Supplement to "Avoiding the curse of dimensionality in dynamic stochastic games": Online appendix

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A1. Entry and exit

Below we show how to add entry and exit to the continuous-time quality ladder model in Section 4 of the main paper. Recall that $\omega^i \in \{1, \dots, M\}$ describes the quality of firm i's product. To model entry and exit, we add $M+1$ to the set of firm i's feasible states and assume that $\omega^i = M+1$ designates firm i as being inactive in the product market game. The state space thus becomes $\Omega = \{1, \dots, M, M+1\}^N$. Once an incumbent firm exits the industry, it transits from state $\omega^i \neq M+1$ to state $(\omega')^i = M+1$. It then becomes a potential entrant that, upon entry, transits from state $\omega^i = M+1$ to state $(\omega')^i \neq M+1$. The transitions are under the control of firms. Specifically, incumbent firm i's action $x^i = (x^{i,1}, x^{i,2})$ is now a vector instead of a scalar, where $x^{i,1}$ is incumbent firm i's investment in quality improvements and $x^{i,2}$ governs exit. It is natural to assume that a potential entrant cannot invest so as to improve the quality of its product before it has actually entered the industry. Thus potential entrant i's action is the scalar $x^{i,2}$ that governs entry. The set of feasible actions is

$$\mathbb{X}^i(\omega) = \begin{cases} [0, \bar{x}^1] \times [0, \bar{x}^2], & \omega^i \neq M+1, \\ [0, \bar{x}^2], & \omega^i = M+1, \end{cases} \tag{A1}$$

where the first line pertains to an incumbent firm and the second line pertains to a potential entrant.

As we have noted in Section 2.1, discrete actions such as entry and exit in EP's framework are computationally challenging. In the remainder of this section, we outline two possibilities for adding entry and exit to the continuous-time quality ladder model in Section 4 that avoid this difficulty. In each case, entry and exit are governed by $x^{i,2}$, a continuous action. The interpretation of $x^{i,2}$ varies with the formulation of entry and exit.

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A1.1 *Entry/exit intensity*

In our first formulation, $x^{i,2} \in [0, \bar{x}^2]$ is firm i's "exit intensity" if it is an incumbent firm or its "entry intensity" if it is a potential entrant. The exit (entry) intensity $x^{i,2}$ translates into a hazard rate $h^2(x^{i,2})$ of exiting (entering) the industry. If an incumbent firm exits the industry, it receives a scrap value $\kappa - x^{i,2}$. Note that we make the scrap value a decreasing function of the exit intensity. That is, if a firm is in a hurry to exit, it receives less for its assets. Hence, $x^{i,2}$ can be thought of as reducing the firm's reservation price for selling its assets.[1] Conversely, if a potential entrant enters the industry, it pays a setup cost $\kappa^e + x^{i,2}$, an increasing function of the entry intensity.

The discrete-time analog to our first formulation of entry and exit has an incumbent firm (potential entrant) choose an exit (entry) intensity in each period that translates into a probability of exiting (entering) the industry. The scrap value (setup cost) is a decreasing (increasing) function of the exit (entry) intensity and thus indirectly also of the exit (entry) probability. The computational difficulty with mixed entry/exit strategies arises because an incumbent firm (potential entrant) that is indifferent between exiting and not exiting (entering and not entering) may choose any value for the exit (entry) probability. We break this indifference and avoid the need for mixed entry/exit strategies by assuming that a higher probability is more costly than a lower probability.

Incumbent firm The details of entry and exit are as follows: Suppose first that firm i is an incumbent firm (i.e., $\omega^i \neq M+1$). Jumps in firm i's state occur according to a Poisson process with hazard rate

$$\phi^i(x, \omega) = h^1(x^{i,1}) + \delta + h^2(x^{i,2}), \tag{A2}$$

where $h^1(x^{i,1})$ is the hazard rate of an investment success, δ is the depreciation hazard, and $h^2(x^{i,2})$ is the hazard rate of exiting the industry. When a jump occurs, firm i's state changes according to the transition probability

$$f^i((\omega')^i|\omega, x) = \begin{cases} \dfrac{h^1(x^{i,1})}{\phi^i(x, \omega)}, & (\omega')^i = \omega^i + 1, \\ \dfrac{\delta}{\phi^i(x, \omega)}, & (\omega')^i = \omega^i - 1, \\ \dfrac{h^2(x^{i,2})}{\phi^i(x, \omega)}, & (\omega')^i = M + 1, \end{cases} \tag{A3}$$

if $\omega^i \in \{2, \dots, M-1\}$.[2] Note that the last line captures the possibility of exit. Upon exit, the incumbent firm receives a scrap value and the instantaneous change in wealth is

$$\Phi^i(x, \omega^i, \omega^{-i}, M+1, (\omega')^{-i}) = \kappa - x^{i,2}.$$

[1] More elaborate specifications are possible, for example, the scrap value of a firm's assets may depend on its state as in $\kappa(\omega^i) - x^{i,2}$, where $\kappa(\omega^i)$ is a (presumably increasing) function of ω^i.

[2] As discussed in Section 4, if $\omega^i = 1$ or if $\omega^i = M$, then the hazard rate and the transition probability need to be adjusted.

Potential entrant Suppose next that firm i is a potential entrant (i.e., $\omega^i = M+1$). Jumps in firm i's state occur according to a Poisson process with hazard rate

$$\phi^i(x, M+1, \omega^{-i}) = h^2(x^{i,2}), \tag{A4}$$

where $h^2(x^{i,2})$ is the hazard rate of entering the industry. When a jump occurs, firm i's state changes according to the transition probability

$$f^i(\omega^e | M+1, \omega^{-i}, x) = 1, \tag{A5}$$

where $\omega^e \in \{1, \dots, M\}$ is the (exogenously given) initial quality of a firm's product. Upon entry, the potential entrant pays a setup cost and the instantaneous change in wealth is

$$\Phi^i(x, M+1, \omega^{-i}, \omega^e, (\omega')^{-i}) = -(\kappa^e + x^{i,2}).$$

Finally, since a potential entrant is inactive in the product market game, its payoff flow is

$$\pi^i(x, M+1, \omega^{-i}) = 0.$$

Bellman equation Suppose first that firm i is an incumbent firm (i.e., $\omega^i \neq M+1$). Its Bellman equation is

$$\begin{aligned}
\rho V^i(\omega) = \max_{(x^{i,1}, x^{i,2}) \in \mathbb{X}^i(\omega)} \; & \pi^i(\omega) - x^{i,1} \\
& + h^1(x^{i,1}) \cdot 1(\omega^i \neq M)(V^i(\omega^i + 1, \omega^{-i}) - V^i(\omega)) \\
& + \sum_{\{j \neq i : \omega^j \neq M+1\}} h^1(X^{j,1}(\omega)) \cdot 1(\omega^j \neq M)(V^i(\omega^j + 1, \omega^{-j}) - V^i(\omega)) \\
& + \sum_{\{j : \omega^j \neq M+1\}} \delta \cdot 1(\omega^j \neq 1)(V^i(\omega^j - 1, \omega^{-j}) - V^i(\omega)) \\
& + h(x^{i,2})(\kappa - x^{i,2} + V^i(M+1, \omega^{-i}) - V^i(\omega)) \\
& + \sum_{\{j \neq i : \omega^j \neq M+1\}} h^2(X^{j,2}(\omega))(V^i(M+1, \omega^{-j}) - V^i(\omega)) \\
& + \sum_{\{j : \omega^j = M+1\}} h^2(X^{j,2}(\omega))(V^i(\omega^e, \omega^{-j}) - V^i(\omega)),
\end{aligned} \tag{A6}$$

where $1(\cdot)$ is the indicator function. In the first line, we use the fact that the payoff function $\pi^i(x, \omega)$ is additively separable in incumbent firm i's investment in quality improvements to write it as $\pi^i(\omega) - x^{i,1}$. The second and third lines pertain to an investment success and the fourth to a depreciation shock. The fifth and sixth lines pertain to exit and the seventh pertains to entry.

The maximization problem on the right-hand side of its Bellman equation determines the investment in quality improvements and the exit intensity of incumbent firm i. For concreteness, let $h^1(x^{i,1}) = \frac{\alpha x^{i,1}}{1+\alpha x^{i,1}}$ be the success hazard as in Section 4.2 and let

$h^2(x^{i,2}) = (x^{i,2})^\eta$ be the hazard rate for exiting the industry. Starting with investment, the first-order condition for an interior solution is

$$-1 + \frac{\alpha}{(1+\alpha x^{i,1})^2} \cdot 1(\omega^i \neq M)(V^i(\omega^i + 1, \omega^{-i}) - V^i(\omega)) = 0.$$

Hence, the investment in quality improvements of incumbent firm i in state ω is

$$x^{i,1}(\omega) = \begin{cases} \min\left\{\bar{x}^1, \frac{1}{\alpha}\left(-1 + \sqrt{\max\{1, \alpha(V^i(\omega^i+1, \omega^{-i}) - V^i(\omega))\}}\right)\right\}, \\ \qquad \omega^i \neq M, \\ 0, \quad \omega^i = M. \end{cases} \tag{A7}$$

Turning to exit, the first-order condition for an interior solution is

$$\eta(x^{i,2})^{\eta-1}(\kappa - x^{i,2} + V^i(M+1, \omega^{-i}) - V^i(\omega)) - (x^{i,2})^\eta = 0.$$

Hence, the exit intensity of incumbent firm i in state ω is

$$x^{i,2}(\omega) = \max\left\{0, \min\left\{\bar{x}^2, \frac{\eta}{1+\eta}(\kappa + V^i(M+1, \omega^{-i}) - V^i(\omega))\right\}\right\}. \tag{A8}$$

Suppose next that firm i is a potential entrant (i.e., $\omega^i = M + 1$). Its Bellman equation is

$$\begin{aligned} \rho V^i(\omega) = \max_{x^{i,2} \in \mathbb{X}^i(\omega)} & \sum_{\{j:\omega^j \neq M+1\}} h^1(X^{j,1}(\omega)) \cdot 1(\omega^j \neq M)(V^i(\omega^j + 1, \omega^{-j}) - V^i(\omega)) \\ & + \sum_{\{j:\omega^j \neq M+1\}} \delta \cdot 1(\omega^j \neq 1)(V^i(\omega^j - 1, \omega^{-j}) - V^i(\omega)) \\ & + \sum_{\{j:\omega^j \neq M+1\}} h^2(X^{j,2}(\omega))(V^i(M+1, \omega^{-j}) - V^i(\omega)) \\ & + h(x^{i,2})(-(\kappa^e + x^{i,2}) + V^i(\omega^e, \omega^{-i}) - V^i(\omega)) \\ & + \sum_{\{j \neq i:\omega^j = M+1\}} h^2(X^{j,2}(\omega))(V^i(\omega^e, \omega^{-j}) - V^i(\omega)). \end{aligned} \tag{A9}$$

The maximization problem on the right-hand side of its Bellman equation determines the entry intensity of potential entrant i. The first-order condition for an interior solution is

$$\eta(x^{i,2})^{\eta-1}(-(\kappa^e + x^{i,2}) + V^i(\omega^e, \omega^{-i}) - V^i(M+1, \omega^{-i})) - (x^{i,2})^\eta = 0.$$

Hence, the entry intensity of potential entrant i in state ω is

$$\begin{aligned} & x^{i,2}(\omega) \\ & \quad = \max\left\{0, \min\left\{\bar{x}^2, \frac{\eta}{1+\eta}(-\kappa^e + V^i(\omega^e, \omega^{-i}) - V^i(M+1, \omega^{-i}))\right\}\right\}. \end{aligned} \tag{A10}$$

Existence The existence of a Markov perfect equilibrium in pure strategies follows from Proposition 1. Assumption 1 is satisfied in light of equation (A1).[3] Assumption 2 requires $\rho > 0$ and our functional forms satisfy Assumption 3. Finally, it is obvious from equations (A7), (A8), and (A10) that firm i's maximization problem always has a unique solution as required by Assumption 4.

A1.2 *Random setup costs/scrap values*

In our second formulation of entry and exit, $x^{i,2} \in [0, 1]$ is the probability that firm i accepts an offer to sell assets and exit the industry if it is an incumbent firm or the probability that firm i accepts an offer to buy assets and enter the industry if it is a potential entrant. If incumbent firm i exits the industry, it receives a scrap value κ^i drawn randomly from a distribution $G(\cdot)$ with continuous and positive density. Scrap values are independently and identically distributed and privately known, that is, while incumbent firm i learns its own scrap value κ^i, those of its rivals remain unknown to it. Note that since the exit decision is conditioned on the privately known scrap value, it is a random variable from the perspective of other firms. Rather than actively searching for a buyer for its assets as in our first formulation, we assume that buyers arrive at incumbent firm i with hazard rate λ. Upon arrival, the buyer makes a take-it-or-leave-it offer. If the offered scrap value κ^i is above a threshold $\bar{\kappa}^i$, then incumbent firm i accepts the offer and exits the industry; otherwise, it rejects the offer and remains in the industry. This decision rule can be represented either with the cutoff scrap value $\bar{\kappa}^i$ itself or with the probability $x^{i,2}$ of incumbent firm i accepting the offer because $x^{i,2} = \int 1(\kappa^i \geq \bar{\kappa}^i)\, dG(\kappa^i) = 1 - G(\bar{\kappa}^i)$, where $1(\cdot)$ is the indicator function, is equivalent to $\bar{\kappa}^i = G^{-1}(1 - x^{i,2})$.[4]

Turning from exit to entry, if potential entrant i enters the industry, it pays a setup cost $\kappa^{e,i}$ drawn randomly from a distribution $G^e(\cdot)$ with continuous and positive density. Like scrap values, setup costs are independently and identically distributed and privately known. Potential entrant i has to acquire assets to enter the industry, and sellers of assets arrive at potential entrant i with hazard rate λ. Upon arrival the seller makes a take-it-or-leave-it offer that potential entrant i accepts if it is below a threshold $\bar{\kappa}^{e,i}$.

The discrete-time analog to our second formulation of entry and exit is developed in Doraszelski and Satterthwaite (2010). There an incumbent firm (potential entrant) randomly draws a scrap value (setup cost) in each period. These draws are independent across firms and periods. Its scrap value (setup cost) is known to the firm but unknown to its rivals. By building on Harsanyi's (1973) insight that a perturbed game of incomplete information can purify the mixed-strategy equilibria of an underlying game of complete information, Doraszelski and Satterthwaite (2010) avoided the need for mixed entry/exit strategies in EP's framework.

While an incumbent firm chooses an exit intensity at each point in time in our first formulation of entry and exit, in our second formulation, an incumbent firm decides whether to accept or reject an offer the instant the offer is made. If instead we allowed

[3] In practice, we set the upper bounds $\bar{x}^1$ and $\bar{x}^2$ large enough to not be binding.

[4] If the support of $G(\cdot)$ is bounded, we define $G^{-1}(0)$ ($G^{-1}(1)$) to be the infimum (supremum) of the support.

the incumbent firm to "sit" on the offer, then its rivals may make inferences about the offer, that is otherwise unknown to them, from observing the reaction of the incumbent firm to a change in the state of the industry. Hence, firms' beliefs about each others' offers become part of the state space.[5] The advantage of our assumption that offers are rejected or accepted right away is thus that the model remains tractable.

The disadvantage is that the possible timings of entry and exit are exogenous (although, of course, whether entry and exit actually take place is endogenous). Again this is similar to the discrete-time version of EP's framework: There every period a firm receives another draw of its setup cost or scrap value and thus another opportunity to enter or exit. Here a firm receives another offer and thus another opportunity to enter or exit *on average* every $\frac{1}{\lambda}$ periods. Moreover, in the continuous-time model, we can ensure that these opportunities arise more frequently by making λ larger. For example, if the unit of time is a year, then the opportunities to enter or exit arise on average once per year if $\lambda = 1$ and once per day if $\lambda = 365$.

Incumbent firm The details of entry and exit are as follows: Suppose first that firm i is an incumbent firm (i.e., $\omega^i \neq M+1$). $\phi^i(x,\omega)$ and $f^i((\omega')^i|\omega, x)$ are as in equations (A2) and (A3), respectively, except that the hazard rate of exiting the industry is given by $h^2(x^{i,2}) = \lambda x^{i,2}$, because exit requires first that a buyer arrives (hazard rate λ) and second that the offer is accepted (probability $x^{i,2}$). Upon exit, the instantaneous change in wealth is

$$\Phi^i(x, \omega^i, \omega^{-i}, M+1, (\omega')^{-i}) = \frac{1}{x^{i,2}} \int_{\kappa^i \geq G^{-1}(1-x^{i,2})} \kappa^i \, dG(\kappa^i),$$

because an optimizing incumbent cares about the expectation of the scrap value conditional on receiving it,

$$\begin{aligned} \mathrm{E}\{\kappa^i | \kappa^i \geq \bar{\kappa}^i\} &= \frac{1}{1-G(\bar{\kappa}^i)} \int_{\kappa^i \geq \bar{\kappa}^i} \kappa^i \, dG(\kappa^i) \\ &= \frac{1}{x^{i,2}} \int_{\kappa^i \geq G^{-1}(1-x^{i,2})} \kappa^i \, dG(\kappa^i), \end{aligned}$$

rather than its unconditional expectation $\mathrm{E}\{\kappa^i\}$.

Potential entrant Suppose next that firm i is a potential entrant (i.e., $\omega^i = M+1$). $\phi^i(x, M+1, \omega^{-i})$ and $f^i(\omega^e | M+1, \omega^{-i}, x)$ are as in equations (A4) and (A5), respectively, except that the hazard rate of entering the industry is given by $h^2(x^{i,2}) = \lambda x^{i,2}$. Upon entry, the instantaneous change in wealth is

$$\Phi^i(x, M+1, \omega^{-i}, \omega^e, (\omega')^{-i}) = -\frac{1}{x^{i,2}} \int_{\kappa^{e,i} \leq G^{e,-1}(x^{i,2})} \kappa^{e,i} \, dG^e(\kappa^{e,i}),$$

[5] A similar difficulty arises in the discrete-time version of EP's framework if the assumption that the draws of scrap values and setup costs are independent across periods is relaxed.

because an optimizing entrant cares about the expectation of the setup cost conditional on paying it,

$$\begin{aligned}\mathrm{E}\{\kappa^{e,i}|\kappa^{e,i} \le \bar{\kappa}^{e,i}\} &= \frac{1}{G^e(\bar{\kappa}^i)} \int_{\kappa^{e,i} \le \bar{\kappa}^{e,i}} \kappa^{e,i}\, dG^e(\kappa^{e,i}) \\ &= \frac{1}{x^{i,2}} \int_{\kappa^{e,i} \le G^{e,-1}(x^{i,2})} \kappa^{e,i}\, dG^e(\kappa^{e,i}),\end{aligned}$$

rather than its unconditional expectation $\mathrm{E}\{\kappa^{e,i}\}$.

Bellman equation Suppose first that firm i is an incumbent firm (i.e., $\omega^i \neq M+1$). Its Bellman equation is similar to equation (A6) in our first formulation of entry and exit except that the fifth line is replaced by

$$\lambda x^{i,2}\left(\frac{1}{x^{i,2}} \int_{\kappa^i \ge G^{-1}(1-x^{i,2})} \kappa^i\, dG(\kappa^i) + V^i(M+1, \omega^{-i}) - V^i(\omega)\right).$$

The investment in quality improvements of incumbent firm i in state ω is given by equation (A7). Turning from investment to exit, the first-order condition

$$\begin{aligned}&-\lambda G^{-1}(1-x^{i,2})G'(G^{-1}(1-x^{i,2}))\frac{-1}{G'(G^{-1}(1-x^{i,2}))} \\ &\quad + \lambda(V^i(M+1, \omega^{-i}) - V^i(\omega)) = 0\end{aligned}$$

implies that the probability of incumbent firm i accepting the offer and exiting the industry in state ω is

$$x^{i,2}(\omega) = 1 - G(V^i(\omega) - V^i(M+1, \omega^{-i})). \tag{A11}$$

Suppose next that firm i is a potential entrant (i.e., $\omega^i = M+1$). Its Bellman equation is similar to equation (A9) except that the fourth line is replaced by

$$\lambda x^{i,2}\left(-\frac{1}{x^{i,2}} \int_{\kappa^{e,i} \le G^{e,-1}(x^{i,2})} \kappa^{e,i}\, dG^e(\kappa^{e,i}) + V^i(\omega^e, \omega^{-i}) - V^i(\omega)\right).$$

The first-order condition

$$\begin{aligned}&-\lambda G^{e,-1}(x^{i,2})G^{e\prime}(G^{e,-1}(x^{i,2}))\frac{1}{G^{e\prime}(G^{e,-1}(x^{i,2}))} \\ &\quad + \lambda(V^i(\omega^e, \omega^{-i}) - V^i(M+1, \omega^{-i})) = 0\end{aligned}$$

implies that the probability of potential entrant i accepting the offer and entering the industry in state ω is

$$x^{i,2}(\omega) = G^e(V^i(\omega^e, \omega^{-i}) - V^i(M+1, \omega^{-i})). \tag{A12}$$

Existence Similar to our first formulation of entry and exit, the existence of a Markov perfect equilibrium in pure strategies follows from Proposition 1. In particular, it is obvious from equations (A7), (A11), and (A12) that firm i's maximization problem always has a unique solution as required by Assumption 4.

REMARK. Both of the above formulations of entry and exit differ from the one proposed by PM1. In the background of their model is an infinite pool of potential entrants. Among these potential entrants, one is selected at random in each period and given a chance to enter the industry. The potential entrant is therefore short-lived and bases its entry decision solely on the value of immediate entry; it does not take into account the value of deferred entry. In addition, PM1 assumed that by exiting the industry, an incumbent firm de facto exits the game. In contrast, we assume that there is a fixed number of firms and that each firm may be either an incumbent firm or a potential entrant at any given point in time. Moreover, when exiting, the firm takes into account the possibility that it may enter the industry at some later point; conversely, when entering, the firm takes into account the possibility that it may exit the industry at some later point. Exiting is thus tantamount to “mothballing” and entering is tantamount to resuming operations. The advantage of this formulation of entry and exit is that it leads to a game with a finite and constant number of players. Whether one uses our formulation or the one proposed by PM1 is immaterial for the purposes of this paper since the computational advantages of continuous time are exactly the same in both.

A2. INDUSTRYWIDE DEPRECIATION SHOCK

PM1 assumed that firm i's state evolves according to the law of motion in equation (8) in the main paper. Below we reformulate the quality ladder model from Section 4 with an industrywide instead of a firm-specific depreciation shock—first in discrete time, then in continuous time.

Discrete-time model

Firm i's state ω^i represents the quality of its product in the present period. The quality of firm i's product in the subsequent period is governed by its investment $x^i \geq 0$ in quality improvements $\tau^i \in \{0, 1\}$ and by depreciation $\eta \in \{0, 1\}$. If the investment is successful, then the quality of firm i increases by one level. The probability of success is $\frac{\alpha x^i}{1+\alpha x^i}$, where $\alpha > 0$ is a measure of the effectiveness of investment. With probability $\delta \in [0, 1]$, the industry is hit by a depreciation shock and the qualities of all firms decrease by one level.

Conditional on the industry not being hit by a depreciation shock, an event denoted by $\eta = 0$, the quality of firm i's product changes according to the transition probability

$$\Pr^i((\omega')^i|\omega, x, \eta = 0) = \begin{cases} \dfrac{\alpha x^i}{1+\alpha x^i}, & (\omega')^i = \omega^i + 1, \\ \dfrac{1}{1+\alpha x^i}, & (\omega')^i = \omega^i, \end{cases}$$

if $\omega^i \in \{1, \dots, M-1\}$ and $\Pr^i(M|M, \omega^{-i}, x, \eta = 0) = 1$. Conditional on the industry being hit by a depreciation shock, an event denoted by $\eta = 1$, the quality of firm i's product changes according to the transition probability

$$\Pr^i((\omega')^i|\omega, x, \eta = 1) = \begin{cases} \dfrac{\alpha x^i}{1+\alpha x^i}, & (\omega')^i = \omega^i, \\ \dfrac{1}{1+\alpha x^i}, & (\omega')^i = \omega^i - 1, \end{cases}$$

if $\omega^i \in \{2, \dots, M\}$ and $\Pr^i(1|1, \omega^{-i}, x, \eta = 1) = 1$. Since the state-to-state transitions are conditionally independent, the law of motion is

$$\Pr(\omega'|\omega, x) = (1-\delta)\prod_{i=1}^{N} \Pr^i((\omega')^i|\omega, x, \eta = 0) + \delta \prod_{i=1}^{N} \Pr^i((\omega')^i|\omega, x, \eta = 1).$$

To see that the expectation over successor states consists of 2^{N+1} terms, note that it is the sum of two conditional expectations, each of which consists of 2^N terms.

Continuous-time model

To make the continuous- and discrete-time models easily comparable, we take the hazard rate for the investment project of firm i being successful to be $\frac{\alpha x^i}{1+\alpha x^i}$ and take the depreciation hazard to be δ.

Jumps in the state of the industry occur according to a Poisson process with hazard rate

$$\phi(x, \omega) = \sum_{i=1}^{N} \frac{\alpha x^i}{1+\alpha x^i} \cdot 1(\omega^i \neq M) + \delta \cdot 1(\omega \neq (1, \dots, 1)),$$

where $1(\cdot)$ is the indicator function. When a jump occurs, the state changes according to the transition probability

$$f(\omega'|\omega, x) = \begin{cases} \dfrac{\alpha x^1}{(1+\alpha x^1)\phi(x, \omega)}, & \omega' = (\omega^1 + 1, \omega^{-1}), \omega^1 \neq M, \\ \vdots & \\ \dfrac{\alpha x^N}{(1+\alpha x^N)\phi(x, \omega)}, & \omega' = (\omega^N + 1, \omega^{-N}), \omega^N \neq M, \\ \dfrac{\delta}{\phi(x, \omega)}, & \omega' = (\max\{\omega^1 - 1, 1\}, \dots, \max\{\omega^N - 1, 1\}), \\ & \omega \neq (1, \dots, 1). \end{cases}$$

Hence, we need to sum over a total of $N+1$ terms in the continuous-time model compared to 2^{N+1} in the discrete-time model.

A3. Stopping rules

Below we provide a more detailed comparison between the ad hoc and the adaptive stopping rule. Tables A1 and A2 assume prespecified tolerances of $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-8}$,

Table A1. Stopping rules.[a]

		Discrete Time					Continuous Time				
		Ad Hoc Rule		Adaptive Rule			Ad Hoc Rule		Adaptive Rule		
Number of Firms	Discount Factor	Terminal Iteration	Distance to Truth	Terminal Iteration	Distance to Truth	Convergence Factor	Terminal Iteration	Distance to Truth	Terminal Iteration	Distance to Truth	Convergence Factor
3	0.925	99	8.16 (−4)	120	7.95 (−5)	0.8962	131	2.43 (−3)	218	7.84 (−5)	0.9676
3	0.98	304	3.04 (−3)	414	9.26 (−5)	0.9694	313	9.93 (−3)	775	1.00 (−4)	0.9899
3	0.99	519	6.21 (−3)	785	9.46 (−5)	0.9848	455	2.01 (−2)	1483	1.27 (−4)	0.9937
3	0.995	923	1.26 (−2)	1551	9.36 (−5)	0.9926	589	4.05 (−2)	2778	1.92 (−4)	0.9953
6	0.925	99	8.16 (−4)	119	8.73 (−5)	0.8913	220	3.84 (−3)	370	8.42 (−5)	0.9777
6	0.98	387	3.03 (−3)	495	9.39 (−5)	0.9685	742	1.78 (−2)	1689	9.17 (−5)	0.9948
6	0.99	743	5.75 (−3)	985	9.53 (−5)	0.9836	1198	3.66 (−2)	3454	8.15 (−5)	0.9978
6	0.995	1362	1.12 (−2)	1906	9.40 (−5)	0.9916	1832	7.41 (−2)	6766	1.04 (−4)	0.9986
9	0.925	100	7.68 (−4)	120	8.20 (−5)	0.8962	232	4.45 (−3)	407	9.18 (−5)	0.9791
9	0.98	386	3.03 (−3)	493	9.42 (−5)	0.9681	1100	2.30 (−2)	2387	9.01 (−5)	0.9961
9	0.99	751	5.73 (−3)	991	9.45 (−5)	0.9835	1927	4.87 (−2)	5091	7.84 (−5)	0.9984
9	0.995	1469	1.11 (−2)	2009	9.40 (−5)	0.9916	3129	1.00 (−1)	10,358	8.10 (−5)	0.9992
12	0.925						227	4.73 (−3)	411	1.02 (−4)	0.9781
12	0.98						1276	2.58 (−2)	2751	8.89 (−5)	0.9966
12	0.99						2447	5.59 (−2)	6185	7.48 (−5)	0.9987
12	0.995						4217	1.16 (−1)	12,994	7.01 (−5)	0.9994

[a] Terminal iteration, distance to truth at terminal iteration, and estimated convergence factor are shown. The prespecified tolerance is 10^{-4}. Quality ladder model with $M = 9$ quality levels per firm. (k) is shorthand for $\times 10^{k}$.

TABLE A2. Stopping rules.[a]

		Discrete Time					Continuous Time				
		Ad Hoc Rule		Adaptive Rule			Ad Hoc Rule		Adaptive Rule		
Number of Firms	Discount Factor	Terminal Iteration	Distance to Truth	Terminal Iteration	Distance to Truth	Convergence Factor	Terminal Iteration	Distance to Truth	Terminal Iteration	Distance to Truth	Convergence Factor
3	0.925	182	8.16 (−8)	203	7.93 (−9)	0.8962	364	2.45 (−7)	446	9.62 (−9)	0.9611
3	0.98	594	3.05 (−7)	703	9.57 (−9)	0.9690	1238	9.88 (−7)	1700	9.85 (−9)	0.9901
3	0.99	1104	6.21 (−7)	1368	9.71 (−9)	0.9845	2320	2.01 (−6)	3394	9.91 (−9)	0.9951
3	0.995	2100	1.27 (−6)	2720	9.90 (−9)	0.9922	4343	4.05 (−6)	6780	9.97 (−9)	0.9975
6	0.925	182	7.65 (−8)	202	8.18 (−9)	0.8962	581	3.86 (−7)	726	9.55 (−9)	0.9747
6	0.98	673	3.06 (−7)	781	9.45 (−9)	0.9681	2395	1.78 (−6)	3325	9.90 (−9)	0.9944
6	0.99	1286	5.77 (−7)	1527	9.66 (−9)	0.9832	4593	3.66 (−6)	6762	9.97 (−9)	0.9973
6	0.995	2408	1.13 (−6)	2946	9.88 (−9)	0.9912	8729	7.41 (−6)	13,638	9.98 (−9)	0.9987
9	0.925	182	7.95 (−8)	202	8.48 (−9)	0.8962	647	4.45 (−7)	819	9.77 (−9)	0.9779
9	0.98	670	3.01 (−7)	776	9.66 (−9)	0.9681	3235	2.30 (−6)	4494	9.93 (−9)	0.9957
9	0.99	1289	5.74 (−7)	1527	9.74 (−9)	0.9830	6447	4.88 (−6)	9470	9.97 (−9)	0.9980
9	0.995	2509	1.12 (−6)	3043	9.85 (−9)	0.9912	12,452	1.00 (−5)	19,366	9.98 (−9)	0.9990
12	0.925						669	4.67 (−7)	855	9.63 (−9)	0.9793
12	0.98						3668	2.58 (−6)	5107	9.94 (−9)	0.9961
12	0.99						7637	5.59 (−6)	11,182	9.97 (−9)	0.9982
12	0.995						15,085	1.16 (−5)	23,305	9.99 (−9)	0.9991

[a]Terminal iteration, distance to truth at terminal iteration, and estimated convergence factor are shown. The prespecified tolerance is 10^{-8}. Quality ladder model with $M = 9$ quality levels per firm. (k) is shorthand for $\times 10^k$.

respectively. As can be seen, the adaptive rule outperforms the ad hoc rule in all cases. Moreover, its performance improves as the iterations progress. In particular, while the adaptive rule with $\varepsilon = 10^{-4}$ prematurely terminates the continuous-time algorithm in some cases, with $\varepsilon = 10^{-8}$ it always terminates once the distance to the true solution is less than the prespecified tolerance.

Two remarks are in order regarding the convergence factor. First, the discrete-time convergence factor is less than the discount factor β because we are using a Gauss–Seidel instead of a Gauss–Jacobi scheme to compute the equilibrium. Second, for any given N and β, the continuous-time convergence factor exceeds its discrete-time counterpart. This is in line with the "iteration penalty" of the continuous-time algorithm.[6]

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[6] Recall that a large convergence factor implies that convergence is slow.