Bidding in Common-Value Auctions with an Unknown Number of Competitors*

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Abstract

This paper studies a first-price common-value auction in which bidders do not know the number of their competitors. In contrast to the case of common-value auctions with a known number of rival bidders, the inference from winning is not monotone, and a "winner's blessing" emerges at low bids. As a result, bidding strategies may not be strictly increasing, but instead may contain atoms. Moreover, an equilibrium fails to exist when the expected number of competitors is large and the bid space is continuous. Therefore, we consider auctions on a grid. On a fine grid, high-signal bidders follow an essentially strictly increasing strategy, whereas low-signal bidders pool on two adjacent bids on the grid. The solutions of a "communication extension" based on Jackson et al. (2002) capture the equilibrium bidding behavior in the limit, as the grid becomes arbitrarily fine.

Keywords: Common-value auctions, random player games, numbers uncertainty, Poisson games, endogenous tie-breaking, nonexistence

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1 Introduction

In most auctions, bidders are uncertain about the number of competitors they face:

- At auction houses such as Christie's and Sotheby's, personal attendance is in decline as bidders prefer to phone in or place their bids online. Therefore, bidders "know even less about who they're bidding against, which in some cases can leave them wondering how high they should go."[1]
- eBay reveals the number of bidders who have placed a bid but does not disclose how many prospective bidders are following the auction. In particular, the platform does not display how many bidders are online to "snipe", that is, to place their bids in the last seconds of the auction (Roth & Ockenfels, 2002).
- In the realm of auction-like trading mechanisms, the continuous order book at the New York Stock Exchange informs market participants about the stream of (un-)filled buy and sell orders, but reveals neither the number nor the identity of (potential) buyers and sellers.

Although uncertainty about the number of competitors, or "numbers uncertainty", is ubiquitous, the subject has received little attention in the literature on auction theory. One reason may be its irrelevance in standard auction formats with independent private values and risk neutrality: by a revenue-equivalence argument, equilibrium bids are just a weighted average of the bids that are optimal when the number of rival bidders is known; see Krishna (2010, Chapter 3.2.2) and Harstad et al. (1990).

By contrast, in a common-value setting, numbers uncertainty significantly alters bidding behavior. Recall that when the number of rival bidders is known, classic results going back to Milgrom & Weber (1982) establish that there exists a unique symmetric equilibrium in first-price and second-price auctions, in which bids are strictly increasing in the bidders' own value estimates. Uniqueness and strict monotonicity facilitate revenue comparison between auction formats, simplify welfare considerations (in general interdependent-value settings), and allow for empirical identification of the bidders' signals. We show that these classic results no longer hold when the number of competitors is uncertain. Equilibria generally are not strictly increasing but contain atoms. The locations of the atoms are often indeterminate, implying equilibrium multiplicity. Moreover, equilibrium payoffs are discontinuous at the atoms,

[1] *The Wall Street Journal*, "Why Auction Rooms Seem Empty These Days", June 15, 2014; cf. Akbarpour & Li (2020).

invalidating standard methods for analyzing bidding behavior in these auctions: with a continuous bid space, equilibrium generally fails to exist.

To model an auction with numbers uncertainty, we start with a canonical common-value first-price auction. The value of the good is binary (*high* or *low*), and bidders receive conditionally independent and identically distributed signals, with higher signals indicating a higher value. All bidders submit their bids simultaneously; the highest bidder wins and pays her bid. Ties are broken uniformly. We only deviate from the textbook setting in assuming that the number of (rival) bidders is not known but Poisson distributed.

Numbers uncertainty affects bidding behavior with common values because it changes the value inference from winning. In a conventional common-value auction with a known number of bidders, the expected value conditional on winning is increasing in the relative position of the bid because a higher bid eases the “winner’s curse”. In fact, there is no winner’s curse at the very top bid. This reduction reinforces price competition and implies the absence of pooling, i.e., of atoms in the bid distribution.

With numbers uncertainty, winning is also informative about the number of rival bidders. In particular, winning with a low bid is more likely when there are fewer competitors, which eases the winner’s curse. Therefore, winning with a low bid is not necessarily bad news about the value of the good. In our model, the inference is U-shaped: intermediate bids are subject to the strongest winner’s curse, while there is no winner’s curse at the bottom or the top (Lemmas 2 and 3).

We show that every equilibrium is nondecreasing in the bidder’s signal (Lemma 1), but the non-monotone inference implies that equilibria cannot be strictly increasing unless the expected number of competitors is small (Proposition 1). Hence, the equilibrium bid distribution contains one or more atoms, as bidders with different signals pool on common bids. Numbers uncertainty incentivizes bidders to pool because pooling shields them against the winner’s curse: under a uniform tie-breaking rule, a bid that ties with positive probability is more likely to win when there are fewer competitors, which reduces the winner’s curse.

The presence of atoms in the bid distribution substantially alters the analysis of the auction. First, the locations of atoms are often indeterminate, so that there may be multiple equilibria. Second, atoms create discontinuities in the bidders’ payoffs, implying that no equilibrium exists when the expected number of bidders is sufficiently large (Proposition 2).

To overcome the nonexistence and study the bidding incentives, we consider equilibria on a finite but fine grid. We characterize the equilibrium bidding behavior in Proposition 3. Qualitatively, any equilibrium on a fine grid with increments $\Delta > 0$ consists of three regions: bidders with high signals essentially follow a strictly increasing strategy (as the grid permits), while bidders with intermediate signals pool on some bid b_p, and bidders with low signals bid one increment below it, $b_p - \Delta$. The equilibria are shaped by a severe winner's curse at b_p, and a "winner's blessing" that arises at bids below b_p, so that, at these bids, the expected value conditional on winning is significantly higher than at b_p. This induces bidders with low signals to compete for the largest bid strictly below b_p. On the grid, this competition leads them to pool on $b_p - \Delta$; on the continuous bid space, the nonexistence of a largest bid below b_p implies the nonexistence of an equilibrium.

We show that bidding on a fine grid can be approximated by the equilibria of a "communication extension" of the auction, based on Lebrun (1996) and Jackson et al. (2002). In the communication extension, bidders submit not only a monetary bid from the continuous bid space but also a message that indicates their "eagerness" to win, which is used to break ties. The communication extension is useful because, in contrast to the case of the standard auction, the limit of any converging sequence of equilibria on ever finer grids corresponds to an equilibrium of the communication extension. Proposition 5 shows that all equilibria of the communication extension share the qualitative features of the equilibria on a fine grid.

In Section 7, we use the communication extension to discuss the implications of numbers uncertainty for the revenue of the seller and the optimal design of auctions, including reserve prices. Moreover, we show that the seller may benefit from running a generalized clock auction with a non-monotone price path that mirrors the non-monotone expected value conditional on winning. Finally, we discuss the assumptions, especially the Poisson distribution, as well as the related literature, including recent contributions by Murto & Välimäki (2019) and Lauermann & Wolinsky (2022).

2 Model

A single, indivisible good is sold in a first-price, sealed-bid auction. The good's value is either high, v_h, or low, v_ℓ, with $v_h > v_\ell \geq 0$, depending on the unknown state of the world $\omega \in \{h, \ell\}$. The state is $\omega = h$ with probability ρ and $\omega = \ell$ with probability

$1-\rho$, where $\rho \in (0,1)$. The number of bidders is a Poisson-distributed random variable with mean η, so that there are k bidders in the auction with probability $e^{-\eta}\frac{\eta^k}{k!}$. The bidders do not observe the realized number of bidders.

Every bidder receives a signal s from the compact set $[\underline{s}, \bar{s}]$. Conditional on the state, the signals are independent and identically distributed according to the cumulative distribution functions F_h and F_ℓ, respectively.[2] Both distributions have continuous densities f_ω, and the likelihood ratio $\frac{f_h(s)}{f_\ell(s)}$ is strictly increasing in s; that is, the signal distribution satisfies the strict monotone likelihood ratio property (MLRP). Furthermore, $0 < \frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})} < \infty$, so that no signal reveals the state. Let $\breve{s}$ denote the unique "neutral" signal, the one for which $\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$.

Having received her signal, each bidder submits a bid b. There is a reserve price at v_ℓ,[3] and it is without loss to exclude bids above v_h, so that $b \in [v_\ell, v_h]$. The bidder with the highest bid wins the auction, receives the good, and pays her bid. Ties are broken uniformly. If there is no bidder, the good is not allocated.

The Poisson distribution has a number of useful properties; a detailed derivation and discussion of Poisson games can be found in Myerson (1998). In particular, when participating in the auction, a bidder does not change her belief regarding the number of other bidders in the auction: this belief is again a Poisson distribution with mean η.[4] Moreover, Myerson (1998, p. 377) argues that in a Poisson game, attention can be restricted to symmetric equilibria.

Accordingly, we consider symmetric strategies, which are measurable functions $\beta : [\underline{s}, \bar{s}] \to \Delta[v_\ell, v_h]$ mapping the signals into the set of probability distributions over bids. Let $\pi_\omega(b;\beta)$ denote the probability of winning the auction with a bid b in state ω, if the rival bidders follow strategy β. Using Bayes' rule, the interim expected utility for a bidder with signal s choosing bid b is

$$U(b|s;\beta) = \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_h(b;\beta)(v_h - b) + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_\ell(b;\beta)(v_\ell - b).$$

[2]This is the "mineral rights" setup (Krishna, 2010). The signal structure excludes general affiliated signals and signals with shifting support.

[3]If $v_\ell > 0$ and there is no reserve price, then the single-crossing condition (Lemma 1) may fail and equilibria may not be monotone; see Murto & Välimäki (2019) and Lauermann & Wolinsky (2022) for examples. As η becomes large, the assumption becomes innocuous because competition drives almost all bids above v_ℓ.

[4]This property is analogous to that of a stationary Poisson process, in which an event does not allow for inferences about the number of other events.

A strategy β^* is a *best response* to a strategy β if, for almost all s, a bid $b \in \operatorname{supp} \beta^*(s)$ implies that $b \in \arg\max_{\hat{b} \in [v_\ell, v_h]} U(\hat{b}|s;\beta)$.

3 Monotonicity of bidding behavior

3.1 Best responses are weakly increasing

In the appendix, bidders' payoffs are shown to satisfy the strict single-crossing condition: for any β, bids $b < b'$, and signals $s < s'$,

$$U(b|s;\beta) \leq U(b'|s;\beta) \Rightarrow U(b|s';\beta) < U(b'|s';\beta). \tag{1}$$

An immediate consequence of this strict single-crossing condition is that best responses are monotone and pure; see, e.g., Athey (2001).

Lemma 1 (Best responses are monotone)**.** *If β is a strategy and $\hat{\beta}$ a best response to it, then $\hat{\beta}$ is pure and weakly increasing.*

The proof of the lemma in Appendix A.1 verifies (1). The key observation is that, for any bid $b \in (v_\ell, v_h)$, the bidder's payoff is positive in state h and negative in ℓ. Hence, bidders value a higher winning probability at a higher bid $b' > b$ only if they believe that the state is more likely to be h.

Given Lemma 1, we restrict our attention to pure and nondecreasing strategies, which we also denote by $\beta : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$. For any such β and any bid b, the set of signals bidding b is an interval (possibly empty). We denote its boundaries by

$$\sigma_-(b) = \inf\{s \colon \beta(s) \geq b\}, \qquad \sigma_+(b) = \sup\{s \colon \beta(s) \leq b\},$$

where we use the convention that $\inf\{\emptyset\} = \bar{s}$ and $\sup\{\emptyset\} = \underline{s}$. Thus, $\sigma(b) = [\sigma_-(b), \sigma_+(b)]$ is the generalized inverse of β.[5] If $\sigma_-(b) < \sigma_+(b)$, then $\beta(s) = b$ for all $s \in \sigma(b)$. In this case, there is an atom in the implied bid distribution at b, and we say that b is a *pooling bid* and $\sigma(b)$ is a *pool*.

[5]In the following, we pretend that all pooling intervals are closed to simplify notation. The specification of bids on the boundaries are irrelevant because (a) the set of boundary signals of nontrivial intervals has zero measure, and (b) by the continuity of the likelihood ratio, a pooling bid is optimal for the boundary signals if it is optimal for the interior signals.

3.2 The inference from winning is non-monotone

Fix some strategy β and some bid b that is not a pooling bid. The bid b wins when $s_{(1)} \leq s = \sigma_+(b)$, where $s_{(1)} = \sup\{s_{-i}\}$ denotes the highest of the competitors' signals. If there are no competitors, we set $s_{(1)} = -\infty$. Thus, the cumulative distribution function of $s_{(1)}$ is $F_{s_{(1)}}(s|\omega) = e^{-\eta(1-F_\omega(s))}$.[6] Since bid b wins if $s_{(1)} \leq s$, its winning probability in state ω is $\pi_\omega(b;\beta) = e^{-\eta(1-F_\omega(s))}$.

A characteristic feature of common-value auctions is that winning is informative about the value of the good. In choosing a non-pooling bid b, all that matters for this inference is the relative position of the bid, which is given by $s = \sigma_+(b)$. The position of s affects the likelihood ratio of winning,

$$\frac{\pi_h(b;\beta)}{\pi_\ell(b;\beta)} = \frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}}, \tag{2}$$

which, in turn, determines the conditional expected value of the good. In particular, $\mathbb{E}[v|\text{win with } b;\beta] = \mathbb{E}[v|s_{(1)} \leq s]$ is strictly increasing in the ratio (2) and is equal to

$$\mathbb{E}[v|s_{(1)} \leq s] = \frac{\rho e^{-\eta(1-F_h(s))} v_h + (1-\rho) e^{-\eta(1-F_\ell(s))} v_\ell}{\rho e^{-\eta(1-F_h(s))} + (1-\rho) e^{-\eta(1-F_\ell(s))}}. \tag{3}$$

Lemma 2. *The conditional expected value $\mathbb{E}[v|s_{(1)} \leq s]$ is strictly decreasing in s when $s < \breve{s}$, has its unique global minimum at $s = \breve{s}$, and is strictly increasing when $s > \breve{s}$.*

Proof: Because the expected value is strictly increasing in the likelihood ratio (2), it is sufficient to show that the likelihood ratio is U-shaped around $\breve{s}$. Note that

$$\frac{\partial}{\partial s} \frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}} = e^{\eta(F_h(s)-F_\ell(s))} \eta [f_h(s) - f_\ell(s)];$$

therefore, (2) is indeed strictly decreasing below $\breve{s}$ and strictly increasing above. ■

The intuition behind the shape is best explained with the help of Figure 1.

First, consider point (i) at the top right, which marks $\mathbb{E}[v|s_{(1)} \leq \bar{s}]$. By definition, the highest signal, $s_{(1)}$, is always smaller than $\bar{s}$, independent of the state. Hence, the event that $s_{(1)} \leq \bar{s}$ is uninformative about the state, and so $\mathbb{E}[v|s_{(1)} \leq \bar{s}] = \mathbb{E}[v]$.

[6] Conditional on state ω, any competitor independently receives a signal larger than s with probability $1 - F_\omega(s)$. By the decomposition and environmental equivalence properties of the Poisson distribution—see Myerson (1998)—bidders believe that the number of rival bidders with signals larger than s is Poisson distributed with mean $\eta(1 - F_\omega(s))$.

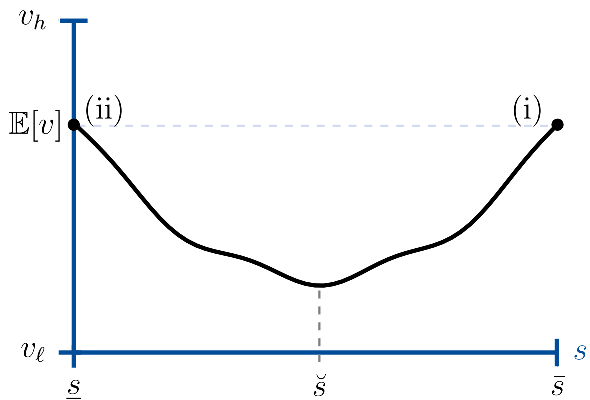


Figure 1: The conditional expected value $\mathbb{E}[v|s_{(1)} \leq s]$ is U-shaped.

Therefore, there is no negative inference at the top, just as in an auction with a known number of competitors.

Second, consider point (ii) at the top left, denoting $\mathbb{E}[v|s_{(1)} \leq \underline{s}]$. The highest signal $s_{(1)}$ equals $\underline{s}$ with zero probability because the signal distribution has no atoms, while there are no competitors and $s_{(1)} = -\infty$ with positive probability; consequently, $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v|s_{(1)} = -\infty]$. Since the distribution of bidders is independent of the state,[7] the event that $s_{(1)} \leq \underline{s}$ is uninformative about the state, and so $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v]$. Thus, there is no winner's curse at the bottom (ii) or at the top (i).

In the middle, where $s \in (\underline{s}, \bar{s})$, the winner's curse comes into play. With positive probability, there are competitors, all of whom received signals below s. Hence, for $s \in (\underline{s}, \bar{s})$, the conditional expected value is smaller than the unconditional one; that is, $\mathbb{E}[v|s_{(1)} \leq s] < \mathbb{E}[v]$, with a global minimum at $\breve{s}$, where $f_h(\breve{s}) = f_\ell(\breve{s})$.[8]

The non-monotone inference is not an artifact of the Poisson distribution but a feature of numbers uncertainty in general. Bidders make an inference from winning not only about the others' signals but also about the number of competitors, which can have opposing effects. In particular, the expected value is again non-monotone if the Poisson distribution is truncated at the bottom, guaranteeing a minimal degree of competition.[9] We discuss the distributional assumption in Section 8.

[7]We discuss a state-dependent bidder distribution in Section 8.

[8]Note that the non-monotonicity occurs with respect to the position of the bid. For a fixed bid, the expected value is monotone in the signal, which is part of the argument for Lemma 1.

[9] Consider a truncated Poisson distribution with at least $\underline{n} \geq 2$ bidders. At the top, the inference from winning is unaffected by the truncation; at the bottom, the winning bidder updates her belief toward $\underline{n} - 1$ rival bidders, all of whom received signal $\underline{s}$. Thus, there is a limited winner's curse at $\underline{s}$, which, however, does not depend on η. In the middle, $s \in (\underline{s}, \bar{s})$, the winner's curse grows in η, so

3.3 A large auction has no strictly increasing equilibrium

The non-monotone inference from winning can substantially affect the equilibrium behavior of bidders. As a benchmark, consider the common-value auction with a known number of bidders, $n \geq 2$. In this setup, the inference is monotone, which implies that there exists a strictly increasing equilibrium and it is unique; see Krishna (2010). With numbers uncertainty, an equilibrium of this form does not exist in general, owing to the non-monotone inference.

Proposition 1. *When η is sufficiently large, no strictly increasing equilibrium exists.*

The proof in Appendix A.2 relies essentially on two observations. First, for large η and any strictly increasing β, the expected value conditional on winning at $\beta(s)$ and on the bidder's own signal s, given by

$$\mathbb{E}[v|\text{win with } \beta(s), s; \beta] = \mathbb{E}[v|s_{(1)} \leq s, s],$$

inherits the U-shape of $\mathbb{E}[v|s_{(1)} \leq s]$. This means that while the inference from the bidder's own signal is monotone increasing, the U-shaped inference from winning turns out to be more relevant for the expected value when η is large. Second, for large η, it must be that $\beta(s) \approx \mathbb{E}[v|s_{(1)} \leq s, s]$ if β is an equilibrium; that is, bids must be close to the expected value conditional on winning, because of bidder competition. However, it cannot be that β is simultaneously close to $\mathbb{E}[v|s_{(1)} \leq s, s]$ and strictly increasing, given that $\mathbb{E}[v|s_{(1)} \leq s, s]$ is decreasing below $\breve{s}$.

The proof formalizes this idea by considering signals $\underline{s} < s' < s'' < \breve{s}$. We show that when β is strictly increasing and η is not too small, either a bidder with signal $\underline{s}$ has an incentive to deviate to $\beta(s')$, or $\beta(s'')$ is too high to be individually rational for a bidder with signal s''. The critical observation of Lemma 6 used for the proof is that for any $R > 1$, when η is sufficiently large,

$$\frac{\pi_h(\beta(s'); \beta)}{\pi_\ell(\beta(s'); \beta)} > R\frac{\pi_h(\beta(s''); \beta)}{\pi_\ell(\beta(s''); \beta)}, \tag{4}$$

so that the expected value conditional on winning with $\beta(s')$ is much higher than with $\beta(s'')$. Hence, if $\beta(s'')$ is low enough to be individually rational, then $\beta(s')$ is

that $\mathbb{E}[v|s_{(1)} \leq s]$ is still U-shaped when η is large.

strictly below the expected value conditional on winning at $\beta(s')$. This gives $\underline{s}$ an incentive to deviate to $\beta(s')$.

3.4 Strictly increasing bidding strategies: an example

For intuition about what it means for η to be "sufficiently large" in Proposition 1, we introduce an example that can be solved numerically.

Example. Let $v_h = 1$, $v_\ell = 0$, and $\rho = \frac{1}{2}$. Assume $s \in [0,1]$ and $f_h(s) = 1$, $f_\ell(s) = 1.5 - s$.

By standard arguments (Krishna, 2010, Chapter 6.4), a strictly increasing equilibrium exists if and only if the unique solution of the following ordinary differential equation (ODE) is strictly increasing:

$$\frac{\partial}{\partial s}\beta(s) = \left(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\right)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \text{with } \beta(\underline{s}) = v_\ell, \tag{5}$$

where $F_{s_{(1)}}(s'|s)$ denotes the expected cumulative distribution function of $s_{(1)}$ conditional on observing s.[10,11]

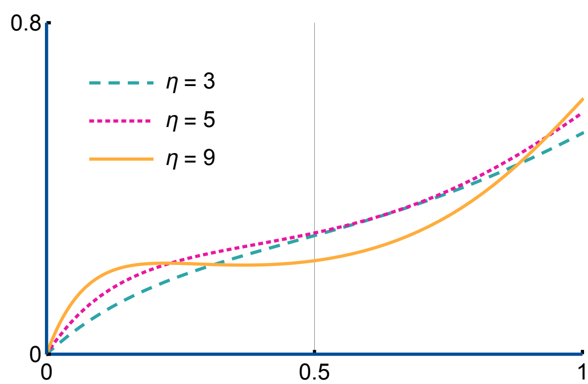


Figure 2: Equilibrium candidates for an example with different η.

In Figure 2, we plot the solution to the ODE (5) for the example, with $\eta \in \{3, 5, 9\}$. One can see that a strictly increasing equilibrium still exists for $\eta = 5$, but this is no longer the case for $\eta = 9$.

[10]So, $F_{s_{(1)}}(s'|s) = \frac{\rho f_h(s)}{\rho f_h(s)+(1-\rho)f_\ell(s)}e^{-\eta(1-F_h(s'))} + \frac{(1-\rho)f_\ell(s)}{\rho f_h(s)+(1-\rho)f_\ell(s)}e^{-\eta(1-F_\ell(s'))}$.

[11]For further discussion, see Lauermann & Speit (2019).

4 Pooling and equilibrium nonexistence

In Section 3, we showed that numbers uncertainty prevents the existence of a strictly increasing equilibrium when η is large. Hence, if an equilibrium exists for large η, it must contain flat parts. We now examine these flat parts to understand why bidders with different signals may have an incentive to pool on the same bid.

4.1 Pooling can ease the winner's curse

Fix some nondecreasing strategy β, and suppose that there is a pooling bid b_p; that is, $\beta(s) = b_p$ for all s from a pool $\sigma(b_p) = [\sigma_-, \sigma_+]$, where we have dropped the argument b_p from $\sigma_{+/-}$ for readability.

The following lemma compares the inference from winning with the pooling bid b_p to the inference from winning with a marginally lower or higher bid.

Lemma 3. *Assume that β is a weakly increasing strategy with a pooling bid b_p; that is, $\sigma(b_p)$ is a nondegenerate interval. Then the following hold:*

1. *If $\sigma_+(b_p) \leq \breve{s}$, then $\mathbb{E}[v|s_{(1)} \leq \sigma_-] > \mathbb{E}[v|\text{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq \sigma_+]$.*
2. *If $\sigma_-(b_p) \geq \breve{s}$, then $\mathbb{E}[v|s_{(1)} \leq \sigma_-] < \mathbb{E}[v|\text{win with } b_p; \beta] < \mathbb{E}[v|s_{(1)} \leq \sigma_+]$.*

The proof is in Appendix A.3.2. Combined with Lemma 2, Lemma 3 implies that the inference from winning is always U-shaped—even if β contains atoms.

To gain intuition for the inference, note that with positive probability, multiple bidders tie on the pooling bid b_p, so that the winner is decided by a uniform tie-break. Consequently, the bid b_p is more likely to win when there are fewer competitors who also choose b_p—that is, who have signals in $[\sigma_-, \sigma_+]$. If those signals are low, meaning that they are more likely to be realized in the low state, this implies that b_p wins less often in the low state than in the high state—a blessing, compared to marginally overbidding b_p. Conversely, if the signals are high, so that they are more likely to be realized in the high state, then the bid b_p wins more often in the low state—an additional winner's curse.[12]

[12] Formally, if $\sigma_+ \leq \breve{s}$, the MLRP implies that $\eta[F_h(\sigma_+) - F_h(\sigma_-)] < \eta[F_\ell(\sigma_+) - F_\ell(\sigma_-)]$, so that winning the tie-break is indeed a blessing; if $\sigma_- \geq \breve{s}$, then all inequalities and the inference from winning the random tie-break are reversed.

4.2 Atoms complicate the equilibrium analysis

In auctions, atoms usually do not occur, because the discretely higher winning probability when overbidding provides a deviation incentive. In common-value auctions with a known number of bidders, this incentive is reinforced by the curse from tying, which fosters competition among bidders. When there is numbers uncertainty and a blessing from tying, that is, if $F_h(\sigma_+) - F_h(\sigma_-) < F_\ell(\sigma_+) - F_\ell(\sigma_-)$, bidders may prefer to pool, trading the lower probability of winning for a higher expected value of the good. Therefore, an equilibrium β may contain atoms at or below $\breve{s}$ but must be strictly increasing above.

One critical feature entailed by atoms in the bid distribution is the existence of discontinuities, both in the winning probabilities and in the expected value conditional on winning. These discontinuities create room for equilibrium multiplicity but can also—together with the U-shape of the expected values around $\breve{s}$—upset equilibrium existence altogether.

4.3 Unless the auction is small, there is no equilibrium

Proposition 2. *For η large enough, no equilibrium exists.*

The proof of the proposition is in Appendix A.4. The idea of the proof is to exclude two collectively exhaustive types of equilibrium candidates: those in which essentially all signals below $\breve{s}$ pool on one common bid and those in which there are two or more bids chosen by signals below $\breve{s}$.

We defer the intuition until after the discussion of the bidding behavior in discrete auctions. The discussion of the discrete auction is instructive because the forces that shape the equilibria on the fine grid are the same that prevent equilibrium existence on the continuous bid space.

5 Equilibrium behavior in discrete auctions

5.1 Equilibria are characterized by two adjacent pools

What does bidding with an uncertain number of competitors look like? To overcome the nonexistence on the continuum and answer this question, we consider an auction with a grid, for which equilibrium exists; see, e.g., Athey (2001).

Specifically, suppose bids are from a grid with d elements and step size $\Delta = \frac{v_h - v_\ell}{d-1}$; that is, the grid of admissible bids is $D = \{v_\ell, v_\ell + \Delta, \ldots, v_\ell + (d-1)\,\Delta, v_h\}$. A strategy is now some function $\beta : [\underline{s}, \bar{s}] \to D$. The same arguments as before imply that every best response is weakly increasing and pure for almost every signal.

Lemma 4. *The auction with a grid has an equilibrium for every d and η, and every equilibrium bidding strategy is weakly increasing.*

Let β^* be an equilibrium for fixed d and η. Our main result characterizes equilibrium bidding for a fine grid and a sufficiently large but fixed η. Roughly speaking, β^* is strictly increasing above $\breve{s}$ (as the grid permits), while there is pooling on two adjacent bids below $\breve{s}$.

Proposition 3. *Fix any $\varepsilon \in (0, \frac{\breve{s} - \underline{s}}{2})$. When η is sufficiently large (given ε) and d is sufficiently large (given ε and η), any equilibrium β^* takes the following form: there are two disjoint, adjacent intervals of signals A, B such that*

(i) $[\underline{s} + \varepsilon, \breve{s} - \varepsilon] \subset A \cup B$;

(ii) $\beta^*(s) = b_p - \Delta$ *for all* $s \in A$, *and* $\beta^*(s) = b_p$ *for all* $s \in B$;

(iii) $\int_A \eta f_\omega(z)dz > \frac{1}{\varepsilon}$, *and* $\int_B \eta f_\omega(z)dz > \frac{1}{\varepsilon}$ *for* $\omega \in \{h, \ell\}$;

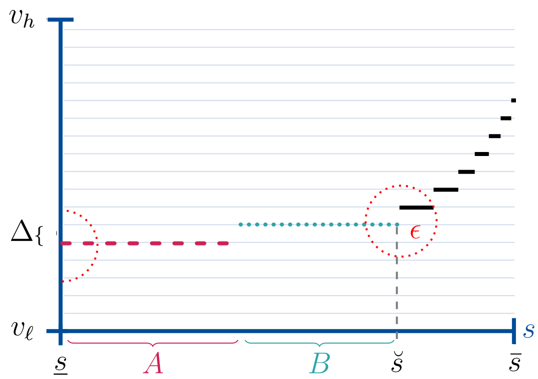
(iv) on $s \in (\breve{s} + \varepsilon, \bar{s}]$, *the expected number of bids on any step of the grid is smaller than* ε.[13]

The result is best summarized with the help of a figure; see Figure 3a.

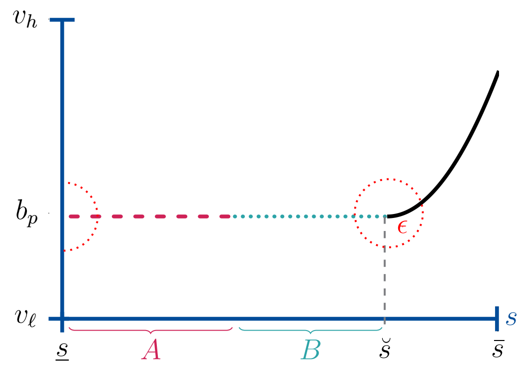
There are two adjacent intervals A and B (pink/dashed and teal/dotted) which span $(\underline{s} + \varepsilon, \breve{s} - \varepsilon)$; this is assertion (i). By assertion (ii), bidders with signals from the interval A pool on bid $b_p - \Delta$, while bidders on the interval B pool on the next higher bid, b_p. The intervals can vary in length as η increases, but the expected number of bidders in both intervals grows without bound—assertion (iii). Finally, by assertion (iv), there are no significant atoms above $\breve{s} + \varepsilon$. In fact, the proof shows that the bidding function above $\breve{s} + \varepsilon$ becomes strictly increasing as the grid becomes fine, $\Delta \to 0$. Observe that the proposition does not assert the uniqueness of equilibrium;

[13] Meaning that for all $s \geq \breve{s} + \epsilon$, it holds that $\eta \int_{\sigma(s)} f_\omega(z)dz < \varepsilon$ for $\omega \in \{h, \ell\}$.

[14] We introduce the communication extension in the next section (Section 6) to capture the limit of the discrete auction when $\Delta \to 0$.



(a) The discrete auction.



(b) The communication extension.[14]

Figure 3: Equilibria of the discrete auction and the communication extension.

in general, there will be multiple equilibria, as discussed later. However, up to the two ϵ neighborhoods, all equilibria must take the aforementioned form.[15]

The proof is Appendix B.1. As the first step of the proof, we exclude those types of equilibria in which almost all signals below $\breve{s}$ pool on some common bid b_p. We do so by showing that when b_p is low enough to be individually rational for $\underline{s}$, bidders with signals close to $\breve{s}$ will have an incentive to overbid b_p when η is large. Therefore, in equilibrium there must be some $s_1 < s_2 \leq \breve{s}$ that bid differently, $\beta(s_1) < \beta(s_2)$. As the second step, we reuse arguments from the proof of Proposition 1 to argue that there can be no "in-between" bid b, with $\beta(s_1) < b < \beta(s_2)$. Specifically, we show that if $\beta(s_2 + \varepsilon)$ is low enough to be individually rational for $s_2 + \varepsilon$, then $\underline{s}$ has an incentive to deviate to any such bid b when η is large. Consequently, such a bid must not exists, and so $\beta(s_1) = \beta(s_2) - \Delta$. Because this argument holds for any s_1 and s_2, the characterization in the proposition follows with $A = \sigma(s_1)$ and $B = \sigma(s_2)$.

5.2 Incentives are shaped by the inference from winning

The equilibria are shaped by the U-shaped inference from winning identified in Lemma 3: For signals above $\breve{s}$, the expected value conditional on winning is strictly increasing. For these signals, incentives and bidding behavior are essentially the same as in an auction without numbers uncertainty. Since there is a curse from tying by Lemma 3, bidders bid away from any atom as the grid becomes dense, so that the

[15] In general, there may be additional small pools in the ϵ neighborhoods. In Section 7, we present a numerical example with exactly two pools.

strategy becomes strictly increasing (Athey, 2001).

The incentives of bidders with "intermediate" signals below $\breve{s}$, that is, those with signals from the interval B, are dominated by an insurance motive. Here, the equilibrium strategy cannot be strictly increasing, because the inference from winning is decreasing by Lemma 3. Instead, bidders mutually insure themselves against winning in the low state by pooling on a common bid b_p, using the blessing from tying and winning the random tie-break, which is more likely in the high state.

Bidders with "low" signals, that is, from the interval A, are in competition for the highest bid below b_p, that is, $b_p - \Delta$. Because of the U-shaped inference, the expected value conditional on winning with b_p is dwarfed by that of winning with any smaller bid. Therefore, bidders who win with a bid below b_p earn strictly positive rents.[16] This sparks a Bertrand competition for the highest bid below b_p among bidders with signals from A. In other words, bids are pinned below b_p, at which the conditional expected value plummets. On the continuous bid space, the largest bid below b_p does not exist; so, no equilibrium exists.

6 Communication extension

The characterization in Section 5 clarifies why the limit of the equilibria of an auction with an increasingly fine grid is not an equilibrium of an auction with a continuous bid space: in the limit, when $\Delta = 0$, intervals A and B bid the same and win with the same probability, so that the utility changes discontinuously. In a sense, the continuum bid space is not rich enough to capture the limit outcome of the finite auction.

To represent this limit and develop a handy tool for further analysis, we introduce an augmented auction with a continuum of bids. In this "communication extension," bidders report an additional message to break ties.[17]

Specifically, the auctioneer chooses a message space, while the bidders choose a bid and a report. The message space is an interval partition M of the signal set $[\underline{s}, \bar{s}]$, a bidding strategy is some $\beta : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$, and a reporting strategy is some $\mu : [\underline{s}, \bar{s}] \to [\underline{s}, \bar{s}]$. Given a partition M and some realized bid–report pairs, the

[16] Individual rationality requires that $b_p \leq \mathbb{E}[v|\text{win with } b_p, s = \inf B]$. For large η, the U-shaped inference implies that $\mathbb{E}[v|\text{win with } b, s'] > \mathbb{E}[v|\text{win with } b_p, s = \inf B]$ for all s' and $b < b_p$, leaving strictly positive rents conditional on winning.

[17] This idea goes back to Lebrun (1996) and Jackson et al. (2002), as discussed below.

outcome is determined as follows. If there is a single highest bidder, then that bidder wins the good. If there are multiple highest bidders, the good is allocated to the one who reports a signal from the highest interval of the partition, and if multiple highest bidders report a signal from the highest interval, ties are broken fairly.

An extended profile (M, β, μ) is a *solution* if (β, μ) is a mutual best response for the bidders given M. A solution is *truthful* if $\mu(s) = s$ for all s.

We now argue that every pointwise limit of a sequence of equilibria on ever finer grids corresponds to a truthful solution.[18] For the result, note that if β is monotone, $\sigma_-(\beta(\cdot))$ is monotone and hence induces an interval partition M of $[\underline{s}, \bar{s}]$,

$$A \in M \iff \sigma_-(\beta(s)) = \sigma_-(\beta(s')) \text{ for all } s, s' \in A.$$

Proposition 4. *Pick some η and suppose β^k is an equilibrium of a discrete auction with $d = k$ steps. Suppose that β^k converges pointwise to some β^* and $\sigma_-\left(\beta^k(\cdot)\right)$ converges pointwise to some $\hat{\sigma}^*_-$ as $k \to \infty$. Let M^* be the partition induced by $\hat{\sigma}^*_-$ and $\mu^*(s) = s$. Then (M^*, β^*, μ^*) is a truthful solution.*

This proof is in Appendix C.1.[19] Proposition 4 shows that the set of solutions of the communication extension contains the set of equilibria of the discretized auction with a vanishing grid. The next result gives a partial converse for our setup: for large η, the truthful outcomes of the communication extension are qualitatively identical to the equilibrium outcomes of a discrete auction with a fine grid.

Proposition 5. *Fix any $\varepsilon \in (0, \frac{\breve{s}-\underline{s}}{2})$. When η is sufficiently large (given ε), any truthful solution (M^*, β^*, μ^*) takes the following form: there are two disjoint, adjacent intervals of signals $A, B \in M^*$ such that*

(i) $[\underline{s} + \varepsilon, \breve{s} - \varepsilon] \subset A \cup B$;

(ii) $\int_A \eta f_\omega(z)dz > \frac{1}{\varepsilon}$, *and* $\int_B \eta f_\omega(z)dz > \frac{1}{\varepsilon}$ *for* $\omega \in \{h, \ell\}$;

(iii) $\beta^*(s) = \beta^*(s')$ *for all* $s, s' \in A \cup B$;

(iv) β^* *is strictly increasing on* $(\breve{s} + \varepsilon, \bar{s}]$.

[18] Every sequence of monotone functions has a subsequence that converges pointwise by Helly's selection theorem.

[19] The result is essentially a special case of the result in Jackson et al. (2002).

The solutions, depicted in Figure 3b, are shaped by the same economic incentives that shape the equilibria of the discrete auction. Indeed, the proof in Appendix C.2 follows the same arguments as the proof of Proposition 3.

For an idea of the proof, consider signals below $\breve{s}$ and pick some s_1, s_2 with $\underline{s} < s_1 < s_2 < \breve{s}$. Reusing arguments form the proof of Proposition 1, for large η, it cannot be that $\beta^*(s_1) < \beta^*(s_2)$. This is because, if $\beta^*(s_2)$ were individually rational, then $\underline{s}$ would have an incentive to submit some in-between bid b', $\beta^*(s_1) < b' < \beta^*(s_2)$. Moreover, the same argument now also implies that there can be no "extended bid" in between. Suppose $b_p = \beta^*(s_1) = \beta^*(s_2)$ but $s_1 \in A$ and $s_2 \in B$ for two non-adjacent intervals $\{A, B\} \in M$, meaning that $s' \notin A \cup B$ for some in-between signal with $s_1 < s' < s_2$. Then a bidder who bids b_p and reports s' wins against bidders in A but loses against bidders in B. For large η, the previous arguments imply that the extended bid (b_p, s') would be a profitable deviation for a bidder with a signal from A or below—in particular, with signal $\underline{s}$. Thus, the intervals A and B must be adjacent for large η.

Remarks: In the online appendix, we give an example with a continuum of solutions, illustrating the equilibrium multiplicity. The equilibrium construction also gives further insights into the equilibrium bidding incentives.

If β^c is an equilibrium of the standard auction with a continuum of bids, then β^c and the partition induced by $\sigma_-(\beta^c(\cdot))$ are a truthful solution. Thus, truthful solutions nest the standard equilibria. However, Proposition 5 implies that, for large η, there is *no* solution (M^*, β^*, μ^*) that corresponds to an equilibrium β^c of the standard auction: on the interval $A \cup B$, the bidding strategy β^* is constant, but A and B are different elements of the partition. Thus, Proposition 5 implies Proposition 2.

Our communication extension is different from the existing ones in Lebrun (1996) and Jackson et al. (2002). In the private-value setting of Lebrun (1996), the message space is essentially unrestricted. In our setting, however, the message space must be restricted; we cannot, for example, allow bidders to simply report some number from $[\underline{s}, \bar{s}]$ and break ties among those who report the highest number. Otherwise, there would always be a report in between what bidders with signals from A and bidders with signals from B report, and bidders with signals from A would have a strict incentive to deviate to this "in-between" report.

Jackson et al. (2002) consider a more flexible tie-breaking rule, allowing the tie-break to depend arbitrarily on the whole vector of reports. This allows them to prove existence in a more general class of games. However, in our setting, this added flexibility can be shown to permit solutions that are qualitatively distinct from any limit of equilibria of the discretized auction. Thus, the tie-breaking rule of Jackson et al. (2002) cannot help us construct a continuum approximation of finite auctions to facilitate the analysis here.

7 Revenue and optimal auction

We want to shed some light on how numbers uncertainty affects classic questions in auction design: in particular, the effects of reserve prices, the auction format, and information disclosure. To this end, we revisit the example from Section 3.4 with $\eta = 5$. When working with strategies that contain pooling bids, we utilize the communication extension.

7.1 The effect of a reserve price on bidding behavior

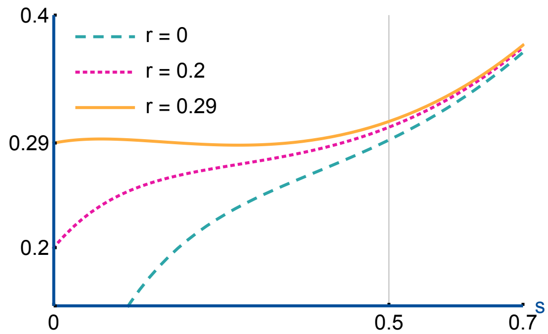
In an auction without numbers uncertainty and with $n \geq 2$ bidders, there is a unique symmetric equilibrium, and it is strictly increasing. Furthermore, any reserve price r either is non-binding or excludes bidders, starting at those with the lowest signals, with higher reserve prices continuously excluding higher signals.

In our setting with numbers uncertainty, the reserve price can raise revenue without exclusion, and exclusion expands discontinuously. Furthermore, reserve prices can induce qualitatively different bidding behavior, determining whether bidders pool or follow a strictly increasing bidding strategy. To illustrate these points, we numerically derive equilibria (solutions) for four reserve prices, $r \in \{0, 0.2, 0.29, 0.35\}$, as shown in Figures 4a and 4b. As the reserve price rises, the revenue increases from 0.426 to 0.433 and 0.436, before dropping to 0.412.

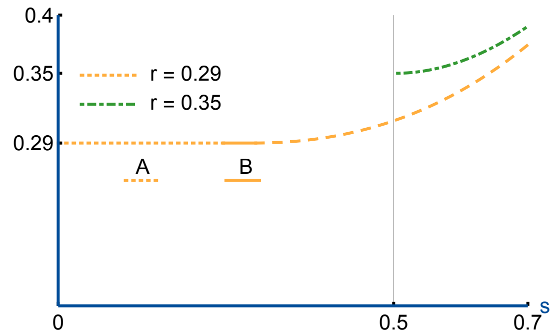
When the reserve price is raised from $r = 0$ to $r = 0.2$, a strictly increasing equilibrium continues to exist, and no bidder is excluded; instead, the reserve price raises the whole bidding strategy. For low r, this is possible because even the lowest signal expects positive rents due to the possibility that there is no competitor.

When the reserve price is raised further to $r = 0.29$, a strictly increasing equilibrium ceases to exist.[20] Roughly speaking, the reserve price acts like a competing bid,

[20]For $r = 0.29$, the ODE (5) has no strictly increasing solution that starts at $\beta(\underline{s}) = r$. There is



(a) At reserve price $r = 0.29$, the solution to the ODE (5) is no longer increasing.



(b) The pooling solution ceases to exist when $r = 0.35$.

Figure 4: Reserve prices discontinuously affect the equilibrium form and participation.

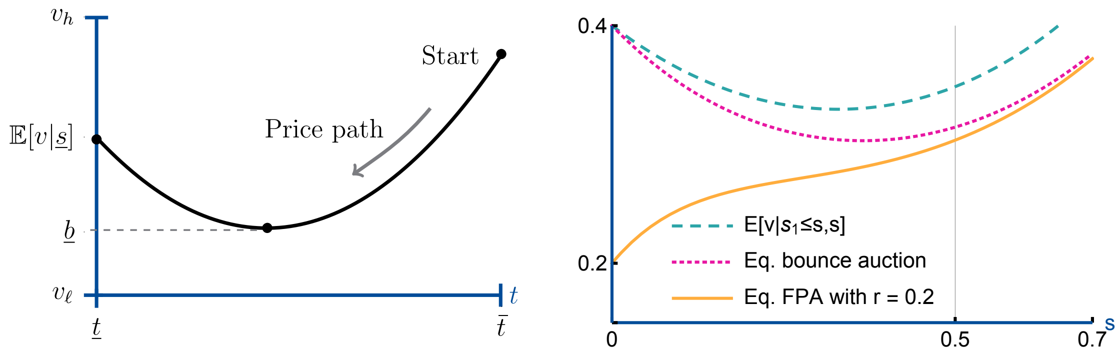
pushing all bids closer to the non-monotone expected value, similarly to a large η. Instead of a strictly increasing equilibrium, there is a solution as described by Proposition 5, in which bidders from the intervals $A = [0, 0.247)$ and $B = [0.247, 0.294)$ all pool to bid $b_p = r = 0.29$ but win with different probabilities. This is shown in Figure 4b.

When r increases further to 0.35, all equilibria (solutions) exclude bidders. In the equilibrium shown in Figure 4b, bidders below $s^* = 0.506$ do not participate, and above s^* the bidding strategy is given by the solution to the ODE (5), starting at $r = \beta(s^*) = \mathbb{E}[v|s_{(1)} \leq s^*, s^*]$. An important implication of the U-shaped inference from winning is that exclusion does not occur continuously from the bottom of the signal space. Instead, it starts binding at intermediate bids and hence at intermediate signals. By monotonicity, this also excludes all bidders with lower signals, so that participation is discontinuous in the reserve price.

7.2 A generalized clock auction can raise revenue

Next, we want to consider how the auction format as a whole might be optimized for numbers uncertainty. Bidding behavior in an auction with numbers uncertainty is driven by the non-monotone inference from winning. In standard auction formats, such as the first-price auction, this non-monotonicity implies that low bids stay away from the expected value conditional on winning, leaving information rents to bidders

also no such equilibrium in which $\beta(s') = r$ for some $s' > \underline{s}$ (exclusion). This is because $\mathbb{E}[v|s_{(1)} \leq s, s] > 0.29$ for all s, so that it is always profitable to participate and bid $r = 0.29$.



(a) The price path of the bounce auction is U-shaped, like the inference from winning.

(b) The bounce auction leaves lower information rents to low signals.

Figure 5: The bounce auction.

with low signals.

To account for the non-monotone inference, we propose a generalized clock auction in which the price first falls, then rises. Any bidder can stop the auction at any time, pay the current price, and receive the good. We call this auction a "bounce auction".

When the expected value conditional on being tied, $\mathbb{E}[v|s_{(1)} \leq s, s]$, is U-shaped, an atomless equilibrium for the bounce auction can be found by solving the ODE (5) with initial value $\beta(\underline{s}) = \mathbb{E}[v|\underline{s}]$, and denoting the bounce by $\underline{b} = \inf \beta(s)$. If we replace s by t and let time run from $\bar{t}$ to $\underline{t}$, the price continuously runs down from $\beta(\bar{t})$ to $\underline{b}$, then back up to $\beta(\underline{t})$. Stopping the auction at $t = s$ represents an atomless equilibrium. This is depicted in Figure 5a.[21]

For our running example, Figure 5b compares the strictly increasing equilibrium of the first-price auction with reserve price $r = 0.2$ to the equilibrium of the bounce auction and the expected value conditional on winning. The figure shows how the bounce auction moves the bids closer to the conditional expected value, reducing the bidders' information rents. Thus, the bounce auction extracts more revenue: 0.439 instead of 0.433 ($r = 0.2$).

By mirroring the U-shape of the inference, the bounce auction reflects a basic economic idea: winning later is good news because it implies that the reason no sale has occurred is that there are no other bidders—rather than that the other bidders are all pessimistic. Thus, bidders are willing to pay more later.

[21] The exact form of the price path is irrelevant. Any continuous and U-shaped price path that starts at or above $\beta(\bar{t})$, ends at or above $\beta(\underline{t})$, and has its minimum at $\underline{b}$ results in the same outcome.

7.3 Information revelation and contingent bidding

Would it be beneficial for the seller to commit to revealing the number of bidders before the auction? In our running example ($\eta = 5$) without a reserve price, doing so raises the seller's expected revenue slightly, from 0.4260 to 0.4262. However, in a related setting, Murto & Välimäki (2019) give an example with the opposite conclusion; thus, it may be interesting to understand what drives the revenue implications more systematically. A related idea in auction design by Harstad et al. (1990) is to allow bidders to submit bids that are contingent on the actual number of bidders. Fully contingent bidding would replicate revealing the number of bidders.

8 Discussion of assumptions

Auction format. When η is large, a second-price auction has no strictly increasing equilibrium either.[22] We further conjecture that the nonexistence on the continuous bid space, as well as the equilibrium characterization on the discrete bid space, extend to the second-price auction.

State-independent competition. One natural modification of our model is state-dependent participation, expressed by a state-dependent mean η_ω. This combines numbers uncertainty with the deterministic but state-dependent participation in Lauermann & Wolinsky (2017). We analyze this general case in our working paper, Lauermann & Speit (2019). Importantly, we show that our results extend if there is a signal with $\frac{f_h(s)}{f_\ell(s)}\frac{\eta_h}{\eta_\ell} = 1$.

Exogenous participation. Numbers uncertainty arises endogenously with a prior entry stage. For example, suppose uninformed potential bidders decide whether to enter at a cost. With entry cost, the expected number of actual participants needs to be bounded, and when the pool of potential bidders is large, symmetric equilibria need to be in mixed strategies. As the pool of potential bidders grows, the distribution of actual participants will converge to a Poisson distribution. If the potential bidders observe their signal first and condition their entry decision on it, then the distribution of actual participants will generally be state-dependent, as discussed above.[23] By

[22] This is because the expected value conditional on being tied is not monotone when η is large; see Lauermann & Speit (2019).

[23] One can show that the implied bidder distribution in a model with signal-dependent entry rates

characterizing the equilibrium outcomes for general bidder distributions, our predictions do not depend on the details of the entry stage. Thereby, our analysis remains valid for other settings with numbers uncertainty.

Distribution of the number of bidders and large auctions. The use of the Poisson distribution simplifies the analysis but has several special properties that may raise concerns: it allows for fewer than two bidders, has unbounded support, and is concentrated around its mean for large η (its standard deviation is just $\sqrt{\eta}$). As discussed in Footnote 9, truncating the distribution at two bidders does not qualitatively change the main insights; the same is true when the distribution is truncated at the top, for a sufficiently large upper bound. However, the concentration of the Poisson distribution around its mean has implications. As η becomes large, the winning bid comes almost surely from a bidder having signal $s > \breve{s}$, where the equilibrium strategy is strictly increasing. For distributions that do not become concentrated, the critical non-monotonicity of the expected value will emerge at the top. As a result, the atoms remain part of the winning bid distribution even in large auctions.

For instance, suppose the number of bidders is either n or n^2 with equal probability, and the signal distribution is as in the example from Section 3.4. For n sufficiently large, the expected value conditional on winning, $\mathbb{E}[v|s_{(1)} \leq s, s]$, will be non-monotone at the top; see Figure 6.[24]

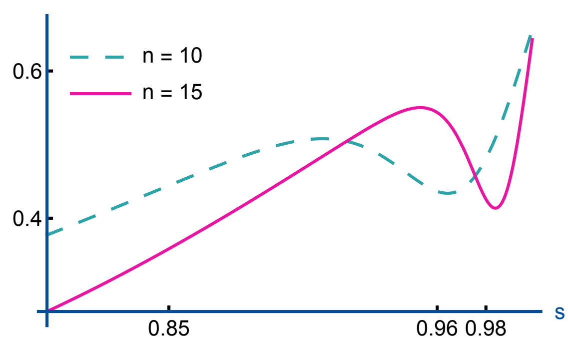


Figure 6: The expected value conditional on winning with either n or n^2 bidders.

In general, one may be interested in the effect of a change in the variance of the number of bidders, holding its mean fixed; this is something the Poisson distribution does not allow for. One may expect that, for a fixed mean, the effects of uncertainty smoothly vanish as the variance decreases, approaching the standard outcome in the limit. Conversely, as the distribution becomes very dispersed, one

is indeed equivalent to the one from some model with exogenous, state-dependent bidder numbers; see Lauermann & Wolinsky (2022, Section 5.3).

[24] When the number of bidders is either 10 or 100, a bidder having signal 0.96 (the local minimizer) wins with probability 0.44, and when the number of bidders is either 15 or 225, the signal $s = 0.98$ wins with probability 0.47.

may suspect that the outcome increasingly diverges from the standard one.

Binary state. When there are more than two states, the inference from winning retains its qualitative shape: at the bottom and at the top, there is no winner's curse, whereas in the middle, winning is bad news about the quality of the good. Thus, no strictly increasing equilibrium can exist when η is large; instead, bidders with low signals must pool. However, when there are more than two states, we cannot exclude decreasing strategies (cf. proof of Lemma 1). Without monotonicity, however, the analysis becomes technically much more challenging.

9 Related literature

There is a small literature on numbers uncertainty with independent types—notably Matthews (1987), McAfee & McMillan (1987), and Harstad et al. (1990)—studying, e.g., the interaction of numbers uncertainty and risk aversion.

Moreover, there is a recent strand of literature on numbers uncertainty in common-value auctions with correlated types. Murto & Välimäki (2019) consider a common-value auction with costly entry.[25] After observing a binary signal, potential bidders decide whether to pay a fee to bid in the auction. When the pool of potential bidders is large, the number of participating bidders is approximately Poisson distributed with a state-dependent mean. Their interest is in the information revelation incentives of the seller; see Section 7.3. They concentrate on parameters for which the entry pattern implies atomless bidding strategies, excluding the effects we are interested in here.[26]

In Lauermann & Wolinsky (2017, 2022) the participation is deterministic but state-dependent due to a solicitation decision by an informed auctioneer. In the paper, atoms are the result of a "participation curse" that arises when there are far fewer bidders in the high than in the low state.

In a double-auction setting with many goods, Harstad et al. (2008) and Atakan & Ekmekci (2021) consider the effect of numbers uncertainty on the information aggregation properties of a kth-price auction (Pesendorfer & Swinkels, 1997). In Harstad et al. (2008), the distribution of bidders is exogenously given. Harstad et al. (2008) find that even if the equilibrium strategy is strictly increasing (which aids aggregation), information aggregation fails unless the numbers uncertainty is negligible. They

[25] For auctions with endogenous entry, see also Levin & Smith (1994) and Harstad (1990).

[26] Basically, in terms of our model, only bidders with the highest signals enter and bid above v_ℓ.

also provide an example in which equilibrium is not strictly increasing, but they do not study this question further. In Atakan & Ekmekci (2021), bidders have a state- and type-dependent outside option so that numbers uncertainty arises endogenously via entry, and participation is correlated with the state. They study how the winning bid in the auction is affected by the opportunity cost of forgoing the outside option.

10 Conclusion

We have studied a canonical common-value auction in which the bidders are uncertain about the number of their competitors. Such "noise" in participation is ubiquitous in auctions, and in price competition more generally, so that the forces studied are present across a wide range of settings.

We find that the numbers uncertainty invalidates classic findings for common-value auctions (Milgrom & Weber, 1982). In particular, it breaks the affiliation between the first order statistic of the signals and the value of the good. As a consequence, bidding strategies generally are not strictly increasing but contain atoms. The locations of the atoms are indeterminate, implying equilibrium multiplicity. Moreover, no equilibrium exists in the standard auction on the continuous bid space when the expected number of bidders is sufficiently large.

Many of the known failures of equilibrium existence in auctions require careful crafting of the setup, and rely on a discrete type space to generate atoms in the bid distribution (Jackson, 2009). By contrast, we have identified a failure of equilibrium existence in an otherwise standard auction setting in which the type space is continuous, and atoms in the bid distribution arise endogenously.

The pooling and the equilibrium multiplicity that arise from numbers uncertainty have interesting implications. For example, even though the model is purely competitive, bidders with low signals behave "cooperatively" to reduce the winner's curse; unlike in the case of a common-value auction with affiliation, they have an incentive to coordinate on certain bids. Consequently, equilibria resemble collusive behavior, even though they are the outcome of independent, utility-maximizing behavior on the bidders' part. Moreover, the presence of atoms in the bid distribution invalidates empirical identification strategies that rely on the bidder's first-order condition (cf. Athey & Haile (2007)) and, hence, on a strictly increasing strategy.

Further analysis may examine more systematically the consequences of pooling and equilibrium multiplicity for classic auction design questions that we touched on in Section 7. In Section 7.2 we discussed how a clock auction with a non-monotone price path can be used to increase revenue given the non-monotone value conditional on winning. The underlying idea is that, when the good has not been sold even after a long delay, a bidder believes that she is the sole participant, rather than that the other bidders are all pessimistic about the value of the good. These considerations for dynamic trade with adverse selection may be worth further study.

Some generalizations may also be worthwhile. We noted that the Poisson distribution becomes highly concentrated on its mean for large numbers. It may be interesting to study the bidding behavior in large auctions for other distributions that are less concentrated. Similarly, one could consider general interdependent values for which the random allocation within a pool has efficiency implications. Lastly, an unknown ratio of goods to buyers might also stem from a random supply of goods (cf. Harstad et al. (2008)). Future research could dig deeper into the implications of such random market tightness more generally.

Appendix A Continuous auction

A.1 Proof of Lemma 1

We prove the strict single-crossing condition, (1). Since $b' > b \geq v_\ell$ it follows that $(v_\ell - b') < (v_\ell - b) \leq 0$. Because the winning probability π_ω is weakly increasing and never zero (the bidder is alone with positive probability), $\pi_\omega(b';\beta) \geq \pi_\omega(b;\beta) \geq \pi_\omega(v_\ell;\beta) > 0$. Together, these observations yield $\pi_\ell(b';\beta)(v_\ell - b') < \pi_\ell(b;\beta)(v_\ell - b) \leq 0$. Hence, $U(b'|s;\beta) \geq U(b|s;\beta)$ requires that $\pi_h(b';\beta)(v_h - b') > \pi_h(b;\beta)(v_h - b)$. Rearranging $U(b'|s;\beta) \geq U(b|s;\beta)$ gives

$$\frac{\rho}{(1-\rho)}\frac{f_h(s)}{f_\ell(s)}[\pi_h(b';\beta)(v_h - b') - \pi_h(b;\beta)(v_h - b)] \geq \pi_\ell(b;\beta)(v_\ell - b) - \pi_\ell(b';\beta)(v_\ell - b').$$

From $s' > s$, we have $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$. Since the left side is strictly positive, it follows that the left side is strictly larger for s' than for s, which is equivalent to the claimed inequality, $U(b'|s',\beta) > U(b|s',\beta)$.

A.2 Proof of Proposition 1

Consider two signals s', s'' with $\underline{s} < s' < s'' < \check{s}$. The proof shows that if η is large enough, and $\beta(s'')$ is low enough to be individually rational for s'', a bidder with signal $\underline{s}$ has an incentive to deviate to $\beta(s')$.

Step 1: Individual rationality. Suppose $\beta(s)$ is optimal for some signal s. Then it must be individually rational:

$$\beta(s) \leq \mathbb{E}[v|\text{win with } \beta(s), s; \beta]. \tag{6}$$

Otherwise, s would be strictly better off bidding v_ℓ, which ensures nonnegative payoffs.[27] The inequality (6) can be written in a convenient "ratio form" as

$$\frac{\beta(s) - v_\ell}{v_h - \beta(s)} \leq \frac{\rho}{1-\rho} \frac{f_h(s)}{f_\ell(s)} \frac{\pi_h(\beta(s); \beta)}{\pi_\ell(\beta(s); \beta)}. \tag{7}$$

Step 2: Competitive bidding. The following lemma shows that as competition becomes fierce, bids must be close to the expected value conditional on winning.

Lemma 5 (Competitive bidding). *Take any two signals s_1 and s_2, with $s_1 < s_2$. For every η, there exists some $C(\eta)$ such that, if β is strictly increasing and s_1 prefers $\beta(s_1)$ to $\beta(s_2)$, that is, if*

$$U(\beta(s_1)|s_1; \beta) \geq U(\beta(s_2)|s_1; \beta),$$

then

$$\frac{\beta(s_2) - v_\ell}{v_h - \beta(s_2)} \geq \frac{\rho}{1-\rho} \frac{f_h(s_1)}{f_\ell(s_1)} \frac{\pi_h(\beta(s_2); \beta)}{\pi_\ell(\beta(s_2); \beta)} C(\eta), \tag{8}$$

and $\lim_{\eta \to \infty} C(\eta) = 1$.

Because the winning probability is much lower at $\beta(s_1)$ than at $\beta(s_2)$, a bidder with signal s_1 will deviate to $\beta(s_2)$ if the payoff conditional on winning there is strictly positive. This requires the bid to be very close to or higher than the expected value conditional on winning. The lemma states this requirement in ratio form, analogously to (7): when $C(\eta) = 1$, the inequality (8) is equivalent to $\beta(s_2) \geq \mathbb{E}[v|s_{(1)} \leq s_2, s_1]$.

Proof of Lemma 5 Expanding $U(\beta(s_1)|s_1; \beta) \geq U(\beta(s_2)|s_1; \beta)$ gives

$$\frac{\rho f_h(s_1)\pi_h(\beta(s_1); \beta)(v_h - \beta(s_1)) + (1-\rho) f_\ell(s_1)\pi_\ell(\beta(s_1); \beta)(v_\ell - \beta(s_1))}{\rho f_h(s_1) + (1-\rho) f_\ell(s_1)}$$

[27] In fact, payoffs are strictly positive at v_ℓ since there is a chance of being the only bidder.

$$\geq \frac{\rho f_h(s_1)\pi_h(\beta(s_2);\beta)(v_h-\beta(s_2))+(1-\rho)f_\ell(s_1)\pi_\ell(\beta(s_2);\beta)(v_\ell-\beta(s_2))}{\rho f_h(s_1)+(1-\rho)f_\ell(s_1)}.$$

Since $\beta(s_1)\geq v_\ell$, a necessary condition for the inequality is that

$$\begin{aligned}&\rho f_h(s_1)\pi_\ell(\beta(s_2);\beta)(v_h-v_\ell)\\ \geq\, &\rho f_h(s_1)\pi_h(\beta(s_2);\beta)(v_h-\beta(s_2))+(1-\rho)f_\ell(s_1)\pi_\ell(\beta(s_2);\beta)(v_\ell-\beta(s_2)).\end{aligned}$$

Rearranging the inequality gives a lower bound on $\beta(s_2)$:

$$\frac{\beta(s_2)-v_\ell}{v_h-\beta(s_2)}\geq \frac{\rho}{1-\rho}\frac{f_h(s_1)}{f_\ell(s_1)}\frac{\pi_h(\beta(s_2);\beta)}{\pi_\ell(\beta(s_2);\beta)}\Big(1-\frac{\pi_h(\beta(s_1);\beta)}{\pi_h(\beta(s_2);\beta)}\frac{v_h-v_\ell}{v_h-\beta(s_2)}\Big). \tag{9}$$

Let

$$C(\eta)=1-\frac{\pi_h(\beta(s_1);\beta)}{\pi_h(\beta(s_2);\beta)}\frac{v_h-v_\ell}{v_h-\mathbb{E}\left[v|s_1\right]}.$$

Note that $C\left(\eta\right)$ is independent of β, since $\pi_h(\beta(s);\beta)$ depends only on s for strictly increasing β. Moreover, $C\left(\eta\right)<1$ for all η because $0<\frac{\pi_h(\beta(s_1);\beta)}{\pi_h(\beta(s_2);\beta)}\leq 1$ and $\frac{v_h-v_\ell}{v_h-\mathbb{E}[v|s_1]}>0$.

Now, the inequality in the lemma holds: First, if $\beta(s_2)<\mathbb{E}\left[v|s_1\right]$, then the bracketed term in (9) is larger than $C(\eta)$. Second, if $\beta(s_2)\geq\mathbb{E}\left[v|s_1\right]$, then this is equivalent to

$$\frac{\beta(s_2)-v_\ell}{v_h-\beta(s_2)}\geq\frac{\rho}{1-\rho}\frac{f_h(s_1)}{f_\ell(s_1)},$$

and so, in particular, $\frac{\beta(s_2)-v_\ell}{v_h-\beta(s_2)}\geq\frac{\rho}{1-\rho}\frac{f_h(s_1)}{f_\ell(s_1)}\frac{\pi_h(\beta(s_2);\beta)}{\pi_\ell(\beta(s_2);\beta)}$, since $\frac{\pi_h(\beta(s_2);\beta)}{\pi_\ell(\beta(s_2);\beta)}\leq 1$. The claim follows because $C(\eta)\leq 1$.

Finally, $\lim_{\eta\to\infty}C\left(\eta\right)=1$ since $\frac{\pi_h(\beta(s_1);\beta)}{\pi_h(\beta(s_2);\beta)}=e^{-\eta(F_h(s_2)-F_h(s_1))}\to 0$. ■

Step 3: Non-monotone expected values. The next lemma shows that when η becomes large, the inference from winning grows arbitrarily strong.

Lemma 6 (U-shaped values)**.** *For any two signals s_1,s_2 with $s_1<s_2<\check{s}$ and any $R>1$, there is some η large enough so that, if β is strictly increasing,*

$$\frac{\pi_h(\beta\left(s_1\right);\beta)}{\pi_\ell(\beta\left(s_1\right);\beta)}>R\frac{\pi_h(\beta\left(s_2\right);\beta)}{\pi_\ell(\beta\left(s_2\right);\beta)}. \tag{10}$$

Proof: Using (2), the ratio of the outer terms of (10) is

$$\frac{\pi_h(\beta\left(s_1\right);\beta)}{\pi_\ell(\beta\left(s_1\right);\beta)}\Big/\frac{\pi_h(\beta\left(s_2\right);\beta)}{\pi_\ell(\beta\left(s_2\right);\beta)}=e^{\eta[(F_\ell(s_2)-F_h(s_2))+(F_\ell(s_1)-F_h(s_1))]}. \tag{11}$$

From $s_1 < s_2 < \breve{s}$, we have $f_\ell(s) > f_h(s)$ for all $s \in (s_1, s_2)$; therefore,

$$(F_\ell(s_2) - F_h(s_2)) + (F_\ell(s_1) - F_h(s_1)) > 0.$$

Hence, the result follows from (11) and $\eta \to \infty$. ■

Combined, the three steps imply the proposition.

Proof of Proposition 1 Pick some strictly increasing bidding strategy β for every η, and a pair of signals s' and s'' with $\underline{s} < s' < s'' < \breve{s}$.

From (7), individual rationality at s'' implies an upper bound for $\beta(s'')$:

$$\frac{\beta(s'') - v_\ell}{v_h - \beta(s'')} \leq \frac{\rho}{1-\rho} \frac{f_h(s'')}{f_\ell(s'')} \frac{\pi_h(\beta(s''); \beta)}{\pi_\ell(\beta(s''); \beta)}. \tag{12}$$

Conversely, (8) from Lemma 5 implies a lower bound on $\beta(s')$ to disincentivize deviations of $\underline{s}$ from $\beta(\underline{s})$ to $\beta(s')$:

$$\frac{\beta(s') - v_\ell}{v_h - \beta(s')} \geq \frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h(\beta(s'); \beta)}{\pi_\ell(\beta(s'); \beta)} C(\eta). \tag{13}$$

We now show that these bounds cannot hold simultaneously when η is large. For both bounds to hold, given $\beta(s') < \beta(s'')$, we must have

$$\frac{\rho}{1-\rho} \frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{\pi_h(\beta(s'); \beta)}{\pi_\ell(\beta(s'); \beta)} C(\eta) \leq \frac{\rho}{1-\rho} \frac{f_h(s'')}{f_\ell(s'')} \frac{\pi_h(\beta(s''); \beta)}{\pi_\ell(\beta(s''); \beta)}. \tag{14}$$

Since $C(\eta) \to 1$ for $\eta \to \infty$ from Lemma 5, this requires that for any $R > \frac{f_h(s'')}{f_\ell(s'')} / \frac{f_h(\underline{s})}{f_\ell(\underline{s})}$,

$$\frac{\pi_h(\beta(s'); \beta)}{\pi_\ell(\beta(s'); \beta)} \leq R \frac{\pi_h(\beta(s''); \beta)}{\pi_\ell(\beta(s''); \beta)}$$

for all η large enough. However, the inequality (10) from Lemma 6 implies that this inequality fails for η large enough. Thus, we have reached a contradiction.

So, if $\beta(s'')$ is individually rational for s'', then (8) from Lemma 5 fails for η large enough; that is, a bidder with signal $\underline{s}$ strictly prefers to bid $\beta(s'')$. It follows that, when η is large, no strictly increasing bidding strategy is an equilibrium. ■

A.3 Auxiliary results for Section 4

A.3.1 Characterization of the winning probability

We show that the winning probability with a pooling bid b_p is

$$\pi_\omega(b_p;\beta) = \frac{\mathbb{P}[s_{(1)} \in \sigma\,|\omega]}{\mathbb{E}[\#s \in \sigma\,|\omega]} = \frac{e^{-\eta(1-F_\omega(\sigma_+))} - e^{-\eta(1-F_\omega(\sigma_-))}}{\eta(F_\omega(\sigma_+) - F_\omega(\sigma_-))}. \tag{15}$$

Let $s_+ = \sigma_+(b_p)$ and $s_- = \sigma_-(b_p)$. Then

$$\begin{aligned}
\pi_\omega(b_p;\beta) &= \mathbb{P}(\text{no bid} > b_p|\omega)\sum_{n=0}^{\infty}\frac{1}{n+1}\mathbb{P}(\text{n competitors bid } b_p|\omega)\\
&= e^{-\eta(1-F_\omega(s_+))}\Big(\sum_{n=0}^{\infty}\frac{1}{n+1}e^{-\eta(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta(F_\omega(s_+)-F_\omega(s_-))]^n}{n!}\Big)\\
&= e^{-\eta(1-F_\omega(s_+))}\Big(\sum_{n=0}^{\infty}e^{-\eta(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta(F_\omega(s_+)-F_\omega(s_-))]^n}{(n+1)!}\Big)\\
&= \frac{e^{-\eta(1-F_\omega(s_+))}}{\eta(F_\omega(s_+)-F_\omega(s_-))}\Big(\sum_{n=1}^{\infty}e^{-\eta(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta(F_\omega(s_+)-F_\omega(s_-))]^n}{n!}\Big)\\
&= \frac{e^{-\eta(1-F_\omega(s_+))}}{\eta(F_\omega(s_+)-F_\omega(s_-))}\Big(1-e^{-\eta(F_\omega(s_+)-F_\omega(s_-))}\Big) = \frac{e^{-\eta(1-F_\omega(s_+))}-e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+)-F_\omega(s_-))}.
\end{aligned}$$

The numerator is $\mathbb{P}[s_{(1)} \in [s_-, s_+]|\omega]$, and the denominator is the expected number of signals from $[s_-, s_+]$ in state ω, i.e. $\mathbb{E}[\#s \in [s_-, s_+]\,|\omega]$.

A.3.2 Proof of Lemma 3

We prove Lemma 3 for the case of $\sigma_+(b_p) = s_+ \leq \breve{s}$; the case of $\sigma_-(b_p) = s_- \geq \breve{s}$ is symmetric and is omitted. In particular, we show that

$$\frac{e^{-\eta(1-F_h(s_-))}}{e^{-\eta(1-F_\ell(s_-))}} > \frac{\pi_h(b_p;\beta)}{\pi_l(b_p;\beta)} > \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}. \tag{16}$$

Let $x_\omega = \mathbb{E}[\#s \in [s_-, s_+]\,|\omega]$; that is,

$$x_h = \eta[F_h(s_+) - F_h(s_-)] \text{ and } x_\ell = \eta[F_\ell(s_+) - F_\ell(s_-)], \tag{17}$$

and note that $s_+ \leq \breve{s}$ implies $x_h < x_\ell$. Further

$$\frac{\pi_\omega(b_p;\beta)}{e^{-\eta(1-F_\omega(s_-))}} = \frac{1}{e^{-\eta(1-F_\omega(s_-))}} \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))} = \frac{e^{x_\omega} - 1}{x_\omega},$$

and so dividing (16) by $\frac{\pi_h(b_p;\beta)}{\pi_l(b_p;\beta)}$ gives

$$\frac{\frac{e^{x_\ell}-1}{x_\ell}}{\frac{e^{x_h}-1}{x_h}} = \frac{\frac{e^{-\eta(1-F_h(s_-))}}{e^{-\eta(1-F_\ell(s_-))}}}{\frac{\pi_h(b_p;\beta)}{\pi_l(b_p;\beta)}} > 1 > \frac{\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}}{\frac{\pi_h(b_p;\beta)}{\pi_l(b_p;\beta)}} = \frac{\frac{1-e^{-x_\ell}}{x_\ell}}{\frac{1-e^{-x_h}}{x_h}}.$$

This holds because $\frac{e^z-1}{z}$ is strictly increasing in z, $\frac{1-e^{-z}}{z}$ is strictly decreasing in z, and $x_h < x_\ell$. Thus, (16) holds for the case of $s_+ \le \check{s}$, as claimed.

A.3.3 Zero-profit condition and U-shaped limit values

We first generalize Lemma 5 to Lemma 7, to allow for weakly increasing β^k and bids that are not in the image of β^k. We then similarly generalize Lemma 6 to Lemma 8.

Lemma 7 (Competitive bidding)**.** *Let (β^k) be a sequence of bidding strategies and $\eta^k \to \infty$. Fix some s_1 and some sequence (b^k), with $b^k > \beta^k(s_1)$ for all k. If*

$$\lim_{k\to\infty} \frac{\pi_h(\beta^k(s_1);\beta^k)}{\pi_h(b^k;\beta^k)} = 0$$

and s_1 prefers $\beta^k(s_1)$ to b^k, that is,

$$U\left(\beta^k(s_1)|s_1;\beta^k\right) \ge U\left(b^k|s_1;\beta^k\right) \text{ for all } k, \tag{18}$$

then there is some sequence (C^k) such that $C^k \to 1$ and

$$\frac{b^k - v_\ell}{v_h - b^k} \ge \frac{\rho}{1-\rho}\frac{f_h(s_1)}{f_\ell(s_1)}\frac{\pi_h(b^k;\beta^k)}{\pi_\ell(b^k;\beta^k)}C^k \text{ for all } k. \tag{19}$$

Proof: By the same argument as in the proof of Lemma 5, (18) implies that

$$\frac{b^k - v_\ell}{v_h - b^k} \ge \frac{\rho}{1-\rho}\frac{f_h(s_1)}{f_\ell(s_1)}\frac{\pi_h(b^k;\beta^k)}{\pi_\ell(b^k;\beta^k)}C^k,$$

for

$$C^k = 1 - \frac{\pi_h(\beta^k(s_1);\beta^k)}{\pi_h(b^k;\beta^k)}\frac{v_h - v_\ell}{v_h - \mathbb{E}[v|s_1]}.$$

Finally, $C^k \to 1$ follows from the hypothesis that $\frac{\pi_h(\beta^k(s_1);\beta^k)}{\pi_h(b^k;\beta^k)} \to 0$. ■

Lemma 8 (U-shaped limit values)**.** *Let* (β^k) *be a sequence of bidding strategies,* $\eta^k \to \infty$*, and* $\left(b_1^k, b_2^k\right)$ *a pair of bids with* $\lim \sigma_+^k\left(b_1^k\right) < \lim \sigma_+^k\left(b_2^k\right) \leq \breve{s}$*. Then, for every* $R > 1$*, for all* k *large enough,*

$$\frac{\pi_h(b_1^k;\beta^k)}{\pi_\ell(b_1^k;\beta^k)} > R\frac{\pi_h(b_2^k;\beta^k)}{\pi_\ell(b_2^k;\beta^k)}. \tag{20}$$

The condition $\lim \sigma_+^k\left(b_1^k\right) < \lim \sigma_+^k\left(b_2^k\right)$ ensures that the winning probability is significantly higher at b_2^k than at b_1^k. The condition $\lim \sigma_+^k\left(b_2^k\right) \leq \breve{s}$ ensures that we are on the decreasing branch of the expected value conditional on winning.

Proof of Lemma 8. If there is an atom at some b^k, then

$$\begin{aligned}\frac{\pi_h(b^k;\beta)}{\pi_\ell(b^k;\beta)} &= \frac{\frac{e^{-\eta(1-F_h(s_+))}-e^{-\eta(1-F_h(s_-))}}{\eta(F_h(s_+)-F_h(s_-))}}{\frac{e^{-\eta(1-F_\ell(s_+))}-e^{-\eta(1-F_\ell(s_-))}}{\eta(F_\ell(s_+)-F_\ell(s_-))}} \\ &= \frac{\eta(F_\ell(s_+)-F_\ell(s_-))}{\eta(F_h(s_+)-F_h(s_-))}\frac{1-e^{-\eta(F_h(s_+)-F_h(s_-))}}{1-e^{-\eta(F_\ell(s_+)-F_\ell(s_-))}}\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}},\end{aligned}$$

and it follows from $\frac{1-e^{-x}}{x}$ being decreasing in x that

$$\min\left\{1, \frac{F_\ell(s_+)-F_\ell(s_-)}{F_h(s_+)-F_h(s_-)}\right\} \leq \frac{\eta(F_\ell(s_+)-F_\ell(s_-))}{\eta(F_h(s_+)-F_h(s_-))}\frac{1-e^{-\eta(F_h(s_+)-F_h(s_-))}}{1-e^{-\eta(F_\ell(s_+)-F_\ell(s_-))}} \leq \max\left\{1, \frac{F_\ell(s_+)-F_\ell(s_-)}{F_h(s_+)-F_h(s_-)}\right\},$$

which is uniformly bounded (we used a similar argument in the proof of Lemma 3). Finally, from $\lim \sigma_+^k\left(b_1^k\right) < \lim \sigma_+^k\left(b_2^k\right) \leq \breve{s}$, we have

$$\lim \frac{e^{-\eta(1-F_h(\sigma_+^k(b_2^k)))}}{e^{-\eta(1-F_\ell(\sigma_+^k(b_2^k)))}} \Bigg/ \frac{e^{-\eta(1-F_h(\sigma_+^k(b_1^k)))}}{e^{-\eta(1-F_\ell(\sigma_+^k(b_1^k)))}} = 0,$$

which was observed in Lemma 6. The claim follows. ■

A.4 Proof of Proposition 2

We consider two cases: first, that for every pair of signals s_1, s_2 with $\underline{s} < s_1 < s_2 < \breve{s}$, and η large enough, $\beta(s_1) = \beta(s_2)$ (Case 1); second, that there exists some pair s_1, s_2 for which $\beta(s_1) < \beta(s_2)$ for all η (Case 2).

The following two subsections show that a sequence of bidding strategies satisfying the assumptions of either case cannot be an equilibrium for large η.

A.4.1 Case 1: Pooling of all signals below $\breve{s}$

The following lemma shows that there can be no equilibrium in which there is a pooling bid b_p for which the pool $\sigma(b_p) = [\sigma_-, \sigma_+]$ starts at some $\sigma_- < \breve{s}$ and extends to some σ_+ close to or beyond $\breve{s}$ when η is large: either b_p is too high to be individually rational for σ_-, or σ_+ will have a strict incentive to marginally overbid.

Lemma 9. *Take any $s_1 < \breve{s}$. Then there exist $s_2 \in (s_1, \breve{s})$ and η^* such that, for all $\eta \geq \eta^*$ and for all β with $\beta(s_1) = \beta(s_2) = b_p$, the following cannot hold simultaneously: $U(b_p|\sigma_-(b_p); \beta) \geq 0$ and*

$$U(b_p|\sigma_+(b_p); \beta) \geq \lim_{\varepsilon \to 0^+} U(b_p + \varepsilon|\sigma_+(b_p); \beta). \tag{21}$$

The main observation of the proof is that, for large enough η,

$$\mathbb{E}[v|\text{win with } b_p, \sigma_-; \beta] < \mathbb{E}[v|s_{(1)} \leq \sigma_+, \sigma_+; \beta], \tag{22}$$

with $\sigma_{+/-} = \sigma_{+/-}(b_p)$. For b_p to be individually rational for σ_-, it must be smaller than the left side of (22). However, when σ_+ marginally overbids b_p, the expected value conditional on winning is equal to the right side of (22). Thus, if b_p is low enough to be individually rational for σ_-, then σ_+ obtains strictly positive profits conditional on marginally overbidding b_p. Since the winning probability at $b_p + \varepsilon$ is significantly larger than at b_p, this will imply that (21) fails, i.e. that σ_+ strictly prefers the deviation.

To gain intuition for (22), recall from Lemma 3 that, for $\sigma_+ \leq \breve{s}$, it holds that $\mathbb{E}[v|\text{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq \sigma_+; \beta]$; that is, there is a winner's blessing at b_p. Hence, (22) shows that this winner's blessing is weaker than the change in the signal inference going from $\sigma_-$to σ_+.

Since it is a central piece of the argument, we now derive (22). First,[28]

$$\frac{\pi_h(b_p)}{\pi_\ell(b_p)} \approx \frac{F_\ell(\sigma_+) - F_\ell(\sigma_-)}{F_h(\sigma_+) - F_h(\sigma_-)} \lim_{\varepsilon \to 0} \frac{\pi_h(b_p + \varepsilon)}{\pi_\ell(b_p + \varepsilon)}. \tag{24}$$

[28] We say that $f(\eta) \approx g(\eta)$ if $\lim_{\eta \to \infty} \frac{f(\eta)}{g(\eta)} = 1$. The claim follows from

$$\frac{\pi_\omega(b_p)}{\lim_{\varepsilon \to 0} \pi_\omega(b_p + \varepsilon)} \eta (F_\omega(\sigma_+) - F_\omega(\sigma_-)) = \frac{e^{-\eta(1-F_\omega(\sigma_+))} - e^{-\eta(1-F_\omega(\sigma_-))}}{e^{-\eta(1-F_\omega(\sigma_+))}}, \tag{23}$$

which converges to 1 as $\eta \to \infty$.

Second, from the MLRP,

$$\frac{f_h(\sigma_-)}{f_\ell(\sigma_-)} < \frac{F_h(\sigma_+) - F_h(\sigma_-)}{F_\ell(\sigma_+) - F_\ell(\sigma_-)}. \tag{25}$$

Hence, combining (24) and (25), we get that $\frac{f_h(\sigma_-)}{f_\ell(\sigma_-)} \frac{\pi_h(b_p)}{\pi_\ell(b_p)} < \lim_{\varepsilon \to 0} \frac{\pi_h(b_p+\varepsilon)}{\pi_h(b_p+\varepsilon)}$. Finally, for σ_+ sufficiently close to or larger than $\breve{s}$, $\frac{f_h(\sigma_+)}{f_\ell(\sigma_+)}$ is close to or larger than 1; thus,

$$\frac{f_h(\sigma_-)}{f_\ell(\sigma_-)} \frac{\pi_h(b_p)}{\pi_\ell(b_p)} < \frac{f_h(\sigma_+)}{f_\ell(\sigma_+)} \lim_{\varepsilon \to 0} \frac{\pi_h(b_p + \varepsilon)}{\pi_h(b_p + \varepsilon)}.$$

This likelihood ratio ordering implies the ordering of the expected values in (22).

Roughly speaking, (24) shows that the strength of the winner's blessing at b_p (relative to overbidding) is proportional to the ratio of the expected numbers of bidders that are tied at b_p, given by $\frac{F_\ell(\sigma_+)-F_\ell(\sigma_-)}{F_h(\sigma_+)-F_h(\sigma_-)}$. This ratio "averages" the inverse likelihood ratios $\frac{f_\ell(s)}{f_h(s)}$ over $s \in [\sigma_-, \sigma_+]$. However, the winner's blessing is weaker than the negative inference from the marginal signal, $\frac{f_h(\sigma_-)}{f_\ell(\sigma_-)}$, by (25).

Proof of Lemma 9. Pick some $s_1 \in (\underline{s}, \breve{s})$. For all s_-, s_+ with $s_- \leq s_1 < s_+$, the MLRP implies that $\frac{f_h(s_-)}{f_\ell(s_-)} \frac{F_\ell(s_+)-F_\ell(s_-)}{F_h(s_+)-F_h(s_-)} < 1$. Hence, we can pick $s_2 \in (s_1, \breve{s})$ close enough to $\breve{s}$ so that, for all $s_- \leq s_1 < s_2 \leq s_+$,

$$\frac{f_h(s_-)}{f_\ell(s_-)} \frac{F_\ell(s_+) - F_\ell(s_-)}{F_h(s_+) - F_h(s_-)} < \frac{f_h(s_+)}{f_\ell(s_+)}. \tag{26}$$

Consider any sequence (β^k) with $\beta^k(s_1) = \beta^k(s_2) = b_p^k$ for some b_p^k with $s_-^k \leq s_1 < s_2 \leq s_+^k$, for $s_-^k = \sigma_-^k(b_p^k)$ and $s_+^k = \sigma_+^k(b_p^k)$, and $\eta^k \to \infty$. We show that, for large k, it cannot hold both that b_p^k is individually rational for s_-^k, i.e. that

$$\frac{b_p^k - v_\ell}{v_h - b_p^k} \leq \frac{\rho}{1-\rho} \frac{f_h(s_-^k)}{f_\ell(s_-^k)} \frac{\pi_h(b_p^k; \beta^k)}{\pi_\ell(b_p^k; \beta^k)}, \tag{27}$$

and that s_+^k does not prefer to overbid the atom, i.e. that

$$U\left(b_p^k | s_+^k; \beta^k\right) \geq \lim_{\varepsilon \to 0^+} U\left(b_p^k + \varepsilon | s_+^k; \beta^k\right). \tag{28}$$

For (28) to hold, Lemma 7 requires that

$$\frac{b_p^k - v_\ell}{v_h - b_p^k} \geq \frac{\rho}{1-\rho} \frac{f_h(s_+^k)}{f_\ell(s_+^k)} \lim_{\varepsilon \to 0^+} \frac{\pi_h(b_p^k + \varepsilon; \beta^k)}{\pi_\ell(b_p^k + \varepsilon; \beta^k)} C^k, \tag{29}$$

for some $C^k \to 1$, given that $\lim_{\varepsilon \to 0^+} \frac{\pi_h(b_p^k;\beta^k)}{\pi_h(b_p^k+\varepsilon;\beta^k)} = \frac{e^{-\eta(1-F_h(s_+^k))} - e^{-\eta(1-F_h(s_-^k))}}{\eta(F_h(s_+^k) - F_h(s_-^k))} / e^{-\eta(1-F_h(s_+^k))} \to 0$. As shown in (24) and Footnote 28, there is a sequence (R^k) such that, for all k,

$$\frac{\pi_h(b_p^k; \beta^k)}{\pi_\ell(b_p^k; \beta^k)} = R^k \frac{F_\ell(s_+^k) - F_\ell(s_-^k)}{F_h(s_+^k) - F_h(s_-^k)} \lim_{\varepsilon \to 0^+} \frac{\pi_h(b_p^k + \varepsilon; \beta^k)}{\pi_\ell(b_p^k + \varepsilon; \beta^k)}, \tag{30}$$

with $R^k \to 1$. Combining (27), (29), and (30), we have

$$\frac{f_h(s_-^k)}{f_\ell(s_-^k)} \frac{F_\ell(s_+^k) - F_\ell(s_-^k)}{F_h(s_+^k) - F_h(s_-^k)} R^k \geq \frac{f_h(s_+^k)}{f_\ell(s_+^k)} C^k.$$

However, since $\lim R^k = \lim C^k = 1$, and $s_{-.}^k \leq s_1$ and $s_2 \leq s_+^k$, this contradicts (26) for large k. Thus, for large k, β^k cannot simultaneously satisfy (27) and (28). ■

A.4.2 Case 2: Some signals below $\breve{s}$ separate

Suppose that there is a pair of signals s_1, s_2 with $\underline{s} < s_1 < s_2 < \breve{s}$ for which $\beta(s_1) < \beta(s_2)$, even for large η. In this case, an argument analogous to the one for Proposition 1 implies that β is not an equilibrium for η large. Specifically, for any bid b' in between, that is, $\beta(s_1) < b' < \beta(s_2)$, a bidder with signal $\underline{s}$ strictly prefers bidding b' to bidding $\beta(\underline{s})$ if the bid $\beta(s_2 + \varepsilon)$ is individually rational for some $s_2 + \varepsilon < \breve{s}$.

Formally, we argue the following. For any $\underline{s} < s_1 < s_2 < \breve{s}$ and any sequence (β^k, η^k), if $\beta^k(s_1) < \beta^k(s_2)$ for all k and $\eta^k \to \infty$, then, for k large enough, β^k is not an equilibrium. For this, take any b^k between $\beta^k(s_1)$ and $\beta^k(s_2)$, i.e., $\beta^k(s_1) < b^k < \beta^k(s_2)$ and any $\hat{s}_2$ with $s_2 < \hat{s}_2 < \breve{s}$. If $\lim \sigma_+^k(\beta^k(\hat{s}_2)) > \breve{s}$, then β^k cannot be an equilibrium by Lemma 9. So, suppose

$$\lim \sigma_+^k(\beta^k(\hat{s}_2)) \leq \breve{s}. \tag{31}$$

We show that if $\beta^k(\hat{s}_2)$ is individually rational for $\hat{s}_2$, then $\underline{s}$ strictly prefers bidding

b^k to bidding $\beta^k(\underline{s})$ for large enough k. Individual rationality for $\hat{s}_2$ requires

$$\frac{\beta^k(\hat{s}_2)-v_\ell}{v_h-\beta^k(\hat{s}_2)} \leq \frac{\rho}{1-\rho}\frac{f_h(\hat{s}_2)}{f_\ell(\hat{s}_2)}\frac{\pi_h(\beta^k(\hat{s}_2);\beta^k)}{\pi_\ell(\beta^k(\hat{s}_2);\beta^k)}.$$

Let

$$C^k = \frac{\frac{f_h(\hat{s}_2)}{f_\ell(\hat{s}_2)}\frac{\pi_h(\beta^k(\hat{s}_2);\beta^k)}{\pi_\ell(\beta^k(\hat{s}_2);\beta^k)}}{\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\pi_h(b^k;\beta^k)}{\pi_\ell(b^k;\beta^k)}},$$

so that, using $b^k < \beta^k(\hat{s}_2)$,

$$\frac{b^k-v_\ell}{v_h-b^k} < \frac{\rho}{1-\rho}\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\pi_h(b^k;\beta^k)}{\pi_\ell(b^k;\beta^k)}C^k \text{ for all } k. \tag{32}$$

Note that

$$\lim \sigma_+^k\left(b^k\right) \leq s_2 < \lim \sigma_+^k\left(\beta^k(\hat{s}_2)\right) \leq \breve{s},$$

where the first inequality is from $b^k < \beta^k(s_2)$, the second from $s_2 < \hat{s}_2 \leq \sigma_+^k\left(\beta^k(\hat{s}_2)\right)$, and the third from (31). Thus, Lemma 8 implies that $\lim C^k = 0$.

Therefore, (19) from Lemma 7 is violated for k large enough, since it requires (32) to hold with the inequality reversed for some $C^k \to 1$. Thus, $\underline{s}$ strictly prefers b^k to $\beta^k(\underline{s})$; hence, β^k is not an equilibrium for large k.

A.4.3 Proof of Proposition 2

Pick any $s_1 < \breve{s}$ and some $s_2 \in (s_1, \breve{s})$ for which Lemma 9 applies. Then, by the lemma, there is some η_1 such that $\beta(s_1) = \beta(s_2)$ implies that β is not an equilibrium for $\eta \geq \eta_1$. Likewise, as just argued in Section A.4.2, there is some η_2 such that $\beta(s_1) < \beta(s_2)$ implies that β is not an equilibrium for $\eta \geq \eta_2$. Hence, there exists no β that is an equilibrium for $\eta \geq \max\{\eta_1, \eta_2\}$.

Appendix B Discrete auction, Section 5

B.1 Proof of Proposition 3

Take some sequences (η^k) and $(d^{k,i})$ with $\lim_{k\to\infty}\eta^k = \infty$ and $\lim_{i\to\infty} d^{k,i} = \infty$ for all k. All other terms are indexed by k, i correspondingly. Given a bidding strategy $\beta^{k,i} : [\underline{s}, \bar{s}] \to \left\{v_\ell, v_\ell + \Delta^{k,i}, \ldots, v_h\right\}$, we abuse notation and write $\sigma_{+/-}^{k,i}(s) = \sigma_{+/-}^{k,i}\left(\beta^{k,i}(s)\right)$, so s is pooled with signals $\left[\sigma_-^{k,i}(s), \sigma_+^{k,i}(s)\right]$. We say $\left(\beta^{k,i}\right)$ is a convergent sequence if, for every k, $\beta^{k,i}$ and $\sigma^{k,i}_{+/-}$ converge pointwise everywhere for $i \to \infty$; that is, for all s and k, $\lim_{i\to\infty} \sigma^{k,i}_{+/-}(s) = \bar{\sigma}^k_{+/-}(s)$ and $\lim_{i\to\infty} \beta^{k,i}(s) = \bar{\beta}^k(s)$, for some $\bar{\sigma}^k_{+/-}$ and $\bar{\beta}^k$. Moreover, for $k \to \infty$, for some $\bar{\sigma}_{+/-}$ and $\bar{\beta}$, $\lim_{k\to\infty} \bar{\sigma}^k_{+/-}(s) = \bar{\sigma}_{+/-}(s)$ and $\lim_{k\to\infty} \bar{\beta}^k(s) = \bar{\beta}(s)$. Since winning probabilities are continuous, the following lemma is immediate.

Lemma 10. *For every convergent sequence of bidding strategies, the following hold:*

- *If $\bar{\sigma}^k_-(s) = \bar{\sigma}^k_+(s)$, then $\lim_{i\to\infty} \pi^{k,i}_\omega\left(\beta^{k,i}(s)\right) = e^{-\eta^k F_\omega\left(1-\bar{\sigma}^k_+(s)\right)}$.*
- *If $\bar{\sigma}^k_-(s) < \bar{\sigma}^k_+(s)$, then, with $\bar{\sigma}^k_{+/-} = \bar{\sigma}^k_{+/-}(s)$,*

$$\lim_{i\to\infty} \pi^{k,i}_\omega\left(\beta^{k,i}(s)\right) = \frac{e^{-\eta^k(1-F_\omega(\bar{\sigma}^k_+))} - e^{-\eta^k(1-F_\omega(\bar{\sigma}^k_-))}}{\eta^k(F_\omega(\bar{\sigma}^k_+) - F_\omega(\bar{\sigma}^k_-))}.$$

B.1.1 Characterization for high signals

We prove that bidders with signals $s > \breve{s}$ do not pool.

Lemma 11. *Suppose $(\beta^{k,i})$ is a convergent sequence of equilibria. Then, for all s, the following hold:*

1. *For all k, if $\bar{\sigma}^k_-(s) \geq \breve{s}$, then $\bar{\sigma}^k_-(s) = \bar{\sigma}^k_+(s) = s$.*
2. *If $\lim_{k\to\infty} \bar{\sigma}^k_-(s) = \breve{s}$, then $\lim_{k\to\infty} \bar{\sigma}^k_+(s) = \breve{s}$.*

For the proof of the lemma, we build on three claims. First, for atoms from bidders with signals $s \geq \breve{s}$, the expected value conditional on winning is strictly below the value from winning when overbidding by one increment.

Claim 1. *For every convergent sequence $(\beta^{k,i})$, every k, and every $s > \breve{s}$, the following holds: if $\breve{s} \leq \bar{\sigma}^k_-(s) < \bar{\sigma}^k_+(s)$ then*

$$\lim_{i\to\infty} \frac{\pi^{k,i}_h\left(\beta^{k,i}(s)\right)}{\pi^{k,i}_\ell\left(\beta^{k,i}(s)\right)} < \lim_{i\to\infty} \frac{\pi^{k,i}_h\left(\beta^{k,i}(s) + \Delta^{k,i}\right)}{\pi^{k,i}_\ell\left(\beta^{k,i}(s) + \Delta^{k,i}\right)}. \tag{33}$$

Proof of Claim 1. For every k and i large enough, $\breve{s} < \sigma^{k,i}_+(s) = \sigma^{k,i}_-\left(\beta^{k,i}(s) + \Delta^{k,i}\right)$; hence, from Lemma 3,

$$\frac{e^{-\eta^k(1-F_h(\sigma^{k,i}_+(s)))}}{e^{-\eta^k(1-F_\ell(\sigma^{k,i}_+(s)))}} \leq \frac{\pi_h(\beta^{k,i}(s) + \Delta^{k,i})}{\pi_\ell(\beta^{k,i}(s) + \Delta^{k,i})}.$$

Moreover, using Lemmas 3 and 10, $\breve{s} \leq \bar{\sigma}_-^k(s) < \bar{\sigma}_+^k(s)$ implies

$$\lim_{i\to\infty} \frac{\pi_h^{k,i}(\beta^{k,i}(s))}{\pi_\ell^{k,i}(\beta^{k,i}(s))} < \frac{e^{-\eta^k(1-F_h(\bar{\sigma}_+^k))}}{e^{-\eta^k(1-F_\ell(\bar{\sigma}_+^k))}},$$

where we drop the argument s from $\bar{\sigma}_{+/-}^k$; these inequalities prove the claim. ■

The next claim shows in particular that the implication of Claim 1 also holds when $\bar{\sigma}_-^k(s) < \breve{s}$ for all k, provided $\lim_{k\to\infty} \bar{\sigma}_-^k(s) = \breve{s}$.

Claim 2. *For every convergent sequence* $(\beta^{k,i})$ *and every* s*, the following holds: if*

$$\breve{s} \leq \lim_{k\to\infty} \bar{\sigma}_-^k(s) < \lim_{k\to\infty} \bar{\sigma}_+^k(s),$$

then, for large enough k*,*

$$\lim_{i\to\infty} \frac{\pi_h^{k,i}\left(\beta^{k,i}(s)\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)\right)} < \lim_{i\to\infty} \frac{\pi_h^{k,i}\left(\beta^{k,i}(s) + \Delta^{k,i}\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s) + \Delta^{k,i}\right)}. \tag{34}$$

Proof of Claim 2. For large enough k and i, $\breve{s} < \sigma_+^{k,i}(s) = \sigma_-^{k,i}\left(\beta^{k,i}(s) + \Delta^{k,i}\right)$. Given Lemmas 3 and 10, it is therefore sufficient to show that, for large enough k,

$$\lim_{i\to\infty} \frac{\pi_h^{k,i}\left(\beta^{k,i}(s)\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)\right)} < \frac{e^{-\eta^k(1-F_h(\bar{\sigma}_+^k))}}{e^{-\eta^k(1-F_\ell(\bar{\sigma}_+^k))}},$$

where $\bar{\sigma}_+^k = \bar{\sigma}_+^k(s)$. Rewriting as in the proof of Lemma 3,

$$\frac{\pi_h^{k,i}\left(\beta^{k,i}(s)\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)\right)} \Bigg/ \frac{e^{-\eta^k(1-F_h(\sigma_+^{k,i}))}}{e^{-\eta^k(1-F_\ell(\sigma_+^{k,i}))}} = \frac{1-e^{-x_h^{k,i}}}{x_h^{k,i}} \Bigg/ \frac{1-e^{-x_\ell^{k,i}}}{x_\ell^{k,i}},$$

with $x_\omega^{k,i} = \eta^k(F_\omega(\sigma_+^{k,i}) - F_\omega(\sigma_-^{k,i}))$. Let $\bar{x}_\omega^k = \lim_{i\to\infty} x_\omega^{k,i} = \eta^k(F_\omega(\bar{\sigma}_+^k) - F_\omega(\bar{\sigma}_-^k))$. The claim now follows from

$$\lim_{i\to\infty} \frac{1-e^{-x_h^{k,i}}}{x_h^{k,i}} \Bigg/ \frac{1-e^{-x_\ell^{k,i}}}{x_\ell^{k,i}} = \frac{F_l(\bar{\sigma}_+^k) - F_\ell(\bar{\sigma}_-^k)}{F_h(\bar{\sigma}_+^k) - F_h(\bar{\sigma}_-^k)} \frac{1-e^{-\bar{x}_h^k}}{1-e^{-\bar{x}_\ell^k}}$$

and

$$\lim_{k\to\infty} \frac{F_l(\bar{\sigma}_+^k) - F_\ell(\bar{\sigma}_-^k)}{F_h(\bar{\sigma}_+^k) - F_h(\bar{\sigma}_-^k)} \frac{1-e^{-\bar{x}_h^k}}{1-e^{-\bar{x}_\ell^k}} = \frac{F_l(\bar{\sigma}_+) - F_\ell(\bar{\sigma}_-)}{F_h(\bar{\sigma}_+) - F_h(\bar{\sigma}_-)} < 1,$$

with $\bar{\sigma}_{+/-} = \lim_{k\to\infty} \bar{\sigma}_{+/-}^k(s)$, where the equality follows from $\bar{x}_\omega^k \to \infty$ (given $\bar{\sigma}_- < \bar{\sigma}_+$) and the strict inequality follows from $\breve{s} \leq \bar{\sigma}_- < \bar{\sigma}_+$. ■

If $(\beta^{k,i})$ is an equilibrium sequence, then for any atom at $\beta^{k,i}(s)$, the expected value conditional on winning must be above the expected value conditional on overbidding by one increment—otherwise, bidders would have an incentive to overbid, as this would increase both the profit conditional on winning and the probability of winning.

Claim 3. *For every sequence of converging equilibria $(\beta^{k,i})$ and for every s and k for which $\bar{\sigma}_-^k(s) < \bar{\sigma}_+^k(s)$,*

$$\lim_{i\to\infty} \frac{\pi_h^{k,i}\left(\beta^{k,i}(s)\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)\right)} \geq \lim_{i\to\infty} \frac{\pi_h^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)}. \tag{35}$$

Proof of Claim 3. If the inequality fails, then

$$\lim_{i\to\infty} \mathbb{E}^{k,i}[v|\text{win with } \beta^{k,i}(s), s] < \lim_{i\to\infty} \mathbb{E}^{k,i}[v|\text{win with } \beta^{k,i}(s)+\Delta^{k,i}, s].$$

Since the winning probability is also strictly higher at $\beta^{k,i}(s)+\Delta^{k,i}$, a bidder with signal s strictly prefers the bid $\beta^{k,i}(s)+\Delta^{k,i}$ to $\beta^{k,i}(s)$, for $\Delta^{k,i}$ small enough. ■

Proof of Lemma 11. For the first part, suppose that $\bar{\sigma}_-^k(s) \geq \breve{s}$ for some k. If the claim of the lemma fails and $\bar{\sigma}_+^k(s) > \bar{\sigma}_-^k(s)$, then (33) implies that (35) fails; hence, $(\beta^{k,i})$ cannot be a sequence of equilibria, which is a contradiction.

For the second part, $\lim_{k\to\infty}\bar{\sigma}_-^k(s) = \breve{s}$. If the claim fails and $\lim_{k\to\infty}\bar{\sigma}_+^k(s) > \breve{s}$ for some convergent sequence of bidding strategies, then (34) implies that (35) fails; hence, $(\beta^{k,i})$ cannot be an equilibrium sequence, a contradiction. ■

B.1.2 Characterization for low signals

The largest upper bound on any atom is $\breve{s}$ (for $k \to \infty$), and if a large atom indeed goes up to $\breve{s}$, then there must be another atom one increment above it:

Lemma 12. *Suppose $(\beta^{k,i})$ is a convergent sequence of equilibria. Then, for all s for which $\lim_{k\to\infty}\bar{\sigma}_-^k(s) < \lim_{k\to\infty}\bar{\sigma}_+^k(s)$, the following hold:*

1. $\lim_{k\to\infty}\bar{\sigma}_+^k(s) \leq \breve{s}$.
2. *If $\lim_{k\to\infty}\bar{\sigma}_+^k(s) = \breve{s}$, then for $s_{+/-}^{k,i} = \sigma_{+/-}^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)$,*

$$\lim_{k\to\infty}\lim_{i\to\infty} \eta^k(F_\omega(s_+^{k,i}) - F_\omega(s_-^{k,i})) = \infty \qquad \textit{for } \omega \in \{\ell, h\}.$$

Proof of Lemma 12: Lemma 9 showed that for any atom that goes up to $\breve{s}$, bidders with signal $\bar{\sigma}_+^k(s)$ would strictly prefer to overbid the atom if this would imply winning against all bidders with signals above $\bar{\sigma}_{+}^{k}(s)$. The reason is that, in that case, the expected value conditional on winning at the higher bid is strictly larger than at the atom. Using the same argument here implies that the likelihood ratio of winning with $\beta^{k,i}(s)+\Delta^{k,i}$ must *not* be equal to the likelihood ratio of winning against all bidders with signals above $\bar{\sigma}_{+}^{k}(s)$; thus, $\beta^{k,i}(s)+\Delta^{k,i}$ is an atom.

Claim 4. *Suppose $(\beta^{k,i})$ is a convergent sequence of equilibria with $\lim_{k\to\infty}\bar{\sigma}_{-}^{k}(s) < \lim_{k\to\infty}\bar{\sigma}_{+}^{k}(s)$ and $\breve{s} \leq \lim_{k\to\infty}\bar{\sigma}_{+}^{k}(s)$ for some s. Then*

$$\lim_{k\to\infty}\lim_{i\to\infty}\frac{\frac{\pi_h^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)}}{\frac{e^{-\eta^k(1-F_h(\sigma_+^{k,i}))}}{e^{-\eta^k(1-F_\ell(\sigma_+^{k,i}))}}} \leq \frac{f_h(\bar{\sigma}_-)}{f_\ell(\bar{\sigma}_-)}\frac{F_\ell(\bar{\sigma}_+)-F_\ell(\bar{\sigma}_-)}{F_h(\bar{\sigma}_+)-F_h(\bar{\sigma}_-)} < 1, \tag{36}$$

with $\bar{\sigma}_{+/-} = \lim_{k\to\infty}\bar{\sigma}_{+/-}^{k}(s)$.

Proof of Claim 4. The claim follows from exactly the same arguments as the proof of Lemma 9. We make the abbreviations

$$b^{k,i} = \beta^{k,i}(s), \quad \pi_{0,\omega}^{k,i} = \pi_\omega^{k,i}\left(\beta^{k,i}(s)\right), \quad \pi_{+,\omega}^{k,i} = \pi_\omega^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right),$$

and we write $\bar{b}^k = \lim_{i\to\infty} b^{k,i}$ (recall pointwise convergence of $(\beta^{k,i})$) and, analogously, $\bar{\pi}_{0,\omega}^{k}$ and $\bar{\pi}_{+,\omega}^{k}$. As usual, we drop the s from all $\sigma_{+/-}^{k,i}$ and $\bar{\sigma}_{+/-}^{k}$.

Using the individual rationality of $b^{k,i}$ for $\sigma_{-}^{k,i}$ and the continuity of the ratios gives

$$\frac{\bar{b}^k - v_\ell}{v_h - \bar{b}^k} \leq \frac{\rho}{1-\rho}\frac{f_h(\bar{\sigma}_-^k)}{f_\ell(\bar{\sigma}_-^k)}\frac{\bar{\pi}_{0,h}^k}{\bar{\pi}_{0,\ell}^k}. \tag{37}$$

Rewriting and evaluating the optimality condition $U(b^{k,i}|\sigma_{+}^{k,i};\beta^{k,i}) \geq U(b^{k,i}+\Delta^{k,i}|\sigma_{+}^{k,i};\beta^{k,i})$ as in the proof of Lemma 9 gives

$$\frac{\bar{b}^k - v_\ell}{v_h - \bar{b}^k} \geq \frac{\rho}{1-\rho}\frac{f_h(\bar{\sigma}_+^k)}{f_\ell(\bar{\sigma}_+^k)}\frac{\bar{\pi}_{+,h}^k}{\bar{\pi}_{+,\ell}^k}C^k, \tag{38}$$

for some C^k with $\lim_{k\to\infty} C^k = 1$. Combining (37) and (38), we have

$$\frac{f_h(\bar{\sigma}_+^k)}{f_\ell(\bar{\sigma}_+^k)}\frac{\bar{\pi}_{+,h}^k}{\bar{\pi}_{+,\ell}^k}C^k \leq \frac{f_h(\bar{\sigma}_-^k)}{f_\ell(\bar{\sigma}_-^k)}\frac{\bar{\pi}_{0,h}^k}{\bar{\pi}_{0,\ell}^k}. \tag{39}$$

Divide both sides by $\frac{e^{-\eta^k(1-F_h(\bar{\sigma}_+^k))}}{e^{-\eta^k(1-F_\ell(\bar{\sigma}_+^k))}}$. Using Lemma 10 and rewriting as in Lemma 3,

$$\frac{\frac{\bar{\pi}^k_{0,h}}{\bar{\pi}^k_{0,\ell}}}{\frac{e^{-\eta^k(1-F_h(\bar{\sigma}^k_+))}}{e^{-\eta^k(1-F_\ell(\bar{\sigma}^k_+))}}} = \frac{F_\ell(\bar{\sigma}^k_+) - F_\ell(\bar{\sigma}^k_-)}{F_h(\bar{\sigma}^k_+) - F_h(\bar{\sigma}^k_-)} \frac{1 - e^{-\eta^k(F_h(\bar{\sigma}^k_+) - F_h(\bar{\sigma}^k_-))}}{1 - e^{-\eta^k(F_\ell(\bar{\sigma}^k_+) - F_\ell(\bar{\sigma}^k_-))}} \approx \frac{F_\ell(\bar{\sigma}^k_+) - F_\ell(\bar{\sigma}^k_-)}{F_h(\bar{\sigma}^k_+) - F_h(\bar{\sigma}^k_-)},$$

where the $\approx$ comes from $\lim_{k\to\infty} \eta^k \left(F_\omega(\bar{\sigma}^k_+) - F_\omega(\bar{\sigma}^k_-)\right) = \infty$. Hence, (39) implies

$$\lim_{k\to\infty} \frac{f_h(\bar{\sigma}^k_+)}{f_\ell(\bar{\sigma}^k_+)} \frac{\frac{\bar{\pi}^k_{+,h}}{\bar{\pi}^k_{+,\ell}}}{\frac{e^{-\eta^k(1-F_h(\bar{\sigma}^k_+))}}{e^{-\eta^k(1-F_\ell(\bar{\sigma}^k_+))}}} C^k \le \lim_{k\to\infty} \frac{f_h(\bar{\sigma}^k_-)}{f_\ell(\bar{\sigma}^k_-)} \frac{F_\ell(\bar{\sigma}^k_+) - F_\ell(\bar{\sigma}^k_-)}{F_h(\bar{\sigma}^k_+) - F_h(\bar{\sigma}^k_-)}. \tag{40}$$

Since $\frac{f_h(\bar{\sigma}^k_+)}{f_\ell(\bar{\sigma}^k_+)} \ge 1$ from the hypothesis $\lim_{k\to\infty} \bar{\sigma}^k_+ \ge \breve{s}$ and $\lim_{k\to\infty} C^k = 1$, the inequality (40) and the MLRP imply the two inequalities in (36). ∎

Proof of Part 1 of Lemma 12.

$$\lim_{k\to\infty} \bar{\sigma}^k_-(s) < \lim_{k\to\infty} \bar{\sigma}^k_+(s) \Rightarrow \lim_{k\to\infty} \bar{\sigma}^k_+(s) \le \breve{s}.$$

The proof is by contradiction. Take a convergent sequence of equilibria $(\beta^{k,i})$ and any s' with $\lim_{k\to\infty} \bar{\sigma}^k_+(s') > \breve{s}$. Without further loss of generality, suppose $\sigma^{k,i}_+(s') > \breve{s}$ for all k, i. We show below that, for any such s', for all k,

$$\lim_{i\to\infty} \frac{\pi^{k,i}_h\left(\beta^{k,i}(s') + \Delta^{k,i}\right)}{\pi^{k,i}_\ell\left(\beta^{k,i}(s') + \Delta^{k,i}\right)} \Bigg/ \frac{e^{-\eta^k(1-F_h(\sigma^{k,i}_+(s')))}}{e^{-\eta^k(1-F_\ell(\sigma^{k,i}_+(s')))}} = 1. \tag{41}$$

Thus, (36) from Claim 4 fails for s' if $\lim \bar{\sigma}^k_+(s') > \breve{s}$, so $\lim \bar{\sigma}^k_+(s') \le \breve{s}$, as claimed.

To prove (41), pick any k and any $s^k > \bar{\sigma}^k_+(s')$. Then $\bar{\sigma}^k_-\left(s^k\right) \ge \bar{\sigma}^k_+(s') > \breve{s}$, and Lemma 11 implies that $\bar{\sigma}^k_+\left(s^k\right) = \bar{\sigma}^k_-\left(s^k\right) = s^k$. Since this is also true for the signal $\frac{s^k + \bar{\sigma}^k_+(s')}{2}$, it follows that $\beta^{k,i}\left(s^k\right) > \beta^{k,i}(s') + \Delta^{k,i}$ for i large. Hence,

$$\begin{aligned}\frac{e^{-\eta^k(1-F_h(\bar{\sigma}^k_+(s')))}}{e^{-\eta^k(1-F_\ell(\bar{\sigma}^k_+(s')))}} &= \lim_{i\to\infty} \frac{e^{-\eta^k(1-F_h(\sigma^{k,i}_+(s')))}}{e^{-\eta^k(1-F_\ell(\sigma^{k,i}_+(s')))}} \le \lim_{i\to\infty} \frac{\pi^{k,i}_h\left(\beta^{k,i}(s') + \Delta^{k,i}\right)}{\pi^{k,i}_\ell\left(\beta^{k,i}(s') + \Delta^{k,i}\right)} \\ &\le \lim_{i\to\infty} \frac{\pi^{k,i}_h\left(\beta^{k,i}\left(s^k\right)\right)}{\pi^{k,i}_\ell\left(\beta^{k,i}\left(s^k\right)\right)} = \frac{e^{-\eta^k(1-F_h(s^k))}}{e^{-\eta^k(1-F_\ell(s^k))}},\end{aligned}$$

where the first equality is from $\sigma^{k,i}_+(s') \to \bar{\sigma}^k_+(s')$, the two inequalities are from Lemma 3 and $\beta^{k,i}\left(s^k\right) > \beta^{k,i}(s') + \Delta^{k,i}$, and the final equality is from $\bar{\sigma}^k_+\left(s^k\right) = \bar{\sigma}^k_-\left(s^k\right) = s^k$. Since s^k can be chosen arbitrarily close to $\bar{\sigma}^k_+(s')$ and the two outer expressions are continuous, (41) holds, as desired.

Proof of Part 2 of Lemma 12.

If $\lim_{k\to\infty} \bar{\sigma}_+^k(s) = \breve{s}$, then for $s_{+/-}^{k,i} = \sigma_{+/-}^{k,i}\left(\beta^{k,i}(s) + \Delta^{k,i}\right)$,

$$\lim_{k\to\infty}\lim_{i\to\infty} x_\omega^{k,i} = \lim_{k\to\infty}\lim_{i\to\infty} \eta^k \left(F_\omega\left(s_+^{k,i}\right) - F_\omega\left(s_-^{k,i}\right)\right) = \infty. \tag{42}$$

As in Lemma 3,

$$\frac{\pi_h^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)}{\pi_\ell^{k,i}\left(\beta^{k,i}(s)+\Delta^{k,i}\right)} \Bigg/ \frac{e^{-\eta^k(1-F_h(\sigma_+^{k,i}))}}{e^{-\eta^k(1-F_\ell(\sigma_+^{k,i}))}} = \frac{x_\ell^{k,i}}{x_h^{k,i}}\frac{e^{x_h^{k,i}}-1}{e^{x_\ell^{k,i}}-1}$$

with $x_\omega^{k,i} = \eta^k\left(F_\omega\left(s_+^{k,i}\right) - F_\omega\left(s_-^{k,i}\right)\right)$, where we note that $\sigma_+^{k,i} = s_-^{k,i}$.

If $\lim_{k\to\infty}\lim_{i\to\infty} x_h^{k,i} = \bar{x}_h < \infty$, then $\lim_{k\to\infty}\lim_{i\to\infty} s_+^{k,i} = \lim_{k\to\infty}\lim_{i\to\infty} s_-^{k,i} = \breve{s}$. Since $\frac{f_h(\breve{s})}{f_\ell(\breve{s})} = 1$ and the likelihood ratio is continuous, this implies $\lim_{k\to\infty}\lim_{i\to\infty} x_\ell^{k,i} = \bar{x}_\ell = \bar{x}_h$. Together,

$$\lim_{k\to\infty}\lim_{i\to\infty}\frac{x_\ell^{k,i}}{x_h^{k,i}}\frac{e^{x_h^{k,i}}-1}{e^{x_\ell^{k,i}}-1} = 1.$$

Thus, the necessary condition (36) from Claim 4 fails if $\bar{x}_h < \infty$. Hence $\bar{x}_h = \infty$ and so $\bar{x}_\ell = \infty$; that is, (42) holds. This completes the proof of the lemma. ■

Lemma 13. *Let* $(\beta^{k,i})$ *be a convergent equilibrium sequence. If* $\underline{s} < s_1 < s_2 < \breve{s}$ *and* $\beta^{k,i}(s_1) < \beta^{k,i}(s_2)$ *for all* k, i, *then, for all large* k *and* i *large enough given* k,

$$\beta^{k,i}(s_2) = \beta^{k,i}(s_1) + \Delta^{k,i}. \tag{43}$$

Proof of Lemma 13. Take any $s_2 < \hat{s}_2 < \breve{s}$. Since $(\beta^{k,i})$ is a sequence of equilibria,

$$U^{k,i}\left(\beta^{k,i}(\hat{s}_2), \hat{s}_2; \beta^{k,i}\right) \geq 0 \text{ and } \lim_{k\to\infty}\bar{\sigma}_+^k(\hat{s}_2) \leq \breve{s},$$

where the first inequality states individual rationality and the second follows from Lemma 12. Now, suppose (43) does not hold. Then, choosing a further subsequence if necessary, there is some sequence of bids $(\hat{b}^{k,i})$ with $\hat{b}^{k,i} \in D^{k,i}$ such that, for all k, i,

$$\beta^{k,i}(s_1) < \hat{b}^{k,i} < \beta^{k,i}(s_2).$$

However, for any fixed i', for k large enough, $\underline{s}$ would deviate to $\hat{b}^{k,i'}$, that is, $U^{k,i'}(\hat{b}^{k,i'}, \underline{s}; \beta^{k,i'}) > U^{k,i'}(\beta^{k,i'}(s_1), \underline{s}; \beta^{k,i'})$. This follows from the same argument as in Case 2 in the proof of Proposition 2. This is because the proof does not utilize the continuum of feasible bids but only compares the payoffs at the bids in question. ■

B.1.3 Proof of Proposition 3

First, we show the characterization holds for convergent sequences of equilibria $(\beta^{k,i})$. Assertion (iv): Take any $s > \breve{s}$. By Lemma 12, $\lim_{k\to\infty} \bar{\sigma}_{-}^{k}(s) > \breve{s}$, and by Lemma 11, this implies that, for all k, $\lim_{i\to\infty} \sigma_{-}^{k,i}(s) = \lim_{i\to\infty} \sigma_{+}^{k,i}(s) = s$. Hence, for any k and i large enough, the expected number of bids that tie with s vanishes.

For assertions (i)–(iii) pick some $s \in (\underline{s}, \breve{s})$.

Case 1: $\bar{\sigma}_{-}(s) = \underline{s}$ and $\bar{\sigma}_{+}(s) = \breve{s}$. The proposition holds with $A = \sigma^{k,i}(s)$ and $B = \sigma^{k,i}\left(\beta^{k,i}(s) + \Delta^{k,i}\right)$. By hypothesis, assertions (i) and (ii) hold. Assertion (iii) (many bidders are pooled) holds for A by $\sigma^{k,i}(s) \to [\underline{s}, \breve{s}]$, and for B by Part 2 of Lemma 12.

Case 2: There are $\underline{s} < s_1 < s_2 < \breve{s}$ such that $\bar{\sigma}_{+}(s_1) \le \bar{\sigma}_{-}(s_2)$. By Lemma 13, for k, i large enough, $\beta^{k,i}(s_2) = \beta^{k,i}(s_1) + \Delta^{k,i}$. Moreover, for all $s_1' \in (\underline{s}, s_1)$, $s_2' \in (s_2, \breve{s})$, and high k, i, $\beta^{k,i}(s_1) = \beta^{k,i}(s_1')$ and $\beta^{k,i}(s_2) = \beta^{k,i}(s_2')$. Hence, assertions (i)–(iii) hold for $A = \sigma^{k,i}(s_1)$ and $B = \sigma^{k,i}(s_2)$.

Since Case 1 and Case 2 are exhaustive, the characterization indeed holds for all convergent sequences of equilibria. Now, take some arbitrary sequence of equilibria $(\beta^{k,i})$. We prove the characterization by contradiction: If the characterization is not true, then, for every K, there is some $k' \ge K$ such that, for every I, there is some $i' \ge I$ such that at least one of the four assertions, (i)–(iv), fails for $\beta^{k',i'}$. Thus, we can pick a new sequence $(\beta^{k',i'})$ of equilibria such that some assertion fails for all k', i'. Of course, this sequence has a convergent subsequence, and, as just shown, all of the assertions hold for k', i' large enough—contradicting the starting hypothesis. ■

Appendix C Communication extension, Section 6

C.1 Proof of Proposition 4

Given the profile (M^*, β^*, μ^*) from the proposition and an extended bid (b, r), let $\pi_\omega^*(b, r)$ denote the winning probability in state ω. Below, we show that for all s,

$$\pi_\omega^*(\beta^*(s), s) = \lim_{k\to\infty} \pi_\omega\left(\beta^k(s); \beta^k\right). \tag{44}$$

Moreover, for every extended bid (b, r), there exists a sequence $b^k \in D^k$ with $b^k \to b$ such that

$$\pi_\omega^*(b, r) = \lim_{k\to\infty} \pi_\omega\left(b^k; \beta^k\right). \tag{45}$$

The displayed equations imply the proposition as follows. Take any s and any (b,r) with a sequence $b^k \to b$ that satisfies (45). Since β^k is an equilibrium, $U^k\left(\beta^k(s)|s;\beta^k\right) \geq U^k\left(b^k|s;\beta^k\right)$ for all b^k. Moreover, (44) and (45) imply that $U^*\left(\beta^*(s), r^*(s)|s;\beta^*,\mu^*\right) = \lim U^k\left(\beta^k(s)|s;\beta^k\right)$ and $U^*\left(b,r|s;\beta^*,\mu^*\right) = \lim U^k\left(b^k|s;\beta^*\right)$. Taken together, these yield $U^*\left(\beta^*(s),\mu^*(s)|s;\beta^*,\mu^*\right) \geq U^*\left(b,r|s;\beta^*,\mu^*\right)$; thus, (β^*,μ^*) is a best response.

Proof of (44): First, consider some s for which $\beta^*(s)$ is not a pooling bid. Then $\beta^*(s-\varepsilon) < \beta^*(s) < \beta^*(s+\varepsilon)$ for all $\varepsilon > 0$, and so the winning probability $\pi^*_\omega\left(\beta^*(s), s\right)$ is sandwiched by $\mathbb{P}\left(s_{(1)} \leq s-\varepsilon|\omega\right)$ and $\mathbb{P}\left(s_{(1)} \leq s+\varepsilon|\omega\right)$. Since $\beta^k(s-\varepsilon) \to \beta^*(s-\varepsilon)$ and $\beta^k(s+\varepsilon) \to \beta^*(s+\varepsilon)$, this is also true for $\pi^k_\omega\left(\beta^k(s);\beta^k\right)$. Choosing ε small implies the claim. Second, consider some s for which $\beta^*(s)$ is a pooling bid. Let $\hat{\sigma}^*_+(s) = \inf\left\{s'|\hat{\sigma}^*_-(s') > \hat{\sigma}^*_-(s)\right\}$, so that s is in the element $m^*(s) = \left[\hat{\sigma}^*_-(s), \hat{\sigma}^*_+(s)\right]$ of the partition M^* (or its equivalent). Given any $\varepsilon > 0$ small enough, pointwise convergence of σ^k_- and σ^k_+ implies that, for all k large enough,

$$\beta^k\left(\hat{\sigma}^*_-(s)-\varepsilon\right) < \beta^k\left(\hat{\sigma}^*_-(s)+\varepsilon\right) = \beta^k(s) = \beta^k\left(\hat{\sigma}^*_+(s)-\varepsilon\right) < \beta^k\left(\hat{\sigma}^*_+(s)+\varepsilon\right),$$

if s is in the interior of $m^*(s)$ (we may ignore the boundary types for this proof). As $\epsilon \to 0$, the winning probability at $(\beta^*(s), s)$ is sandwiched and the result follows.

Proof of (45): Suppose b is not a pooling bid according to β^*, so that r is irrelevant. The cases $b \leq \beta^*(\underline{s})$ and $b \geq \beta^*(\bar{s})$ are immediate. If $\beta^*(\underline{s}) < b < \beta^*(\bar{s})$, we have $\beta^*(s'-\varepsilon) < b < \beta^*(s'+\varepsilon)$ for some s' and all ε small enough. Hence, the winning probability at b is sandwiched between $\mathbb{P}\left(s_{(1)} \leq s'-\varepsilon|\omega\right)$ and $\mathbb{P}\left(s_{(1)} \leq s'+\varepsilon|\omega\right)$, and the claim follows as before. Second, suppose b is a pooling bid according to β^*. With σ^*_- and σ^*_+ the (generalized) inverse of β^*, we are done if $\sigma^*_-(b) \leq r \leq \sigma^*_+(b)$, since then $(b,r) = (\beta^*(s'), s')$ for some s' and (44) applies. So, suppose $\sigma^*_+(b) < r$. Then (b,r) wins if and only if $s_{(1)} \leq \sigma^*_+(b)$. Moreover, for ε small enough $\beta^*\left(\sigma^*_+(b)+\varepsilon\right) > b = \beta^*\left(\sigma^*_+(b)-\varepsilon\right)$. It follows that there is a sequence $b^k \to b$ for which $\beta^k\left(\sigma^*_+(b)+\varepsilon\right) > b^k > \beta^k\left(\sigma^*_+(b)-\varepsilon\right)$ for all k large enough, implying the claim by sandwiching the winning probabilities as before. An analogous argument applies if $r < \sigma^*_-(b)$. ■

C.2 Proof of Proposition 5

Given some $\varepsilon > 0$ and $\eta^k \to \infty$, let $\left(M^k,\beta^k,\mu^k\right)$ be a sequence of truthful solutions. We argue that, for k large enough, $\left(M^k,\beta^k,\mu^k\right)$ satisfies the properties of Proposition

5. To do so, we retrace the proof of Proposition 3. Let $m^k(s) \in M^k$ denote the element of the partition containing s, with $\hat{\sigma}_-^k(s) = \inf\left\{s' | m^k(s') = m^k(s)\right\}$ and $\hat{\sigma}_+^k(s) = \sup\left\{s' | m^k(s') = m^k(s)\right\}$.

We already argued the analogue of Lemma 13 in the main text: for any $\underline{s} < s_1 < s_2 < \breve{s}$ and any k large enough, if r is in between the signals, that is, if $s_1 < r < s_2$, then either $r \in m^k(s_1)$ or $r \in m^k(s_2)$. Thus, either $m^k(s_1) = m^k(s_2)$ or the two elements $m^k(s_1)$ and $m^k(s_2)$ are adjacent.

The analogue of Lemma 11 (no pooling of signals above $\breve{s}$) is that (i) for all k, if $\hat{\sigma}_-^k(s) \geq \breve{s}$ then $m^k(s) = s$, and (ii) $\lim_{k\to\infty} \hat{\sigma}_-^k(s) \geq \breve{s}$ implies $\lim_{k\to\infty} \hat{\sigma}_+^k(s) = \breve{s}$. Like the original lemma, this absence of pooling above $\breve{s}$ follows from Lemma 3.

The analogue of Lemma 12 is that, whenever $\lim_{k\to\infty} \hat{\sigma}_-^k(s) < \lim_{k\to\infty} \hat{\sigma}_+^k(s)$, we have the following: (i) $\lim_{k\to\infty} \hat{\sigma}_+^k(s) \leq \breve{s}$, and (ii) if $\lim_{k\to\infty} \hat{\sigma}_+^k(s) = \breve{s}$, then for all k large enough, there is some $B^k \in M^k$ for which B^k is adjacent to $A^k = m^k(s)$ and $\lim_{k\to\infty} \int_{B^k} \eta^k f_\omega(z)dz = \infty$. For this, we argue the following analogue of Claim 4. Let $\lim_{\delta\to 0^+} \frac{\pi_h^k\left(\beta^k(s), \hat{\sigma}_+^k + \delta\right)}{\pi_\ell^k\left(\beta^k(s), \hat{\sigma}_+^k + \delta\right)} = L^k$ be the likelihood ratio conditional on overbidding the extended bid $\left(\beta^k(s), s\right)$ by bidding $\left(\beta^k(s), \hat{\sigma}_+^k + \delta\right)$. Then, by exactly the same arguments as in the original proof, individual rationality of $\left(\beta^k\left(\hat{\sigma}_-^k\right), \hat{\sigma}_-^k\right)$ and optimality of $\left(\beta^k\left(\hat{\sigma}_+^k\right), \hat{\sigma}_+^k\right)$ require that

$$\lim_{k\to\infty} \frac{L^k}{\frac{e^{-\eta^k(1-F_h(\hat{\sigma}_+^k))}}{e^{-\eta^k(1-F_\ell(\hat{\sigma}_+^k))}}} \leq \frac{f_h(\hat{\sigma}_-)}{f_\ell(\hat{\sigma}_-)} \frac{F_\ell(\hat{\sigma}_+) - F_\ell(\hat{\sigma}_-)}{F_h(\hat{\sigma}_+) - F_h(\hat{\sigma}_-)} < 1, \tag{46}$$

with $\hat{\sigma}_{+/-} = \lim_{k\to\infty} \hat{\sigma}_{+/-}^k(s)$. From this, it follows that there must be some significant pool B^k that is directly adjacent to $m^k(s)$. By the analogue of Lemma 11, it cannot be that B^k starts above $\breve{s}$; hence (i) holds: $\lim_{k\to\infty} \hat{\sigma}_+^k(s) \leq \breve{s}$. Moreover, if $\lim_{k\to\infty} \hat{\sigma}_+^k(s) = \breve{s}$, then (46) requires that $\lim_{k\to\infty} \int_{B^k} \eta^k f_\omega(z)dz = \infty$ for $B^k = m^k\left(\hat{\sigma}_+^k + \delta\right)$, for all δ small enough.

Finally, the analogues of Lemmas 11–13 imply Proposition 5.[29] Thus, as claimed,

[29] Assertion (iv): Take any $\breve{s} + \varepsilon$. By Lemma 12, for k large enough, $\hat{\sigma}_-^k(\breve{s} + \varepsilon) > \breve{s}$, and so, by Lemma 11, $m^k(s) = s$ for all $s \geq \breve{s} + \varepsilon$, implying that β^k is strictly increasing above $\breve{s} + \varepsilon$. Assertions (i)–(iii): Pick some $s \in (\underline{s}, \breve{s})$. Case 1: $\lim_{k\to\infty} \hat{\sigma}_-^k(s) = \underline{s}$ and $\lim_{k\to\infty} \hat{\sigma}_+^k(s) = \breve{s}$. Let $A^k = m^k(s)$. By Lemma 12, there is an adjacent element B^k that becomes large, as required. Case 2: There are $\underline{s} < s_1 < s_2 < \breve{s}$ such that $\hat{\sigma}_+^k(s_1) \leq \hat{\sigma}_-^k(s_2)$ for all k large enough. The desired properties then follow from Lemma 13. Since all sequences of solutions can be partitioned into subsequences that satisfy either Case 1 or Case 2, these cases are exhaustive.

Proposition 5 indeed follows from the same arguments as Proposition 3. ■

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