, dw \right| \\
&\quad = \sup_{x\in \operatorname{supp}(X)} \left| \int \left(\frac{1}{2\pi} \frac{i}{\theta_1(\zeta)} \int_\zeta^{\pm\infty} (-i\xi)^\lambda \kappa(h_1\xi) \exp(-i\xi x) \phi_V(\xi, w)\, d\xi \right) \right. \\
&\quad\quad \left. \times\, s(x,w)\, dw \right| \\
&\quad \preceq \frac{1}{|\theta_1(\zeta)|} 1_{\{|\zeta|\le h_1^{-1}\}} \int_\zeta^{h_1^{-1}} |\xi|^\lambda \left(\sup_{x\in \operatorname{supp}(X)} \left| \int \phi_V(\xi, w) s(x,w)\, dw \right| \right) d\xi,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int \Psi^+_{V,\lambda,X_1,s}(\zeta, h_1)\, d\zeta \\
&\quad \preceq \int_0^{h_1^{-1}} (1+|\zeta|)^{-\gamma_\theta} \exp(-\alpha_\theta|\zeta|^{\nu_\theta}) \left(\int_0^{h_1^{-1}} |\xi|^\lambda (1+|\xi|)^{\gamma_{\phi s}} \exp(\alpha_{\phi s}|\xi|^{\nu_{\phi s}})\, d\xi \right) d\zeta \\
&\quad \preceq (1+h_1^{-1})^{\gamma_{\phi s}+\lambda-\gamma_\theta+2} \exp(-\alpha_\theta (h_1^{-1})^{\nu_\theta}) \exp(\alpha_{\phi s}(h_1^{-1})^{\nu_{\phi s}}).
\end{aligned}
$$

Third,

$$
\begin{aligned}
&\Psi^+_{V,\lambda,\chi_V,s}(\zeta, h_1) \\
&\quad \equiv \sup_{x\in \operatorname{supp}(X)} \left| \int \Psi_{V,\lambda,\chi_V}(\zeta, x, w, h_1) s(x,w)\, dw \right| \\
&\quad = \sup_{x\in \operatorname{supp}(X)} \left| \int \left(\frac{1}{2\pi} (-i\zeta)^\lambda \kappa(h_1\zeta) \exp(-i\zeta x) \frac{\phi_V(\zeta, w)}{\chi_V(\zeta, w)} \right) s(x,w)\, dw \right| \\
&\quad \le |\zeta|^\lambda 1_{\{|\zeta|\le h_1^{-1}\}} \left(\sup_{x\in \operatorname{supp}(X)} \left| \int \phi_V(\zeta, w) s(x,w)\, dw \right| \right) \left(\frac{1}{\inf_{w\in \operatorname{supp}(W)} |\chi_V(\zeta, w)|} \right),
\end{aligned}
$$

so that

$$
\begin{aligned}
&h_2^{-1}\int \Psi^{+}_{V,\lambda,\chi_V,s}(\zeta,h_1)\,d\zeta\\
&\quad\preceq h_2^{-1}\int_0^{h_1^{-1}} |\zeta|^{\lambda}\big(1+|\zeta|\big)^{-\gamma_\theta}\exp\big(-\alpha_\theta|\zeta|^{\nu_\theta}\big)\big(1+|\zeta|\big)^{\gamma_{\phi s}}\exp\big(\alpha_{\phi s}|\zeta|^{\nu_{\phi s}}\big)\,d\zeta\\
&\quad\preceq h_2^{-1}\big(1+h_1^{-1}\big)^{\gamma_{\phi s}-\gamma_\theta+\lambda+1}\exp\big(-\alpha_\theta\big(h_1^{-1}\big)^{\nu_\theta}\big)\exp\big(\alpha_{\phi s}\big(h_1^{-1}\big)^{\nu_{\phi s}}\big).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\Psi^{+}_{V,\lambda,f_W,s}(\zeta,h_1)\\
&\quad\equiv \sup_{x\in\operatorname{supp}(X)}\left|\int \Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)s(x,w)\,dw\right|\\
&\quad=\sup_{x\in\operatorname{supp}(X)}\left|\int\left(-\frac{1}{2\pi}(-i\zeta)^{\lambda}\kappa(h_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{f_W(w)}\right)s(x,w)\,dw\right|\\
&\quad\preceq|\zeta|^{\lambda}1_{\{|\zeta|\le h_1^{-1}\}}\left(\sup_{x\in\operatorname{supp}(X)}\left|\int\phi_V(\zeta,w)s(x,w)\,dw\right|\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
h_2^{-1}\int\Psi^{+}_{V,\lambda,f_W,s}(\zeta,h_1)\,d\zeta&\preceq h_2^{-1}\int_0^{h_1^{-1}}|\zeta|^{\lambda}\big(1+|\zeta|\big)^{\gamma_{\phi s}}\exp\big(\alpha_{\phi s}|\zeta|^{\nu_{\phi s}}\big)\,d\zeta\\
&\preceq h_2^{-1}\big(1+h_1^{-1}\big)^{\gamma_{\phi s}+\lambda+1}\exp\big(\alpha_{\phi s}\big(h_1^{-1}\big)^{\nu_{\phi s}}\big).
\end{aligned}
$$

Putting the four terms together gives the desired result. □

Proof of Theorem C.1. (i) Since we have, by Assumption C.3(i),

$$
\begin{aligned}
&\max_{j=1,\dots,J}\sup_{(x,w)\in\operatorname{supp}(X,W)}\big|\hat g_{V_j,\lambda_j}(x,w,h_n)-g_{V_j,\lambda_j}(x,w,h_{1n})\big|\\
&\quad=\max_{j=1,\dots,J}\sup_{(x,w)\in\operatorname{supp}(X,W)}\big|B_{V_j,\lambda_j}(x,w,h_{1n})+L_{V_j,\lambda_j}(x,w,h_n)+R_{V_j,\lambda_j}(x,w,h_n)\big|\\
&\quad=o(\alpha_{1n})+o_p\big(\alpha_{1n}^{1/2}\big)+o_p(\alpha_{1n})=o_p\big(\alpha_{1n}^{1/2}\big),
\end{aligned}
$$

the remainder term in eq. (S7) is $o_p((\alpha_{1n}^{1/2})^2)=o_p(\alpha_{1n})$ by letting $\tilde g_{V_j,\lambda_j}(x,w)=\hat g_{V_j,\lambda_j}(x,w,h_n)$. We also have

$$
\begin{aligned}
&\left|\sum_{j=1}^{J}\int\big(\hat g_{V_j,\lambda_j}(x,w,h_n)-g_{V_j,\lambda_j}(x,w)\big)s_j(x,w)\,dw\right|\\
&\quad\preceq\big\|\hat g_{V_j,\lambda_j}(x,w,h_n)-g_{V_j,\lambda_j}(x,w)\big\|_{\infty},
\end{aligned}
$$

since $\sup_{x\in\operatorname{supp}(X)}\int|s_j(x,w)|\,dw<\infty$. Then the result immediately follows.

(ii) Since we have, by Assumption C.3(ii) and a similar argument in part (i),

$$\max_{j=1,\dots,J}\sup_{(x,w)\in\text{supp}(X,W)}\left|\hat{g}_{V_j,\lambda_j}(x,w,h_n)-g_{V_j,\lambda_j}(x,w,h_{1n})\right|=o_p\big(\alpha_{2n}^{1/2}\big),$$

the remainder term in eq. (S7) is $o_p((\alpha_{2n}^{1/2})^2)=o_p(\alpha_{2n})$ by letting $\tilde{g}_{V_j,\lambda_j}(x,w)=\hat{g}_{V_j,\lambda_j}(x, w,h_n)$. We have

$$\begin{aligned}
&\sum_{j=1}^{J}\int\big(\hat{g}_{V_j,\lambda_j}(x,w,h_n)-g_{V_j,\lambda_j}(x,w)\big)s_j(x,w)\,dw\\
&\quad=\sum_{j=1}^{J}\int B_{V_j,\lambda_j}(x,w,h_{1n})s_j(x,w)\,dw+\sum_{j=1}^{J}\int L_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\\
&\qquad+\sum_{j=1}^{J}\int R_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\sup_{x\in\text{supp}(X)}\left|\sum_{j=1}^{J}\int B_{V_j,\lambda_j}(x,w,h_{1n})s_j(x,w)\,dw\right|\\
&\quad\le\sup_{x\in\text{supp}(X)}\sum_{j=1}^{J}\left|\int B_{V_j,\lambda_j}(x,w,h_{1n})s_j(x,w)\,dw\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sup_{x\in\text{supp}(X)}\left|\int B_{V,\lambda}(x,w,h_{1n})s(x,w)\,dw\right|\\
&\quad=\sup_{x\in\text{supp}(X)}\left|\int\bigg(\frac{1}{2\pi}\int\kappa(h_1\zeta)(-i\zeta)^{\lambda}\phi_V(\zeta,w)\exp(-i\zeta x)\,d\zeta\right.\\
&\qquad\left.-\frac{1}{2\pi}\int(-i\zeta)^{\lambda}\phi_V(\zeta,w)\exp(-i\zeta x)\,d\zeta\bigg)s(x,w)\,dw\right|\\
&\quad\le\frac{1}{\pi}\int_{\bar{\xi}/h_1}^{\infty}\big|\big(\kappa(h_1\zeta)-1\big)\big||\zeta|^{\lambda}\bigg(\sup_{x\in\text{supp}(X)}\left|\int\phi_V(\zeta,w)s(x,w)\,dw\right|\bigg)\,d\zeta\\
&\quad\preceq\int_{\bar{\xi}/h_1}^{\infty}|\zeta|^{\lambda}\big(1+|\zeta|\big)^{\gamma_{\phi s}}\exp\big(\alpha_{\phi s}|\zeta|^{\nu_{\phi s}}\big)\,d\zeta\\
&\quad=O\big((\bar{\xi}/h_1)^{\gamma_{\phi s}+\lambda+1}\exp\big(\alpha_{\phi s}(\bar{\xi}/h_1)^{\nu_{\phi s}}\big)\big)\\
&\quad=O\big(\big(h_1^{-1}\big)^{\gamma_{\lambda,B,s}}\exp\big(\alpha_{B,s}\big(h_1^{-1}\big)^{\nu_{B,s}}\big)\big).
\end{aligned}$$

Thus, we have

$$\sup_{x\in\mathrm{supp}(X)}\left|\sum_{j=1}^{J}\int B_{V_j,\lambda_j}(x,w,h_1)s_j(x,w)\,dw\right| = O\big((h_1^{-1})^{\gamma_{\lambda,B,s}}\exp\big(\alpha_{B,s}(h_1^{-1})^{\nu_{B,s}}\big)\big).$$

For the second term,

$$\begin{aligned}
&\sup_{x\in\mathrm{supp}(X)}\left|\sum_{j=1}^{J}\int L_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\right| \\
&\quad\le \sup_{x\in\mathrm{supp}(X)}\sum_{j=1}^{J}\left|\int L_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sup_{x\in\mathrm{supp}(X)}\left|\int L_{V,\lambda}(x,w,h_n)s(x,w)\,dw\right| \\
&\quad= \sup_{x\in\mathrm{supp}(X)}\left|\int\int\big[\Psi_{V,\lambda,1}(\zeta,x,w,h_1)\big(\hat{E}\big[e^{i\zeta X_2}\big]-E\big[e^{i\zeta X_2}\big]\big)\right. \\
&\qquad+\Psi_{V,\lambda,X_1}(\zeta,x,w,h_1)\big(\hat{E}\big[X_1e^{i\zeta X_2}\big]-E\big[X_1e^{i\zeta X_2}\big]\big) \\
&\qquad+\Psi_{V,\lambda,\chi_V}(\zeta,x,w,h_1)\big(\hat{E}\big[Ve^{i\zeta X_2}k_{h_2}(W-w)\big]-E\big[Ve^{i\zeta X_2}k_{h_2}(W-w)\big]\big) \\
&\qquad\left.+\Psi_{V,\lambda,f_W}(\zeta,x,w,h_1)\big(\hat{E}\big[k_{h_2}(W-w)\big]-E\big[k_{h_2}(W-w)\big]\big)\big]\,d\zeta s(x,w)\,dw\right| \\
&\quad\le\int\Big[\Psi^{+}_{V,\lambda,1,s}(\zeta,h_1)\big|\hat{E}\big[e^{i\zeta X_2}\big]-E\big[e^{i\zeta X_2}\big]\big| \\
&\qquad+\Psi^{+}_{V,\lambda,X_1,s}(\zeta,h_1)\big|\hat{E}\big[X_1e^{i\zeta X_2}\big]-E\big[X_1e^{i\zeta X_2}\big]\big| \\
&\qquad+h_2^{-1}\Psi^{+}_{V,\lambda,\chi_V,s}(\zeta,h_1)\Big(\sup_{w\in\mathrm{supp}(W)}\big|\hat{E}\big[Ve^{i\zeta X_2}h_2k_{h_2}(W-w)\big] \\
&\qquad-E\big[Ve^{i\zeta X_2}h_2k_{h_2}(W-w)\big]\big|\Big) \\
&\qquad+h_2^{-1}\Psi^{+}_{V,\lambda,f_W,s}(\zeta,h_1)\Big(\sup_{w\in\mathrm{supp}(W)}\big|\hat{E}\big[h_2k_{h_2}(W-w)\big] \\
&\qquad-E\big[h_2k_{h_2}(W-w)\big]\big|\Big)\Big]\,d\zeta.
\end{aligned}$$

Then we have

$$\begin{aligned}
&E\left[\sup_{x\in\mathrm{supp}(X)}\left|\int L_{V,\lambda}(x,w,h_n)s(x,w)\,dw\right|\right] \\
&\quad\le\int\Big[\Psi^{+}_{V,\lambda,1,s}(\zeta,h_1)\big\{E\big(\big|\hat{E}\big[e^{i\zeta X_2}-E\big[e^{i\zeta X_2}\big]\big]\big|^2\big)\big\}^{1/2} \\
&\qquad+\Psi^{+}_{V,\lambda,X_1,s}(\zeta,h_1)\big\{E\big(\big|\hat{E}\big[X_1e^{i\zeta X_2}-E\big[X_1e^{i\zeta X_2}\big]\big]\big|^2\big)\big\}^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+ h_2^{-1}\Psi^{+}_{V,\lambda,\chi_V,s}(\zeta,h_1)\Big\{E\Big(\Big|\sup_{w\in\operatorname{supp}(W)}\big(\hat{E}\big[Ve^{i\zeta X_2}h_2k_{h_2}(W-w)\\
&- E\big[Ve^{i\zeta X_2}h_2k_{h_2}(W-w)\big]\big]\big)\Big|^2\Big)\Big\}^{1/2}\\
&+ h_2^{-1}\Psi^{+}_{V,\lambda,f_W,s}(\zeta,h_2)\Big\{E\Big(\Big|\sup_{w\in\operatorname{supp}(W)}\big(\hat{E}\big[h_2k_{h_2}(W-w)\\
&- E\big[h_2k_{h_2}(W-w)\big]\big]\big)\Big|^2\Big)\Big\}^{1/2}\Big]\,d\zeta\\
&\preceq n^{-1/2}\Big(\sum_{A=1,X_1}\int \Psi^{+}_{V,\lambda,A,s}(\zeta,h_1)\,d\zeta + h_2^{-1}\sum_{B=\chi_V,f_W}\int \Psi^{+}_{V,\lambda,B,s}(\zeta,h_1)\,d\zeta\Big)\\
&= n^{-1/2}\Psi^{+}_{V,\lambda,s}(h),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi^{+}_{V,\lambda,s}(h) = O\Big(&\max\big\{\big(1+h_1^{-1}\big)^{\gamma_1+1}, h_2^{-1}\big\}\big(1+h_1^{-1}\big)^{\gamma_{\phi s}+\lambda-\gamma_\theta+1}\\
&\times \exp\big((\alpha_{\phi s}1_{\{\nu_{\phi s}\ge\nu_\theta\}} - \alpha_\theta 1_{\{\nu_{\phi s}\le\nu_\theta\}})\big(h_1^{-1}\big)^{\max\{\nu_\theta,\nu_{\phi s}\}}\big)\Big)
\end{aligned}
$$

from Lemma C.5. It follows by Markov's inequality that

$$
\begin{aligned}
&\sup_{x\in\operatorname{supp}(X)}\Big|\sum_{j=1}^{J}\int L_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\Big|\\
&= O_p\Big(n^{-1/2}\big(\max\big\{\big(1+h_1^{-1}\big)^{\gamma_1+1}, h_2^{-1}\big\}\big)\big(1+h_1^{-1}\big)^{\gamma_{\phi s}+\lambda-\gamma_\theta+1}\\
&\quad\times \exp\big((\alpha_{\phi s}1_{\{\nu_{\phi s}\ge\nu_\theta\}} - \alpha_\theta 1_{\{\nu_{\phi s}\le\nu_\theta\}})\big(h_1^{-1}\big)^{\max\{\nu_\theta,\nu_{\phi s}\}}\big)\Big).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\sup_{x\in\operatorname{supp}(X)}\Big|\sum_{j=1}^{J}\int R_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\Big|\\
&\le \sup_{x\in\operatorname{supp}(X)}\sum_{j=1}^{J}\Big|\int R_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\Big|\\
&= \sup_{x\in\operatorname{supp}(X)}\sum_{j=1}^{J}\Big|\int \sum_{i=1}^{22} R_{ij}s_j(x,w)\,dw\Big|.
\end{aligned}
$$

We obtain upper bounds for the terms

$$
\sup_{x\in\operatorname{supp}(X)}\Big|\int R_i s(x,w)\,dw\Big|,\quad i=1,\ldots,22,
$$

using the fact that the integral of R_1 dominates them, which is similar to Lemma B.6. For brevity, we only provide a bound on this integral:

$$
\begin{aligned}
&\sup_{x\in\mathrm{supp}(X)}\left|\int R_1 s(x,w)\,dw\right| \\
&\preceq \int_0^\infty |\zeta|^\lambda \big|\kappa(h_1\zeta)\big|\left(\sup_{x\in\mathrm{supp}(X)}\left|\int \phi_V(\zeta,w)s(x,w)\,dw\right|\right) \\
&\quad\times \exp\big(\big|\delta\bar{Q}_{X_1}(\zeta)\big|\big)\left(\int_0^\zeta \big|\delta\hat{q}_{X_1}(\xi)\big|\,d\xi\right)^2 d\zeta \\
&\preceq \exp\big(o_p(1)\big)\int_0^\infty |\zeta|^\lambda \big|\kappa(h_1\zeta)\big|\left(\sup_{x\in\mathrm{supp}(X)}\left|\int \phi_V(\zeta,w)s(x,w)\,dw\right|\right) \\
&\quad\times \int_0^\zeta \left|\left(\frac{\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)}-\frac{\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\right)\left(1+\frac{\delta\hat{\theta}_1(\xi)}{\theta_1(\xi)}\right)^{-1}\right| d\xi \\
&\quad\times \int_0^\zeta \left|\left(\frac{\delta\hat{\theta}_{X_1}(\xi)}{\theta_1(\xi)}-\frac{\theta_{X_1}(\xi)\delta\hat{\theta}_1(\xi)}{(\theta_1(\xi))^2}\right)\left(1+\frac{\delta\hat{\theta}_1(\xi)}{\theta_1(\xi)}\right)^{-1}\right| d\xi\zeta \\
&= \exp\big(o_p(1)\big)\Upsilon(h)\hat{\Phi}_n^2\big|1+o_p(1)\big|^{-2} \\
&\quad\times \int_0^\infty \left(\int_\xi^\infty |\zeta|^\lambda\big|\kappa(h_1\zeta)\big|\left(\sup_{x\in\mathrm{supp}(X)}\left|\int \phi_V(\zeta,w)s(x,w)\,dw\right|\right)d\zeta\right) \\
&\quad\times \left(\frac{1}{|\theta_1(\xi)|}+\frac{|\theta_{X_1}(\xi)|}{|\theta_1(\xi)|^2}\right)d\xi \\
&\preceq \Upsilon(h)\hat{\Phi}_n^2\Psi^+_{V,\lambda,s}(h),
\end{aligned}
$$

where Fubini's theorem is used in the third line. Note that

$$
\begin{aligned}
&\Upsilon(h)\hat{\Phi}_n^2\Psi^+_{V,\lambda,s}(h) \\
&= \big(\Upsilon(h)\hat{\Phi}_n^2 n^{1/2}\big)n^{-1/2}\Psi^+_{V,\lambda,s}(h) \\
&= o_p\Big(n^{-1/2}\big(\max\big\{\big(1+h_1^{-1}\big)^{\delta_{L,s}},h_2^{-1}\big\}\big)\big(h_1^{-1}\big)^{\gamma_{\lambda,L,s}}\exp\big(\alpha_{L,s}\big(h_1^{-1}\big)^{\nu_{L,s}}\big)\Big).
\end{aligned}
$$

Because all other terms are also bounded by the upper bound for $\sup_{x\in\mathrm{supp}(X)}|\int R_1\times s(x,w)\,dw|$, we have

$$
\begin{aligned}
&\sup_{x\in\mathrm{supp}(X)}\left|\sum_{j=1}^J\int R_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\right| \\
&= o_p\Big(n^{-1/2}\big(\max\big\{\big(1+h_1^{-1}\big)^{\delta_{L,s}},h_2^{-1}\big\}\big)\big(h_1^{-1}\big)^{\gamma_{\lambda,L,s}}\exp\big(\alpha_{L,s}\big(h_1^{-1}\big)^{\nu_{L,s}}\big)\Big).
\end{aligned}
$$

Collecting these gives the desired result. □

Proof of Theorem C.2. By Assumption C.4, we have

$$\begin{aligned}
&\max_{j=1,\dots,J} \sup_{(x,w)\in \operatorname{supp}(X,W)} \left|\hat{g}_{V_j,\lambda_j}(x,w,h_n) - g_{V_j,\lambda_j}(x,w,h_{1n})\right| \\
&= \max_{j=1,\dots,J} \sup_{(x,w)\in \operatorname{supp}(X,W)} \left|B_{V_j,\lambda_j}(x,w,h_{1n}) + L_{V_j,\lambda_j}(x,w,h_n) + R_{V_j,\lambda_j}(x,w,h_n)\right| \\
&= o_p\big(n^{-1/4}\big).
\end{aligned}$$

Thus, the remainder term in eq. (S8) is $o_p((n^{-1/4})^2) + o_p((n^{-1/4})^2) = o_p(n^{-1/2})$ when we let $\tilde{g}_{V_j,\lambda_j}(x,w) = \hat{g}_{V_j,\lambda_j}(x,w,h_n)$ and $\tilde{f}_W(w) = \hat{f}_W(w)$. We have

$$\begin{aligned}
&\sum_{j=1}^{J} \int\int \big(\hat{g}_{V_j,\lambda_j}(x,w,h_n) - g_{V_j,\lambda_j}(x,w)\big) s_j(x,w)\,dw\,dx \\
&= \sum_{j=1}^{J} \int\int L_{V_j,\lambda_j}(x,w,h_n) s_j(x,w)\,dw\,dx \\
&\quad + \sum_{j=1}^{J} \int\int \big(B_{V_j,\lambda_j}(x,w,h_{1n}) + R_{V_j,\lambda_j}(x,w,h_n)\big) s_j(x,w)\,dw\,dx.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left|\sum_{j=1}^{J} \int\int \big(B_{V_j,\lambda_j}(x,w,h_{1n}) + R_{V_j,\lambda_j}(x,w,h_n)\big) s_j(x,w)\,dw\,dx\right| \\
&\le \Big(\max_{j=1,\dots,J} \sup_{(x,w)\in(X,W)} \left|B_{V_j,\lambda_j}(x,w,h_{1n}) + R_{V_j,\lambda_j}(x,w,h_n)\right|\Big) \\
&\quad \times \sum_{j=1}^{J} \int\int \left|s_j(x,w)\right|\,dw\,dx \\
&= o_p\big(n^{-1/2}\big),
\end{aligned}$$

since

$$\max_{j=1,\dots,J} \sup_{(x,w)\in(X,W)} \max\big\{\left|B_{V_j,\lambda_j}(x,w,h_{1n})\right|, \left|R_{V_j,\lambda_j}(x,w,h_n)\right|\big\} = o_p(n^{-1/2})$$

and $\int\int |s_j(x,w)|\,dw\,dx < \infty$. Therefore, we have

$$\begin{aligned}
&b\big(\hat{g}(h_n), \hat{f}(h_n)\big) - b(g,f) \\
&= \sum_{j=1}^{J} \int\int L_{V_j,\lambda_j}(x,w,h_n) s_j(x,w)\,dw\,dx \\
&\quad + \int\int \big(\hat{f}_W(w) - f_W(w)\big) s_{J+1}(x,w)\,dw\,dx + o_p\big(n^{-1/2}\big).
\end{aligned}$$

We also note that

$$
\begin{aligned}
&\sum_{j=1}^{J}\int\int L_{V_j,\lambda_j}(x,w,h_n)s_j(x,w)\,dw\,dx \\
&\quad+\int\int\big(\hat{f}_W(w)-f_W(w)\big)s_{J+1}(x,w)\,dw\,dx \\
&=\Bigg\{\lim_{\tilde{h}\to 0}\sum_{j=1}^{J}\int\int L_{V_j,\lambda_j}(x,w,\tilde{h})s_j(x,w)\,dw\,dx \\
&\quad+\lim_{\tilde{h}_2\to 0}\int\int\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)s_{J+1}(x,w)\,dw\,dx\Bigg\} \qquad \text{(S15)}\\
&\quad+\Bigg\{\lim_{\tilde{h}_2\to 0}\int\int\big(L_{V_j,\lambda_j}(x,w,h_n)-L_{V_j,\lambda_j}(x,w,\tilde{h})\big)s_j(x,w)\,dw\,dx \\
&\quad+\lim_{\tilde{h}_2\to 0}\int\int\big\{\big(\hat{E}\big[k_{h_{2n}}(W-w)\big]-E\big[k_{h_{2n}}(W-w)\big]\big) \\
&\quad-\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)\big\}s_{J+1}(x,w)\,dw\,dx\Bigg\}.
\end{aligned}
$$

We will show that the first term in the right-hand side is a standard sample average, while the second term is asymptotically negligible. By the definition of $L_{V_j,\lambda_j}(x,w,\tilde{h})$ in Lemma 4.1 and the fact that because the assumption that $\bar{\Psi}_{V,\lambda,s}<\infty$ ensures the integrand is absolutely integrable for any given sample, integrals and limits can be interchanged, so we have

$$
\begin{aligned}
&\lim_{\tilde{h}\to 0}\sum_{j=1}^{J}\int\int L_{V_j,\lambda_j}(x,w,\tilde{h})s_j(x,w)\,dw\,dx \\
&\quad+\lim_{\tilde{h}_2\to 0}\int\int\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)s_{J+1}(x,w)\,dw\,dx \\
&=\sum_{j=1}^{J}\int\Bigg\{\lim_{\tilde{h}_1\to 0}\int\int\Psi_{V_j,\lambda_j,1}(\zeta,x,w,\tilde{h}_1)s_j(x,w)\,dw\,dx\big(\hat{E}\big[e^{i\zeta X_2}\big]-E\big[e^{i\zeta X_2}\big]\big) \\
&\quad+\lim_{\tilde{h}_1\to 0}\int\int\Psi_{V_j,\lambda_j,X_1}(\zeta,x,w,\tilde{h}_1)s_j(x,w)\,dw\,dx\big(\hat{E}\big[X_1e^{i\zeta X_2}\big]-E\big[X_1e^{i\zeta X_2}\big]\big) \\
&\quad+\lim_{\tilde{h}\to 0}\int\int\Psi_{V_j,\lambda_j,\chi V_j}(\zeta,x,w,\tilde{h}_1)s_j(x,w) \qquad \text{(S16)}\\
&\quad\times\big(\hat{E}\big[V_je^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]-E\big[V_je^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&\quad+\lim_{\tilde{h}\to 0}\int\int\Psi_{V_j,\lambda_j,f_W}(\zeta,x,w,\tilde{h}_1)s_j(x,w)
\end{aligned}
$$

$$\times \left(\hat{E}\left[k_{\tilde{h}_2}(W-w)\right] - E\left[k_{\tilde{h}_2}(W-w)\right]\right) dw\,dx\Big\} d\zeta$$
$$+ \int \lim_{\tilde{h}_2 \to 0} \int \left(\hat{E}\left[k_{\tilde{h}_2}(W-w)\right] - E\left[k_{\tilde{h}_2}(W-w)\right]\right) s_{J+1}(x,w)\, dw\,dx.$$

For the first term in the integrand of eq. (S16), we have by Lebesgue's dominated convergence theorem and Fubini's theorem,

$$\begin{aligned}
&\lim_{\tilde{h}_1 \to 0} \int\int \Psi_{V,\lambda,1}(\zeta, x, w, \tilde{h}_1) s(x,w)\, dw\,dx \\
&= \lim_{\tilde{h}_1 \to 0} \int\int \Big\{ -\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2} \int_\zeta^{\pm\infty} (-i\xi)^\lambda \kappa(\tilde{h}_1\xi) \exp(-i\xi x)\phi_V(\xi, w)\, d\xi \\
&\quad - \frac{1}{2\pi}(-i\zeta)^\lambda \kappa(\tilde{h}_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta, w)}{\theta_1(\zeta)}\Big\} s(x,w)\, dw\,dx \\
&= -\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2} \\
&\quad \times \int_\zeta^{\pm\infty}\left(\int \exp(-i\xi x)\int s(x,w)\phi_V(\xi,w)\, dw\,dx\right)(-i\xi)^\lambda \lim_{\tilde{h}_1\to 0}\kappa(\tilde{h}_1\xi)\, d\xi \\
&\quad - \frac{1}{2\pi}\frac{(-i\zeta)^\lambda}{\theta_1(\zeta)}\left(\int \exp(-i\zeta x)\int s(x,w)\phi_V(\zeta,w)\, dw\,dx\right)\lim_{\tilde{h}_1\to 0}\kappa(\tilde{h}_1\zeta) \\
&= -\frac{1}{2\pi}\frac{i\theta_{X_1}(\zeta)}{(\theta_1(\zeta))^2}\int_\zeta^{\pm\infty}\left(\int \exp(-i\xi x)\int s(x,w)\phi_V(\xi,w)\, dw\,dx\right)(-i\xi)^\lambda\, d\xi \\
&\quad - \frac{1}{2\pi}\frac{(-i\zeta)^\lambda}{\theta_1(\zeta)}\left(\int \exp(-i\zeta x)\int s(x,w)\phi_V(\zeta,w)\, dw\,dx\right) \\
&\equiv \Psi_{V,\lambda,1,s}(\zeta).
\end{aligned}$$

Similarly, for the second term, we have

$$\begin{aligned}
&\lim_{\tilde{h}_1 \to 0} \int\int \Psi_{V,\lambda,X_1}(\zeta, x, w, \tilde{h}_1) s(x,w)\, dw\,dx \\
&= \lim_{\tilde{h}_1 \to 0} \int\int \Big\{ \frac{1}{2\pi}\frac{i}{\theta_1(\zeta)} \\
&\quad \times \int_\zeta^{\pm\infty} (-i\xi)^\lambda \kappa(\tilde{h}_1\xi)\exp(-i\xi x)\phi_V(\xi,w)\, d\xi\Big\} s(x,w)\, dw\,dx \\
&= \frac{1}{2\pi}\frac{i}{\theta_1(\zeta)}\int_\zeta^{\pm\infty}\left(\int \exp(-i\xi x)\int s(x,w)\phi_V(\xi,w)\, dw\,dx\right)(-i\xi)^\lambda\, d\xi \\
&\equiv \Psi_{V,\lambda,X_1,s}(\zeta).
\end{aligned}$$

We also note that for the third term,

$$\begin{aligned}
&\lim_{\tilde{h}\to 0}\int\int \Psi_{V,\lambda,\chi_V}(\zeta,x,w,\tilde{h}_1)s(x,w) \\
&\quad\times\big(\hat{E}\big[Ve^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]-E\big[Ve^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&=\lim_{\tilde{h}\to 0}\int\int\left\{\frac{1}{2\pi}(-i\zeta)^{\lambda}\kappa(\tilde{h}_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{\chi_V(\zeta,w)}\right\}s(x,w) \\
&\quad\times\big(\hat{E}\big[Ve^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]-E\big[Ve^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&=\frac{1}{2\pi}(-i\zeta)^{\lambda}\int\exp(-i\zeta x)\lim_{\tilde{h}_2\to 0}\int\frac{1}{\chi_V(\zeta,w)}s(x,w)\phi_V(\zeta,w) \\
&\quad\times\big(\hat{E}\big[Ve^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]-E\big[Ve^{i\zeta X_2}k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&\equiv\hat{E}\big[\mathcal{Z}_{V,\lambda,\chi_V}(s,\zeta;V,X_2,W)\big]-E\big[\mathcal{Z}_{V,\lambda,\chi_V}(s,\zeta;V,X_2,W)\big],
\end{aligned}$$

and for the fourth term,

$$\begin{aligned}
&\lim_{\tilde{h}\to 0}\int\int \Psi_{V,\lambda,f_W}(\zeta,x,w,\tilde{h}_1)s(x,w)\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&=\lim_{\tilde{h}\to 0}\int\int\left\{\frac{-1}{2\pi}(-i\zeta)^{\lambda}\kappa(\tilde{h}_1\zeta)\exp(-i\zeta x)\frac{\phi_V(\zeta,w)}{f_W(w)}\right\}s(x,w) \\
&\quad\times\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&=-\frac{1}{2\pi}(-i\zeta)^{\lambda}\int\exp(-i\zeta x)\lim_{\tilde{h}_2\to 0}\int\frac{1}{f_W(w)}s(x,w)\phi_V(\zeta,w) \\
&\quad\times\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)\,dw\,dx \\
&\equiv\hat{E}\big[\mathcal{Z}_{V,\lambda,f_W}(s,\zeta;W)\big]-E\big[\mathcal{Z}_{V,\lambda,f_W}(s,\zeta;W)\big],
\end{aligned}$$

where $\mathcal{Z}_{V,\lambda,\chi_V}(s,\zeta;V,X_2,W)$ and $\mathcal{Z}_{V,\lambda,f_W}(s,\zeta;W)$ are defined in front of the theorem.

Thus it follows that

$$\begin{aligned}
&\lim_{\tilde{h}\to 0}\sum_{j=1}^{J}\int\int L_{V_j,\lambda_j}(x,w,\tilde{h})s_j(x,w)\,dw\,dx \\
&\qquad+\lim_{\tilde{h}_2\to 0}\int\int\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)s_{J+1}(x,w)\,dw\,dx \\
&=\sum_{j=1}^{J}\int\big\{\Psi_{V_j,\lambda_j,1,s_j}(\zeta)\big(\hat{E}\big[e^{i\zeta X_2}\big]-E\big[e^{i\zeta X_2}\big]\big) \\
&\quad+\Psi_{V_j,\lambda_j,X_1,s_j}(\zeta)\big(\hat{E}\big[X_1e^{i\zeta X_2}\big]-E\big[X_1e^{i\zeta X_2}\big]\big) \\
&\quad+\big(\hat{E}\big[\mathcal{Z}_{V_j,\lambda_j,\chi_{V_j}}(s_j,\zeta;V_j,X_2,W)\big]-E\big[\mathcal{Z}_{V_j,\lambda_j,\chi_{V_j}}(s_j,\zeta;V_j,X_2,W)\big]\big)
\end{aligned}$$

$$+\big(\hat{E}\big[\mathcal{Z}_{V_j,\lambda_j,f_W}(s_j,\zeta;W)\big]-E\big[\mathcal{Z}_{V_j,\lambda_j,f_W}(s_j,\zeta;W)\big]\big)\big\}\,d\zeta$$
$$+\int\lim_{\tilde{h}_2\to 0}\int\big(\hat{E}\big[k_{\tilde{h}_2}(W-w)\big]-E\big[k_{\tilde{h}_2}(W-w)\big]\big)s_{J+1}(x,w)\,dw\,dx$$
$$=\hat{E}\Bigg[\sum_{j=1}^{J}\psi_{V_j,\lambda_j}(s_j;V_j,X_1,X_2,W)+\psi_f(s_{J+1};W)\Bigg]=\hat{E}\big[\psi_s(V,X_1,X_2,W)\big],$$

as defined just before the theorem. The assumption that $\bar{\Psi}_{V,\lambda,s}<\infty$ ensures that for some $C<\infty$,

$$\big|\psi_s(v,x_1,x_2,\tilde{w})\big|\le C\max\{1,|x_1|\}\bar{\Psi}_{V,\lambda,s}.$$

Since $E[X_1^2]<\infty$ by Assumption 4.2 and since $E[|\psi_s(V,X_1,X_2,W)|^2]<\infty$, the Lindeberg–Levy central limit theorem gives that $\hat{E}[\psi_s(V,X_1,X_2,W)]$ is $\sqrt{n}$ consistent and asymptotically normal.

The second term of eq. (S15) can be shown to be $o_p(n^{-1/2})$ because it can be written as an h_n-dependent sample average $\hat{E}[\bar{\psi}_s(V,X_1,X_2,W,h_n)]$, where $\bar{\psi}_s(V,X_1,X_2,W,h_n)$ is such that $\lim_{h_n\to 0}E[|\bar{\psi}_s(V,X_1,X_2,W,h_n)|^2]=0$. The similar procedure works for the case of $\hat{E}[\psi_s(V,X_1,X_2,W)]$ by replacing $\kappa(\tilde{h}_1\xi)$ with $(\kappa(h_{1n}\xi)-\kappa(\tilde{h}_1\xi))$ and $k_{\tilde{h}_2}(\cdot)$ with $(k_{h_{2n}}(\cdot)-k_{\tilde{h}_2}(\cdot))$, and taking the limit as $h_n\equiv(h_{1n},h_{2n})\to 0$ and $\tilde{h}\equiv(\tilde{h}_1,\tilde{h}_2)\to 0$. □

Proof of Theorem C.3. We prove the theorem by applying Theorem C.1 and straightforward Taylor expansions. (i) From the definitions of $\hat{\beta}_m(x)$ and $\beta_m(x)$, we have

$$\sup_{x\in\mathbb{M}}\big|\hat{\beta}_m(x)-\beta_m(x)\big|$$
$$=\sup_{x\in\mathbb{M}}\bigg|\int_{S^w_{\hat{\beta}(\cdot,h_n)}}\big(\hat{\beta}(x,w,h_n)-\beta(x,w)\big)m(w)\,dw\bigg|$$
$$=\sup_{x\in\mathbb{M}}\bigg|\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\int_{S^w_{\hat{\beta}(\cdot,h_n)}}m(w)s_{V,\lambda}(x,w)\big(\hat{g}_{V,\lambda}(x,w,h_n)-g_{V,\lambda}(x,w)\big)\,dw\bigg|$$
$$+o_p(1)$$
$$=O\big(\tau^{-3}\big(h_{1n}^{-1}\big)^{\gamma_{1,B,s}}\exp\big(\alpha_{B,s}\big(h_{1n}^{-1}\big)^{\nu_{B,s}}\big)\big)$$
$$+O_p\big(\tau^{-3}n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_{L,s}},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{1,L,s}}\exp\big(\alpha_{L,s}\big(h_{1n}^{-1}\big)^{\nu_{L,s}}\big)\big),$$

where the last equality is attained by Theorem C.1.

(ii) Similarly, from the definitions of $\hat{\beta}_m(x)$ and $\beta_m(x)$, we have

$$\sup_{x\in\mathbb{M}}\big|\hat{\beta}_{mf_W}(x)-\beta_{mf_W}(x)\big|$$
$$=O\big(\tau^{-3}\big(h_{1n}^{-1}\big)^{\gamma_{1,B,s}}\exp\big(\alpha_{B,s}\big(h_{1n}^{-1}\big)^{\nu_{B,s}}\big)\big)$$
$$+O_p\big(\tau^{-3}n^{-1/2}\big(\max\big\{\big(h_{1n}^{-1}\big)^{\delta_{L,s}},h_{2n}^{-1}\big\}\big)\big(h_{1n}^{-1}\big)^{\gamma_{1,L,s}}\exp\big(\alpha_{L,s}\big(h_{1n}^{-1}\big)^{\nu_{L,s}}\big)\big).$$

(iii) Similarly, from the definitions of $\hat{\beta}_{mf_{W|X}}(x)$ and $\beta_{mf_{W|X}}(x)$, we have

$$\begin{aligned}
&\sup_{x\in\mathbb{M}}\left|\hat{\beta}_{mf_{W|X}}(x)-\beta_{mf_{W|X}}(x)\right| \\
&\quad = O\left(\tau^{-3}\left(h_{1n}^{-1}\right)^{\gamma_{1,B,s}}\exp\left(\alpha_{B,s}\left(h_{1n}^{-1}\right)^{\nu_{B,s}}\right)\right) \\
&\qquad + O_p\left(\tau^{-3}n^{-1/2}\left(\max\left\{\left(h_{1n}^{-1}\right)^{\delta_{L,s}}, h_{2n}^{-1}\right\}\right)\left(h_{1n}^{-1}\right)^{\gamma_{1,L,s}}\exp\left(\alpha_{L,s}\left(h_{1n}^{-1}\right)^{\nu_{L,s}}\right)\right). \qquad \square
\end{aligned}$$

PROOF OF THEOREM C.4. Again, Theorem C.2 and Taylor expansions deliver the result. (i) From the definition of $\hat{\beta}_{\tilde{m}}$ and $\beta_{\tilde{m}}$, we have

$$\begin{aligned}
\hat{\beta}_{\tilde{m}}-\beta_{\tilde{m}} &= \int_{S^x_{\hat{\beta}(\cdot,h_n)}}\int_{S^w_{\hat{\beta}(\cdot,h_n)}}\left(\hat{\beta}(x,w,h_n)-\beta(x,w)\right)\tilde{m}(x,w)\,dw\,dx \\
&= \sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\int_{S^x_{\hat{\beta}(\cdot,h_n)}}\int_{S^w_{\hat{\beta}(\cdot,h_n)}}\tilde{m}(x,w)s_{V,\lambda}(x,w) \\
&\qquad \times\left(\hat{g}_{V,\lambda}(x,w,h_n)-g_{V,\lambda}(x,w)\right)dw\,dx \\
&\qquad + o_p\left(n^{-1/2}\right) \\
&= \hat{E}\left[\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_1;V,X_1,X_2,W)\right]+o_p\left(n^{-1/2}\right),
\end{aligned}$$

where P_1 is defined in front of the theorem. Let

$$\psi_{\beta_{\tilde{m}}}(v,x_1,x_2,\tilde{w}) \equiv \sum_{V\in 1\{,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(\tilde{m}s_{V,\lambda};v,x_1,x_2,\tilde{w}).$$

The result is immediate by applying Theorem C.2.

(ii) Similarly, from the definitions of $\hat{\beta}_{\tilde{m}f_{W|X}}$ and $\beta_{\tilde{m}f_{W|X}}$, we get

$$\begin{aligned}
&\hat{\beta}_{\tilde{m}f_{W|X}}-\beta_{\tilde{m}f_{W|X}} \\
&\quad = \hat{E}\left[\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_2;V,X_1,X_2,W)+\psi_{1,0}(P_3;1,X_1,X_2,W)\right. \\
&\qquad \left. -\psi_{1,0}(P_4;1,X_1,X_2,W)+\psi_f(P_5;W)\right]+o_p\left(n^{-1/2}\right),
\end{aligned}$$

where P_2–P_5 are defined in front of the theorem. Let

$$\begin{aligned}
\psi_{\beta_{\tilde{m}f_{W|X}}}(v,x_1,x_2,\tilde{w}) \equiv &\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_2;v,x_1,x_2,\tilde{w})+\psi_{1,0}(P_3;1,x_1,x_2,\tilde{w}) \\
&-\psi_{1,0}(P_4;1,x_1,x_2,\tilde{w})+\psi_f(P_5;\tilde{w}).
\end{aligned}$$

The result is immediate by applying Theorem C.2.

(iii) Similarly, from the definitions of $\hat{\beta}_{\tilde{m}f_{W,X}}$ and $\beta_{\tilde{m}f_{W,X}}$, we get

$$\hat{\beta}_{\tilde{m}f_{W,X}} - \beta_{\tilde{m}f_{W,X}} = \hat{E}\Bigg[\sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_6; V, X_1, X_2, W) + \psi_{1,0}(P_7; 1, X_1, X_2, W) + \psi_f(P_8; W)\Bigg] + o_p\big(n^{-1/2}\big),$$

where P_6–P_8 are defined in front of the theorem. Let

$$\psi_{\beta_{\tilde{m}f_{W,X}}}(v, x_1, x_2, \tilde{w}) \equiv \sum_{V\in\{1,Y\}}\sum_{\lambda=0,1}\psi_{V,\lambda}(P_6; v, x_1, x_2, \tilde{w}) + \psi_{1,0}(P_7; 1, x_1, x_2, \tilde{w}) + \psi_f(P_8; \tilde{w}).$$

The result is immediate by applying Theorem C.2. □

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