# TECHNICAL APPENDIX I SOURCES OF MACROECONOMIC FLUCTUATIONS: A REGIME-SWITCHING DSGE APPROACH (NOT INTENDED FOR PUBLICATION) 

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In this appendix, we derive the optimizing decisions, describe the stationary equilibrium, and derive the log-linearized equilibrium conditions in the paper entitled "Sources of Macroeconomic Fluctuations: A Regime-Switching DSGE Approach" by Liu, Waggoner, and Zha.

## I. The optimizing decisions

I.1. Households' optimizing decisions. Each household chooses consumption, investment, new capital stock, capacity utilization, and next-period bond to solve the following utility maximizing problem:

$$
\begin{equation*}
\operatorname{Max}_{\left\{C_{t}, I_{t}, K_{t}, u_{t}, B_{t+1}\right\}} \mathrm{E} \sum_{t=0}^{\infty} \beta^{t} A_{t}\left\{\log \left(C_{t}-b C_{t-1}\right)-\frac{\psi}{1+\eta} L_{t+i}^{d}(h)^{1+\eta}\right\} \tag{1}
\end{equation*}
$$

subject to

$$
\begin{gather*}
\bar{P}_{t} C_{t}+\frac{\bar{P}_{t}}{Q_{t}}\left(I_{t}+a\left(u_{t}\right) K_{t-1}\right)+\mathrm{E}_{t} D_{t, t+1} B_{t+1} \leq W_{t}(h) L_{t}^{d}(h)+\bar{P}_{t} r_{k t} u_{t} K_{t-1}+\Pi_{t}+B_{t}+T_{t}  \tag{2}\\
K_{t}=\left(1-\delta_{t}\right) K_{t-1}+\left[1-S\left(\frac{I_{t}}{I_{t-1}}\right)\right] I_{t} \tag{3}
\end{gather*}
$$

Denote by $\mu_{t}$ the Lagrangian multiplier for the budget constraint (2) and by $\mu_{k t}$ the Lagrangian multiplier for the capital accumulation equation (3). The first order conditions for the utility-maximizing problem are given by

$$
\begin{align*}
A_{t} U_{c t} & =\mu_{t} \bar{P}_{t}  \tag{4}\\
D_{t, t+1} & =\beta \frac{\mu_{t+1}}{\mu_{t}}  \tag{5}\\
\frac{\mu_{t} \bar{P}_{t}}{Q_{t}} & =\mu_{k t}\left\{1-S\left(\lambda_{I t}\right)-S^{\prime}\left(\lambda_{I t}\right) \lambda_{I t}\right\}+\beta \mathrm{E}_{t} \mu_{k, t+1} S^{\prime}\left(\lambda_{I, t+1}\right)\left(\lambda_{I, t+1}\right)^{2}  \tag{6}\\
\mu_{k t} & =\beta \mathrm{E}_{t}\left[\mu_{k, t+1}\left(1-\delta_{t+1}\right)+\mu_{t+1} \bar{P}_{t+1} r_{k, t+1} u_{t+1}-\frac{\mu_{t+1} \bar{P}_{t+1}}{Q_{t+1}} a\left(u_{t+1}\right)\right] \tag{7}
\end{align*}
$$

$$
\begin{equation*}
r_{k t}=\frac{a^{\prime}\left(u_{t}\right)}{Q_{t}} \tag{8}
\end{equation*}
$$

where $\lambda_{I t} \equiv I_{t} / I_{t-1}$.
Let $q_{k t} \equiv Q_{t} \frac{\mu_{k t}}{\mu_{t} P_{t}}$ denote the shadow price of capital stock (in units of investment goods). Then, (4) and (6) imply that

$$
\begin{equation*}
\frac{1}{Q_{t}}=\frac{q_{k t}}{Q_{t}}\left\{1-S\left(\lambda_{I t}\right)-S^{\prime}\left(\lambda_{I t}\right) \lambda_{I t}\right\}+\beta \mathrm{E}_{t} \frac{q_{k, t+1}}{Q_{t+1}} \frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}} S^{\prime}\left(\lambda_{I, t+1}\right)\left(\lambda_{I, t+1}\right)^{2} \tag{9}
\end{equation*}
$$

Thus, in the absence of adjustment cost or in the steady-state equilibrium where $S\left(\lambda_{I}\right)=S^{\prime}\left(\lambda_{I}\right)=0$, we have $q_{k t}=1$. One can interpret $q_{k t}$ as Tobin's Q.

By eliminating the Lagrangian multipliers $\mu_{t}$ and $\mu_{k t}$, the capital Euler equation (7) can be rewritten as

$$
\begin{equation*}
\frac{q_{k t}}{Q_{t}}=\beta \mathrm{E}_{t} \frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}}\left[\left(1-\delta_{t+1}\right) \frac{q_{k, t+1}}{Q_{t+1}}+r_{k, t+1} u_{t+1}-\frac{a\left(u_{t+1}\right)}{Q_{t+1}}\right] \tag{10}
\end{equation*}
$$

The cost of acquiring a marginal unit of capital is $q_{k t} / Q_{t}$ today (in consumption unit). The benefit of having this extra unit of capital consists of the expected discounted future resale value and the rental value net of utilization cost.

By eliminating the Lagrangian multiplier $\mu_{t}$, the first-order condition with respect to bond holding can be written as

$$
\begin{equation*}
D_{t, t+1}=\beta \frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}} \frac{\bar{P}_{t}}{\bar{P}_{t+1}} \tag{11}
\end{equation*}
$$

Denote by $R_{t}=\left[\mathrm{E}_{t} D_{t, t+1}\right]^{-1}$ the interest rate for a one-period risk-free nominal bond. Then we have

$$
\begin{equation*}
\frac{1}{R_{t}}=\beta \mathrm{E}_{t}\left[\frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}} \frac{\bar{P}_{t}}{\bar{P}_{t+1}}\right] \tag{12}
\end{equation*}
$$

In each period $t$, a fraction $\xi_{w}$ of households re-optimize their nominal wage setting decisions. Those households who can re-optimize wage setting chooses the nominal wage $W_{t}(h)$ to maximize

$$
\begin{array}{r}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \beta^{i} \xi_{w}^{i} A_{t+i}\left[\log \left(C_{t+i}-b C_{t+i-1}\right)-\frac{\psi}{1+\eta} L_{t+i}^{d}(h)^{1+\eta}\right]+ \\
\mu_{t+i}\left[W_{t}(h) \chi_{t, t+i}^{w} L_{t+i}^{d}(h)+m_{t+i}\right] \tag{14}
\end{array}
$$

where the labor demand schedule is given by

$$
\begin{equation*}
L_{t+i}^{d}(h)=\left(\frac{W_{t}(h) \chi_{t, t+i}^{w}}{\bar{W}_{t+i}}\right)^{-\theta_{w t}} L_{t+i}, \quad \theta_{w t}=\frac{\mu_{w t}}{\mu_{w t}-1}, \tag{15}
\end{equation*}
$$

the term $m_{t}$ is given by

$$
m_{t}=\bar{P}_{t} r_{k t} u_{t} K_{t-1}+\Pi_{t}+B_{t}+T_{t}-\bar{P}_{t} C_{t}-\frac{\bar{P}_{t}}{Q_{t}}\left(I_{t}+a\left(u_{t}\right) K_{t-1}\right)-\mathrm{E}_{t} D_{t, t+1} B_{t+1}
$$

and the term $\chi_{t, t+i}^{w}$ is given by

$$
\chi_{t, t+i}^{w} \equiv \begin{cases}\Pi_{k=1}^{i} \pi_{t+k-1}^{\gamma_{w}} \pi^{1-\gamma_{w}} \lambda_{t, t+i}^{*} & \text { if } i \geq 1  \tag{16}\\ 1 & \text { if } i=0\end{cases}
$$

where $\lambda_{t, t+i}^{*} \equiv \frac{\lambda_{t+i}^{*}}{\lambda_{t}^{*}}$.
The first-order condition for the wage-setting problem is given by

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{-A_{t+i} \psi L_{t+i}^{d}(h)^{\eta} \frac{\partial L_{t+i}^{d}(h)}{\partial W_{t}(h)}+\mu_{t+i}\left(1-\theta_{w, t+i}\right) \chi_{t, t+i}^{w} L_{t+i}^{d}(h)\right\}=0 \tag{17}
\end{equation*}
$$

where

$$
\frac{\partial L_{t+i}^{d}(h)}{\partial W_{t}(h)}=-\theta_{w, t+i} \frac{L_{t+i}^{d}(h)}{W_{t}(h)}=-\frac{\mu_{w, t+i}}{\mu_{w, t+i}-1} \frac{L_{t+i}^{d}(h)}{W_{t}(h)}
$$

Factoring out the common terms and rearranging, we obtain

$$
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \frac{\mu_{t+i}}{\mu_{t}} L_{t+i}^{d}(h) \frac{1}{\mu_{w, t+i}-1}\left\{\mu_{w, t+i} \frac{\psi A_{t+i} L_{t+i}^{d}(h)^{\eta}}{\mu_{t+i}}-\chi_{t, t+i}^{w} W_{t}(h)\right\}=0
$$

Let $M R S_{t}(h) \equiv \frac{\psi A_{t} L_{t}^{d}(h)^{\eta}}{\mu_{t}}$ denote the marginal rate of substitution between leisure and income. Then, using (11), we can rewrite the first-order condition for wage setting as

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \xi_{w}^{i} D_{t, t+i} L_{t+i}^{d}(h) \frac{1}{\mu_{w, t+i}-1}\left\{\mu_{w, t+i} M R S_{t+i}(h)-\chi_{t, t+i}^{w} W_{t}(h)\right\}=0 \tag{18}
\end{equation*}
$$

I.2. Firms' optimizing decisions. Pricing decisions are staggered across firms. In each period, a fraction $\xi_{p}$ of firms can re-optimize their pricing decisions and the other fraction $1-\xi_{p}$ of firms mechanically update their prices according to the rule

$$
\begin{equation*}
P_{t}(j)=\pi_{t-1}^{\gamma_{p}} \pi^{1-\gamma_{p}} P_{t-1}(j), \tag{19}
\end{equation*}
$$

If a firm can re-optimize, it chooses $P_{t}(j)$ to solve

$$
\begin{equation*}
\operatorname{Max}_{P_{t}(j)} \quad \mathrm{E}_{t} \sum_{i=0}^{\infty} \xi_{p}^{i} D_{t, t+i}\left[P_{t}(j) \chi_{t, t+i}^{p} Y_{t+i}^{d}(j)-V_{t+i}(j)\right] \tag{20}
\end{equation*}
$$

subject to

$$
\begin{equation*}
Y_{t+i}^{d}(j)=\left(\frac{P_{t}(j) \chi_{t, t+i}^{p}}{\bar{P}_{t+i}}\right)^{-\frac{\mu_{p, t+i}}{\mu_{p, t+i}-1}} Y_{t+i} \tag{21}
\end{equation*}
$$

where $V_{t+i}(j)$ is the cost function and the term $\chi_{t, t+i}^{p}$ comes from the price-updating rule (19) and is given by

$$
\chi_{t, t+i}^{p}= \begin{cases}\Pi_{k=1}^{i} \pi_{t+k-1}^{\gamma_{p}} \pi^{1-\gamma_{p}} & \text { if } i \geq 1  \tag{22}\\ 1 & \text { if } i=0\end{cases}
$$

The first order condition for the profit-maximizing problem yields the optimal pricing rule

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \xi_{p}^{i} D_{t, t+i} Y_{t+i}^{d}(j) \frac{1}{\mu_{p, t+i}-1}\left[\mu_{p, t+i} \Phi_{t+i}(j)-P_{t}(j) \chi_{t, t+i}^{p}\right]=0 \tag{23}
\end{equation*}
$$

where $\Phi_{t+i}(j)=\partial V_{t+i}(j) / \partial Y_{t+i}^{d}(j)$ denotes the marginal cost function. In the absence of markup shocks, $\mu_{p t}$ would be a constant and (23) implies that the optimal price is a markup over an average of the marginal costs for the periods in which the price will remain effective. Clearly, if $\xi_{p}=0$ for all $t$, that is, if prices are perfectly flexible, then the optimal price would be a markup over the contemporaneous marginal cost.

Cost-minimizing implies that the marginal cost function is given by

$$
\begin{equation*}
\Phi_{t}(j)=\left[\tilde{\alpha}\left(\bar{P}_{t} r_{k t}\right)^{\alpha_{1}}\left(\frac{\bar{W}_{t}}{Z_{t}}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} Y_{t}(j)^{\frac{1}{\alpha_{1}+\alpha_{2}}-1} \tag{24}
\end{equation*}
$$

where $\tilde{\alpha} \equiv \alpha_{1}^{-\alpha_{1}} \alpha_{2}^{-\alpha_{2}}$ and $r_{k t}$ denotes the real rental rate of capital input. The conditional factor demand functions are given by

$$
\begin{align*}
\bar{W}_{t} & =\Phi_{t}(j) \alpha_{2} \frac{Y_{t}(j)}{L_{t}^{f}(j)}  \tag{25}\\
\bar{P}_{t} r_{k t} & =\Phi_{t}(j) \alpha_{1} \frac{Y_{t}(j)}{K_{t}^{f}(j)} \tag{26}
\end{align*}
$$

It follows that

$$
\begin{equation*}
\frac{\bar{W}_{t}}{\bar{P}_{t} r_{k t}}=\frac{\alpha_{2}}{\alpha_{1}} \frac{K_{t}^{f}(j)}{L_{t}^{f}(j)}, \quad \forall j \in[0,1] . \tag{27}
\end{equation*}
$$

I.3. Market clearing. In equilibrium, markets for bond, composite labor, capital stock, and composite goods all clear. Bond market clearing implies that $B_{t}=0$ for all $t$. Labor market clearing implies that $\int_{0}^{1} L_{t}^{f}(j) d j=L_{t}$. Capital market clearing implies that $\int_{0}^{1} K_{t}^{f}(j) d j=u_{t} K_{t-1}$. Composite goods market clearing implies that

$$
\begin{equation*}
C_{t}+\frac{1}{Q_{t}}\left[I_{t}+a\left(u_{t}\right) K_{t-1}\right]+G_{t}=Y_{t} \tag{28}
\end{equation*}
$$

where aggregate output is related to aggregate primary factors through the aggregate production function

$$
\begin{equation*}
G_{p t} Y_{t}=\left(u_{t} K_{t-1}\right)^{\alpha_{1}}\left(Z_{t} L_{t}\right)^{\alpha_{2}} \tag{29}
\end{equation*}
$$

with $G_{p t} \equiv \int_{0}^{1}\left(\frac{P_{t}(j)}{P_{t}}\right)^{-\frac{\mu_{p t}}{\mu_{p t}-1} \frac{1}{\alpha_{1}+\alpha_{2}}} d j$ measuring the price dispersion.

## II. Stationary Equilibrium conditions

Since both the neutral technology and the investment-specific technology are growing over time, we transform the appropriate variables to induce stationarity. In particular, we denote by $\tilde{X}_{t}$ the stationary counterpart of the variable $X_{t}$ and we make the following transformations:

$$
\begin{array}{r}
\tilde{Y}_{t}=\frac{Y_{t}}{\lambda_{t}^{*}}, \quad \tilde{C}_{t}=\frac{C_{t}}{\lambda_{t}^{*}}, \quad \tilde{I}_{t}=\frac{I_{t}}{Q_{t} \lambda_{t}^{*}}, \quad \tilde{G}_{t}=\frac{G_{t}}{\lambda_{t}^{*}}, \quad \tilde{K}_{t}=\frac{K_{t}}{Q_{t} \lambda_{t}^{*}}, \\
\tilde{w}_{t}=\frac{\bar{W}_{t}}{\bar{P}_{t} \lambda_{t}^{*}}, \quad \tilde{r}_{k t}=r_{k t} Q_{t}, \quad \tilde{U}_{c t}=U_{c t} \lambda_{t}^{*},
\end{array}
$$

where the underlying trend for output is given by

$$
\lambda_{t}^{*} \equiv\left(Z_{t}^{\alpha_{2}} Q_{t}^{\alpha_{1}}\right)^{\frac{1}{1-\alpha_{1}}}
$$

II.1. Stationary pricing decisions. In terms of the stationary variables, we can rewrite the optimal pricing decision (23) as

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} A_{t+i} \tilde{U}_{c, t+i} \tilde{Y}_{t+i}^{d}(j) \frac{1}{\mu_{p, t+i}-1}\left[\mu_{p, t+i} \phi_{t+i}(j)-p_{t}^{*} Z_{t, t+i}^{p}\right]=0 \tag{30}
\end{equation*}
$$

In this equation, $\tilde{Y}_{t+i}^{d}(j)=\frac{Y_{t+i}^{d}(j)}{\lambda_{t+i}^{*}}$ denotes the detrended output demand; $p_{t}^{*} \equiv \frac{P_{t}(j)}{P_{t}}$ denotes the relative price for optimizing firms, which does not have a $j$ index since all optimizing firms make identical pricing decisions in a symmetric equilibrium; the term $Z_{t, t+i}^{p}$ is defined as

$$
\begin{equation*}
Z_{t, t+i}^{p}=\frac{\chi_{t, t+i}^{p}}{\prod_{k=1}^{i} \pi_{t+k}} \tag{31}
\end{equation*}
$$

and finally, the term $\phi_{t+i}(j) \equiv \frac{\Phi_{t+i}(j)}{P_{t+i}}$ denotes the real unit cost function, which is given by

$$
\begin{align*}
\phi_{t+i}(j) & =\left[\tilde{\alpha}\left(\frac{\tilde{r}_{k, t+i}}{Q_{t+i}}\right)^{\alpha_{1}}\left(\tilde{w}_{t+i} \frac{\lambda_{t+i}^{*}}{Z_{t+i}}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} Y_{t+i}^{d}(j)^{\frac{1}{\alpha_{1}+\alpha_{2}}-1} \\
& =\left[\tilde{\alpha}\left(\tilde{r}_{k, t+i}\right)^{\alpha_{1}}\left(\tilde{w}_{t+i}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} \tilde{Y}_{t+i}^{d}(j)^{\frac{1}{\alpha_{1}+\alpha_{2}}-1} . \tag{32}
\end{align*}
$$

The demand schedule $\tilde{Y}_{t+i}^{d}(j)$ for the optimizing firm $j$ is related to the relative price and aggregate output through

$$
\begin{align*}
\tilde{Y}_{t+i}^{d}(j) & =\left[\frac{P_{t}(j) \chi_{t, t+i}^{p}}{\bar{P}_{t+i}}\right]^{-\theta_{p, t+i}} \tilde{Y}_{t+i} \\
& =\left[p_{t}^{*} \frac{\bar{P}_{t}}{\bar{P}_{t+i}} \chi_{t, t+i}^{p}\right]^{-\theta_{p, t+i}} \tilde{Y}_{t+i} \\
& =\left[p_{t}^{*} Z_{t, t+i}^{p}\right]^{-\theta_{p, t+i}} \tilde{Y}_{t+i} . \tag{33}
\end{align*}
$$

Combining (32) and (33), we have

$$
\begin{equation*}
\phi_{t+i}(j)=\tilde{\phi}_{t+i}\left[p_{t}^{*} Z_{t, t+i}^{p}\right]^{-\theta_{p, t+i} \bar{\alpha}}\left(\tilde{Y}_{t+i}\right)^{\bar{\alpha}} \tag{34}
\end{equation*}
$$

where $\bar{\alpha} \equiv \frac{1-\alpha_{1}-\alpha_{2}}{\alpha_{1}+\alpha_{2}}$ and

$$
\begin{equation*}
\tilde{\phi}_{t+i} \equiv\left[\tilde{\alpha}\left(\tilde{r}_{k, t+i}\right)^{\alpha_{1}}\left(\tilde{w}_{t+i}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} . \tag{35}
\end{equation*}
$$

Given these relations, we can rewrite the optimal pricing rule (30) in terms of stationary variables

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} \frac{A_{t+i} \tilde{U}_{c, t+i} \tilde{Y}_{t+i}^{d}(j)}{\mu_{p, t+i}-1}\left[\mu_{p, t+i} \tilde{\phi}_{t+i}\left[p_{t}^{*} Z_{t, t+i}^{p}\right]^{-\theta_{p, t+i} \bar{\alpha}}\left(\tilde{Y}_{t+i}\right)^{\bar{\alpha}}-p_{t}^{*} Z_{t, t+i}^{p}\right]=0 \tag{36}
\end{equation*}
$$

where $\tilde{\phi}$ is defined in (35).
II.2. Stationary wage setting decision. Using (4) and (11), we can rewrite the optimal wage-setting decision (18) as

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \frac{A_{t+i} U_{c, t+i}}{A_{t} U_{c t}} \frac{\bar{P}_{t}}{\bar{P}_{t+i}} L_{t+i}^{d}(h) \frac{1}{\mu_{w, t+i}-1}\left[\mu_{w, t+i} \psi \frac{L_{t+i}^{d}(h)^{\eta}}{U_{c, t+i}} \bar{P}_{t+i}-W_{t}(h) \chi_{t, t+i}^{w}\right]=0 \tag{37}
\end{equation*}
$$

where the labor demand schedule $L_{t+i}^{d}(h)$ is related to aggregate variables through

$$
\begin{align*}
L_{t+i}^{d}(h) & =\left[\frac{W_{t}(h) \chi_{t, t+i}^{w}}{\bar{W}_{t+i}}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{38}\\
& =\left[w_{t}^{*} \frac{\bar{W}_{t}}{\bar{W}_{t+i}} \chi_{t, t+i}^{w}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{39}\\
& =\left[w_{t}^{*} \frac{\tilde{w}_{t} \bar{P}_{t} \lambda_{t}^{*}}{\tilde{w}_{t+i} \bar{P}_{t+i} \lambda_{t+i}^{*}} \chi_{t, t+i}^{w}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{40}\\
& =\left[\frac{w_{t}^{*} \tilde{w}_{t}}{\tilde{w}_{t+i}} \frac{\chi_{t, t+i}^{w}}{\prod_{k=1}^{i} \pi_{t+k} \lambda_{t, t+i}^{*}}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{41}\\
& \equiv\left[\frac{w_{t}^{*} \tilde{w}_{t}}{\tilde{w}_{t+i}} Z_{t, t+i}^{w}\right]^{-\theta_{w, t+i}} L_{t+i}, \tag{42}
\end{align*}
$$

with $Z_{t, t+i}^{w}$ defined as

$$
\begin{equation*}
Z_{t, t+i}^{w}=\frac{\chi_{t, t+i}^{w}}{\prod_{k=1}^{i} \pi_{t+k} \lambda_{t, t+i}^{*}} \tag{43}
\end{equation*}
$$

Further, we can rewrite the individual optimal nominal wage $W_{t}(h)$ as

$$
W_{t}(h)=w_{t}^{*} \bar{W}_{t}=w_{t}^{*} \tilde{w}_{t} \bar{P}_{t} \lambda_{t}^{*}
$$

Given these relations, we can rewrite the wage setting rule (37) in terms of the stationary variables. With some cancelations, we obtain
$\mathrm{E}_{t} \sum_{i=0}^{\infty} \prod_{k=1}^{i}\left(\beta \xi_{w}\right)^{i} \frac{A_{t+i} \tilde{U}_{c, t+i} L_{t+i}^{d}(h)}{\mu_{w, t+i}-1}\left\{\mu_{w, t+i} \psi\left[\frac{w_{t}^{*} \tilde{w}_{t}}{\tilde{w}_{t+i}} Z_{t, t+i}^{w}\right]^{-\eta \theta_{w, t+i}} \frac{L_{t+i}^{\eta}}{\tilde{U}_{c, t+i}}-w_{t}^{*} \tilde{w}_{t} Z_{t, t+i}^{w}\right\}=0$.
II.3. Other stationary equilibrium conditions. We now rewrite the rest of the equilibrium conditions in terms of stationary variables.

First, the optimal investment decision equation (9) can be written as

$$
\begin{equation*}
1=q_{k t}\left\{1-S\left(\lambda_{I t}\right)-S^{\prime}\left(\lambda_{I t}\right) \lambda_{I t}\right\}+\beta \mathrm{E}_{t} q_{k, t+1} \frac{\lambda_{t}^{*} Q_{t}}{\lambda_{t+1}^{*} Q_{t+1}} \frac{A_{t+1} \tilde{U}_{c, t+1}}{A_{t} \tilde{U}_{c t}} S^{\prime}\left(\lambda_{I, t+1}\right)\left(\lambda_{I, t+1}\right)^{2}, \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{I t}=\frac{I_{t}}{I_{t-1}}=\frac{\tilde{I}_{t} Q_{t} \lambda_{t}^{*}}{\tilde{I}_{t-1} Q_{t-1} \lambda_{t-1}^{*}} \tag{46}
\end{equation*}
$$

Second, the capital Euler equation (10) can be written as

$$
\begin{equation*}
q_{k t}=\beta \mathrm{E}_{t} \frac{A_{t+1} \tilde{U}_{c, t+1}}{A_{t} \tilde{U}_{c t}} \frac{\lambda_{t}^{*} Q_{t}}{\lambda_{t+1}^{*} Q_{t+1}}\left[\left(1-\delta_{t+1}\right) q_{k, t+1}+\tilde{r}_{k, t+1} u_{t+1}-a\left(u_{t+1}\right)\right] . \tag{47}
\end{equation*}
$$

Third, the optimal capacity utilization decision (8) is equivalent to

$$
\begin{equation*}
\tilde{r}_{k t}=a^{\prime}\left(u_{t}\right) . \tag{48}
\end{equation*}
$$

Fourth, the intertemporal bond Euler equation (12) can be written as

$$
\begin{equation*}
\frac{1}{R_{t}}=\beta \mathrm{E}_{t}\left[\frac{\lambda_{t}^{*}}{\lambda_{t+1}^{*}} \frac{A_{t+1} \tilde{U}_{c, t+1}}{A_{t} \tilde{U}_{c t}} \frac{1}{\pi_{t+1}}\right] \tag{49}
\end{equation*}
$$

Fifth, the law of motion for capital stock in (3) can be written as

$$
\begin{equation*}
\tilde{K}_{t}=\left(1-\delta_{t}\right) \frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} \tilde{K}_{t-1}+\left[1-S\left(\lambda_{I t}\right)\right] \tilde{I}_{t} \tag{50}
\end{equation*}
$$

Sixth, the aggregate resource constraint is given by

$$
\begin{equation*}
\tilde{C}_{t}+\tilde{I}_{t}+\frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} a\left(u_{t}\right) \tilde{K}_{t-1}+\tilde{G}_{t}=\tilde{Y}_{t} \tag{51}
\end{equation*}
$$

Seventh, the aggregate production function (29) can be written as

$$
\begin{equation*}
G_{p t} \tilde{Y}_{t}=\left[\frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} u_{t} \tilde{K}_{t-1}\right]^{\alpha_{1}} L_{t}^{\alpha_{2}} \tag{52}
\end{equation*}
$$

Eighth, firms' cost-minimizing implies that, in the stationary equilibrium, we have

$$
\begin{equation*}
\frac{\tilde{w}_{t}}{\tilde{r}_{k t}}=\frac{\alpha_{2}}{\alpha_{1}} \frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} \frac{u_{t} \tilde{K}_{t-1}}{L_{t}} . \tag{53}
\end{equation*}
$$

Finally, we rewrite the interest rate rule here for convenience of referencing:

$$
\begin{equation*}
R_{t}=\kappa R_{t-1}^{\rho_{r}}\left[\left(\frac{\pi_{t}}{\pi^{*}\left(s_{t}\right)}\right)^{\phi_{\pi}} \tilde{Y}_{t}^{\phi_{y}}\right]^{1-\rho_{r}} e^{\sigma_{r t} \varepsilon_{r t}} \tag{54}
\end{equation*}
$$

## III. Steady State

A deterministic steady state is an equilibrium in which all stochastic shocks are shut off. Our model contains a non-standard "shock": the Markov regime switching in monetary policy regime and the shock regime. In computing the steady-state equilibrium, we shut off all shocks, including the regime shocks. Since there is a mapping between any finite-state Markov switching process and a vector AR(1) process (Hamilton, 1994), shutting off the regime shocks in the steady state is equivalent to setting the innovations in the $\mathrm{AR}(1)$ process to its unconditional mean (which is zero). In such a steady state, all stationary variables are constant.

In the steady state, $p^{*}=1$ and $Z^{p}=1$, so that the price setting rule (36) reduces to

$$
\begin{equation*}
\frac{1}{\mu_{p}}=\left[\tilde{\alpha} \tilde{r}_{k}^{\alpha_{1}} \tilde{w}^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} \tilde{Y}^{\bar{\alpha}} \tag{55}
\end{equation*}
$$

That is, the real marginal cost is constant and equals the inverse markup.
Similarly, in the steady state, $w^{*}=1$ and $Z^{w}=1$, so that the wage setting rule (44) reduces to

$$
\begin{equation*}
\tilde{w}=\mu_{w} \frac{\psi L^{\eta}}{\tilde{U}_{c}} \tag{56}
\end{equation*}
$$

which says that the real wage is a constant markup over the marginal rate of substitution between leisure and consumption.

Given that the steady-state markup, and thus the steady-state real marginal cost, is a constant, the conditional factor demand function (26) for capital input together with the capital market clearing condition imply that

$$
\begin{equation*}
\tilde{r}_{k}=\frac{\alpha_{1}}{\mu_{p}} \frac{\tilde{Y} \lambda_{q} \lambda^{*}}{\tilde{K}} . \tag{57}
\end{equation*}
$$

The rest of the steady-state equilibrium conditions for the private sector come from (45) -(53) and are summarized below:

$$
\begin{align*}
1 & =q_{k}  \tag{58}\\
\frac{\lambda_{q} \lambda^{*}}{\beta} & =1-\delta+\tilde{r}_{k},  \tag{59}\\
\tilde{r}_{k} & =a^{\prime}(1),  \tag{60}\\
R & =\frac{\lambda^{*}}{\beta} \pi  \tag{61}\\
\frac{\tilde{I}}{\tilde{K}} & =1-\frac{1-\delta}{\lambda_{q} \lambda^{*}}  \tag{62}\\
\tilde{Y} & =\tilde{C}+\tilde{I}+\tilde{G}  \tag{63}\\
\tilde{Y} & =\left(\frac{\tilde{K}}{\lambda_{q} \lambda^{*}}\right)^{\alpha_{1}} L^{\alpha_{2}},  \tag{64}\\
\frac{\tilde{w}}{\tilde{r}_{k}} & =\frac{1}{\lambda_{q} \lambda^{*}} \frac{\alpha_{2}}{\alpha_{1}} \frac{\tilde{K}}{L} \tag{65}
\end{align*}
$$

## IV. Linearized equilibrium conditions

We now describe our procedure to linearize the stationary equilibrium conditions around the deterministic steady state.
IV.1. Linearizing the price setting rule. Log-linearizing the price setting rule (36) around the steady state, we get

$$
\begin{array}{r}
\mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{y}_{t+i}^{d}(h)-\frac{\mu_{p}}{\mu_{p}-1} \hat{\mu}_{p, t+i}+\hat{\mu}_{p, t+i}+\right. \\
\left.\hat{\tilde{\phi}}_{t+i}-\theta_{p} \bar{\alpha}\left[\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right]+\bar{\alpha} \hat{y}_{t+i}\right\} \\
\approx \mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{y}_{t+i}^{d}(h)-\frac{\mu_{p}}{\mu_{p}-1} \hat{\mu}_{p, t+i}+\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right\},
\end{array}
$$

where

$$
\begin{equation*}
\hat{\tilde{\phi}}_{t+i}=\frac{1}{\alpha_{1}+\alpha_{2}}\left[\alpha_{1} \hat{r}_{k, t+i}+\alpha_{2} \hat{w}_{t+i}\right] . \tag{66}
\end{equation*}
$$

Collecting terms to get

$$
\begin{aligned}
& \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left\{\hat{\mu}_{p, t+i}+\hat{\tilde{\phi}}_{t+i}-\theta_{p} \bar{\alpha}\left[\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right]+\bar{\alpha} \hat{y}_{t+i}\right\} \\
& \approx \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left\{\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right\}
\end{aligned}
$$

Further simplifying

$$
\frac{1+\theta_{p} \bar{\alpha}}{1-\beta \xi_{p}} \hat{p}_{t}^{*}=\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left\{\hat{\mu}_{p, t+i}+\hat{\tilde{\phi}}_{t+i}+\bar{\alpha} \hat{y}_{t+i}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t+i}^{p}\right\}
$$

Denote $\hat{m} c_{t+i} \equiv \hat{\tilde{\phi}}_{t+i}+\bar{\alpha} \hat{y}_{t+i}$. Expanding the infinite sum in the above equation, we get

$$
\begin{aligned}
\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \hat{p}_{t}^{*} & =\hat{\mu}_{p t}+\hat{m} c_{t}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t}^{p} \\
& +\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+1}+\hat{m} c_{t+1}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t+1}^{p}\right] \\
& +\left(\beta \xi_{p}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+2}+\hat{m} c_{t+2}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t+2}^{p}\right]+\ldots
\end{aligned}
$$

Forwarding this relation one period to get

$$
\begin{aligned}
\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \hat{p}_{t+1}^{*} & =\hat{\mu}_{p, t+1}+\hat{m} c_{t+1}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t+1, t+1}^{p} \\
& +\beta \xi_{p} \mathrm{E}_{t+1}\left[\hat{\mu}_{p, t+2}+\hat{m} c_{t+2}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t+1, t+2}^{p}\right] \\
& +\left(\beta \xi_{p}\right)^{2} \mathrm{E}_{t+1}\left[\hat{\mu}_{p, t+3}+\hat{m} c_{t+3}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t+1, t+3}^{p}\right]+\ldots
\end{aligned}
$$

Moving the $Z_{t, t+i}^{p}$ terms to the left, we have

$$
\begin{aligned}
& \frac{1+}{1-} \bar{\alpha} \theta_{p} \\
& \left.\quad \hat{p}_{p}^{*}+\left(1+\bar{\alpha} \theta_{p}\right) \mathrm{E}_{t}\left[\hat{Z}_{t, t}^{p}+\beta \xi_{p} \hat{Z}_{t, t+1}^{p}+\ldots\right]=\hat{\mu}_{p t}+\hat{m} \hat{\mu}_{p, t+1}+\hat{m} c_{t+1}\right] \\
& \quad+\left(\beta \xi_{p}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+2}+\hat{m} c_{t+2}\right]+\ldots \\
& \quad=\hat{\mu}_{p t}+\hat{m} c_{t} \\
& \quad+\beta \xi_{p}\left[\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \mathrm{E}_{t} \hat{p}_{t+1}^{*}+\left(1+\bar{\alpha} \theta_{p}\right) \mathrm{E}_{t}\left[\hat{Z}_{t+1, t+1}^{p}+\beta \xi_{p} \hat{Z}_{t+1, t+2}^{p}+\ldots\right]\right]
\end{aligned}
$$

Since $\hat{Z}_{t, t}^{p}=0$, we have

$$
\begin{align*}
\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \hat{p}_{t}^{*} & =\hat{\mu}_{p t}+\hat{m} c_{t}+\beta \xi_{p} \frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \mathrm{E}_{t} \hat{p}_{t+1}^{*} \\
& +\left(1+\bar{\alpha} \theta_{p}\right) \beta \xi_{p} \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{p}-\hat{Z}_{t, t+i+1}^{p}\right] \tag{67}
\end{align*}
$$

Using the definition of $Z_{t, t+i}^{p}$ in (31), we obtain

$$
\begin{aligned}
\hat{Z}_{t, t+i+1}^{p} & =-\left[\hat{\pi}_{t+i+1}-\gamma_{p} \hat{\pi}_{t+i}+\cdots+\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}\right] \\
\hat{Z}_{t+1, t+i+1}^{p} & =-\left[\hat{\pi}_{t+i+1}-\gamma_{p} \hat{\pi}_{t+i}+\cdots+\hat{\pi}_{t+2}-\gamma_{p} \hat{\pi}_{t+1}\right] .
\end{aligned}
$$

Thus,

$$
\hat{Z}_{t+1, t+i+1}^{p}-\hat{Z}_{t, t+i+1}^{p}=\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}
$$

and the $Z^{p}$ terms in (67) can be reduced to

$$
\sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{p}-\hat{Z}_{t, t+i+1}^{p}\right]=\frac{1}{1-\beta \xi_{p}}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}\right]
$$

Substituting this result into (67), we obtain

$$
\begin{equation*}
\hat{p}_{t}^{*}=\frac{1-\beta \xi_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right)+\beta \xi_{p} \mathrm{E}_{t} \hat{p}_{t+1}^{*}+\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p t} \hat{\pi}_{t}\right] \tag{68}
\end{equation*}
$$

This completes log-linearizing the optimal price setting equation. We now log-linearize the price index relation. In an symmetric equilibrium, the price index relation is given by

$$
\begin{equation*}
1=\xi_{p}\left[\frac{1}{\pi_{t}} \pi_{t-1}^{\gamma_{p}} \pi^{1-\gamma_{p}}\right]^{\frac{1}{1-\mu_{p t}}}+\left(1-\xi_{p}\right)\left(p_{t}^{*}\right)^{\frac{1}{1-\mu_{p t}}} \tag{69}
\end{equation*}
$$

the linearized version of which is given by

$$
\begin{equation*}
\hat{p}_{t}^{*}=\frac{\xi_{p}}{1-\xi_{p}}\left(\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}\right) \tag{70}
\end{equation*}
$$

Using (70) to substitute out the $\hat{p}_{t}^{*}$ in (68), we obtain

$$
\begin{aligned}
& \frac{\xi_{p}}{1-\xi_{p}}\left[\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}\right] \\
& \quad=\frac{1-\beta \xi_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right) \\
& \quad+\beta \xi_{p} \frac{\xi_{p}}{1-\xi_{p}} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}\right]+\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}\right]
\end{aligned}
$$

or

$$
\begin{equation*}
\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}=\frac{\kappa_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right)+\beta \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}\right] \tag{71}
\end{equation*}
$$

where the real marginal cost is given by

$$
\begin{equation*}
\hat{m} c_{t}=\frac{1}{\alpha_{1}+\alpha_{2}}\left[\alpha_{1} \hat{r}_{k, t+i}+\alpha_{2} \hat{w}_{t+i}\right]+\bar{\alpha} \hat{y}_{t} . \tag{72}
\end{equation*}
$$

and the term $\kappa_{p}$ is given by

$$
\kappa_{p} \equiv \frac{\left(1-\beta \xi_{p}\right)\left(1-\xi_{p}\right)}{\xi_{p}}
$$

This completes the derivation of the price Phillips curve.
IV.2. Linearizing the optimal wage setting rule. Log-linearizing this wage decision rule, we get

$$
\begin{array}{r}
\mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{l}_{t+i}^{d}(h)-\frac{\mu_{w}}{\mu_{w}-1} \hat{\mu}_{w, t+i}+\hat{\mu}_{w, t+i}-\right. \\
\left.\eta \theta_{w}\left[\hat{w}_{t}^{*}+\hat{w}_{t}-\hat{w}_{t+i}+\hat{Z}_{t, t+i}^{w}\right]+\eta \hat{l}_{t+i}-\hat{u}_{c, t+i}\right\} \\
\approx \mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{l}_{t+i}^{d}(h)-\frac{\mu_{w}}{\mu_{w}-1} \hat{\mu}_{w, t+i}+\hat{w}_{t}^{*}+\hat{w}_{t}+\hat{Z}_{t, t+i}^{w}\right\} .
\end{array}
$$

Collecting terms to get

$$
\begin{array}{r}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{\hat{\mu}_{w, t+i}-\eta \theta_{w}\left[\hat{w}_{t}^{*}+\hat{w}_{t}-\hat{w}_{t+i}+\hat{Z}_{t, t+i}^{w}\right]+\eta \hat{l}_{t+i}-\hat{u}_{c, t+i}\right\} \\
\\
\approx \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{\hat{w}_{t}^{*}+\hat{w}_{t}+\hat{Z}_{t, t+i}^{w}\right\}
\end{array}
$$

Further simplifying
$\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right)=\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{\hat{\mu}_{w, t+i}+\eta \hat{l}_{t+i}-\hat{u}_{c, t+i}+\eta \theta_{w} \hat{w}_{t+i}-\left(1+\eta \theta_{w}\right) \hat{Z}_{t, t+i}^{w}\right\}$.

Denote $m \hat{r} s_{t+i} \equiv \eta \hat{l}_{t+i}-\hat{u}_{c, t+i}$. Expanding the infinite sum in the above equation, we get

$$
\begin{aligned}
\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right) & =\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t}-\hat{Z}_{t, t}^{w}\right) \\
& +\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+1}+m \hat{r} s_{t+1}-\hat{w}_{t+1}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+1}-\hat{Z}_{t, t+1}^{w}\right)\right] \\
& +\left(\beta \xi_{w}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+2}+m \hat{r} s_{t+2}-\hat{w}_{t+2}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+2}-\hat{Z}_{t, t+2}^{w}\right)\right]+\ldots
\end{aligned}
$$

Forwarding this relation one period to get

$$
\begin{aligned}
\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right) & =\hat{\mu}_{w, t+1}+m \hat{r} s_{t+1}-\hat{w}_{t+1}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+1}-\hat{Z}_{t+1, t+1}^{w}\right) \\
& +\beta \xi_{w} \mathrm{E}_{t+1}\left[\hat{\mu}_{w, t+2}+m \hat{r} s_{t+2}-\hat{w}_{t+2}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+2}-\hat{Z}_{t+1, t+2}^{w}\right)\right] \\
& +\left(\beta \xi_{w}\right)^{2} \mathrm{E}_{t+1}\left[\hat{\mu}_{w, t+3}+m \hat{r} s_{t+3}-\hat{w}_{t+3}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+3}-\hat{Z}_{t+1, t+3}^{w}\right)\right]+\ldots
\end{aligned}
$$

Moving the $Z_{t, t+i}^{w}$ terms to the left, we have

$$
\begin{aligned}
& \frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right)+\left(1+\eta \theta_{w}\right) \mathrm{E}_{t}\left[\hat{Z}_{t, t}^{w}+\beta \xi_{w} \hat{Z}_{t, t+1}^{w}+\ldots\right]=\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right) \hat{w}_{t} \\
& \quad+\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+1}+m \hat{r} s_{t+1}-\hat{w}_{t+1}+\left(1+\eta \theta_{w}\right) \hat{w}_{t+1}\right] \\
& \quad+\left(\beta \xi_{w}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+2}+m \hat{r} s_{t+2}-\hat{w}_{t+2}+\left(1+\eta \theta_{w}\right) \hat{w}_{t+2}\right]+\ldots \\
& \quad=\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right) \hat{w}_{t} \\
& \quad+\beta \xi_{w} \mathrm{E}_{t}\left[\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right)+\left(1+\eta \theta_{w}\right)\left[\hat{Z}_{t+1, t+1}^{w}+\beta \xi_{w} \hat{Z}_{t+1, t+2}^{w}+\ldots\right]\right]
\end{aligned}
$$

Since $\hat{Z}_{t, t}^{w}=0$, we have

$$
\begin{align*}
\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right) & =\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right) \hat{w}_{t}+\beta \xi_{w} \frac{1+\eta \theta_{w}}{1-\beta \xi_{w}} \mathrm{E}_{t}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right) \\
& +\left(1+\eta \theta_{w}\right) \beta \xi_{w} \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{w}-\hat{Z}_{t, t+i+1}^{w}\right] \tag{73}
\end{align*}
$$

Using the definition of $Z_{t, t+i}^{w}$ in (43), we obtain

$$
\begin{aligned}
\hat{Z}_{t, t+i+1}^{w} & =-\left[\hat{\pi}_{t+i+1}-\gamma_{w} \hat{\pi}_{t+i}+\cdots+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right] \\
\hat{Z}_{t+1, t+i+1}^{w} & =-\left[\hat{\pi}_{t+i+1}-\gamma_{w} \hat{\pi}_{t+i}+\cdots+\hat{\pi}_{t+2}-\gamma_{w} \hat{\pi}_{t+1}\right] .
\end{aligned}
$$

Thus,

$$
\hat{Z}_{t+1, t+i+1}^{w}-\hat{Z}_{t, t+i+1}^{w}=\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t},
$$

and the $Z^{w}$ terms in (73) can be reduced to

$$
\sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{w}-\hat{Z}_{t, t+i+1}^{w}\right]=\frac{1}{1-\beta \xi_{w}}\left[\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right]
$$

Substituting this result into (73), we obtain

$$
\begin{equation*}
\hat{w}_{t}^{*}+\hat{w}_{t}=\frac{1-\beta \xi_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+\left(1-\beta \xi_{w}\right) \hat{w}_{t}+\beta \xi_{w} \mathrm{E}_{t}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right)+\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right] \tag{74}
\end{equation*}
$$

This completes log-linearizing the wage decision equation. We now log-linearize the wage index relation. In an symmetric equilibrium, the wage index relation is given by

$$
\begin{equation*}
1=\xi_{w}\left[\frac{\tilde{w}_{t-1}}{\tilde{w}_{t}} \frac{1}{\pi_{t}} \pi_{t-1}^{\gamma_{w}} \pi^{1-\gamma_{w}}\right]^{\frac{1}{1-\mu_{w t}}}+\left(1-\xi_{w}\right)\left(w_{t}^{*}\right)^{\frac{1}{1-\mu_{w t}}} \tag{75}
\end{equation*}
$$

the linearized version of which is given by

$$
\begin{equation*}
\left.\hat{w}_{t}^{*}=\frac{\xi_{w}}{1-\xi_{w}}\left(\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}\right)\right] . \tag{76}
\end{equation*}
$$

Using (76) to substitute out the $\hat{w}_{t}^{*}$ in (74), we obtain

$$
\begin{aligned}
\hat{w}_{t} & +\frac{\xi_{w}}{1-\xi_{w}}\left[\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}\right] \\
& =\frac{1-\beta \xi_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+\left(1-\beta \xi_{w}\right) \hat{w}_{t} \\
& +\beta \xi_{w} \mathrm{E}_{t}\left\{\hat{w}_{t+1}+\frac{\xi_{w}}{1-\xi_{w}}\left[\hat{w}_{t+1}-\hat{w}_{t}+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right]\right\}+\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right]
\end{aligned}
$$

or

$$
\begin{align*}
& \hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}=\frac{\kappa_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+ \\
& \quad \beta \mathrm{E}_{t}\left[\hat{w}_{t+1}-\hat{w}_{t}+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right], \tag{77}
\end{align*}
$$

where $\kappa_{w} \equiv \frac{\left(1-\beta \xi_{w}\right)\left(1-\xi_{w}\right)}{\xi_{w}}$.
To help understand the economics behind this equation, we define the nominal wage inflation as

$$
\begin{equation*}
\pi_{t}^{w}=\frac{\bar{W}_{t}}{\bar{W}_{t-1}}=\frac{\tilde{w}_{t} \bar{P}_{t} \lambda_{t}^{*}}{\tilde{w}_{t-1} \bar{P}_{t-1} \lambda_{t-1}^{*}}=\frac{\tilde{w}_{t}}{\tilde{w}_{t-1}} \pi_{t} \lambda_{t-1, t}^{*} \tag{78}
\end{equation*}
$$

The log-linearized version is given by

$$
\hat{\pi}_{t}^{w}=\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}+\Delta \hat{\lambda}_{t}^{*}
$$

where $\Delta x_{t}=x_{t}-x_{t-1}$ is the first-difference operator and $\hat{\lambda}_{t}^{*}=\frac{1}{1-\alpha_{1}}\left(\alpha_{1} \hat{q}_{t}+\alpha_{2} \hat{z}_{t}\right)$. Thus, the optimal wage decision (77) is equivalent to

$$
\begin{gather*}
\hat{\pi}_{t}^{w}-\gamma_{w} \hat{\pi}_{t-1}=\frac{\kappa_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+\beta \mathrm{E}_{t}\left(\hat{\pi}_{t+1}^{w}-\gamma_{w} \hat{\pi}_{t}\right) \\
+\frac{1}{1-\alpha_{1}}\left[\alpha_{1}\left(\Delta \hat{z}_{t}-\beta \mathrm{E}_{t} \Delta \hat{z}_{t+1}\right)+\alpha_{2}\left(\Delta \hat{q}_{t}-\beta \mathrm{E}_{t} \Delta \hat{q}_{t+1}\right)\right] \tag{79}
\end{gather*}
$$

This nominal-wage Phillips curve relation parallels that of the price-Phillips curve and has similar interpretations.
IV.3. Linearizing other stationary equilibrium conditions. Taking total differentiation in the investment decision equation (45) and using the steady-state conditions that $S\left(\lambda_{I}\right)=S^{\prime}\left(\lambda_{I}\right)=0$, we obtain

$$
\begin{equation*}
\hat{q}_{k t}=S^{\prime \prime}\left(\lambda_{I}\right) \lambda_{I}^{2}\left[\hat{\lambda}_{I t}-\beta \mathrm{E}_{t} \hat{\lambda}_{I, t+1}\right] \tag{80}
\end{equation*}
$$

which, combined with the definition of the investment growth rate

$$
\begin{equation*}
\hat{\lambda}_{I t}=\Delta \hat{i}_{t}+\frac{1}{1-\alpha_{1}}\left[\Delta \hat{q}_{t}+\alpha_{2} \Delta \hat{z}_{t}\right] \tag{81}
\end{equation*}
$$

implies the linearized investment decision equation in the text.

Taking total differentiation in the capital Euler equation (47) and using the steadystate conditions that $\tilde{q}_{k}=1, u=1, a(1)=0, \tilde{r}_{k}=a^{\prime}(1)$, and $\frac{\beta}{\lambda_{I}}\left(1-\delta+\tilde{r}_{k}\right)=1$, we obtain

$$
\begin{equation*}
\hat{q}_{k t}=\mathrm{E}_{t}\left\{\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\Delta \hat{\lambda}_{t+1}^{*}-\Delta \hat{q}_{t+1}+\frac{\beta}{\lambda_{I}}\left[(1-\delta) \hat{q}_{k, t+1}-\delta \hat{\delta}_{t+1}+\tilde{r}_{k} \hat{r}_{k, t+1}\right]\right\} \tag{82}
\end{equation*}
$$

which, upon substituting the expressions for the $\Delta \hat{\lambda}_{t}^{*}$ and $\Delta \hat{q}_{t}$, implies the linearized capital Euler equation in the text.

The linearized capacity utilization decision equation (48) is given by

$$
\begin{equation*}
\hat{r}_{k t}=\sigma_{u} \hat{u}_{t} \tag{83}
\end{equation*}
$$

where $\sigma_{u} \equiv \frac{a^{\prime \prime}(1)}{a^{\prime}(1)}$ is the curvature parameter for the capacity utility function $a(u)$ evaluated at the steady state.

The linearized intertemporal bond Euler equation (49) is given by

$$
\begin{equation*}
0=\mathrm{E}_{t}\left[\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\Delta \hat{\lambda}_{t+1}^{*}+\hat{R}_{t}-\hat{\pi}_{t+1}\right] \tag{84}
\end{equation*}
$$

which, along with the definition of the exogenous term $\Delta \hat{\lambda}_{t+1}^{*}$, implies the linearized bond Euler equation in the text.

Log-linearize the capital law of motion (50) leads to

$$
\begin{equation*}
\hat{k}_{t}=\frac{1-\delta}{\lambda_{I}}\left[\hat{k}_{t-1}-\Delta \hat{\lambda}_{t}^{*}-\Delta \hat{q}_{t}\right]-\frac{\delta}{\lambda_{I}} \hat{\delta}_{t}+\frac{\tilde{I}}{\tilde{K}} \hat{i}_{t}, \tag{85}
\end{equation*}
$$

which implies the linearized capital law of motion in the text.
To obtain the linearized resource constraint, we take total differentiation of (51) to obtain

$$
\begin{equation*}
\hat{y}_{t}=c_{y} \hat{c}_{t}+i_{y} \hat{i}_{t}+u_{y} \hat{u}_{t}+g_{y} \hat{g}_{t}, \tag{86}
\end{equation*}
$$

where $c_{y}=\frac{\tilde{C}}{\tilde{Y}}, i_{y}=\frac{\tilde{I}}{\tilde{Y}}, u_{y}=\frac{\tilde{r}_{k} \tilde{K}}{\tilde{Y} \lambda_{I}}$, and $g_{y}=\frac{\tilde{G}}{\tilde{Y}}$.
Log-linearizing the aggregate production function (52), we get

$$
\begin{align*}
\hat{y}_{t} & =\alpha_{1}\left[\hat{k}_{t-1}+\hat{u}_{t}-\Delta \hat{\lambda}_{t}^{*}-\Delta \hat{q}_{t}\right]+\alpha_{2} \hat{l}_{t} \\
& =\alpha_{1}\left[\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)\right]+\alpha_{2} \hat{l}_{t} \tag{87}
\end{align*}
$$

The linearized version of the factor demand relation (53) is given by

$$
\begin{align*}
\hat{w}_{t} & =\hat{r}_{k t}+\hat{k}_{t-1}+\hat{u}_{t}-\Delta \hat{\lambda}_{t}^{*}-\Delta \hat{q}_{t}-\hat{l}_{t} \\
& =\hat{r}_{k t}+\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)-\hat{l}_{t} \tag{88}
\end{align*}
$$

Finally, linearizing the interest rate rule (54) gives

$$
\begin{equation*}
\hat{R}_{t}=\rho_{r} \hat{R}_{t-1}+\left(1-\rho_{r}\right)\left[\phi_{\pi}\left(\hat{\pi}_{t}-\hat{\pi}^{*}\left(s_{t}\right)\right)+\phi_{y} \hat{y}_{t}\right]+\sigma_{r t} \varepsilon_{r t}, \tag{89}
\end{equation*}
$$

where

$$
\hat{\pi}^{*}\left(s_{t}\right) \equiv \log \pi^{*}\left(s_{t}\right)-\log \pi .
$$

Note that, with regime-switching inflation target, we have

$$
\hat{\pi}^{*}\left(s_{t}\right)=\mathbf{1}\left\{s_{t}=1\right\} \hat{\pi}^{*}(1)+\mathbf{1}\left\{s_{t}=2\right\} \hat{\pi}^{*}(2)=\left[\hat{\pi}^{*}(1), \hat{\pi}^{*}(2)\right] e_{s_{t}},
$$

where

$$
e_{s_{t}}=\left[\begin{array}{l}
\mathbf{1}\left\{s_{t}=1\right\} \\
\mathbf{1}\left\{s_{t}=2\right\}
\end{array}\right] .
$$

It is useful to use the result that the random vector $e_{s_{t}}$ follows an $\mathrm{AR}(1)$ process:

$$
e_{s_{t}}=Q e_{s_{t-1}}+v_{t}
$$

where $Q$ is the Markov transition matrix of the regime and $\mathrm{E}_{t-1} v_{t}=0$.
IV.4. Summary of linearized equilibrium conditions. We now summarize the linearized equilibrium conditions to be used for solving and estimating the model. These conditions are listed below.

$$
\begin{align*}
\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}= & \frac{\kappa_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right)+\beta \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}\right],  \tag{90}\\
\hat{w}_{t}-\hat{w}_{t-1}+ & \hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}=\frac{\kappa_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+ \\
& \beta \mathrm{E}_{t}\left[\hat{w}_{t+1}-\hat{w}_{t}+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}\right] .  \tag{91}\\
\hat{q}_{k t}= & S^{\prime \prime}\left(\lambda_{I}\right) \lambda_{I}^{2}\left\{\Delta \hat{i}_{t}+\frac{1}{1-\alpha_{1}}\left(\Delta \hat{q}_{t}+\alpha_{2} \Delta \hat{z}_{t}\right)\right. \\
& \left.-\beta \mathrm{E}_{t}\left[\Delta \hat{i}_{t+1}+\frac{1}{1-\alpha_{1}}\left(\Delta \hat{q}_{t+1}+\alpha_{2} \Delta \hat{z}_{t+1}\right)\right]\right\}  \tag{92}\\
\hat{q}_{k t}= & \mathrm{E}_{t}\left\{\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\frac{1}{1-\alpha_{1}}\left[\alpha_{2} \Delta \hat{z}_{t+1}+\Delta \hat{q}_{t+1}\right]\right. \\
& \left.+\frac{\beta}{\lambda_{I}}\left[(1-\delta) \hat{q}_{k, t+1}-\delta \hat{\delta}_{t+1}+\tilde{r}_{k} \hat{r}_{k, t+1}\right]\right\},  \tag{93}\\
\hat{r}_{k t}= & \sigma_{u} \hat{u}_{t},  \tag{94}\\
0= & \mathrm{E}_{t}\left[\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\frac{1}{1-\alpha_{1}}\left[\alpha_{2} \Delta \hat{z}_{t+1}+\alpha_{1} \Delta \hat{q}_{t+1}\right]+\hat{R}_{t}-\hat{\pi}_{t+1}\right],  \tag{95}\\
\hat{k}_{t}= & \frac{1-\delta}{\lambda_{I}}\left[\hat{k}_{t-1}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)\right]-\frac{\delta}{\lambda_{I}} \hat{\delta}_{t}+\left(1-\frac{1-\delta}{\lambda_{I}}\right) \hat{i}_{t},  \tag{96}\\
\hat{y}_{t}= & c_{y} \hat{c}_{t}+i_{y} \hat{i}_{t}+u_{y} \hat{u}_{t}+g_{y} \hat{g}_{t},  \tag{97}\\
\hat{y}_{t}= & \alpha_{1}\left[\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)\right]+\alpha_{2} \hat{l}_{t}, \tag{98}
\end{align*}
$$

$$
\begin{align*}
\hat{w}_{t} & =\hat{r}_{k t}+\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)-\hat{l}_{t}  \tag{99}\\
\hat{R}_{t} & =\rho_{r} \hat{R}_{t-1}+\left(1-\rho_{r}\right)\left[\phi_{\pi}\left(\hat{\pi}_{t}-\hat{\pi}^{*}\left(s^{t}\right)\right)+\phi_{y} \hat{y}_{t}\right]+\sigma_{r t} \varepsilon_{r t} \tag{100}
\end{align*}
$$

where

$$
\begin{align*}
\hat{m} c_{t}= & \frac{1}{\alpha_{1}+\alpha_{2}}\left[\alpha_{1} \hat{r}_{k t}+\alpha_{2} \hat{w}_{t}\right]+\bar{\alpha} \hat{y}_{t}  \tag{101}\\
m \hat{r} s_{t}= & \eta \hat{l}_{t}-\hat{U}_{c t}  \tag{102}\\
\hat{U}_{c t}= & \frac{\beta b\left(1-\rho_{a}\right)}{\lambda_{*}-\beta b} \hat{a}_{t}-\frac{\lambda_{*}}{\left(\lambda_{*}-b\right)\left(\lambda_{*}-\beta b\right)}\left[\lambda_{*} \hat{c}_{t}-b\left(\hat{c}_{t-1}-\Delta \hat{\lambda}_{t}^{*}\right)\right] \\
& +\frac{\beta b}{\left(\lambda_{*}-b\right)\left(\lambda_{*}-\beta b\right)}\left[\lambda_{*} \mathrm{E}_{t}\left(\hat{c}_{t+1}+\Delta \hat{\lambda}_{t+1}^{*}\right)-b \hat{c}_{t}\right]  \tag{103}\\
\hat{\pi}^{*}\left(s_{t}\right)= & {\left[\hat{\pi}^{*}(1), \hat{\pi}^{*}(2)\right] e_{s_{t}}, \quad e_{s_{t}}=Q e_{s_{t-1}}+v_{t} } \tag{104}
\end{align*}
$$

and the steady-state variables are given by

$$
\begin{align*}
\tilde{r}_{k} & =\frac{\lambda_{I}}{\beta}-(1-\delta)  \tag{106}\\
u_{y} & \equiv \frac{\tilde{r}_{k} \tilde{K}}{\tilde{Y} \lambda_{I}}=\frac{\alpha_{1}}{\mu_{p}}  \tag{107}\\
i_{y} & =\left[\lambda_{I}-(1-\delta)\right] \frac{\alpha_{1}}{\mu_{p} \tilde{r}_{k}},  \tag{108}\\
c_{y} & =1-i_{y}-g_{y} \tag{109}
\end{align*}
$$

with $\lambda_{I} \equiv\left(\lambda_{q} \lambda_{z}^{\alpha_{2}}\right)^{\frac{1}{1-\alpha_{1}}}, \lambda_{*} \equiv\left(\lambda_{z}^{\alpha_{2}} \lambda_{q}^{\alpha_{1}}\right)^{\frac{1}{1-\alpha_{1}}}, \Delta \hat{\lambda}_{t}^{*} \equiv \frac{1}{1-\alpha_{1}}\left(\alpha_{1} \Delta \hat{q}_{t}+\alpha_{2} \Delta \hat{z}_{t}\right)$, and $g_{y}$ calibrated to match the average ratio of government spending to real GDP.

Recall that $\theta_{p} \equiv \frac{\mu_{p}}{\mu_{p}-1}, \Delta x_{t}=x_{t}-x_{t-1}, \kappa_{p} \equiv \frac{\left(1-\beta \xi_{p}\right)\left(1-\xi_{p}\right)}{\xi_{p}}, \bar{\alpha} \equiv \frac{1-\alpha_{1}-\alpha_{2}}{\alpha_{1}+\alpha_{2}}, \theta_{w} \equiv \frac{\mu_{w}}{\mu_{w}-1}$, $\kappa_{w} \equiv \frac{\left(1-\beta \xi_{w}\right)\left(1-\xi_{w}\right)}{\xi_{w}}$, and $\hat{\pi}_{t}^{w}=\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}+\Delta \hat{\lambda}_{t}^{*}$,

To compute the equilibrium, we eliminate $\hat{u}_{t}$ by using (97), leaving 10 equations (90)-(96) and (98)-(100) with 10 variables $\hat{\pi}_{t}, \hat{w}_{t}, \hat{i}_{t}, \hat{q}_{k t}, \hat{r}_{k t}, \hat{c}_{t}, \hat{k}_{t}, \hat{y}_{t}, \hat{l}_{t}$, and $\hat{R}_{t}$. Out of these 10 variables, we have 7 observable variables, that is, all but $\hat{q}_{k t}, \hat{r}_{k t}$, and $\hat{k}_{t}$, for our estimation. We also include the biased technology shock $\hat{q}_{t}$ in the set of observable variables.

## V. GEnERAL SETUP FOR ESTIMATION

In this section, we describe our empirical strategy in general terms so that the method can be applied to any state-space-form model.

Consider a regime-switching DSGE model with $s_{t}$ following a Markov-switching process. Let $\theta$ be a vector of all the model parameters except the transition matrix for $s_{t}$. Let $y_{t}$ be an $n \times 1$ vector of observable variables. In our case, $n=8$. The vector $y_{t}$ is connected to the state vector $f_{t}$. For our regime-switching DSGE model, this
state-space representation implies a non-standard Kalman-filter problem as discussed in Kim and Nelson (1999).

Let $\left(Y_{t}, \theta, Q, S_{t}\right)$ be a collection of random variables where

$$
\begin{aligned}
Y_{t} & =\left(y_{1}, \cdots, y_{t}\right) \in\left(\mathbb{R}^{n}\right)^{t}, \\
\theta & =\left(\theta_{i}\right)_{i \in H} \in\left(\mathbb{R}^{r}\right)^{h}, \\
Q & =\left(q_{i, j}\right)_{(i, j) \in H \times H} \in \mathbb{R}^{h^{2}}, \\
S_{t} & =\left(s_{0}, \cdots, s_{t}\right) \in H^{t+1}, \\
S_{t+1}^{T} & =\left(s_{t+1}, \cdots, s_{T}\right) \in H^{T-t},
\end{aligned}
$$

and $H$ is a finite set with $h$ elements and is usually taken to be the set $\{1, \cdots, h\}$. Because $s_{t}$ represents a composite regime, $h$ can be greater than the actual number of regimes at time $t$. The matrix $Q$ is the Markov transition matrix and $q_{i, j}$ is the probability that $s_{t}$ is equal to $i$ given that $s_{t-1}$ is equal to $j$. The matrix $Q$ is restricted to satisfy

$$
q_{i, j} \geq 0 \text { and } \sum_{i \in H} q_{i, j}=1
$$

The object $\theta$ is a vector of all the model parameters except the elements in $Q$. The object $S_{t}$ represents a sequence of unobserved regimes or states. We assume that $\left(Y_{t}, \theta, Q, S_{t}\right)$ has a joint density function $p\left(Y_{t}, \theta, Q, S_{t}\right)$, where we use the Lebesgue measure on $\left(\mathbb{R}^{n}\right)^{t} \times\left(\mathbb{R}^{r}\right)^{h} \times \mathbb{R}^{h^{2}}$ and the counting measure on $H^{t+1}$. This density satisfies the following key condition.

## Condition 1.

$$
p\left(s_{t} \mid Y_{t-1}, \theta, Q, S_{t-1}\right)=q_{s_{t}, s_{t-1}}
$$

for $t>0$.
V.1. Propositions for Hamilton filter. Given $p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right)$ for all $t$, the following propositions follow from Condition 1 (Hamilton, 1989; Chib, 1996; Sims, Waggoner, and Zha, 2008).

Proposition 1.

$$
p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)=\sum_{s_{t-1} \in H} q_{s_{t}, s_{t-1}} p\left(s_{t-1} \mid Y_{t-1}, \theta, Q\right)
$$

for $t>0$.

Proposition 2.

$$
p\left(s_{t} \mid Y_{t}, \theta, Q\right)=\frac{p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)}{\sum_{s_{t-1} \in H} p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)}
$$

for $t>0$.
Proposition 3.

$$
p\left(s_{t} \mid Y_{t}, \theta, Q, s_{t+1}\right)=p\left(s_{t} \mid Y_{T}, \theta, Q, S_{t+1}^{T}\right)
$$

for $0 \leq t<T$.
V.2. Likelihood. We follow the standard assumption in the literature that the initial data $Y_{0}$ is taken as given. Using Kim and Nelson (1999)'s Kalman-filter updating procedure, we obtain the conditional likelihood function at time $t$

$$
\begin{equation*}
p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) . \tag{110}
\end{equation*}
$$

It follows from the rules of conditioning that

$$
\begin{aligned}
p\left(y_{t}, \mid Y_{t-1}, \theta, Q\right) & =\sum_{s_{t} \in H} p\left(y_{t}, s_{t} \mid Y_{t-1}, \theta, Q\right) \\
& =\sum_{s_{t} \in H} p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)
\end{aligned}
$$

Using (110) and the above equation, one can show that the likelihood function of $Y_{T}$ is

$$
\begin{align*}
p\left(Y_{T} \mid \theta, Q\right) & =\prod_{t=1}^{T} p\left(y_{t} \mid Y_{t-1}, \theta, Q\right) \\
& =\prod_{t=1}^{T}\left[\sum_{s_{t} \in H} p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)\right] . \tag{111}
\end{align*}
$$

We assume that $p\left(s_{0} \mid Y_{0}, \theta, Q\right)=\frac{1}{h}$ for every $s_{0} \in H .{ }^{1}$ Given this initial condition, the likelihood function (111) can be evaluated recursively, using Propositions 1 and 2.
V.3. Posterior distributions. The prior for all the parameters is denoted by $p(\theta, Q)$, which will be discussed further in the main text of the article. By the Bayes rule, it follows from (111) that the posterior distribution of $(\theta, Q)$ is

$$
\begin{equation*}
p\left(\theta, Q \mid Y_{T}\right) \propto p(\theta, Q) p\left(Y_{T} \mid \theta, Q\right) \tag{112}
\end{equation*}
$$

The posterior density $p\left(\theta, Q \mid Y_{T}\right)$ is unknown and complicated; the Monte Carlo Markov Chain (MCMC) simulation directly from this distribution can be inefficient

[^0]and problematic. One can, however, use the idea of Gibbs sampling to obtain the empirical joint posterior density $p\left(\theta, Q, S_{T} \mid Y_{T}\right)$ by sampling alternately from the following conditional posterior distributions:
\[

$$
\begin{aligned}
& p\left(S_{T} \mid Y_{T}, \theta, Q\right) \\
& p\left(Q \mid Y_{T}, S_{T}, \theta\right) \\
& p\left(\theta \mid Y_{T}, Q, S_{T}\right)
\end{aligned}
$$
\]

One can use the Metropolis-Hastings sampler to sample from the conditional posterior distributions $p\left(\theta \mid Y_{T}, Q, S_{T}\right)$ and $p\left(Q \mid Y_{T}, S_{T}, \theta\right)$. To simulate from the distribution $p\left(S_{T} \mid Y_{T}, \theta, Q\right)$, we can see from the rules of conditioning that

$$
\begin{align*}
p\left(S_{T} \mid Y_{T}, \theta, Q\right) & =p\left(s_{T} \mid Y_{T}, \theta, Q\right) p\left(S_{T-1} \mid Y_{T}, \theta, Q, S_{T}^{T}\right) \\
& =p\left(s_{T} \mid Y_{T}, \theta, Q\right) \prod_{t=0}^{T-1} p\left(s_{t} \mid Y_{T}, \theta, Q, S_{t+1}^{T}\right) \tag{113}
\end{align*}
$$

where $S_{t+1}^{T}=\left\{s_{t+1}, \cdots, s_{T}\right\}$. From Proposition 3,

$$
\begin{align*}
p\left(s_{t} \mid Y_{T}, \theta, Q, S_{t+1}^{T}\right) & =p\left(s_{t} \mid Y_{t}, \theta, Q, s_{t+1}\right) \\
& =\frac{p\left(s_{t}, s_{t+1} \mid Y_{t}, \theta, Q\right)}{p\left(s_{t+1} \mid Y_{t}, \theta, Q\right)} \\
& =\frac{p\left(s_{t+1} \mid Y_{t}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t}, \theta, Q\right)}{p\left(s_{t+1} \mid Y_{t}, \theta, Q\right)}  \tag{114}\\
& =\frac{q_{s_{t+1}, s_{t}} p\left(s_{t} \mid Y_{t}, \theta, Q\right)}{p\left(s_{t+1} \mid Y_{t}, \theta, Q\right)} .
\end{align*}
$$

The conditional density $p\left(s_{t} \mid Y_{T}, Z_{T}, \theta, Q, S_{t+1}^{T}\right)$ is straightforward to evaluate according to Propositions 1 and 2.

To draw $S_{T}$, we use the backward recursion by drawing the last state $s_{T}$ from the terminal density $p\left(s_{T} \mid Y_{T}, \theta, Q\right)$ and then drawing $s_{t}$ recursively given the path $S_{t+1}^{T}$ according to (114). It can be seen from (113) that draws of $S_{T}$ this way come from $\operatorname{Pr}\left(S_{T} \mid Y_{T}, \theta\right)$.
V.4. Marginal posterior density of $s_{t}$. The smoothed probability of $s_{t}$ given the values of the parameters and the data can be evaluated through backward recursions.

Starting with $s_{T}$ and working backward, we can calculate the probability of $s_{t}$ conditional on $Y_{T}, \theta, Q$ by using the following fact

$$
\begin{aligned}
p\left(s_{t} \mid Y_{T}, \theta, Q\right) & =\sum_{s_{t+1} \in H} p\left(s_{t}, s_{t+1} \mid Y_{T}, \theta, Q\right) \\
& =\sum_{s_{t+1} \in H} p\left(s_{t} \mid Y_{T}, \theta, Q, s_{t+1}\right) p\left(s_{t+1} \mid Y_{T}, \theta, Q\right)
\end{aligned}
$$

where $p\left(s_{t} \mid Y_{t}, \theta, Q, s_{t+1}\right)$ can be evaluated according to (114).

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# TECHNICAL APPENDIX II SOURCES OF MACROECONOMIC FLUCTUATIONS: A REGIME-SWITCHING DSGE APPROACH (NOT INTENDED FOR PUBLICATION) 

ZHENG LIU, DANIEL F. WAGGONER, AND TAO ZHA

This technical appendix differs from Technical Appendix I by allowing the price and wage indexation rules and the interest rate rule to reflect regime changes in the inflation target. Under this alternative specification, we derive the optimizing decisions, describe the stationary equilibrium, and derive the log-linearized equilibrium conditions in the paper entitled "Sources of Macroeconomic Fluctuations: A Regime-Switching DSGE Approach" by Liu, Waggoner, and Zha.

For a quick reference, the equations affected by the dynamic indexation rules and the dynamic Taylor rule include (16), (19), (22), (68), (69), (67), (70), (71), (72), (73), (76), (78), (79), (80), (82), (93), (94), and (103).

## I. The optimizing Decisions

I.1. Households' optimizing decisions. Each household chooses consumption, investment, new capital stock, capacity utilization, and next-period bond to solve the following utility maximizing problem:

$$
\begin{equation*}
\operatorname{Max}_{\left\{C_{t}, I_{t}, K_{t}, u_{t}, B_{t+1}\right\}} \mathrm{E} \sum_{t=0}^{\infty} \beta^{t} A_{t}\left\{\log \left(C_{t}-b C_{t-1}\right)-\frac{\psi}{1+\eta} L_{t+i}^{d}(h)^{1+\eta}\right\} \tag{1}
\end{equation*}
$$

subject to

$$
\begin{gather*}
\bar{P}_{t} C_{t}+\frac{\bar{P}_{t}}{Q_{t}}\left(I_{t}+a\left(u_{t}\right) K_{t-1}\right)+\mathrm{E}_{t} D_{t, t+1} B_{t+1} \leq W_{t}(h) L_{t}^{d}(h)+\bar{P}_{t} r_{k t} u_{t} K_{t-1}+\Pi_{t}+B_{t}+T_{t},  \tag{2}\\
K_{t}=\left(1-\delta_{t}\right) K_{t-1}+\left[1-S\left(\frac{I_{t}}{I_{t-1}}\right)\right] I_{t} \tag{3}
\end{gather*}
$$

Denote by $\mu_{t}$ the Lagrangian multiplier for the budget constraint (2) and by $\mu_{k t}$ the Lagrangian multiplier for the capital accumulation equation (3). The first order conditions for the utility-maximizing problem are given by

$$
\begin{align*}
A_{t} U_{c t} & =\mu_{t} \bar{P}_{t}  \tag{4}\\
D_{t, t+1} & =\beta \frac{\mu_{t+1}}{\mu_{t}}  \tag{5}\\
\frac{\mu_{t} \bar{P}_{t}}{Q_{t}} & =\mu_{k t}\left\{1-S\left(\lambda_{I t}\right)-S^{\prime}\left(\lambda_{I t}\right) \lambda_{I t}\right\}+\beta \mathrm{E}_{t} \mu_{k, t+1} S^{\prime}\left(\lambda_{I, t+1}\right)\left(\lambda_{I, t+1}\right)^{2}  \tag{6}\\
\mu_{k t} & =\beta \mathrm{E}_{t}\left[\mu_{k, t+1}\left(1-\delta_{t+1}\right)+\mu_{t+1} \bar{P}_{t+1} r_{k, t+1} u_{t+1}-\frac{\mu_{t+1} \bar{P}_{t+1}}{Q_{t+1}} a\left(u_{t+1}\right)\right]  \tag{7}\\
r_{k t} & =\frac{a^{\prime}\left(u_{t}\right)}{Q_{t}} \tag{8}
\end{align*}
$$

where $\lambda_{I t} \equiv I_{t} / I_{t-1}$.
Let $q_{k t} \equiv Q_{t} \frac{\mu_{k t}}{\mu_{t} P_{t}}$ denote the shadow price of capital stock (in units of investment goods). Then, (4) and (6) imply that

$$
\begin{equation*}
\frac{1}{Q_{t}}=\frac{q_{k t}}{Q_{t}}\left\{1-S\left(\lambda_{I t}\right)-S^{\prime}\left(\lambda_{I t}\right) \lambda_{I t}\right\}+\beta \mathrm{E}_{t} \frac{q_{k, t+1}}{Q_{t+1}} \frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}} S^{\prime}\left(\lambda_{I, t+1}\right)\left(\lambda_{I, t+1}\right)^{2} \tag{9}
\end{equation*}
$$

Thus, in the absence of adjustment cost or in the steady-state equilibrium where $S\left(\lambda_{I}\right)=S^{\prime}\left(\lambda_{I}\right)=0$, we have $q_{k t}=1$. One can interpret $q_{k t}$ as Tobin's Q.

By eliminating the Lagrangian multipliers $\mu_{t}$ and $\mu_{k t}$, the capital Euler equation (7) can be rewritten as

$$
\begin{equation*}
\frac{q_{k t}}{Q_{t}}=\beta \mathrm{E}_{t} \frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}}\left[\left(1-\delta_{t+1}\right) \frac{q_{k, t+1}}{Q_{t+1}}+r_{k, t+1} u_{t+1}-\frac{a\left(u_{t+1}\right)}{Q_{t+1}}\right] \tag{10}
\end{equation*}
$$

The cost of acquiring a marginal unit of capital is $q_{k t} / Q_{t}$ today (in consumption unit). The benefit of having this extra unit of capital consists of the expected discounted future resale value and the rental value net of utilization cost.

By eliminating the Lagrangian multiplier $\mu_{t}$, the first-order condition with respect to bond holding can be written as

$$
\begin{equation*}
D_{t, t+1}=\beta \frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}} \frac{\bar{P}_{t}}{\bar{P}_{t+1}} \tag{11}
\end{equation*}
$$

Denote by $R_{t}=\left[\mathrm{E}_{t} D_{t, t+1}\right]^{-1}$ the interest rate for a one-period risk-free nominal bond. Then we have

$$
\begin{equation*}
\frac{1}{R_{t}}=\beta \mathrm{E}_{t}\left[\frac{A_{t+1} U_{c, t+1}}{A_{t} U_{c t}} \frac{\bar{P}_{t}}{\bar{P}_{t+1}}\right] \tag{12}
\end{equation*}
$$

In each period $t$, a fraction $\xi_{w}$ of households re-optimize their nominal wage setting decisions. Those households who can re-optimize wage setting chooses the nominal
wage $W_{t}(h)$ to maximize

$$
\begin{array}{r}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \beta^{i} \xi_{w}^{i} A_{t+i}\left[\log \left(C_{t+i}-b C_{t+i-1}\right)-\frac{\psi}{1+\eta} L_{t+i}^{d}(h)^{1+\eta}\right]+ \\
\mu_{t+i}\left[W_{t}(h) \chi_{t, t+i}^{w} L_{t+i}^{d}(h)+m_{t+i}\right] \tag{14}
\end{array}
$$

where the labor demand schedule is given by

$$
\begin{equation*}
L_{t+i}^{d}(h)=\left(\frac{W_{t}(h) \chi_{t, t+i}^{w}}{\bar{W}_{t+i}}\right)^{-\theta_{w t}} L_{t+i}, \quad \theta_{w t}=\frac{\mu_{w t}}{\mu_{w t}-1} \tag{15}
\end{equation*}
$$

the term $m_{t}$ is given by

$$
m_{t}=\bar{P}_{t} r_{k t} u_{t} K_{t-1}+\Pi_{t}+B_{t}+T_{t}-\bar{P}_{t} C_{t}-\frac{\bar{P}_{t}}{Q_{t}}\left(I_{t}+a\left(u_{t}\right) K_{t-1}\right)-\mathrm{E}_{t} D_{t, t+1} B_{t+1}
$$

and the term $\chi_{t, t+i}^{w}$ is given by

$$
\chi_{t, t+i}^{w} \equiv \begin{cases}\Pi_{k=1}^{i} \pi_{t+k-1}^{\gamma_{w}} \pi^{*}\left(s_{t+k}\right)^{1-\gamma_{w}} \lambda_{t, t+i}^{*} & \text { if } i \geq 1  \tag{16}\\ 1 & \text { if } i=0\end{cases}
$$

where $\lambda_{t, t+i}^{*} \equiv \frac{\lambda_{t+i}^{*}}{\lambda_{t}^{*}}$ and $\pi^{*}\left(s_{t}\right)$ is the regime-dependent inflation target.
The first-order condition for the wage-setting problem is given by

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{-A_{t+i} \psi L_{t+i}^{d}(h)^{\eta} \frac{\partial L_{t+i}^{d}(h)}{\partial W_{t}(h)}+\mu_{t+i}\left(1-\theta_{w, t+i}\right) \chi_{t, t+i}^{w} L_{t+i}^{d}(h)\right\}=0 \tag{17}
\end{equation*}
$$

where

$$
\frac{\partial L_{t+i}^{d}(h)}{\partial W_{t}(h)}=-\theta_{w, t+i} \frac{L_{t+i}^{d}(h)}{W_{t}(h)}=-\frac{\mu_{w, t+i}}{\mu_{w, t+i}-1} \frac{L_{t+i}^{d}(h)}{W_{t}(h)}
$$

Factoring out the common terms and rearranging, we obtain

$$
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \frac{\mu_{t+i}}{\mu_{t}} L_{t+i}^{d}(h) \frac{1}{\mu_{w, t+i}-1}\left\{\mu_{w, t+i} \frac{\psi A_{t+i} L_{t+i}^{d}(h)^{\eta}}{\mu_{t+i}}-\chi_{t, t+i}^{w} W_{t}(h)\right\}=0
$$

Let $M R S_{t}(h) \equiv \frac{\psi A_{t} L_{t}^{d}(h)^{\eta}}{\mu_{t}}$ denote the marginal rate of substitution between leisure and income. Then, using (11), we can rewrite the first-order condition for wage setting as

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \xi_{w}^{i} D_{t, t+i} L_{t+i}^{d}(h) \frac{1}{\mu_{w, t+i}-1}\left\{\mu_{w, t+i} M R S_{t+i}(h)-\chi_{t, t+i}^{w} W_{t}(h)\right\}=0 \tag{18}
\end{equation*}
$$

I.2. Firms' optimizing decisions. Pricing decisions are staggered across firms. In each period, a fraction $\xi_{p}$ of firms can re-optimize their pricing decisions and the other fraction $1-\xi_{p}$ of firms mechanically update their prices according to the rule

$$
\begin{equation*}
P_{t}(j)=\pi_{t-1}^{\gamma_{p}} \pi^{*}\left(s_{t}\right)^{1-\gamma_{p}} P_{t-1}(j), \tag{19}
\end{equation*}
$$

If a firm can re-optimize, it chooses $P_{t}(j)$ to solve

$$
\begin{equation*}
\operatorname{Max}_{P_{t}(j)} \quad \mathrm{E}_{t} \sum_{i=0}^{\infty} \xi_{p}^{i} D_{t, t+i}\left[P_{t}(j) \chi_{t, t+i}^{p} Y_{t+i}^{d}(j)-V_{t+i}(j)\right] \tag{20}
\end{equation*}
$$

subject to

$$
\begin{equation*}
Y_{t+i}^{d}(j)=\left(\frac{P_{t}(j) \chi_{t, t+i}^{p}}{\bar{P}_{t+i}}\right)^{-\frac{\mu_{p, t+i}}{\mu_{p, t+i}-1}} Y_{t+i} \tag{21}
\end{equation*}
$$

where $V_{t+i}(j)$ is the cost function and the term $\chi_{t, t+i}^{p}$ comes from the price-updating rule (19) and is given by

$$
\chi_{t, t+i}^{p}= \begin{cases}\Pi_{k=1}^{i} \pi_{t+k-1}^{\gamma_{p}} \pi^{*}\left(s_{t+k}\right)^{1-\gamma_{p}} & \text { if } i \geq 1  \tag{22}\\ 1 & \text { if } i=0\end{cases}
$$

The first order condition for the profit-maximizing problem yields the optimal pricing rule

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \xi_{p}^{i} D_{t, t+i} Y_{t+i}^{d}(j) \frac{1}{\mu_{p, t+i}-1}\left[\mu_{p, t+i} \Phi_{t+i}(j)-P_{t}(j) \chi_{t, t+i}^{p}\right]=0 \tag{23}
\end{equation*}
$$

where $\Phi_{t+i}(j)=\partial V_{t+i}(j) / \partial Y_{t+i}^{d}(j)$ denotes the marginal cost function. In the absence of markup shocks, $\mu_{p t}$ would be a constant and (23) implies that the optimal price is a markup over an average of the marginal costs for the periods in which the price will remain effective. Clearly, if $\xi_{p}=0$ for all $t$, that is, if prices are perfectly flexible, then the optimal price would be a markup over the contemporaneous marginal cost.

Cost-minimizing implies that the marginal cost function is given by

$$
\begin{equation*}
\Phi_{t}(j)=\left[\tilde{\alpha}\left(\bar{P}_{t} r_{k t}\right)^{\alpha_{1}}\left(\frac{\bar{W}_{t}}{Z_{t}}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} Y_{t}(j)^{\frac{1}{\alpha_{1}+\alpha_{2}}-1} \tag{24}
\end{equation*}
$$

where $\tilde{\alpha} \equiv \alpha_{1}^{-\alpha_{1}} \alpha_{2}^{-\alpha_{2}}$ and $r_{k t}$ denotes the real rental rate of capital input. The conditional factor demand functions are given by

$$
\begin{align*}
\bar{W}_{t} & =\Phi_{t}(j) \alpha_{2} \frac{Y_{t}(j)}{L_{t}^{f}(j)}  \tag{25}\\
\bar{P}_{t} r_{k t} & =\Phi_{t}(j) \alpha_{1} \frac{Y_{t}(j)}{K_{t}^{f}(j)} \tag{26}
\end{align*}
$$

It follows that

$$
\begin{equation*}
\frac{\bar{W}_{t}}{\bar{P}_{t} r_{k t}}=\frac{\alpha_{2}}{\alpha_{1}} \frac{K_{t}^{f}(j)}{L_{t}^{f}(j)}, \quad \forall j \in[0,1] \tag{27}
\end{equation*}
$$

I.3. Market clearing. In equilibrium, markets for bond, composite labor, capital stock, and composite goods all clear. Bond market clearing implies that $B_{t}=0$ for all $t$. Labor market clearing implies that $\int_{0}^{1} L_{t}^{f}(j) d j=L_{t}$. Capital market clearing implies that $\int_{0}^{1} K_{t}^{f}(j) d j=u_{t} K_{t-1}$. Composite goods market clearing implies that

$$
\begin{equation*}
C_{t}+\frac{1}{Q_{t}}\left[I_{t}+a\left(u_{t}\right) K_{t-1}\right]+G_{t}=Y_{t} \tag{28}
\end{equation*}
$$

where aggregate output is related to aggregate primary factors through the aggregate production function

$$
\begin{equation*}
G_{p t} Y_{t}=\left(u_{t} K_{t-1}\right)^{\alpha_{1}}\left(Z_{t} L_{t}\right)^{\alpha_{2}} \tag{29}
\end{equation*}
$$

with $G_{p t} \equiv \int_{0}^{1}\left(\frac{P_{t}(j)}{P_{t}}\right)^{-\frac{\mu_{p t}}{\mu_{p t}-1} \frac{1}{\alpha_{1}+\alpha_{2}}} d j$ measuring the price dispersion.

## II. Stationary Equilibrium conditions

Since both the neutral technology and the investment-specific technology are growing over time, we transform the appropriate variables to induce stationarity. In particular, we denote by $\tilde{X}_{t}$ the stationary counterpart of the variable $X_{t}$ and we make the following transformations:

$$
\begin{array}{r}
\tilde{Y}_{t}=\frac{Y_{t}}{\lambda_{t}^{*}}, \quad \tilde{C}_{t}=\frac{C_{t}}{\lambda_{t}^{*}}, \quad \tilde{I}_{t}=\frac{I_{t}}{Q_{t} \lambda_{t}^{*}}, \quad \tilde{G}_{t}=\frac{G_{t}}{\lambda_{t}^{*}}, \quad \tilde{K}_{t}=\frac{K_{t}}{Q_{t} \lambda_{t}^{*}}, \\
\tilde{w}_{t}=\frac{\bar{W}_{t}}{\bar{P}_{t} \lambda_{t}^{*}}, \quad \tilde{r}_{k t}=r_{k t} Q_{t}, \quad \tilde{U}_{c t}=U_{c t} \lambda_{t}^{*}
\end{array}
$$

where the underlying trend for output is given by

$$
\lambda_{t}^{*} \equiv\left(Z_{t}^{\alpha_{2}} Q_{t}^{\alpha_{1}}\right)^{\frac{1}{1-\alpha_{1}}}
$$

II.1. Stationary pricing decisions. In terms of the stationary variables, we can rewrite the optimal pricing decision (23) as

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} A_{t+i} \tilde{U}_{c, t+i} \tilde{Y}_{t+i}^{d}(j) \frac{1}{\mu_{p, t+i}-1}\left[\mu_{p, t+i} \phi_{t+i}(j)-p_{t}^{*} Z_{t, t+i}^{p}\right]=0 \tag{30}
\end{equation*}
$$

In this equation, $\tilde{Y}_{t+i}^{d}(j)=\frac{Y_{t i}^{d}(j)}{\lambda_{t+i}^{*}}$ denotes the detrended output demand; $p_{t}^{*} \equiv \frac{P_{t}(j)}{P_{t}}$ denotes the relative price for optimizing firms, which does not have a $j$ index since all
optimizing firms make identical pricing decisions in a symmetric equilibrium; the term $Z_{t, t+i}^{p}$ is defined as

$$
\begin{equation*}
Z_{t, t+i}^{p}=\frac{\chi_{t, t+i}^{p}}{\prod_{k=1}^{i} \pi_{t+k}} \tag{31}
\end{equation*}
$$

and finally, the term $\phi_{t+i}(j) \equiv \frac{\Phi_{t+i}(j)}{P_{t+i}}$ denotes the real unit cost function, which is given by

$$
\begin{align*}
\phi_{t+i}(j) & =\left[\tilde{\alpha}\left(\frac{\tilde{r}_{k, t+i}}{Q_{t+i}}\right)^{\alpha_{1}}\left(\tilde{w}_{t+i} \frac{\lambda_{t+i}^{*}}{Z_{t+i}}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} Y_{t+i}^{d}(j)^{\frac{1}{\alpha_{1}+\alpha_{2}}-1} \\
& =\left[\tilde{\alpha}\left(\tilde{r}_{k, t+i}\right)^{\alpha_{1}}\left(\tilde{w}_{t+i}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} \tilde{Y}_{t+i}^{d}(j)^{\frac{1}{\alpha_{1}+\alpha_{2}}-1} . \tag{32}
\end{align*}
$$

The demand schedule $\tilde{Y}_{t+i}^{d}(j)$ for the optimizing firm $j$ is related to the relative price and aggregate output through

$$
\begin{align*}
\tilde{Y}_{t+i}^{d}(j) & =\left[\frac{P_{t}(j) \chi_{t, t+i}^{p}}{\bar{P}_{t+i}}\right]^{-\theta_{p, t+i}} \tilde{Y}_{t+i} \\
& =\left[p_{t}^{*} \frac{\bar{P}_{t}}{\bar{P}_{t+i}} \chi_{t, t+i}^{p}\right]^{-\theta_{p, t+i}} \tilde{Y}_{t+i} \\
& =\left[p_{t}^{*} Z_{t, t+i}^{p}\right]^{-\theta_{p, t+i}} \tilde{Y}_{t+i} . \tag{33}
\end{align*}
$$

Combining (32) and (33), we have

$$
\begin{equation*}
\phi_{t+i}(j)=\tilde{\phi}_{t+i}\left[p_{t}^{*} Z_{t, t+i}^{p}\right]^{-\theta_{p, t+i} \bar{\alpha}}\left(\tilde{Y}_{t+i}\right)^{\bar{\alpha}} \tag{34}
\end{equation*}
$$

where $\bar{\alpha} \equiv \frac{1-\alpha_{1}-\alpha_{2}}{\alpha_{1}+\alpha_{2}}$ and

$$
\begin{equation*}
\tilde{\phi}_{t+i} \equiv\left[\tilde{\alpha}\left(\tilde{r}_{k, t+i}\right)^{\alpha_{1}}\left(\tilde{w}_{t+i}\right)^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} . \tag{35}
\end{equation*}
$$

Given these relations, we can rewrite the optimal pricing rule (30) in terms of stationary variables

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} \frac{A_{t+i} \tilde{U}_{c, t+i} \tilde{Y}_{t+i}^{d}(j)}{\mu_{p, t+i}-1}\left[\mu_{p, t+i} \tilde{\phi}_{t+i}\left[p_{t}^{*} Z_{t, t+i}^{p}\right]^{-\theta_{p, t+i} \bar{\alpha}}\left(\tilde{Y}_{t+i}\right)^{\bar{\alpha}}-p_{t}^{*} Z_{t, t+i}^{p}\right]=0 \tag{36}
\end{equation*}
$$

where $\tilde{\phi}$ is defined in (35).
II.2. Stationary wage setting decision. Using (4) and (11), we can rewrite the optimal wage-setting decision (18) as

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \frac{A_{t+i} U_{c, t+i}}{A_{t} U_{c t}} \frac{\bar{P}_{t}}{\bar{P}_{t+i}} L_{t+i}^{d}(h) \frac{1}{\mu_{w, t+i}-1}\left[\mu_{w, t+i} \psi \frac{L_{t+i}^{d}(h)^{\eta}}{U_{c, t+i}} \bar{P}_{t+i}-W_{t}(h) \chi_{t, t+i}^{w}\right]=0 \tag{37}
\end{equation*}
$$

where the labor demand schedule $L_{t+i}^{d}(h)$ is related to aggregate variables through

$$
\begin{align*}
L_{t+i}^{d}(h) & =\left[\frac{W_{t}(h) \chi_{t, t+i}^{w}}{\bar{W}_{t+i}}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{38}\\
& =\left[w_{t}^{*} \frac{\bar{W}_{t}}{\bar{W}_{t+i}} \chi_{t, t+i}^{w}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{39}\\
& =\left[w_{t}^{*} \frac{\tilde{w}_{t} \bar{P}_{t} \lambda_{t}^{*}}{\tilde{w}_{t+i} \bar{P}_{t+i} \lambda_{t+i}^{*}} \chi_{t, t+i}^{w}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{40}\\
& =\left[\frac{w_{t}^{*} \tilde{w}_{t}}{\tilde{w}_{t+i}} \frac{\chi_{t, t+i}^{w}}{\prod_{t=1}^{i} \pi_{t+k} \lambda_{t, t+i}^{*}}\right]^{-\theta_{w, t+i}} L_{t+i}  \tag{41}\\
& \equiv\left[\frac{w_{t}^{*} \tilde{w}_{t}}{\tilde{w}_{t+i}} Z_{t, t+i}^{w}\right]^{-\theta_{w, t+i}} L_{t+i}, \tag{42}
\end{align*}
$$

with $Z_{t, t+i}^{w}$ defined as

$$
\begin{equation*}
Z_{t, t+i}^{w}=\frac{\chi_{t, t+i}^{w}}{\prod_{k=1}^{i} \pi_{t+k} \lambda_{t, t+i}^{*}} \tag{43}
\end{equation*}
$$

Further, we can rewrite the individual optimal nominal wage $W_{t}(h)$ as

$$
W_{t}(h)=w_{t}^{*} \bar{W}_{t}=w_{t}^{*} \tilde{w}_{t} \bar{P}_{t} \lambda_{t}^{*}
$$

Given these relations, we can rewrite the wage setting rule (37) in terms of the stationary variables. With some cancelations, we obtain

$$
\begin{equation*}
\mathrm{E}_{t} \sum_{i=0}^{\infty} \prod_{k=1}^{i}\left(\beta \xi_{w}\right)^{i} \frac{A_{t+i} \tilde{U}_{c, t+i} L_{t+i}^{d}(h)}{\mu_{w, t+i}-1}\left\{\mu_{w, t+i} \psi\left[\frac{w_{t}^{*} \tilde{w}_{t}}{\tilde{w}_{t+i}} Z_{t, t+i}^{w}\right]^{-\eta \theta_{w, t+i}} \frac{L_{t+i}^{\eta}}{\tilde{U}_{c, t+i}}-w_{t}^{*} \tilde{w}_{t} Z_{t, t+i}^{w}\right\}=0 \tag{44}
\end{equation*}
$$

II.3. Other stationary equilibrium conditions. We now rewrite the rest of the equilibrium conditions in terms of stationary variables.

First, the optimal investment decision equation (9) can be written as

$$
\begin{equation*}
1=q_{k t}\left\{1-S\left(\lambda_{I t}\right)-S^{\prime}\left(\lambda_{I t}\right) \lambda_{I t}\right\}+\beta \mathrm{E}_{t} q_{k, t+1} \frac{\lambda_{t}^{*} Q_{t}}{\lambda_{t+1}^{*} Q_{t+1}} \frac{A_{t+1} \tilde{U}_{c, t+1}}{A_{t} \tilde{U}_{c t}} S^{\prime}\left(\lambda_{I, t+1}\right)\left(\lambda_{I, t+1}\right)^{2}, \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{I t}=\frac{I_{t}}{I_{t-1}}=\frac{\tilde{I}_{t} Q_{t} \lambda_{t}^{*}}{\tilde{I}_{t-1} Q_{t-1} \lambda_{t-1}^{*}} \tag{46}
\end{equation*}
$$

Second, the capital Euler equation (10) can be written as

$$
\begin{equation*}
q_{k t}=\beta \mathrm{E}_{t} \frac{A_{t+1} \tilde{U}_{c, t+1}}{A_{t} \tilde{U}_{c t}} \frac{\lambda_{t}^{*} Q_{t}}{\lambda_{t+1}^{*} Q_{t+1}}\left[\left(1-\delta_{t+1}\right) q_{k, t+1}+\tilde{r}_{k, t+1} u_{t+1}-a\left(u_{t+1}\right)\right] . \tag{47}
\end{equation*}
$$

Third, the optimal capacity utilization decision (8) is equivalent to

$$
\begin{equation*}
\tilde{r}_{k t}=a^{\prime}\left(u_{t}\right) \tag{48}
\end{equation*}
$$

Fourth, the intertemporal bond Euler equation (12) can be written as

$$
\begin{equation*}
\frac{1}{R_{t}}=\beta \mathrm{E}_{t}\left[\frac{\lambda_{t}^{*}}{\lambda_{t+1}^{*}} \frac{A_{t+1} \tilde{U}_{c, t+1}}{A_{t} \tilde{U}_{c t}} \frac{1}{\pi_{t+1}}\right] . \tag{49}
\end{equation*}
$$

Fifth, the law of motion for capital stock in (3) can be written as

$$
\begin{equation*}
\tilde{K}_{t}=\left(1-\delta_{t}\right) \frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} \tilde{K}_{t-1}+\left[1-S\left(\lambda_{I t}\right)\right] \tilde{I}_{t} . \tag{50}
\end{equation*}
$$

Sixth, the aggregate resource constraint is given by

$$
\begin{equation*}
\tilde{C}_{t}+\tilde{I}_{t}+\frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} a\left(u_{t}\right) \tilde{K}_{t-1}+\tilde{G}_{t}=\tilde{Y}_{t} \tag{51}
\end{equation*}
$$

Seventh, the aggregate production function (29) can be written as

$$
\begin{equation*}
G_{p t} \tilde{Y}_{t}=\left[\frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} u_{t} \tilde{K}_{t-1}\right]^{\alpha_{1}} L_{t}^{\alpha_{2}} \tag{52}
\end{equation*}
$$

Eighth, firms' cost-minimizing implies that, in the stationary equilibrium, we have

$$
\begin{equation*}
\frac{\tilde{w}_{t}}{\tilde{r}_{k t}}=\frac{\alpha_{2}}{\alpha_{1}} \frac{\lambda_{t-1}^{*} Q_{t-1}}{\lambda_{t}^{*} Q_{t}} \frac{u_{t} \tilde{K}_{t-1}}{L_{t}} \tag{53}
\end{equation*}
$$

Finally, the interest rate rule is given by

$$
\begin{equation*}
R_{t}=\kappa R_{t-1}^{\rho_{r}}\left[r \pi^{*}\left(s_{t}\right)\left(\frac{\pi_{t}}{\pi^{*}\left(s_{t}\right)}\right)^{\phi_{\pi}} \tilde{Y}_{t}^{\phi_{y}}\right]^{1-\rho_{r}} e^{\sigma_{r t} \varepsilon_{r t}}, \tag{54}
\end{equation*}
$$

where $r$ is the steady-state real interest rate and $\kappa$ is a constant that captures the steady-state value of $\tilde{Y}^{-\phi_{y}\left(1-\rho_{r}\right)}$.

## III. Steady State

A deterministic steady state is an equilibrium in which all stochastic shocks are shut off. Our model contains a non-standard "shock": the Markov regime switching in monetary policy regime and the shock regime. In computing the steady-state equilibrium, we shut off all shocks, including the regime shocks. Since there is a mapping between any finite-state Markov switching process and a vector AR(1) process (Hamilton, 1994), shutting off the regime shocks in the steady state is equivalent to setting the innovations in the $\operatorname{AR}(1)$ process to its unconditional mean (which is zero). In such a steady state, all stationary variables are constant.

In the steady state, $p^{*}=1$ and $Z^{p}=1$, so that the price setting rule (36) reduces to

$$
\begin{equation*}
\frac{1}{\mu_{p}}=\left[\tilde{\alpha} \tilde{r}_{k}^{\alpha_{1}} \tilde{w}^{\alpha_{2}}\right]^{\frac{1}{\alpha_{1}+\alpha_{2}}} \tilde{Y}^{\bar{\alpha}} \tag{55}
\end{equation*}
$$

That is, the real marginal cost is constant and equals the inverse markup.
Similarly, in the steady state, $w^{*}=1$ and $Z^{w}=1$, so that the wage setting rule (44) reduces to

$$
\begin{equation*}
\tilde{w}=\mu_{w} \frac{\psi L^{\eta}}{\tilde{U}_{c}} \tag{56}
\end{equation*}
$$

which says that the real wage is a constant markup over the marginal rate of substitution between leisure and consumption.

Given that the steady-state markup, and thus the steady-state real marginal cost, is a constant, the conditional factor demand function (26) for capital input together with the capital market clearing condition imply that

$$
\begin{equation*}
\tilde{r}_{k}=\frac{\alpha_{1}}{\mu_{p}} \frac{\tilde{Y} \lambda_{q} \lambda^{*}}{\tilde{K}} \tag{57}
\end{equation*}
$$

The rest of the steady-state equilibrium conditions for the private sector come from (45) -(53) and are summarized below:

$$
\begin{align*}
1 & =q_{k},  \tag{58}\\
\frac{\lambda_{q} \lambda^{*}}{\beta} & =1-\delta+\tilde{r}_{k},  \tag{59}\\
\tilde{r}_{k} & =a^{\prime}(1),  \tag{60}\\
R & =\frac{\lambda^{*}}{\beta} \pi  \tag{61}\\
\frac{\tilde{I}}{\tilde{K}} & =1-\frac{1-\delta}{\lambda_{q} \lambda^{*}}  \tag{62}\\
\tilde{Y} & =\tilde{C}+\tilde{I}+\tilde{G},  \tag{63}\\
\tilde{Y} & =\left(\frac{\tilde{K}}{\lambda_{q} \lambda^{*}}\right)^{\alpha_{1}} L^{\alpha_{2}},  \tag{64}\\
\frac{\tilde{w}}{\tilde{r}_{k}} & =\frac{1}{\lambda_{q} \lambda^{*}} \frac{\alpha_{2}}{\alpha_{1}} \frac{\tilde{K}}{L} \tag{65}
\end{align*}
$$

## IV. Linearized equilibrium conditions

We now describe our procedure to linearize the stationary equilibrium conditions around the deterministic steady state.
IV.1. Linearizing the price setting rule. Log-linearizing the price setting rule (36) around the steady state, we get

$$
\begin{array}{r}
\mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{y}_{t+i}^{d}(h)-\frac{\mu_{p}}{\mu_{p}-1} \hat{\mu}_{p, t+i}+\hat{\mu}_{p, t+i}+\right. \\
\left.\hat{\tilde{\phi}}_{t+i}-\theta_{p} \bar{\alpha}\left[\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right]+\bar{\alpha} \hat{y}_{t+i}\right\} \\
\approx \mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{y}_{t+i}^{d}(h)-\frac{\mu_{p}}{\mu_{p}-1} \hat{\mu}_{p, t+i}+\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right\},
\end{array}
$$

where

$$
\begin{equation*}
\hat{\tilde{\phi}}_{t+i}=\frac{1}{\alpha_{1}+\alpha_{2}}\left[\alpha_{1} \hat{r}_{k, t+i}+\alpha_{2} \hat{w}_{t+i}\right] \tag{66}
\end{equation*}
$$

Collecting terms to get

$$
\begin{array}{r}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left\{\hat{\mu}_{p, t+i}+\hat{\tilde{\phi}}_{t+i}-\theta_{p} \bar{\alpha}\left[\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right]+\bar{\alpha} \hat{y}_{t+i}\right\} \\
\\
\approx \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left\{\hat{p}_{t}^{*}+\hat{Z}_{t, t+i}^{p}\right\}
\end{array}
$$

Further simplifying

$$
\frac{1+\theta_{p} \bar{\alpha}}{1-\beta \xi_{p}} \hat{p}_{t}^{*}=\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left\{\hat{\mu}_{p, t+i}+\hat{\tilde{\phi}}_{t+i}+\bar{\alpha} \hat{y}_{t+i}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t+i}^{p}\right\}
$$

Denote $\hat{m} c_{t+i} \equiv \hat{\tilde{\phi}}_{t+i}+\bar{\alpha} \hat{y}_{t+i}$. Expanding the infinite sum in the above equation, we get

$$
\begin{aligned}
\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \hat{p}_{t}^{*} & =\hat{\mu}_{p t}+\hat{m} c_{t}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t}^{p} \\
& +\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+1}+\hat{m} c_{t+1}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t+1}^{p}\right] \\
& +\left(\beta \xi_{p}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+2}+\hat{m} c_{t+2}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t, t+2}^{p}\right]+\ldots
\end{aligned}
$$

Forwarding this relation one period to get

$$
\begin{aligned}
\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \hat{p}_{t+1}^{*} & =\hat{\mu}_{p, t+1}+\hat{m} c_{t+1}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t+1, t+1}^{p} \\
& +\beta \xi_{p} \mathrm{E}_{t+1}\left[\hat{\mu}_{p, t+2}+\hat{m} c_{t+2}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t+1, t+2}^{p}\right] \\
& +\left(\beta \xi_{p}\right)^{2} \mathrm{E}_{t+1}\left[\hat{\mu}_{p, t+3}+\hat{m} c_{t+3}-\left(1+\theta_{p} \bar{\alpha}\right) \hat{Z}_{t+1, t+3}^{p}\right]+\ldots
\end{aligned}
$$

Moving the $Z_{t, t+i}^{p}$ terms to the left, we have

$$
\begin{aligned}
& \frac{1+}{1-} \bar{\alpha} \theta_{p} \hat{p}_{t}^{*}+\left(1+\bar{\alpha} \theta_{p}\right) \mathrm{E}_{t}\left[\hat{Z}_{t, t}^{p}+\beta \xi_{p} \hat{Z}_{t, t+1}^{p}+\ldots\right]=\hat{\mu}_{p t}+\hat{m} c_{t} \\
& \quad+\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+1}+\hat{m} c_{t+1}\right] \\
& \quad+\left(\beta \xi_{p}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{p, t+2}+\hat{m} c_{t+2}\right]+\ldots \\
& \quad=\hat{\mu}_{p t}+\hat{m} c_{t} \\
& \quad+\beta \xi_{p}\left[\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \mathrm{E}_{t} \hat{p}_{t+1}^{*}+\left(1+\bar{\alpha} \theta_{p}\right) \mathrm{E}_{t}\left[\hat{Z}_{t+1, t+1}^{p}+\beta \xi_{p} \hat{Z}_{t+1, t+2}^{p}+\ldots\right]\right]
\end{aligned}
$$

Since $\hat{Z}_{t, t}^{p}=0$, we have

$$
\begin{align*}
\frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \hat{p}_{t}^{*} & =\hat{\mu}_{p t}+\hat{m} c_{t}+\beta \xi_{p} \frac{1+\bar{\alpha} \theta_{p}}{1-\beta \xi_{p}} \mathrm{E}_{t} \hat{p}_{t+1}^{*} \\
& +\left(1+\bar{\alpha} \theta_{p}\right) \beta \xi_{p} \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{p}-\hat{Z}_{t, t+i+1}^{p}\right] \tag{67}
\end{align*}
$$

Using the definition of $Z_{t, t+i}^{p}$ in (31), we obtain

$$
\begin{gather*}
\hat{Z}_{t, t+i+1}^{p}=-\left[\hat{\pi}_{t+i+1}-\gamma_{p} \hat{\pi}_{t+i}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+i+1}^{*}+\cdots+\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right],  \tag{68}\\
\hat{Z}_{t+1, t+i+1}^{p}=-\left[\hat{\pi}_{t+i+1}-\gamma_{p} \hat{\pi}_{t+i}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+i+1}^{*}+\cdots+\hat{\pi}_{t+2}-\gamma_{p} \hat{\pi}_{t+1}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+2}^{*}\right] . \tag{69}
\end{gather*}
$$

Thus,

$$
\hat{Z}_{t+1, t+i+1}^{p}-\hat{Z}_{t, t+i+1}^{p}=\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}
$$

and the $Z^{p}$ terms in (67) can be reduced to

$$
\sum_{i=0}^{\infty}\left(\beta \xi_{p}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{p}-\hat{Z}_{t, t+i+1}^{p}\right]=\frac{1}{1-\beta \xi_{p}}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right]
$$

Substituting this result into (67), we obtain

$$
\begin{equation*}
\hat{p}_{t}^{*}=\frac{1-\beta \xi_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right)+\beta \xi_{p} \mathrm{E}_{t} \hat{p}_{t+1}^{*}+\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p t} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right] \tag{70}
\end{equation*}
$$

This completes log-linearizing the optimal price setting equation. We now log-linearize the price index relation. In an symmetric equilibrium, the price index relation is given by

$$
\begin{equation*}
1=\xi_{p}\left[\frac{1}{\pi_{t}} \pi_{t-1}^{\gamma_{p}} \pi^{*}\left(s_{t}\right)^{1-\gamma_{p}}\right]^{\frac{1}{1-\mu_{p t}}}+\left(1-\xi_{p}\right)\left(p_{t}^{*}\right)^{\frac{1}{1-\mu_{p t}}} \tag{71}
\end{equation*}
$$

the linearized version of which is given by

$$
\begin{equation*}
\hat{p}_{t}^{*}=\frac{\xi_{p}}{1-\xi_{p}}\left(\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}-\left(1-\gamma_{p}\right) \hat{\pi}_{t}^{*}\right) . \tag{72}
\end{equation*}
$$

Using (72) to substitute out the $\hat{p}_{t}^{*}$ in (70), we obtain

$$
\begin{aligned}
& \frac{\xi_{p}}{1-\xi_{p}}\left[\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}-\left(1-\gamma_{p}\right) \hat{\pi}_{t}^{*}\right] \\
& \quad=\frac{1-\beta \xi_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right) \\
& \quad+\beta \xi_{p} \frac{\xi_{p}}{1-\xi_{p}} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right]+\beta \xi_{p} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right]
\end{aligned}
$$

or

$$
\begin{equation*}
\hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}-\left(1-\gamma_{p}\right) \hat{\pi}_{t}^{*}=\frac{\kappa_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right)+\beta \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right] \tag{73}
\end{equation*}
$$

where the real marginal cost is given by

$$
\begin{equation*}
\hat{m} c_{t}=\frac{1}{\alpha_{1}+\alpha_{2}}\left[\alpha_{1} \hat{r}_{k, t+i}+\alpha_{2} \hat{w}_{t+i}\right]+\bar{\alpha} \hat{y}_{t} \tag{74}
\end{equation*}
$$

and the term $\kappa_{p}$ is given by

$$
\kappa_{p} \equiv \frac{\left(1-\beta \xi_{p}\right)\left(1-\xi_{p}\right)}{\xi_{p}}
$$

This completes the derivation of the price Phillips curve.
IV.2. Linearizing the optimal wage setting rule. Log-linearizing this wage decision rule, we get

$$
\begin{array}{r}
\mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{l}_{t+i}^{d}(h)-\frac{\mu_{w}}{\mu_{w}-1} \hat{\mu}_{w, t+i}+\hat{\mu}_{w, t+i}-\right. \\
\left.\eta \theta_{w}\left[\hat{w}_{t}^{*}+\hat{w}_{t}-\hat{w}_{t+i}+\hat{Z}_{t, t+i}^{w}\right]+\eta \hat{l}_{t+i}-\hat{u}_{c, t+i}\right\} \\
\approx \mathrm{E}_{t} \ln \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i} \exp \left\{\hat{a}_{t+i}+\hat{u}_{c, t+i}+\hat{l}_{t+i}^{d}(h)-\frac{\mu_{w}}{\mu_{w}-1} \hat{\mu}_{w, t+i}+\hat{w}_{t}^{*}+\hat{w}_{t}+\hat{Z}_{t, t+i}^{w}\right\} .
\end{array}
$$

Collecting terms to get

$$
\begin{array}{r}
\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{\hat{\mu}_{w, t+i}-\eta \theta_{w}\left[\hat{w}_{t}^{*}+\hat{w}_{t}-\hat{w}_{t+i}+\hat{Z}_{t, t+i}^{w}\right]+\eta \hat{l}_{t+i}-\hat{u}_{c, t+i}\right\} \\
\\
\approx \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{\hat{w}_{t}^{*}+\hat{w}_{t}+\hat{Z}_{t, t+i}^{w}\right\}
\end{array}
$$

Further simplifying
$\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right)=\mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left\{\hat{\mu}_{w, t+i}+\eta \hat{l}_{t+i}-\hat{u}_{c, t+i}+\eta \theta_{w} \hat{w}_{t+i}-\left(1+\eta \theta_{w}\right) \hat{Z}_{t, t+i}^{w}\right\}$.

Denote $m \hat{r} s_{t+i} \equiv \eta \hat{l}_{t+i}-\hat{u}_{c, t+i}$. Expanding the infinite sum in the above equation, we get

$$
\begin{aligned}
\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right) & =\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t}-\hat{Z}_{t, t}^{w}\right) \\
& +\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+1}+m \hat{r} s_{t+1}-\hat{w}_{t+1}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+1}-\hat{Z}_{t, t+1}^{w}\right)\right] \\
& +\left(\beta \xi_{w}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+2}+m \hat{r} s_{t+2}-\hat{w}_{t+2}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+2}-\hat{Z}_{t, t+2}^{w}\right)\right]+\ldots
\end{aligned}
$$

Forwarding this relation one period to get

$$
\begin{aligned}
\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right) & =\hat{\mu}_{w, t+1}+m \hat{r} s_{t+1}-\hat{w}_{t+1}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+1}-\hat{Z}_{t+1, t+1}^{w}\right) \\
& +\beta \xi_{w} \mathrm{E}_{t+1}\left[\hat{\mu}_{w, t+2}+m \hat{r} s_{t+2}-\hat{w}_{t+2}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+2}-\hat{Z}_{t+1, t+2}^{w}\right)\right] \\
& +\left(\beta \xi_{w}\right)^{2} \mathrm{E}_{t+1}\left[\hat{\mu}_{w, t+3}+m \hat{r} s_{t+3}-\hat{w}_{t+3}+\left(1+\eta \theta_{w}\right)\left(\hat{w}_{t+3}-\hat{Z}_{t+1, t+3}^{w}\right)\right]+\ldots
\end{aligned}
$$

Moving the $Z_{t, t+i}^{w}$ terms to the left, we have

$$
\begin{aligned}
& \frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right)+\left(1+\eta \theta_{w}\right) \mathrm{E}_{t}\left[\hat{Z}_{t, t}^{w}+\beta \xi_{w} \hat{Z}_{t, t+1}^{w}+\ldots\right]=\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right) \hat{w}_{t} \\
& \quad+\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+1}+m \hat{r} s_{t+1}-\hat{w}_{t+1}+\left(1+\eta \theta_{w}\right) \hat{w}_{t+1}\right] \\
& \quad+\left(\beta \xi_{w}\right)^{2} \mathrm{E}_{t}\left[\hat{\mu}_{w, t+2}+m \hat{r} s_{t+2}-\hat{w}_{t+2}+\left(1+\eta \theta_{w}\right) \hat{w}_{t+2}\right]+\ldots \\
& \quad=\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right) \hat{w}_{t} \\
& \quad+\beta \xi_{w} \mathrm{E}_{t}\left[\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right)+\left(1+\eta \theta_{w}\right)\left[\hat{Z}_{t+1, t+1}^{w}+\beta \xi_{w} \hat{Z}_{t+1, t+2}^{w}+\ldots\right]\right]
\end{aligned}
$$

Since $\hat{Z}_{t, t}^{w}=0$, we have

$$
\begin{align*}
\frac{1+\eta \theta_{w}}{1-\beta \xi_{w}}\left(\hat{w}_{t}^{*}+\hat{w}_{t}\right) & =\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}+\left(1+\eta \theta_{w}\right) \hat{w}_{t}+\beta \xi_{w} \frac{1+\eta \theta_{w}}{1-\beta \xi_{w}} \mathrm{E}_{t}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right) \\
& +\left(1+\eta \theta_{w}\right) \beta \xi_{w} \mathrm{E}_{t} \sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{w}-\hat{Z}_{t, t+i+1}^{w}\right] . \tag{75}
\end{align*}
$$

Using the definition of $Z_{t, t+i}^{w}$ in (43), we obtain

$$
\begin{aligned}
\hat{Z}_{t, t+i+1}^{w} & =-\left[\hat{\pi}_{t+i+1}-\gamma_{w} \hat{\pi}_{t+i}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+i+1}^{*}+\cdots+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right] \\
\hat{Z}_{t+1, t+i+1}^{w} & =-\left[\hat{\pi}_{t+i+1}-\gamma_{w} \hat{\pi}_{t+i}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+i+1}^{*}+\cdots+\hat{\pi}_{t+2}-\gamma_{w} \hat{\pi}_{t+1}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+2}^{*}\right] .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
\hat{Z}_{t+1, t+i+1}^{w}-\hat{Z}_{t, t+i+1}^{w}=\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*} \tag{76}
\end{equation*}
$$

and the $Z^{w}$ terms in (75) can be reduced to

$$
\sum_{i=0}^{\infty}\left(\beta \xi_{w}\right)^{i}\left[\hat{Z}_{t+1, t+i+1}^{w}-\hat{Z}_{t, t+i+1}^{w}\right]=\frac{1}{1-\beta \xi_{w}}\left[\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right]
$$

Substituting this result into (75), we obtain

$$
\begin{equation*}
\hat{w}_{t}^{*}+\hat{w}_{t}=\frac{1-\beta \xi_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+\left(1-\beta \xi_{w}\right) \hat{w}_{t}+\beta \xi_{w} \mathrm{E}_{t}\left(\hat{w}_{t+1}^{*}+\hat{w}_{t+1}\right)+\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right] . \tag{77}
\end{equation*}
$$

This completes log-linearizing the wage decision equation. We now log-linearize the wage index relation. In an symmetric equilibrium, the wage index relation is given by

$$
\begin{equation*}
1=\xi_{w}\left[\frac{\tilde{w}_{t-1}}{\tilde{w}_{t}} \frac{1}{\pi_{t}} \pi_{t-1}^{\gamma_{w}} \pi^{*}\left(s_{t}\right)^{1-\gamma_{w}}\right]^{\frac{1}{1-\mu_{w t}}}+\left(1-\xi_{w}\right)\left(w_{t}^{*}\right)^{\frac{1}{1-\mu_{w t}}} \tag{78}
\end{equation*}
$$

the linearized version of which is given by

$$
\begin{equation*}
\left.\hat{w}_{t}^{*}=\frac{\xi_{w}}{1-\xi_{w}}\left(\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}\right)-\left(1-\gamma_{w}\right) \hat{\pi}_{t}^{*}\right] . \tag{79}
\end{equation*}
$$

Using (79) to substitute out the $\hat{w}_{t}^{*}$ in (77), we obtain

$$
\begin{aligned}
\hat{w}_{t} & +\frac{\xi_{w}}{1-\xi_{w}}\left[\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}-\left(1-\gamma_{w}\right) \hat{\pi}_{t}^{*}\right] \\
& =\frac{1-\beta \xi_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+\left(1-\beta \xi_{w}\right) \hat{w}_{t} \\
& +\beta \xi_{w} \mathrm{E}_{t}\left\{\hat{w}_{t+1}+\frac{\xi_{w}}{1-\xi_{w}}\left[\hat{w}_{t+1}-\hat{w}_{t}+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right]\right\} \\
& +\beta \xi_{w} \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right]
\end{aligned}
$$

or

$$
\begin{gather*}
\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}-\left(1-\gamma_{w}\right) \hat{\pi}_{t}^{*}=\frac{\kappa_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+ \\
\quad \beta \mathrm{E}_{t}\left[\hat{w}_{t+1}-\hat{w}_{t}+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right], \tag{80}
\end{gather*}
$$

where $\kappa_{w} \equiv \frac{\left(1-\beta \xi_{w}\right)\left(1-\xi_{w}\right)}{\xi_{w}}$.
To help understand the economics behind this equation, we define the nominal wage inflation as

$$
\begin{equation*}
\pi_{t}^{w}=\frac{\bar{W}_{t}}{\bar{W}_{t-1}}=\frac{\tilde{w}_{t} \bar{P}_{t} \lambda_{t}^{*}}{\tilde{w}_{t-1} \bar{P}_{t-1} \lambda_{t-1}^{*}}=\frac{\tilde{w}_{t}}{\tilde{w}_{t-1}} \pi_{t} \lambda_{t-1, t}^{*} \tag{81}
\end{equation*}
$$

The log-linearized version is given by

$$
\hat{\pi}_{t}^{w}=\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}+\Delta \hat{\lambda}_{t}^{*}
$$

where $\Delta x_{t}=x_{t}-x_{t-1}$ is the first-difference operator and $\hat{\lambda}_{t}^{*}=\frac{1}{1-\alpha_{1}}\left(\alpha_{1} \hat{q}_{t}+\alpha_{2} \hat{z}_{t}\right)$. Thus, the optimal wage decision (80) is equivalent to

$$
\begin{array}{r}
\hat{\pi}_{t}^{w}-\gamma_{w} \hat{\pi}_{t-1}-\left(1-\gamma_{w}\right) \hat{\pi}_{t}^{*}=\frac{\kappa_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+\beta \mathrm{E}_{t}\left(\hat{\pi}_{t+1}^{w}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right) \\
+\frac{1}{1-\alpha_{1}}\left[\alpha_{1}\left(\Delta \hat{z}_{t}-\beta \mathrm{E}_{t} \Delta \hat{z}_{t+1}\right)+\alpha_{2}\left(\Delta \hat{q}_{t}-\beta \mathrm{E}_{t} \Delta \hat{q}_{t+1}\right)\right] .(8 \tag{82}
\end{array}
$$

This nominal-wage Phillips curve relation parallels that of the price-Phillips curve and has similar interpretations.
IV.3. Linearizing other stationary equilibrium conditions. Taking total differentiation in the investment decision equation (45) and using the steady-state conditions that $S\left(\lambda_{I}\right)=S^{\prime}\left(\lambda_{I}\right)=0$, we obtain

$$
\begin{equation*}
\hat{q}_{k t}=S^{\prime \prime}\left(\lambda_{I}\right) \lambda_{I}^{2}\left[\hat{\lambda}_{I t}-\beta \mathrm{E}_{t} \hat{\lambda}_{I, t+1}\right] \tag{83}
\end{equation*}
$$

which, combined with the definition of the investment growth rate

$$
\begin{equation*}
\hat{\lambda}_{I t}=\Delta \hat{i}_{t}+\frac{1}{1-\alpha_{1}}\left[\Delta \hat{q}_{t}+\alpha_{2} \Delta \hat{z}_{t}\right] \tag{84}
\end{equation*}
$$

implies the linearized investment decision equation in the text.
Taking total differentiation in the capital Euler equation (47) and using the steadystate conditions that $\tilde{q}_{k}=1, u=1, a(1)=0, \tilde{r}_{k}=a^{\prime}(1)$, and $\frac{\beta}{\lambda_{I}}\left(1-\delta+\tilde{r}_{k}\right)=1$, we obtain

$$
\begin{equation*}
\hat{q}_{k t}=\mathrm{E}_{t}\left\{\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\Delta \hat{\lambda}_{t+1}^{*}-\Delta \hat{q}_{t+1}+\frac{\beta}{\lambda_{I}}\left[(1-\delta) \hat{q}_{k, t+1}-\delta \hat{\delta}_{t+1}+\tilde{r}_{k} \hat{r}_{k, t+1}\right]\right\} \tag{85}
\end{equation*}
$$

which, upon substituting the expressions for the $\Delta \hat{\lambda}_{t}^{*}$ and $\Delta \hat{q}_{t}$, implies the linearized capital Euler equation in the text.

The linearized capacity utilization decision equation (48) is given by

$$
\begin{equation*}
\hat{r}_{k t}=\sigma_{u} \hat{u}_{t} \tag{86}
\end{equation*}
$$

where $\sigma_{u} \equiv \frac{a^{\prime \prime}(1)}{a^{\prime}(1)}$ is the curvature parameter for the capacity utility function $a(u)$ evaluated at the steady state.

The linearized intertemporal bond Euler equation (49) is given by

$$
\begin{equation*}
0=\mathrm{E}_{t}\left[\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\Delta \hat{\lambda}_{t+1}^{*}+\hat{R}_{t}-\hat{\pi}_{t+1}\right] \tag{87}
\end{equation*}
$$

which, along with the definition of the exogenous term $\Delta \hat{\lambda}_{t+1}^{*}$, implies the linearized bond Euler equation in the text.

Log-linearize the capital law of motion (50) leads to

$$
\begin{equation*}
\hat{k}_{t}=\frac{1-\delta}{\lambda_{I}}\left[\hat{k}_{t-1}-\Delta \hat{\lambda}_{t}^{*}-\Delta \hat{q}_{t}\right]-\frac{\delta}{\lambda_{I}} \hat{\delta}_{t}+\frac{\tilde{I}}{\tilde{K}^{\prime}} \hat{i}_{t}, \tag{88}
\end{equation*}
$$

which implies the linearized capital law of motion in the text.
To obtain the linearized resource constraint, we take total differentiation of (51) to obtain

$$
\begin{equation*}
\hat{y}_{t}=c_{y} \hat{c}_{t}+i_{y} \hat{i}_{t}+u_{y} \hat{u}_{t}+g_{y} \hat{g}_{t}, \tag{89}
\end{equation*}
$$

where $c_{y}=\frac{\tilde{C}}{\tilde{Y}}, i_{y}=\frac{\tilde{I}}{\tilde{Y}}, u_{y}=\frac{\tilde{r}_{k} \tilde{K}}{\tilde{Y} \lambda_{I}}$, and $g_{y}=\frac{\tilde{G}}{\tilde{Y}}$.
Log-linearizing the aggregate production function (52), we get

$$
\begin{align*}
\hat{y}_{t} & =\alpha_{1}\left[\hat{k}_{t-1}+\hat{u}_{t}-\Delta \hat{\lambda}_{t}^{*}-\Delta \hat{q}_{t}\right]+\alpha_{2} \hat{l}_{t} \\
& =\alpha_{1}\left[\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)\right]+\alpha_{2} \hat{l}_{t} \tag{90}
\end{align*}
$$

The linearized version of the factor demand relation (53) is given by

$$
\begin{align*}
\hat{w}_{t} & =\hat{r}_{k t}+\hat{k}_{t-1}+\hat{u}_{t}-\Delta \hat{\lambda}_{t}^{*}-\Delta \hat{q}_{t}-\hat{l}_{t} \\
& =\hat{r}_{k t}+\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)-\hat{l}_{t} \tag{91}
\end{align*}
$$

Finally, linearizing the interest rate rule (54) gives

$$
\begin{equation*}
\hat{R}_{t}=\rho_{r} \hat{R}_{t-1}+\left(1-\rho_{r}\right)\left[\phi_{\pi} \hat{\pi}_{t}+\left(1-\phi_{\pi}\right) \hat{\pi}_{t}^{*}+\phi_{y} \hat{y}_{t}\right]+\sigma_{r t} \varepsilon_{r t}, \tag{92}
\end{equation*}
$$

where

$$
\hat{\pi}_{t}^{*} \equiv \log \pi^{*}\left(s_{t}\right)-\log \pi
$$

Note that, with regime-switching inflation target, we have

$$
\hat{\pi}_{t}^{*}=\mathbf{1}\left\{s_{t}=1\right\} \hat{\pi}^{*}(1)+\mathbf{1}\left\{s_{t}=2\right\} \hat{\pi}^{*}(2)=\left[\hat{\pi}^{*}(1), \hat{\pi}^{*}(2)\right] e_{s_{t}},
$$

where

$$
e_{s_{t}}=\left[\begin{array}{l}
\mathbf{1}\left\{s_{t}=1\right\} \\
\mathbf{1}\left\{s_{t}=2\right\}
\end{array}\right]
$$

It is useful to use the result that the random vector $e_{s_{t}}$ follows an $\mathrm{AR}(1)$ process:

$$
e_{s_{t}}=Q e_{s_{t-1}}+v_{t}
$$

where $Q$ is the Markov transition matrix of the regime and $\mathrm{E}_{t-1} v_{t}=0$.
IV.4. Summary of linearized equilibrium conditions. We now summarize the linearized equilibrium conditions to be used for solving and estimating the model. Note that

$$
E_{t} \hat{\pi}_{t+1}^{*}=\left[\hat{\pi}^{*}(1) \hat{\pi}^{*}(2)\right] E_{t} e_{s_{t+1}}=\left[\hat{\pi}^{*}(1) \hat{\pi}^{*}(2)\right] Q e_{s_{t}}
$$

The log-linearized equations are listed below.

$$
\begin{align*}
& \hat{\pi}_{t}-\gamma_{p} \hat{\pi}_{t-1}-\left(1-\gamma_{p}\right) \hat{\pi}_{t}^{*}=\frac{\kappa_{p}}{1+\bar{\alpha} \theta_{p}}\left(\hat{\mu}_{p t}+\hat{m} c_{t}\right)+\beta \mathrm{E}_{t}\left[\hat{\pi}_{t+1}-\gamma_{p} \hat{\pi}_{t}-\left(1-\gamma_{p}\right) \hat{\pi}_{t+1}^{*}\right],  \tag{93}\\
& \hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}-\gamma_{w} \hat{\pi}_{t-1}-\left(1-\gamma_{w}\right) \hat{\pi}_{t}^{*}=\frac{\kappa_{w}}{1+\eta \theta_{w}}\left(\hat{\mu}_{w t}+m \hat{r} s_{t}-\hat{w}_{t}\right)+ \\
& \beta \mathrm{E}_{t}\left[\hat{w}_{t+1}-\hat{w}_{t}+\hat{\pi}_{t+1}-\gamma_{w} \hat{\pi}_{t}-\left(1-\gamma_{w}\right) \hat{\pi}_{t+1}^{*}\right] .  \tag{94}\\
& \hat{q}_{k t}=S^{\prime \prime}\left(\lambda_{I}\right) \lambda_{I}^{2}\left\{\Delta \hat{i}_{t}+\frac{1}{1-\alpha_{1}}\left(\Delta \hat{q}_{t}+\alpha_{2} \Delta \hat{z}_{t}\right)\right. \\
& \left.-\beta \mathrm{E}_{t}\left[\Delta \hat{i}_{t+1}+\frac{1}{1-\alpha_{1}}\left(\Delta \hat{q}_{t+1}+\alpha_{2} \Delta \hat{z}_{t+1}\right)\right]\right\}  \tag{95}\\
& \hat{q}_{k t}=\mathrm{E}_{t}\left\{\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\frac{1}{1-\alpha_{1}}\left[\alpha_{2} \Delta \hat{z}_{t+1}+\Delta \hat{q}_{t+1}\right]\right. \\
& \left.+\frac{\beta}{\lambda_{I}}\left[(1-\delta) \hat{q}_{k, t+1}-\delta \hat{\delta}_{t+1}+\tilde{r}_{k} \hat{r}_{k, t+1}\right]\right\},  \tag{96}\\
& \hat{r}_{k t}=\sigma_{u} \hat{u}_{t},  \tag{97}\\
& 0=\mathrm{E}_{t}\left[\Delta \hat{a}_{t+1}+\Delta \hat{U}_{c, t+1}-\frac{1}{1-\alpha_{1}}\left[\alpha_{2} \Delta \hat{z}_{t+1}+\alpha_{1} \Delta \hat{q}_{t+1}\right]+\hat{R}_{t}-\hat{\pi}_{t+1}\right] \text { (98) } \\
& \hat{k}_{t}=\frac{1-\delta}{\lambda_{I}}\left[\hat{k}_{t-1}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)\right]-\frac{\delta}{\lambda_{I}} \hat{\delta}_{t}+\left(1-\frac{1-\delta}{\lambda_{I}}\right) \hat{i}_{t}, \quad \text { (99) } \\
& \hat{y}_{t}=c_{y} \hat{c}_{t}+i_{y} \hat{i}_{t}+u_{y} \hat{u}_{t}+g_{y} \hat{g}_{t},  \tag{100}\\
& \hat{y}_{t}=\alpha_{1}\left[\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)\right]+\alpha_{2} \hat{l}_{t},  \tag{101}\\
& \hat{w}_{t}=\hat{r}_{k t}+\hat{k}_{t-1}+\hat{u}_{t}-\frac{1}{1-\alpha_{1}}\left(\alpha_{2} \Delta \hat{z}_{t}+\Delta \hat{q}_{t}\right)-\hat{l}_{t},  \tag{102}\\
& \hat{R}_{t}=\rho_{r} \hat{R}_{t-1}+\left(1-\rho_{r}\right)\left[\phi_{\pi} \hat{\pi}_{t}+\left(1-\phi_{\pi}\right) \hat{\pi}_{t}^{*}+\phi_{y} \hat{y}_{t}\right]+\sigma_{r t} \varepsilon_{r t}, \tag{103}
\end{align*}
$$

where

$$
\begin{align*}
\hat{m} c_{t}= & \frac{1}{\alpha_{1}+\alpha_{2}}\left[\alpha_{1} \hat{r}_{k t}+\alpha_{2} \hat{w}_{t}\right]+\bar{\alpha} \hat{y}_{t}  \tag{104}\\
\dot{m r} s_{t}= & \eta \hat{l}_{t}-\hat{U}_{c t},  \tag{105}\\
\hat{U}_{c t}= & \frac{\beta b\left(1-\rho_{a}\right)}{\lambda_{*}-\beta b} \hat{a}_{t}-\frac{\lambda_{*}}{\left(\lambda_{*}-b\right)\left(\lambda_{*}-\beta b\right)}\left[\lambda_{*} \hat{c}_{t}-b\left(\hat{c}_{t-1}-\Delta \hat{\lambda}_{t}^{*}\right)\right] \\
& +\frac{\beta b}{\left(\lambda_{*}-b\right)\left(\lambda_{*}-\beta b\right)}\left[\lambda_{*} \mathrm{E}_{t}\left(\hat{c}_{t+1}+\Delta \hat{\lambda}_{t+1}^{*}\right)-b \hat{c}_{t}\right]  \tag{106}\\
\hat{\pi}_{t}^{*}= & {\left[\hat{\pi}^{*}(1), \hat{\pi}^{*}(2)\right] e_{s_{t}}, \quad e_{s_{t}}=Q e_{s_{t-1}}+v_{t} } \tag{107}
\end{align*}
$$

and the steady-state variables are given by

$$
\begin{align*}
\tilde{r}_{k} & =\frac{\lambda_{I}}{\beta}-(1-\delta)  \tag{109}\\
u_{y} & \equiv \frac{\tilde{r}_{k} \tilde{K}}{\tilde{Y} \lambda_{I}}=\frac{\alpha_{1}}{\mu_{p}}  \tag{110}\\
i_{y} & =\left[\lambda_{I}-(1-\delta)\right] \frac{\alpha_{1}}{\mu_{p} \tilde{r}_{k}}  \tag{111}\\
c_{y} & =1-i_{y}-g_{y} \tag{112}
\end{align*}
$$

with $\lambda_{I} \equiv\left(\lambda_{q} \lambda_{z}^{\alpha_{2}}\right)^{\frac{1}{1-\alpha_{1}}}, \lambda_{*} \equiv\left(\lambda_{z}^{\alpha_{2}} \lambda_{q}^{\alpha_{1}}\right)^{\frac{1}{1-\alpha_{1}}}, \Delta \hat{\lambda}_{t}^{*} \equiv \frac{1}{1-\alpha_{1}}\left(\alpha_{1} \Delta \hat{q}_{t}+\alpha_{2} \Delta \hat{z}_{t}\right)$, and $g_{y}$ calibrated to match the average ratio of government spending to real GDP.

Recall that $\theta_{p} \equiv \frac{\mu_{p}}{\mu_{p}-1}, \Delta x_{t}=x_{t}-x_{t-1}, \kappa_{p} \equiv \frac{\left(1-\beta \xi_{p}\right)\left(1-\xi_{p}\right)}{\xi_{p}}, \bar{\alpha} \equiv \frac{1-\alpha_{1}-\alpha_{2}}{\alpha_{1}+\alpha_{2}}, \theta_{w} \equiv \frac{\mu_{w}}{\mu_{w}-1}$, $\kappa_{w} \equiv \frac{\left(1-\beta \xi_{w}\right)\left(1-\xi_{w}\right)}{\xi_{w}}$, and $\hat{\pi}_{t}^{w}=\hat{w}_{t}-\hat{w}_{t-1}+\hat{\pi}_{t}+\Delta \hat{\lambda}_{t}^{*}$,

To compute the equilibrium, we eliminate $\hat{u}_{t}$ by using (100), leaving 10 equations (93)-(99) and (101)-(103) with 10 variables $\hat{\pi}_{t}, \hat{w}_{t}, \hat{i}_{t}, \hat{q}_{k t}, \hat{r}_{k t}, \hat{c}_{t}, \hat{k}_{t}, \hat{y}_{t}, \hat{l}_{t}$, and $\hat{R}_{t}$. Out of these 10 variables, we have 7 observable variables, that is, all but $\hat{q}_{k t}, \hat{r}_{k t}$, and $\hat{k}_{t}$, for our estimation. We also include the biased technology shock $\hat{q}_{t}$ in the set of observable variables.

## V. General setup for estimation

In this section, we describe our empirical strategy in general terms so that the method can be applied to any state-space-form model.

Consider a regime-switching DSGE model with $s_{t}$ following a Markov-switching process. Let $\theta$ be a vector of all the model parameters except the transition matrix for $s_{t}$. Let $y_{t}$ be an $n \times 1$ vector of observable variables. In our case, $n=8$. The vector $y_{t}$ is connected to the state vector $f_{t}$. For our regime-switching DSGE model, this state-space representation implies a non-standard Kalman-filter problem as discussed in Kim and Nelson (1999).

Let $\left(Y_{t}, \theta, Q, S_{t}\right)$ be a collection of random variables where

$$
\begin{aligned}
Y_{t} & =\left(y_{1}, \cdots, y_{t}\right) \in\left(\mathbb{R}^{n}\right)^{t} \\
\theta & =\left(\theta_{i}\right)_{i \in H} \in\left(\mathbb{R}^{r}\right)^{h} \\
Q & =\left(q_{i, j}\right)_{(i, j) \in H \times H} \in \mathbb{R}^{h^{2}} \\
S_{t} & =\left(s_{0}, \cdots, s_{t}\right) \in H^{t+1}, \\
S_{t+1}^{T} & =\left(s_{t+1}, \cdots, s_{T}\right) \in H^{T-t},
\end{aligned}
$$

and $H$ is a finite set with $h$ elements and is usually taken to be the set $\{1, \cdots, h\}$. Because $s_{t}$ represents a composite regime, $h$ can be greater than the actual number of regimes at time $t$. The matrix $Q$ is the Markov transition matrix and $q_{i, j}$ is the probability that $s_{t}$ is equal to $i$ given that $s_{t-1}$ is equal to $j$. The matrix $Q$ is restricted to satisfy

$$
q_{i, j} \geq 0 \text { and } \sum_{i \in H} q_{i, j}=1
$$

The object $\theta$ is a vector of all the model parameters except the elements in $Q$. The object $S_{t}$ represents a sequence of unobserved regimes or states. We assume that
$\left(Y_{t}, \theta, Q, S_{t}\right)$ has a joint density function $p\left(Y_{t}, \theta, Q, S_{t}\right)$, where we use the Lebesgue measure on $\left(\mathbb{R}^{n}\right)^{t} \times\left(\mathbb{R}^{r}\right)^{h} \times \mathbb{R}^{h^{2}}$ and the counting measure on $H^{t+1}$. This density satisfies the following key condition.

Condition 1.

$$
p\left(s_{t} \mid Y_{t-1}, \theta, Q, S_{t-1}\right)=q_{s_{t}, s_{t-1}}
$$

for $t>0$.
V.1. Propositions for Hamilton filter. Given $p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right)$ for all $t$, the following propositions follow from Condition 1 (Hamilton, 1989; Chib, 1996; Sims, Waggoner, and Zha, 2008).

Proposition 1.

$$
p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)=\sum_{s_{t-1} \in H} q_{s_{t}, s_{t-1}} p\left(s_{t-1} \mid Y_{t-1}, \theta, Q\right)
$$

for $t>0$.
Proposition 2.

$$
p\left(s_{t} \mid Y_{t}, \theta, Q\right)=\frac{p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)}{\sum_{s_{t-1} \in H} p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)}
$$

for $t>0$.
Proposition 3.

$$
p\left(s_{t} \mid Y_{t}, \theta, Q, s_{t+1}\right)=p\left(s_{t} \mid Y_{T}, \theta, Q, S_{t+1}^{T}\right)
$$

for $0 \leq t<T$.
V.2. Likelihood. We follow the standard assumption in the literature that the initial data $Y_{0}$ is taken as given. Using Kim and Nelson (1999)'s Kalman-filter updating procedure, we obtain the conditional likelihood function at time $t$

$$
\begin{equation*}
p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) \tag{113}
\end{equation*}
$$

It follows from the rules of conditioning that

$$
\begin{aligned}
p\left(y_{t}, \mid Y_{t-1}, \theta, Q\right) & =\sum_{s_{t} \in H} p\left(y_{t}, s_{t} \mid Y_{t-1}, \theta, Q\right) \\
& =\sum_{s_{t} \in H} p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)
\end{aligned}
$$

Using (113) and the above equation, one can show that the likelihood function of $Y_{T}$ is

$$
\begin{align*}
p\left(Y_{T} \mid \theta, Q\right) & =\prod_{t=1}^{T} p\left(y_{t} \mid Y_{t-1}, \theta, Q\right) \\
& =\prod_{t=1}^{T}\left[\sum_{s_{t} \in H} p\left(y_{t} \mid Y_{t-1}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t-1}, \theta, Q\right)\right] . \tag{114}
\end{align*}
$$

We assume that $p\left(s_{0} \mid Y_{0}, \theta, Q\right)=\frac{1}{h}$ for every $s_{0} \in H .{ }^{1}$ Given this initial condition, the likelihood function (114) can be evaluated recursively, using Propositions 1 and 2.
V.3. Posterior distributions. The prior for all the parameters is denoted by $p(\theta, Q)$, which will be discussed further in the main text of the article. By the Bayes rule, it follows from (114) that the posterior distribution of $(\theta, Q)$ is

$$
\begin{equation*}
p\left(\theta, Q \mid Y_{T}\right) \propto p(\theta, Q) p\left(Y_{T} \mid \theta, Q\right) \tag{115}
\end{equation*}
$$

The posterior density $p\left(\theta, Q \mid Y_{T}\right)$ is unknown and complicated; the Monte Carlo Markov Chain (MCMC) simulation directly from this distribution can be inefficient and problematic. One can, however, use the idea of Gibbs sampling to obtain the empirical joint posterior density $p\left(\theta, Q, S_{T} \mid Y_{T}\right)$ by sampling alternately from the following conditional posterior distributions:

$$
\begin{aligned}
& p\left(S_{T} \mid Y_{T}, \theta, Q\right) \\
& p\left(Q \mid Y_{T}, S_{T}, \theta\right) \\
& p\left(\theta \mid Y_{T}, Q, S_{T}\right)
\end{aligned}
$$

One can use the Metropolis-Hastings sampler to sample from the conditional posterior distributions $p\left(\theta \mid Y_{T}, Q, S_{T}\right)$ and $p\left(Q \mid Y_{T}, S_{T}, \theta\right)$. To simulate from the distribution $p\left(S_{T} \mid Y_{T}, \theta, Q\right)$, we can see from the rules of conditioning that

$$
\begin{align*}
p\left(S_{T} \mid Y_{T}, \theta, Q\right) & =p\left(s_{T} \mid Y_{T}, \theta, Q\right) p\left(S_{T-1} \mid Y_{T}, \theta, Q, S_{T}^{T}\right) \\
& =p\left(s_{T} \mid Y_{T}, \theta, Q\right) \prod_{t=0}^{T-1} p\left(s_{t} \mid Y_{T}, \theta, Q, S_{t+1}^{T}\right) \tag{116}
\end{align*}
$$

[^1]where $S_{t+1}^{T}=\left\{s_{t+1}, \cdots, s_{T}\right\}$. From Proposition 3,
\[

$$
\begin{align*}
p\left(s_{t} \mid Y_{T}, \theta, Q, S_{t+1}^{T}\right) & =p\left(s_{t} \mid Y_{t}, \theta, Q, s_{t+1}\right) \\
& =\frac{p\left(s_{t}, s_{t+1} \mid Y_{t}, \theta, Q\right)}{p\left(s_{t+1} \mid Y_{t}, \theta, Q\right)} \\
& =\frac{p\left(s_{t+1} \mid Y_{t}, \theta, Q, s_{t}\right) p\left(s_{t} \mid Y_{t}, \theta, Q\right)}{p\left(s_{t+1} \mid Y_{t}, \theta, Q\right)}  \tag{117}\\
& =\frac{q_{s_{t+1}, s_{t}} p\left(s_{t} \mid Y_{t}, \theta, Q\right)}{p\left(s_{t+1} \mid Y_{t}, \theta, Q\right)}
\end{align*}
$$
\]

The conditional density $p\left(s_{t} \mid Y_{T}, Z_{T}, \theta, Q, S_{t+1}^{T}\right)$ is straightforward to evaluate according to Propositions 1 and 2.

To draw $S_{T}$, we use the backward recursion by drawing the last state $s_{T}$ from the terminal density $p\left(s_{T} \mid Y_{T}, \theta, Q\right)$ and then drawing $s_{t}$ recursively given the path $S_{t+1}^{T}$ according to (117). It can be seen from (116) that draws of $S_{T}$ this way come from $\operatorname{Pr}\left(S_{T} \mid Y_{T}, \theta\right)$.
V.4. Marginal posterior density of $s_{t}$. The smoothed probability of $s_{t}$ given the values of the parameters and the data can be evaluated through backward recursions. Starting with $s_{T}$ and working backward, we can calculate the probability of $s_{t}$ conditional on $Y_{T}, \theta, Q$ by using the following fact

$$
\begin{aligned}
p\left(s_{t} \mid Y_{T}, \theta, Q\right) & =\sum_{s_{t+1} \in H} p\left(s_{t}, s_{t+1} \mid Y_{T}, \theta, Q\right) \\
& =\sum_{s_{t+1} \in H} p\left(s_{t} \mid Y_{T}, \theta, Q, s_{t+1}\right) p\left(s_{t+1} \mid Y_{T}, \theta, Q\right)
\end{aligned}
$$

where $p\left(s_{t} \mid Y_{t}, \theta, Q, s_{t+1}\right)$ can be evaluated according to (117).

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[^0]:    ${ }^{1}$ The conventional assumption for $p\left(s_{0} \mid \theta, Q\right)$ is the ergodic distribution of $Q$, if it exists. This convention, however, precludes the possibility of allowing for an absorbing regime or state.

[^1]:    ${ }^{1}$ The conventional assumption for $p\left(s_{0} \mid \theta, Q\right)$ is the ergodic distribution of $Q$, if it exists. This convention, however, precludes the possibility of allowing for an absorbing regime or state.

