

## Supplement to “Likelihood-ratio-based confidence sets for the timing of structural breaks”

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This supplement gives proofs for propositions and corollaries in the main text.

### APPENDIX: PROOFS

**PROOF OF PROPOSITION 1.** Following [Qu and Perron \(2007a\)](#), [Qu and Perron \(2007b\)](#), we consider the  $j$ th break date  $\tau_j$  without loss of generality. The log-profile likelihood ratio subject to the restrictions  $g(\beta, \Sigma) = 0$  under the null hypothesis  $H_0 : \tau_j = \tau_j^0$  and the alternative hypothesis  $H_1 : \tau_j \neq \tau_j^0$  is given by

$$\begin{aligned}
 LR_j(\tau_j^0) &= -2[l_j^r(\tau_j^0, \hat{\beta}(\tau_j^0), \hat{\Sigma}(\tau_j^0)) - l_j^r(\hat{\tau}_j, \hat{\beta}, \hat{\Sigma})] \\
 &= -2\{l_j^r(\tau_j^0, \hat{\beta}(\tau_j^0), \hat{\Sigma}(\tau_j^0)) - l_j(\tau_j^0, \beta_j^0, \Sigma_j^0)\} \\
 &\quad - \underbrace{\max_{\beta_j, \Sigma_j} l_j^r(\tau_j^0, \beta_j, \Sigma_j)}_{\max_{\beta_j, \Sigma_j} l_j^r(\tau_j^0, \beta_j, \Sigma_j)} \\
 &\quad + 2\{l_j^r(\hat{\tau}_j, \hat{\beta}, \hat{\Sigma}) - l_j(\tau_j^0, \beta_j^0, \Sigma_j^0)\} \\
 &\quad + \underbrace{\max_{\tau_j, \beta_j, \Sigma_j} l_j^r(\tau_j, \beta_j, \Sigma_j)}_{\max_{\tau_j, \beta_j, \Sigma_j} l_j^r(\tau_j, \beta_j, \Sigma_j)} \\
 &= \max_{\tau_j} lr_j(\tau_j, \beta_j^0, \Sigma_j^0) + o_p(1),
 \end{aligned} \tag{A.1}$$

where the maximization is taken over  $C_M$ . The second and third lines in (A.1) result from adding and subtracting the log-likelihood at the true values  $l_j(\tau_j^0, \beta_j^0, \Sigma_j^0)$  to the first line.<sup>1</sup> The equality of the second and third lines and the fourth line in (A.1) follows from Theorem 1 in [Qu and Perron \(2007a\)](#).

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<sup>1</sup>Note that  $l_j^r(\hat{\tau}_j, \hat{\beta}, \hat{\Sigma}) = l_j^r(\hat{\tau}_j)$  in (4).

We focus on the term  $lr_j(\tau_j, \beta_j^0, \Sigma_j^0) = -2[l_j(\tau_j^0, \beta_j^0, \Sigma_j^0) - l_j(\tau_j, \beta_j^0, \Sigma_j^0)]$  in the fourth line of (A.1) so as to find the asymptotic distribution of  $LR_j(\tau_j^0)$ . Letting  $lr_j(\tau_j, \beta_j^0, \Sigma_j^0) = lr_j(\tau_j - \tau_j^0)$  and  $r = \tau_j - \tau_j^0$ ,

$$lr_j(r) = 0 \quad \text{for } r = 0,$$

$$lr_j(r) = 2 \left( -\frac{r}{2} (\log|\Sigma_j^0| - \log|\Sigma_{j+1}^0|) - \frac{1}{2} \sum_{t=\tau_j^0+r}^{\tau_j^0} (y_t - x_t' \beta_{j+1}^0) (\Sigma_{j+1}^0)^{-1} (y_t - x_t' \beta_{j+1}^0) - (y_t - x_t' \beta_j^0) (\Sigma_j^0)^{-1} (y_t - x_t' \beta_j^0) \right) \quad \text{for } r < 0,$$

$$lr_j(r) = 2 \left( -\frac{r}{2} (\log|\Sigma_j^0| - \log|\Sigma_{j+1}^0|) - \frac{1}{2} \sum_{t=\tau_j^0+1}^{\tau_j^0+r} (y_t - x_t' \beta_j^0) (\Sigma_j^0)^{-1} (y_t - x_t' \beta_j^0) - (y_t - x_t' \beta_{j+1}^0) (\Sigma_{j+1}^0)^{-1} (y_t - x_t' \beta_{j+1}^0) \right) \quad \text{for } r > 0.$$

Then letting  $s = v_T^2(\tau_j - \tau_j^0)$ , with  $v_T$  defined in Assumption 7, the proof of Theorem 3 in Qu and Perron (2007b) shows that for  $s \leq 0$ ,

$$lr_j \left( \left[ \frac{s}{v_T^2} \right] \right) \Rightarrow 2 \left( -\frac{|s|}{2} \Xi_{1,j} + \Lambda_{1,j} W_{1,j}(s) \right), \quad (\text{A.2})$$

and for  $s > 0$ ,

$$lr_j \left( \left[ \frac{s}{v_T^2} \right] \right) \Rightarrow 2 \left( -\frac{|s|}{2} \Xi_{2,j} + \Lambda_{2,j} W_{2,j}(s) \right), \quad (\text{A.3})$$

where

$$\Lambda_{1,j} = \left( \frac{1}{4} \text{vec}(A_{1,j})' \Omega_{1,j}^0 \text{vec}(A_{1,j}) + \delta_j' \Pi_{1,j} \delta_j \right)^{1/2}, \quad (\text{A.4})$$

$$\Lambda_{2,j} = \left( \frac{1}{4} \text{vec}(A_{2,j})' \Omega_{2,j}^0 \text{vec}(A_{2,j}) + \delta_j' \Pi_{2,j} \delta_j \right)^{1/2}, \quad (\text{A.5})$$

$$\Xi_{1,j} = \left( \frac{1}{2} \text{tr}(A_{1,j}^2) + \delta_j' Q_{1,j} \delta_j \right), \quad (\text{A.6})$$

$$\Xi_{2,j} = \left( \frac{1}{2} \text{tr}(A_{2,j}^2) + \delta_j' Q_{2,j} \delta_j \right). \quad (\text{A.7})$$

Note that  $W_{1,j}(0) = W_{2,j}(0) = 0$  because  $W_{1,j}(s)$  and  $W_{2,j}(s)$  are independent and starting at  $s = 0$ .

Qu and Perron (2007a) derive a Bai-type distribution of  $\hat{\tau} - \tau_0$  by taking the arg max of (A.2) and (A.3) over  $C_M$  and using the continuous mapping theorem. Here, instead, we are deriving the distribution of the likelihood ratio by taking the max of (A.2) and (A.3) over  $C_M$ . Thus, under the null hypothesis  $H_0: \tau_j = \tau_j^0$ , we have

$$\text{LR}_j(\tau_j^0) \Rightarrow \max_s \begin{cases} 2\left(-\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_j(s)\right) & \text{for } s \leq 0, \\ 2\left(-\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_j(s)\right) & \text{for } s > 0, \end{cases}$$

where we can simplify this expression to relate it to a known distribution from Bhattacharya and Brockwell (1976). Let  $\text{LR}_j(\tau_j^0) = \xi = \max[\xi_1, \xi_2]$ , where  $\xi_1 = \max_{s \leq 0} 2\left(-\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_j(s)\right)$  and  $\xi_2 = \max_{s > 0} 2\left(-\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_j(s)\right)$ . By a change in variables  $s = (\Lambda_{1,j}^2/\Xi_{1,j}^2)v$  and the distributional equality with  $W(a^2x) \equiv aW(x)$ , for  $s \leq 0$ ,

$$\begin{aligned} \xi_1 &= \sup_{s \leq 0} 2\left(-\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_j(s)\right) \\ &= \max_{v \leq 0} \frac{\Lambda_{1,j}^2}{\Xi_{1,j}} 2\left(-\frac{|v|}{2} + W_j(v)\right) = 2\omega_{1,j} \times \bar{\xi}_1, \end{aligned} \tag{A.8}$$

where  $\bar{\xi}_1 = \max_{v \leq 0} \left(-\frac{|v|}{2} + W_j(v)\right)$  and

$$\frac{\Lambda_{1,j}^2}{\Xi_{1,j}} = \frac{\Lambda_{1,j}^2 v_T^2}{\Xi_{1,j} v_T^2} = \frac{\Gamma_{1,j}^2}{\Psi_{1,j}} \equiv \omega_{1,j}.$$

Similarly, for  $s > 0$  with  $s = (\Lambda_{2,j}^2/\Xi_{2,j}^2)v$ ,

$$\begin{aligned} \xi_2 &= \max_{s > 0} 2\left(-\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_j(s)\right) \\ &= \max_{v > 0} \frac{\Lambda_{2,j}^2}{\Xi_{2,j}} 2\left(-\frac{|v|}{2} + W_j(v)\right) = 2\omega_{2,j} \times \bar{\xi}_2, \end{aligned} \tag{A.9}$$

where  $\bar{\xi}_2 = \max_{v > 0} \left(-\frac{|v|}{2} + W_j(v)\right)$  and

$$\frac{\Lambda_{2,j}^2}{\Xi_{2,j}} = \frac{\Lambda_{2,j}^2 v_T^2}{\Xi_{2,j} v_T^2} = \frac{\Gamma_{2,j}^2}{\Psi_{2,j}} \equiv \omega_{2,j}.$$

Thus, we have the simplified expression for the distribution of the likelihood ratio under the null hypothesis:

$$\text{LR}_j(\tau_j^0) \Rightarrow \max_s \begin{cases} 2\omega_{1,j} \left(-\frac{|v|}{2} + W_j(v)\right) & \text{for } v \leq 0 \\ 2\omega_{2,j} \left(-\frac{|v|}{2} + W_j(v)\right) & \text{for } v > 0. \end{cases}$$

Bhattacharya and Brockwell (1976) show that  $\bar{\xi}_1$  and  $\bar{\xi}_2$  in (A.8) and (A.9) are independent and identically distributed exponential random variables with respective distribution functions  $P(\bar{\xi}_1 \leq x) = 1 - \exp(-x)$  for  $x \leq 0$  and  $P(\bar{\xi}_2 \leq x) = 1 - \exp(-x)$  for  $x > 0$ . Thus,

$$\begin{aligned} P(\xi \leq x) &= P(\max[2\omega_{1,j}\bar{\xi}_1, 2\omega_{2,j}\bar{\xi}_2] \leq x) \\ &= P(2\omega_{1,j}\bar{\xi}_1 \leq x)P(2\omega_{2,j}\bar{\xi}_2 \leq x) \\ &= P\left(\bar{\xi}_1 \leq \frac{x}{2\omega_{1,j}}\right)P\left(\bar{\xi}_2 \leq \frac{x}{2\omega_{2,j}}\right) \\ &= \left(1 - \exp\left(-\frac{x}{2\omega_{1,j}}\right)\right)\left(1 - \exp\left(-\frac{x}{2\omega_{2,j}}\right)\right). \end{aligned}$$

Then using the distribution of the profile likelihood ratio for the break date  $\tau_j$ , we can construct a  $1 - \alpha$  confidence set  $C_{j,1-\alpha} = \{\tau_j \mid \text{LR}_j(\tau_j) \leq \kappa_{\alpha,j}\}$  by inverting the  $\alpha$ -level likelihood ratio test. The probability of coverage  $C_{j,1-\alpha}$  for any  $\tau_j^0$  is given by  $P_{\tau_j^0}(\tau_j^0 \in C_{j,1-\alpha})$ , where we can easily calculate a critical value  $\kappa_{\alpha,j}$  such that

$$\begin{aligned} P_{\tau_j^0}(\tau_j^0 \in C_{j,1-\alpha}) &= (1 - \exp(-\kappa_{\alpha,j}/2\omega_{1,j}))\left(1 - \exp(-\kappa_{\alpha,j}/2\omega_{2,j})\right) \\ &= 1 - \alpha. \end{aligned} \tag{A.10}$$

Note that  $\kappa_{\alpha,j}$  will be unique because for all  $\kappa > 0$ , the CDF is a strictly increasing function  $\frac{d(1 - \exp(-\kappa/2\omega_{1,j}))(1 - \exp(-\kappa/2\omega_{2,j}))}{d\kappa} > 0$ .  $\square$

LEMMA 1. *Under the null hypothesis  $H_0 : \tau = \tau_0$ , if  $lr(\hat{\tau} - \tau_0) \Rightarrow \bar{\xi} = \max_v(-\frac{1}{2}|v| + W(v))$  for  $v \in (-\infty, \infty)$ , then  $E_{\tau_0}[\lambda\{\tau \mid lr(\hat{\tau} - \tau) \leq x\}] = 4(1 - \exp(-x))\{x - \frac{1}{2}(1 - \exp(-x))\}$ , where  $\lambda$  denotes a Lebesgue measure.*

PROOF. As shown in Bhattacharya and Brockwell (1976), the CDF of  $\bar{\xi} = \max_v(-\frac{1}{2}|v| + W(v))$  is given by  $P(\bar{\xi} \leq x) = (1 - \exp(-x))^2$ . Then letting  $C_{1-\alpha} = \{\tau \mid lr(\hat{\tau} - \tau) \leq \kappa_\alpha\}$ , Siegmund (1986) shows that the expected length for a  $1 - \alpha$  confidence set  $C_{1-\alpha}$  is given by

$$\begin{aligned} E_{\tau_0}[\lambda\{C_{1-\alpha}\}] &= E_{\tau_0}[\lambda\{\tau \mid \tau \in C_{1-\alpha}\}] \\ &= \int_{-\infty}^{\infty} P_{\tau_0}(\tau \in C_{1-\alpha}) d\tau \\ &= 4(1 - \alpha)^{1/2} \left\{ -\log[1 - (1 - \alpha)^{1/2}] - \frac{1}{2}(1 - \alpha)^{1/2} \right\}. \end{aligned} \tag{A.11}$$

See Siegmund (1986) for more details.

Because we can find a critical value  $\kappa_\alpha$  such that

$$P(\bar{\xi} \leq \kappa_\alpha) = (1 - \exp(-\kappa_\alpha))^2 = 1 - \alpha,$$

it implies that

$$\kappa_\alpha = -\log[1 - (1 - \alpha)^{1/2}]. \quad (\text{A.12})$$

Then, by substituting (A.12) into (A.11), we can express the expected length for a  $1 - \alpha$  confidence set as a function of the critical value  $\kappa_\alpha$  rather than the level  $1 - \alpha$  as

$$E_{\tau_j^0}[\lambda\{C_{1-\alpha}\}] = 4(1 - \exp(-\kappa_\alpha)) \left\{ \kappa_\alpha - \frac{1}{2}(1 - \exp(-\kappa_\alpha)) \right\}. \quad (\text{A.13})$$

□

**PROOF OF PROPOSITION 2.** For the general case, as in our setup under Assumptions 1–8, first consider the period before the true  $j$ th break date,  $\tau_j - \tau_j^0 \leq 0$  (i.e.,  $v \leq 0$ ). Given a critical value  $\kappa_{\alpha,j}$ , the expected length of a  $1 - \alpha$  confidence set in the segment  $\tau_j - \tau_j^0 \leq 0$  can be shown to be

$$\begin{aligned} & E_{\tau_j^0}[\lambda\{\tau_j \mid \text{LR}_j(\tau_j) \leq \kappa_{\alpha,j}, \hat{\tau}_j - \tau_j \leq 0\}] \\ &= E_{\tau_j^0} \left[ \lambda \left\{ \tau_j \mid \frac{\text{LR}_j(\tau_j)}{2\omega_{1,j}} \leq (\kappa_{\alpha,j}/2\omega_{1,j}), \hat{\tau}_j - \tau_j \leq 0 \right\} \right] \\ &= \underbrace{(\Gamma_{1,j}^2/\Psi_{1,j}^2)}_{(i)} \\ & \quad \times \underbrace{2(1 - \exp(-\kappa_{\alpha,j}/2\omega_{1,j})) \left\{ \kappa_{\alpha,j}/2\omega_{1,j} - \frac{1}{2}(1 - \exp(-\kappa_{\alpha,j}/2\omega_{1,j})) \right\}}_{(ii)}. \end{aligned} \quad (\text{A.14})$$

The expression (i) in the third line of (A.14) is used for rescaling because the expected length of the confidence set is measured on  $v \in (-\infty, 0]$  and

$$\begin{aligned} \tau_j - \tau_j^0 &= r = s/v_T^2 \\ &= (\Lambda_{1,j}^2/\Xi_{1,j}^2)v/v_T^2 \\ &= (\Lambda_{1,j}^2v_T^2/\Xi_{1,j}^2v_T^4)v \\ &= (\Gamma_{1,j}^2/\Psi_{1,j}^2)v. \end{aligned} \quad (\text{A.15})$$

Note that from Proposition 1, the second line in (A.14) implies that

$$\frac{\text{LR}_j(\tau_j)}{2\omega_{1,j}} \Rightarrow \bar{\xi} = \max_v \left( -\frac{1}{2}|v| + W_j(v) \right) \quad \text{for } v \leq 0. \quad (\text{A.16})$$

Thus, the expression (ii) in the fourth line of (A.14) is calculated for  $P(\bar{\xi} \leq \frac{\kappa_{\alpha,j}}{2\omega_{1,j}})$  by substituting the critical value  $\kappa_{\alpha,j}/2\omega_{1,j}$  into half of the expected length in Lemma 1 given that we are considering  $v \leq 0$ . The expected length for  $v > 0$  is calculated in a similar fashion such that the expected length for the entire  $1 - \alpha$  likelihood-ratio-based confi-

dence set is given by

$$\begin{aligned} & 2(\Gamma_{1,j}^2/\Psi_{1,j}^2)(1 - \exp(-\kappa_{\alpha,j}/2\omega_{1,j})) \left\{ \kappa_{\alpha,j}/2\omega_{1,j} - \frac{1}{2}(1 - \exp(-\kappa_{\alpha,j}/2\omega_{1,j})) \right\} \\ & + 2(\Gamma_{2,j}^2/\Psi_{2,j}^2)(1 - \exp(-\kappa_{\alpha,j}/2\omega_{2,j})) \\ & \times \left\{ \kappa_{\alpha,j}/2\omega_{2,j} - \frac{1}{2}(1 - \exp(-\kappa_{\alpha,j}/2\omega_{2,j})) \right\}. \end{aligned}$$

Note that as either  $\omega_{1,j}$  or  $\omega_{2,j}$  gets larger (i.e., the magnitude of a structural break is larger), the expected length becomes shorter because there is more precise information about the timing of the structural break.  $\square$

**PROOF OF COROLLARY 1.** If there is no break in variance,  $\Sigma_j = \Sigma$  for all  $j$  and  $B_{1,j} = B_{2,j} = 0$ . In addition, if the errors form a martingale difference sequence,  $\Pi_{1,j} = Q_{1,j}$  and  $\Pi_{2,j} = Q_{2,j}$ . From these simplifications,  $\omega_{1,j} = \omega_{2,j} = 1$ ,  $(\frac{\Gamma_{1,j}}{\Psi_{1,j}})^2 = \frac{1}{\Delta\beta_j' Q_1 \Delta\beta_j}$ , and  $(\frac{\Gamma_{2,j}}{\Psi_{2,j}})^2 = \frac{1}{\Delta\beta_j' Q_2 \Delta\beta_j}$ . Then, by substituting these values into the critical value and the expected length in Proposition 1, we can find the results in Corollary 1. The results in Remarks 1 and 2 follow in the same way.  $\square$

**PROOF OF COROLLARY 2.** If there is no break in conditional mean,  $\Delta\beta_j = 0$  and, in addition, if the standardized errors,  $\eta_t$ , are identically Normally distributed,  $\eta_t \eta_t'$  has a Wishart distribution with  $\text{var}(\text{vec}(\eta_t \eta_t')) = I_{n^2} + K_n$ , where  $K_n$  is the commutation matrix. Then  $\Omega_{1,j} = \Omega_{2,j} = \Omega = I_{n^2} + K_n$ . Furthermore, because  $K_n$  is an idempotent matrix,

$$\begin{aligned} & \text{vec}(B_{1,j})' \Omega^0 \text{vec}(B_{1,j})/4 \\ & = \text{vec}(B_{1,j})' (I_{n^2} + K_n) \text{vec}(B_{1,j})/4 \\ & = \text{vec}(B_{1,j})' \text{vec}(B_{1,j})/2. \end{aligned}$$

Thus,

$$\begin{aligned} \omega_{1,j} & = \frac{\Gamma_{1,j}^2}{\Psi_{1,j}} \\ & = \frac{\frac{1}{4} \text{vec}(B_{1,j})' \Omega_{1,j}^0 \text{vec}(B_{1,j})}{\frac{1}{2} \text{tr}(B_{1,j}^2)} \\ & = \frac{\frac{1}{2} \text{vec}(B_{1,j})' \text{vec}(B_{1,j})}{\frac{1}{2} \text{tr}(B_{1,j}^2)} \\ & = 1 \end{aligned}$$

because  $\text{vec}(B_{1,j})' \text{vec}(B_{1,j}) = \text{tr}(B_{1,j}^2)$ . Similarly,  $\omega_{2,j} = 1$ . Then  $\frac{\Gamma_{1,j}^2}{\Psi_{1,j}^2} = \frac{2}{\text{tr}(B_{1,j}^2)}$  and  $\frac{\Gamma_{2,j}^2}{\Psi_{2,j}^2} = \frac{2}{\text{tr}(B_{2,j}^2)}$ .  $\square$

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