Supplement to: Bidding in Common-Value Auctions with an Unknown Number of Competitors

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This file contains supplementary material for ”Bidding in Common-Value Auctions with an Unknown Number of Competitors” that shows how to numerically construct solutions of the communication extension.

Appendix D Numerical analysis

D.1 Preamble

The Mathematica code for all numerical solutions is available online and included in this submission as a ZIP file. The code is separated according to the figure where it appears, plus two files which contain the detailed constructions of the numerical examples in Sections D.3.1 and D.3.2. Each file includes extensive documentation.

For the figures, the files are labeled as follows:

1. [Figure 2.nb] Equilibria of the First-Price Auction
2. [Figure 5a.nb] Equilibria with Reserve Prices $r = 0, 0.2, 0.29$
3. [Figure 5b.nb] Equilibria with Reserve Prices $r = 0.29, 0.35$
4. [Figure 6b.nb] The Bounce Auction
5. [Figure 7.nb] Other Distributions of Population Size

For the numerical examples in Sections D.3.1 and D.3.2, the files are:

1. [Eta 5.nb] Solution Construction with $\eta = 5$
2. [Eta 11.nb] Solution Construction with $\eta = 11$

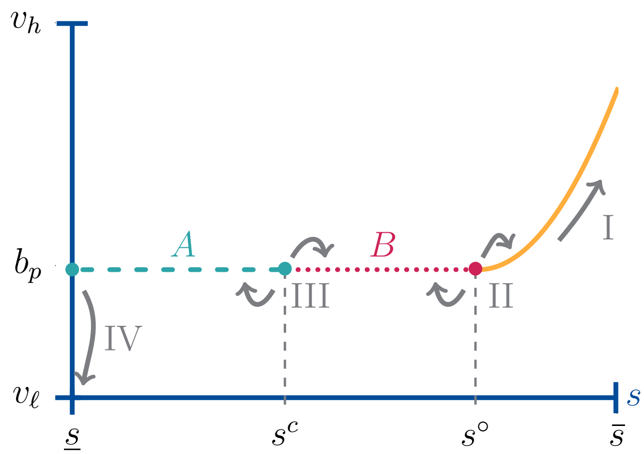

Figure 7: Illustration of solution structure: two pools, with boundaries $s^c < s^\circ$.

D.2 Solution construction and multiplicity

We use the communication extension and construct truthful solutions of the form depicted in Figure 7, allowing for a reserve price $r \geq v_\ell$. For some s^c and s°:

- $M = \{[\underline{s}, s^c)\} \cup \{[s^c, s^\circ)\} \cup \{\{s\}: s > s^\circ\}$.
- Signals from $A = [\underline{s}, s^c)$ bid b_p and report truthfully, so that they pool with other signals from A.
- Signals from $B = [s^c, s^\circ)$ bid b_p and report truthfully, so that they pool with other signals from B.
- Signals above s° follow a strictly increasing bidding strategy.

To construct a solution of this form, we need to find signals $s^c < s^\circ$ and a bidding strategy β, especially a pooling bid b_p, such that the following hold:

I. The bidding strategy β is constant at b_p up to s° and strictly increasing afterward.

II. Signal s° is indifferent between pooling with B (bidding b_p and reporting $s \in [s^c, s^\circ)$) and bidding marginally more.

III. Signal s^c is indifferent between pooling with B (bidding b_p and reporting $s \in [s^c, s^\circ)$) and pooling with A (bidding b_p and reporting $s \in [\underline{s}, s^c)$).

IV. Pooling with A (bidding b_p and reporting $s \in [\underline{s}, s^c))$ is individually rational for signal $\underline{s}$, and there is no profitable downward deviation to a bid $r \geq v_\ell$.

If all of these conditions are satisfied, by monotonicity, no signal has a profitable deviation and individual rationality is satisfied, yielding a truthful solution.

Preview. In Sections D.3.1 and D.3.2, we apply the solution construction to our running example. As we will see, there is a whole range of solutions. For $\eta = 11$ and $r = v_\ell$, Figure 8 plots the areas of (s^c, s°) where the implied incentive conditions hold in different colors. The intersection of all areas and the line correspond to the combinations of (s^c, s°) that form solutions. We highlight this in section in red.

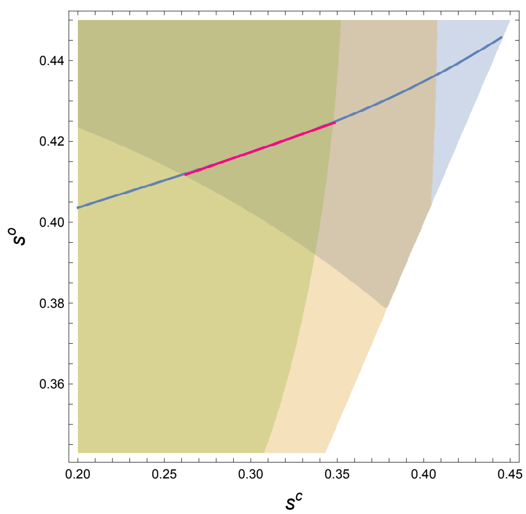

Figure 8: Solution combinations of s^c, s° for $\eta = 11$ are given by the red segment.

Condition I. By standard arguments, when the bidding strategy is strictly increasing with $\beta(s^\circ) = b_p$, then β is the unique solution of the ODE

$$\frac{\partial}{\partial s}\beta(s) = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \text{with } \beta(s^\circ) = b_p,$$

where $F_{s_{(1)}}(s'|s)$ denotes the expected cumulative distribution function of $s_{(1)}$ conditional on observing s. We verify this in the working paper version. Further, we observe that the solution is indeed strictly increasing if $b_p \leq \mathbb{E}[v|s_{(1)} = s^\circ, s^\circ]$ and $\mathbb{E}[v|s_{(1)} = s^\circ, s^\circ]$ is strictly increasing above s°.

We note that $b_p \leq \mathbb{E}[v|s_{(1)} = s^\circ, s^\circ]$ if and only if

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{f_h(s^\circ)^2}{f_\ell(s^\circ)^2} \frac{e^{-\eta(1-F_h(s^\circ))}}{e^{-\eta(1-F_\ell(s^\circ))}}. \tag{48}$$

Further, $\mathbb{E}[v|s_{(1)} = s^\circ, s^\circ]$ is strictly increasing above s° if and only if

$$\frac{f_h(s)^2 e^{-\eta(1-F_h(s))}}{f_\ell(s)^2 e^{-\eta(1-F_\ell(s))}}$$

is strictly increasing above s°. Taking the derivative, this holds if and only if

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\Big(\frac{f_h(s)}{f_\ell(s)}\Big)^{-1} - \eta[f_h(s) - f_\ell(s)] > 0. \tag{49}$$

Condition II. Signal s° is indifferent between pooling with B (bidding b_p and reporting $s \in [s^c, s^\circ)$) and bidding marginally more if[30]

$$\begin{aligned}&\lim_{\epsilon \to 0} \mathbb{P}[\text{win with } b_p + \epsilon | s^\circ](\mathbb{E}[v|\text{win with } b_p + \epsilon, s^\circ] - b_p)\\ &= \mathbb{P}[\text{win pooling with } B|s^\circ](\mathbb{E}[v|\text{win pooling with } B, s^\circ] - b_p),\end{aligned}$$

which can be rearranged to

$$\begin{aligned} b_p &= \lim_{\epsilon \to 0} \mathbb{E}[v|\text{win with } b_p + \epsilon \text{ but not pooling with } B, s^\circ] \\ \iff \frac{b_p - v_\ell}{v_h - b_p} &= \frac{\rho}{1-\rho} \frac{f_h(s^\circ)}{f_\ell(s^\circ)} \frac{e^{-\eta(1-F_h(s^\circ))} - \frac{e^{-\eta(1-F_h(s^\circ))} - e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ) - F_h(s^c)]}}{e^{-\eta(1-F_\ell(s^\circ))} - \frac{e^{-\eta(1-F_\ell(s^\circ))} - e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ) - F_\ell(s^c)]}}. \end{aligned} \tag{50}$$

Condition III. Signal s^c is indifferent between pooling with B (bidding b_p and reporting $s \in [s^c, s^\circ)$) and pooling with A (bidding b_p and reporting $s \in [\underline{s}, s^c)$) if

$$\begin{aligned}&\mathbb{P}[\text{win pooling with } A|s^c](\mathbb{E}[v|\text{win pooling with } A, s^c] - b_p)\\ &= \mathbb{P}[\text{win pooling with } B|s^c](\mathbb{E}[v|\text{win pooling with } B, s^c] - b_p),\end{aligned}$$

[30] Reporting a higher type $> s^\circ$ is an equivalent deviation that wins with the same probabilities.

which can be rearranged to

$$b_p = \mathbb{E}[v|\text{win pooling with } B \text{ but not pooling with } A, s^c]$$

$$\iff \frac{b_p - v_\ell}{v_h - b_p} = \frac{\rho}{1-\rho}\frac{f_h(s^c)}{f_\ell(s^c)}\frac{\frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]} - \frac{e^{-\eta(1-F_h(s^c))}-e^{-\eta}}{\eta F_h(s^c)}}{\frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]} - \frac{e^{-\eta(1-F_\ell(s^c))}-e^{-\eta}}{\eta F_\ell(s^c)}}. \tag{51}$$

Condition IV. Pooling with A (bidding b_p and reporting $s \in [\underline{s}, s^c)$) does not violate individual rationality (IR) for signal $\underline{s}$ if

$$b_p \leq \mathbb{E}[v|\text{win pooling with } A, \underline{s}]$$

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho}\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\frac{e^{-\eta(1-F_h(s^c))}-e^{-\eta}}{\eta F_h(s^c)}}{\frac{e^{-\eta(1-F_\ell(s^c))}-e^{-\eta}}{\eta F_\ell(s^c)}}. \tag{52}$$

Further, there is no profitable downward deviation if

$$\mathbb{P}[\text{win pooling with } A|\underline{s}](\mathbb{E}[v|\text{win pooling with } A, \underline{s}] - b_p) \geq \mathbb{P}[\text{alone}|\underline{s}](\mathbb{E}[v|\text{alone}, \underline{s}] - r),$$

where $r \geq 0$ is the reserve price. The latter can be rearranged to

$$b_p \leq \mathbb{E}[v|\text{win pooling with } A, \underline{s}] - \frac{\mathbb{P}[\text{alone}|\underline{s}]}{\mathbb{P}[\text{win pooling with } A|\underline{s}]}(\mathbb{E}[v|\underline{s}] - r). \tag{53}$$

Overview. Combined, Conditions I–IV give rise to the following sufficient conditions on s^c, s°, and b_p:

$$s^c < s^\circ, \tag{47}$$

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho}\frac{f_h(s^\circ)^2}{f_\ell(s^\circ)^2}\frac{e^{-\eta(1-F_h(s^\circ))}}{e^{-\eta(1-F_\ell(s^\circ))}}, \tag{48}$$

$$0 < 2\left(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\right)\left(\frac{f_h(s)}{f_\ell(s)}\right)^{-1} - \eta[f_h(s) - f_\ell(s)] \quad \forall s > s^\circ, \tag{49}$$

$$\frac{b_p - v_\ell}{v_h - b_p} = \frac{\rho}{1-\rho}\frac{f_h(s^\circ)}{f_\ell(s^\circ)}\frac{e^{-\eta(1-F_h(s^\circ))} - \frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]}}{e^{-\eta(1-F_\ell(s^\circ))} - \frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]}}, \tag{50}$$

$$\frac{b_p - v_\ell}{v_h - b_p} = \frac{\rho}{1-\rho}\frac{f_h(s^c)}{f_\ell(s^c)}\frac{\frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]} - \frac{e^{-\eta(1-F_h(s^c))}-e^{-\eta}}{\eta F_h(s^c)}}{\frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]} - \frac{e^{-\eta(1-F_\ell(s^c))}-e^{-\eta}}{\eta F_\ell(s^c)}}, \tag{51}$$

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\frac{e^{-\eta(1-F_h(s^c))}-e^{-\eta}}{\eta F_h(s^c)}}{\frac{e^{-\eta(1-F_\ell(s^c))}-e^{-\eta}}{\eta F_\ell(s^c)}}, \tag{52}$$

and the downward deviation

$$b_p \leq \mathbb{E}[v|\text{win pooling with } A, \underline{s}] - \frac{\mathbb{P}[\text{alone}|\underline{s}]}{\mathbb{P}[\text{win pooling with } A|\underline{s}]}(\mathbb{E}[v|\underline{s}] - r). \tag{53}$$

Using (50) to rewrite the left sides of (48), (51), and (52) gives

$$s^c < s^\circ, \tag{47}$$

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\Big(\frac{f_h(s)}{f_\ell(s)}\Big)^{-1} - \eta[f_h(s) - f_\ell(s)] > 0 \quad \forall s > s^\circ, \tag{49}$$

$$\frac{f_h(s^\circ)}{f_\ell(s^\circ)} \frac{e^{-\eta(1-F_h(s^\circ))} - \frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]}}{e^{-\eta(1-F_\ell(s^\circ))} - \frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]}} \leq \frac{f_h(s^\circ)^2}{f_\ell(s^\circ)^2} \frac{e^{-\eta(1-F_h(s^\circ))}}{e^{-\eta(1-F_\ell(s^\circ))}}, \tag{54}$$

$$\frac{f_h(s^\circ)}{f_\ell(s^\circ)} \frac{e^{-\eta(1-F_h(s^\circ))} - \frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]}}{e^{-\eta(1-F_\ell(s^\circ))} - \frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]}} =$$

$$\frac{f_h(s^c)}{f_\ell(s^c)} \frac{\frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]} - \frac{e^{-\eta(1-F_h(s^c))}-e^{-\eta}}{\eta F_h(s^c)}}{\frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]} - \frac{e^{-\eta(1-F_\ell(s^c))}-e^{-\eta}}{\eta F_\ell(s^c)}}, \tag{55}$$

$$\frac{f_h(s^\circ)}{f_\ell(s^\circ)} \frac{e^{-\eta(1-F_h(s^\circ))} - \frac{e^{-\eta(1-F_h(s^\circ))}-e^{-\eta(1-F_h(s^c))}}{\eta[F_h(s^\circ)-F_h(s^c)]}}{e^{-\eta(1-F_\ell(s^\circ))} - \frac{e^{-\eta(1-F_\ell(s^\circ))}-e^{-\eta(1-F_\ell(s^c))}}{\eta[F_\ell(s^\circ)-F_\ell(s^c)]}} \leq \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\frac{e^{-\eta(1-F_h(s^c))}-e^{-\eta}}{\eta F_h(s^c)}}{\frac{e^{-\eta(1-F_\ell(s^c))}-e^{-\eta}}{\eta F_\ell(s^c)}}, \tag{56}$$

and the downward deviation

$$b_p \leq \mathbb{E}[v|\text{win pooling with } A, \underline{s}] - \frac{\mathbb{P}[\text{alone}|\underline{s}]}{\mathbb{P}[\text{win pooling with } A|\underline{s}]}(\mathbb{E}[v|\underline{s}] - r). \tag{53}$$

D.3 Numerical example

We revisit the example from the main paper in which $v_h = 1$, $v_\ell = 0$, with $\rho = \frac{1}{2}$, $s \in [0,1]$, $f_h(s) = 1$, and $f_\ell(s) = 1.5 - s$. We construct solutions with exactly two pools for $\eta = 5$ and $\eta = 11$.

D.3.1 Solutions where $\eta = 5$

We show that for $r = 0$, there is no solution of the desired form. With $r > 0$, on the other hand, there is a range of solutions, of which the specific r selects one.

First, consider the condition (49). With $\eta = 5$, the equation $2(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\left(\frac{f_h(s)}{f_\ell(s)}\right)^{-1} - \eta[f_h(s) - f_\ell(s)]$ has only one root at ≈ 0.193774. Therefore, the condition (49) becomes

$$s^\circ > 0.193774. \tag{57}$$

When combined with (47), (54), (55), and (56), this describes a set of candidate solution pairs (s^c, s°).

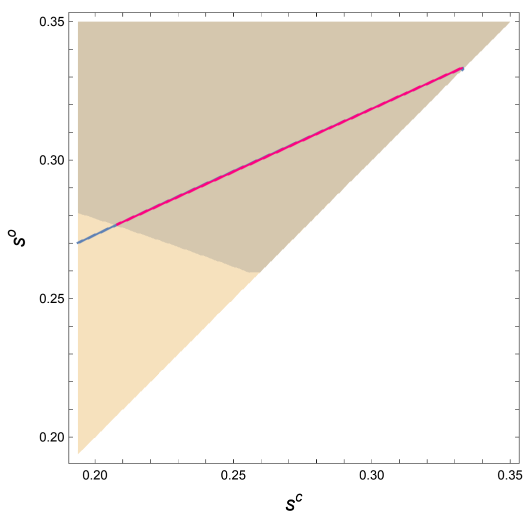

Figure 9: Potential solution combinations of s^c, s° for $\eta = 5$ are given by the red segment.

We plot these in Figure 9, with s^c on the x-axis and s° on the y-axis. The inequality (57) holds everywhere, the inequality (47) above the 45°-line. The region (54) is colored yellow, while the region (56) is colored blue (but shown as brown, since it is a strict subset of the yellow region). The equality (55) is depicted by the line. Every point (s^c, s°) on the line within the intersection of the blue and yellow regions represents a potential solution. We highlight this segment in red.

To finish constructing the solution, we must deter a deviation by $\underline{s}$ to the lowest possible bid, i.e. ensure that inequality (53) holds. This turns out to require a positive reserve price: for any candidate solution pair (s^c, s°), when $r = 0$, signal $\underline{s}$ has a profitable downward deviation.

Instead, we construct solutions where the reserve price is binding, $r = b_p$. Naturally, there can be no profitable downward deviation in this case, and IR of $\underline{s}$ is satisfied by (56). Thus, with the appropriate $r = b_p$, every point on the line within the intersection of the blue and yellow regions represents a solution.

As an example, let us finish constructing one of these solutions: specifically, the one where $s^c = 0.24656$. Then, by (55), it follows that $s^\circ = 0.294468$. By construction, this point is on the line (55) and clearly within the intersection of the sets (47), (54), (56), and (57). Plugging both values into (50) gives $b_p = r = 0.29$. The solution is pictured in Figure 10.

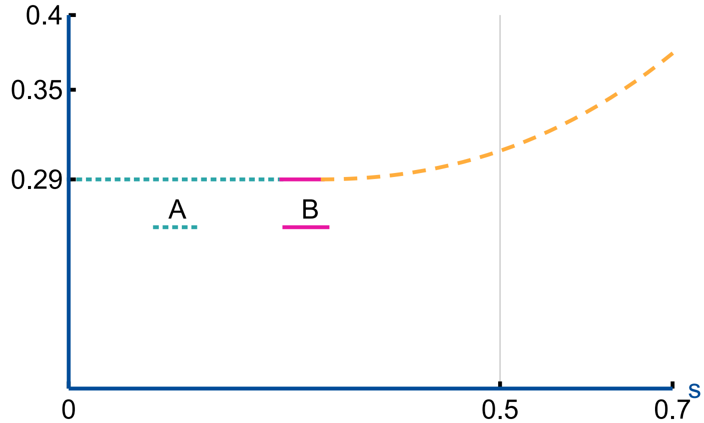

Figure 10: Solution with $b_p = r = 0.29$.

D.3.2 Solution where $\eta = 11$

When η is sufficiently large, no reserve price is needed, and there is a continuum of solutions. We show this for the case where $\eta = 11$.

First, consider the condition (49). With $\eta = 11$, the equation $2(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)})\left(\frac{f_h(s)}{f_\ell(s)}\right)^{-1} - \eta[f_h(s) - f_\ell(s)]$ has only one root at ≈ 0.342871. Therefore, the condition (49) becomes

$$s^\circ > 0.342871. \tag{58}$$

When combined with (47), (54), (55), and (56), this describes a set of candidate solution pairs (s^c, s°). Further, by setting $r = v_\ell = 0$ and replacing b_p by $\mathbb{E}[v|\text{win pooling with } B \text{ but not pooling with } A, s^c]$, the inequality (53) also becomes a function of s^c and s°:

$$\begin{aligned} &\mathbb{E}[v|\text{win pooling with } B \text{ but not pooling with } A, s^c] \\ &\qquad \leq \mathbb{E}[v|\text{win pooling with } A, \underline{s}] - \frac{\mathbb{P}[\text{alone}|\underline{s}]}{\mathbb{P}[\text{win pooling with } A|\underline{s}]}\mathbb{E}[v|\underline{s}]. \end{aligned} \tag{59}$$

We plot the solution candidates in Figure 8, with s^c on the x-axis and s° on the y-axis. The inequality (58) holds everywhere, the inequality (47) outside the white region. The region (54) is colored yellow, the region (56) blue. The region in which the inequality (59) holds is colored green. The equality (55) is depicted by the line. Every point (s^c, s°) on the line within the intersection of the blue, yellow, and green regions represents a solution. We highlight this section in red. Thus, for $\eta = 11$ and no reserve price ($r = 0$), there is a continuum of solutions with exactly two atoms at the bottom.

As an example, let us finish constructing one of these solutions: specifically, the one where $s^c = 0.3$. Then, by (55), it follows that $s^\circ = 0.417276$. By construction, this point is on the line (55) and clearly within the intersection of the sets (47), (53), (54), (56), and (59). Plugging s^c and s° into (50) gives $b_p = 0.18283$. The solution is pictured in Figure 11.

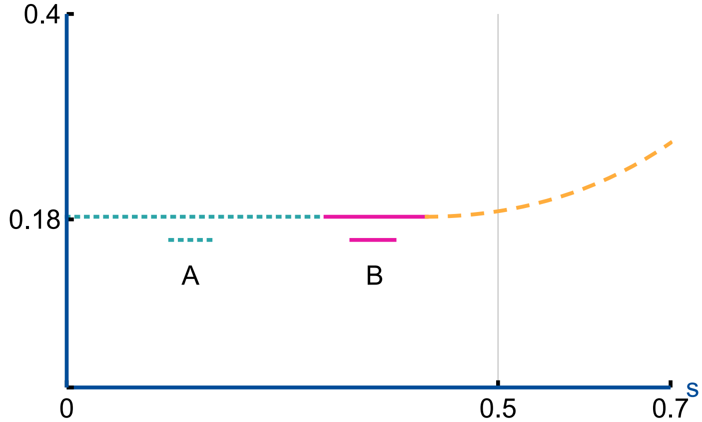

Figure 11: Solution with $r = 0$ and $b_p = 0.18283$.