SUPPLEMENTAL APPENDIX FOR "WEALTH INEQUALITY IN A LOW RATE ENVIRONMENT"

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APPENDIX E: ESTIMATING THE SUFFICIENT STATISTIC IN 1985

So far, we have estimated our sufficient statistic using 2015 as the reference year. Formally, this sufficient statistic answers the following question: in a counterfactual world in which required returns on wealth were a bit higher, by how much lower would Pareto inequality be?

In our empirical application, however, we are interested in the effect of a non-infinitesimal change in the required rate of return (i.e., a 2 pp. decline). In theory, the effect of such a large change in the interest rate can be obtained by integrating our sufficient statistic over the path from $r_0 = 7\%$ to $r_1 = 5\%$:

$$\log\theta(r_1) - \log\theta(r_0) = \int_{r_0}^{r_1} \partial_r \log\theta\big|_{r=r'} \, \mathrm{d}r', \tag{1}$$

where $\partial_r \log\theta\big|_{r=r'}$ denotes the derivative of log Pareto inequality with respect to the required rate of return when it is equal to r'. Along this path, the composition of individuals at the top, as well as the extent to which they use external financing, would change. As a result, the sufficient statistic would change. In this sense, using the sufficient statistic using 2015 as a reference year only gives a first-order approximation for the effect of a change in required returns on Pareto inequality. Using the trapezoidal rule to approximate the integral, a second-order approximation is

$$\log\theta(r_1) - \log\theta(r_0) \approx \frac{1}{2}\Big(\partial_r \log\theta\big|_{r=r_0} + \partial_r \log\theta\big|_{r=r_1} \Big). \tag{2}$$

To quantify these second-order effects, we now estimate our sufficient statistic using 1985 as the reference year. Assuming that the only difference between 1985 and 2015 is due to changes in required returns, the average of two quantities gives a second-order approximation for the effect of the change in required returns on Pareto inequality. We use the same methodology as before. We start from 1985 Forbes list and we categorize the top 100 individuals into entrepreneurs, rentiers and financiers. Table E.I presents our results. Relative to 2015, we find much fewer financiers, and, as a result, slightly more entrepreneurs and rentiers. Moreover, entrepreneurs are much more likely to own a private firm in 1985, while they were more likely to own a public firm in 2015. This makes it a bit harder to measure the sufficient statistic as we typically know much less about these private firms. Still, we now do our best to construct the sufficient statistic in 1985 using the same methodology as the one discussed as in Appendix C.1. One difference is that we will replace the aggregate net worth of households (Board of

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Governors of the Federal Reserve System (US), 2023b) by the aggregate net worth of households and non profits (Board of Governors of the Federal Reserve System (US), 2023a), as only the latter one is available in 1985.

TABLE E.I

INDIVIDUALS IN THE TOP 100 IN 1985 (FORBES LIST)

Group	Count
Entrepreneurs	79
Public corporations	20
Private corporations	59
Rentiers	13
Financiers	8

Notes. "Entrepreneurs" are defined as individuals who are invested in non-financial firms that they (or a family member) founded; "Rentiers" are defined as individuals who are no longer invested in the firm that they (or a family member) founded; "Financiers" are defined as individuals who are invested in a financial firm that they (or a family member) founded.

Because S-1 forms were not available electronically for this time period, we directly use micro-files (via the Columbia Business School library) to compute the number of shares owned by founders at IPO, N_{t_0}. We report the results in Table E.II. Overall, we find that top entrepreneurs in 1985 were much more reliant on debt financing than equity financing. In particular, we find that the average equity payout yield in 1985 is -0.6%, which is much higher than the -2.2% estimated in 2015. Mechanically, this comes from the combination of the fact that (i) there are much fewer public firms in the sample (and we assign a 0% equity payout yield to private firms) and (ii) the public firms in our sample tend to be have been less reliant on external equity financing over their lifetime relative to 2015. One reason could be that required returns on wealth were higher in 1985, and firms rationally respond by investing less. Another reason may be that the venture capital industry was much less developed then, which means that young firms faced larger frictions in issuing equity.

In contrast, we find that the average market leverage is 1.71, which is higher than the 1.43 estimated in 2015. This is consistent with the overall evolution of leverage in the nonfinancial corporate sector (see Hall, 2001). Finally, we find that their lifetime average growth rates of wealth is lower, which reflects the fact that wealth inequality was lower in 1985.

TABLE E.II

SUMMARY STATISTICS IN 1985

	Obs.	Average	Percentiles				
			Min	p25	p50	p75	Max
Equity payout yield	79	–0.6%	–8.7%	0.0%	0.0%	0.0%	3.3%
Dividend yield	79	0.2%	0.0%	0.0%	0.0%	0.1%	4.6%
Buyback yield	79	–0.8%	–9.0%	–0.4%	0.0%	0.0%	0.0%
Market leverage	79	1.71	0.86	1.71	1.71	1.71	4.31
Growth rate of wealth	79	0.22	0.06	0.14	0.18	0.25	0.87

Notes. This table reports the lifetime average dividend yield, buyback yield, market leverage, and growth rate for the top 100 U.S. individuals in 1985. The construction of each variable is detailed in Appendix C.1. Data are from Forbes, Compustat, and SEC S-1 filings.

We now use these results to produce an estimate of our sufficient statistic for the reference year 1985. Using a duration of 35 years, we find that the sufficient statistic in 1985 is -4.9, which is similar to the sufficient statistic of -4.2 estimated using 2015 data. On the one hand, successful entrepreneurs in 1985 relied less on equity financing. On the other hand, they used a larger amount of debt financing and they experienced lower growth rates, which magnifies the effect of a given percentage point change in the growth rate of their wealth on log Pareto inequality.

Table E.III reports the sufficient statistic for the reference years 1985 and 2015 as well as the average of these two numbers. We find an average of -4.5, which is similar to our our first-order approximation using only data from 2015, which gave -4.2.

To improve this second-order approximation, one could also take into account the fact that the duration of a firms is decreasing in r. In particular, the Gordon growth model implies that the derivative of duration with respect to the required return is equal to minus duration squared.[1] This suggests that an average duration of 35 years in our time sample is consistent with a duration of $35 - 0.5 \times 35^2 \times 0.01 \approx 23$ years in 1985 and $35 + 0.5 \times 35^2 \times 0.01 \approx 47$ years in 2015. The second row of Table E.III reports the sufficient statistic using these heterogeneous durations for 1985 and 2015. The average of these two numbers, which can be seen as second-order approximation of the effect of interest rates on Pareto inequality, gives -4.8, which is a bit larger than our first-order approximation using only data from 2015, which gave -4.2.

TABLE E.III

ESTIMATES FOR $\partial_r \log\theta$ OBTAINED BY COMBINING DATA FROM 1985 AND 2015

	Estimate		
	1985	2015	Average
Constant duration (35 years in 1985 and in 2015)	–4.9	–4.2	–4.5
Increasing duration (23 years in 1985 and 47 years in 2015)	–4.3	–5.2	–4.8

APPENDIX F: SOLUTION ALGORITHM FOR MODEL EXPERIMENT

We solve the equilibrium transition dynamics of the model in four steps:

1. Solve for the initial steady-state capital $(K_{0,0}, K_{1,0})$ associated with the required return r_0;
2. Solve for the evolution of the aggregates $(q_{0,t}, q_{1,t}, K_{0,t}, K_{1,t})_{t\geq 0}$ that is consistent with an initial condition $(K_{0,0}, K_{1,0})$ and a sequence of unanticipated and permanent shocks $(\mathrm{d}r_t)_{t\geq 0}$;
3. Back out the sequence of foreign saving shocks $(\mathrm{d}S_{F,t})_{t\geq 0}$ that is consistent with market clearing;
4. We solve for the evolution of the entrepreneur wealth distribution $(p_t(W))_{t\geq 0}$.

Note that the algorithm works for both the inelastic capital baseline and the elastic capital extension. We describe the four steps in the algorithm in the four sections below.

[1]Consider a firm with a positive cash flow stream that grows on average at rate g. Using a constant required return r, the duration of the firm is $1/(r-g)$. As a result, the derivative of the duration with respect to the interest rate is $-1/(r-g)^2$.

F.1. *Steady-states*

Steady-state solution. To solve for the steady-state (q_0, q_1, K_0, K_1) given a constant required return r_0, we simply solve the following system of equations using a root-finding numerical algorithm:

$$0 = -\alpha(K_0+K_1)^{\alpha-1}(1-\pi)^{1-\alpha} + (r-\underline{g}_0) - \tau(\psi q_1 - 1) + (r+\tau-\underline{g}_0)(q_0-0) - \frac{1}{2\chi}(q_0-1)^2,$$

$$0 = -\alpha(K_0+K_1)^{\alpha-1}(1-\pi)^{1-\alpha} + (r-\underline{g}_1) + (r-\underline{g}_1)(q_1-1) - \frac{1}{2\chi}(q_1-1)^2,$$

$$0 = \left(\underline{g}_0 + \frac{1}{\chi}(q_0-1) - \tau - \eta\right)K_0 + \eta\pi\overline{K}, \qquad 0 = \left(\underline{g}_1 + \frac{1}{\chi}(q_1-1) - \eta\right)K_1 + \tau\psi K_0.$$

The first two equations correspond to the HJB equations for the firm problem and the last two equations correspond to the laws of motion for aggregate capital.

F.2. *Computing the trajectory of aggregates*

To solve for the trajectory of aggregates in the presence of a sequence of MIT shocks, we repeatedly solve for perfect foresight transition paths. We first describe how to solve for these paths numerically.

Perfect foresight transition path. Consider an arbitrary period t. Suppose that, at the beginning of period t, the economy has capital $(K_{0,t}, K_{1,t})$ and a constant required return r_t going forward. To solve for the perfect foresight transition dynamics of $(q_{0,t+s}, q_{1,t+s}, K_{0,t+s}, K_{1,t+s})_{s\geq 0}$, we use the method proposed by Achdou et al. (2022). The idea is to assume that the economy will be in steady-state at time $t+T$, for some large value $T>0$. We discretize the time interval $[0,T]$ into equi-spaced intervals $\mathbb{T} = \{0, \Delta t, 2\Delta t, \ldots, T\}$. Let $n \in \mathbb{T}$ denote a point on the grid. The algorithm has two steps.

(a) Backward step. First, we solve for the terminal values $(q_{0,t+T}, q_{1,t+T})$ as the steady-state values implied by the required return r_t (see Section F.1). Then, given a guess for the path of capital $\{K_{0,n}, K_{1,n}\}_{n\in\mathcal{S}}$, we solve for $\{q_{0,n}, q_{1,n}\}_{\in\mathbb{T}}$ backward using the following recursion

$$q_{0,n} = \frac{q_{0,n+1} + \big(\text{MPK}_n - \iota_0(g_{0,n+1}) + \tau(\psi q_{1,n+1} - q_{0,n+1})\big)\Delta t}{1 + (r_{n+1} - g_{0,n+1})\Delta t},$$

$$q_{1,n} = \frac{q_{1,n+1} + \big(\text{MPK}_n - \iota_1(g_{1,n+1})\big)\Delta t}{1 + (r_{n+1} - g_{1,n+1})\Delta t},$$

where $g_{s,n} = \underline{g}_s + \frac{1}{\chi}(q_{s,n+1} - 1)$ for $s = 0, 1$.

(b) Forward step. Given a guess for $\{q_{0,n}, q_{1,n}\}_{n\in\mathbb{T}}$ and initial values $(K_{0,t}, K_{1,t})$, solve for $\{K_{0,n}, K_{1,n}\}_{n\in\mathbb{T}}$ forward using the following recursion

$$K_{0,n+1} = (1 + (g_{0,n+1} - \tau - \eta)\Delta t)K_{0,n} + \eta\overline{K}\pi\Delta t,$$

$$K_{1,n+1} = (1 + (g_{1,n+1} - \eta)\Delta t)K_{1,n} + \tau\psi K_{0,n}\Delta t.$$

We iterate over both steps until the path $(q_{0,t+s}, q_{1,t+s}, K_{0,t+s}, K_{1,t+s})_{s\geq 0}$ converges.

Sequence of permanent required return MIT shocks. Our model experiment consists of feeding a sequence of permanent MIT shocks to the required return. The idea is that, at every t, the required return path gets revised to a constant $r_{t+s} = r_t$ for all $s \geq 0$. Given a required return sequence $(r_t)_{t\geq 0}$, we implement the following algorithm. The goal is to solve for $(q_{0,t}, q_{1,t}, K_{0,t}, K_{1,t})_{t=0}^{T}$, or $\{q_{0,n}, q_{1,n}, K_{0,n}, K_{1,n}\}_{n\in\mathbb{T}}$ in grid notation. The algorithm has two steps.

(a) Initial period ($n = 0$). We solve the steady-state associated with r_0 (see Section F.1) and collect (K_0, K_1), which we store as $(K_{0,0}, K_{1,0})$.

(b) Subsequent periods ($0 < n \leq T/\Delta t$). We solve the perfect foresight transition path associated with a constant required return r_n and initial state $(K_{0,n}, K_{1,n})$ (see paragraph above titled "Perfect foresight transition path") and collect the initial values of of (q_0, q_1) in the perfect foresight transition path, which we store as $(q_{0,n}, q_{1,n})$, and next period's value for (K_0, K_1), which we store as $(K_{0,n+1}, K_{1,n+1})$.

We thus obtain a full transition path $\{q_{0,n}, q_{1,n}, K_{0,n}, K_{1,n}\}_{n\in\mathbb{T}}$ consistent with the sequence of permanent MIT required return shocks.

F.3. *Computing the trajectory of foreign savings*

Once we have the evolution of aggregates, we can compute the path for aggregate wealth $(W_t)_{t\geq 0}$, entrepreneur wealth $(W_{E,t})_{t\geq 0}$, and worker financial wealth $(W_{L,t})_{t\geq 0}$ using the formulas in Appendix D.1. First, we compute the target foreigner wealth as a residual $W_{F,t} = W_t - W_{E,t} - W_{L,t}$. The goal is to find a sequence of foreigner saving rate $(S_{F,t})_{t\geq 0}$, or $\{S_{F,n}\}_{n\in\mathbb{T}}$ in grid notation, such that the law of motion for foreign wealth holds at every period. The idea is that, if the financial market clears, then the product market clears (by Walras law). We thus compute $\{S_{F,n}\}_{n\in\mathbb{T}}$ as a residual:

$$S_{F,n} = \frac{W_{F,n+1} - W_{F,n}\dfrac{q_{M,n+1}}{q_{M,n}}}{\Delta t} + (\eta - r_t) W_{F,n} \qquad \forall n \in \mathbb{T}.$$

F.4. *Computing the wealth distribution*

Finally, we approximate the evolution of the wealth distribution of entrepreneurs $(p(W,t))_{t\geq 0}$ over a finite wealth grid $\mathcal{W}$ and time grid $\mathbb{T}$. Since the wealth distribution is unbounded (i.e., it has a Pareto tail), we use the "Pareto extrapolation" algorithm proposed by Gouin-Bonenfant and Toda (2022). In short, the idea is to approximate the wealth distribution over a finite grid, while accounting for movements of agents in and out of the grid. The method has been shown to be very accurate and robust to grid choices.

APPENDIX: REFERENCES

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