Supplemental web appendix

In this supplemental web appendix I describe the variance estimators used in the Monte Carlo experiments reported in the main text. Graham (2020a) and Graham (2020b) both discuss variance estimation under dyadic dependence and provide references to the primary literature. Equation numbering continues in sequence with that established in the main paper.

D Variance estimation

We have that $\Sigma_{1n}^{c} = \mathbb{C}_n\left(s_{ij,n}s_{ik,n}\right)$ for $j \neq i$. For each of the $i = 1, \ldots, N$ consumers there are $\binom{M}{2} = \frac{2}{M(M-1)}$ pairs of products j and k, yielding a sample covariance of

$$\hat{\Sigma}_{1n}^{c} = \frac{2}{NM\left(M-1\right)} \sum_{i=1}^{N} \sum_{j=1}^{M-1} \sum_{k=j+1}^{M} \hat{s}_{ij,n}\hat{s}_{ik,n}'. \tag{56}$$

A similar argument gives

$$\hat{\Sigma}_{1n}^{p} = \frac{2}{MN\left(N-1\right)} \sum_{j=1}^{M} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \hat{s}_{ij,n}\hat{s}_{kj,n}'. \tag{57}$$

The 'dense', Wald-based, confidence intervals whose coverage properties are analyzed by Monte Carlo are based on the limit distribution for $n^{1/2}S_n$ given in equation (31) of the main text (with (56), (57) and ϕ_n replacing their populating/limiting values). Under dense asymptotics it is also the case that $\hat{\Gamma}_n \stackrel{def}{\equiv} H_n\left(\hat{\theta}\right)$ converges to, say, Γ_0, without rescaling by n. From these two observations a simple sandwich variance estimator can be constructed and inference based on the approximation (see, for example, Graham (2020a)):

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) \stackrel{approx}{\sim} \mathcal{N}\left(0, \hat{\Gamma}_n^{-1}\hat{\Omega}_n^{\mathrm{D}}\hat{\Gamma}_n^{-1}\right), \tag{58}$$

with $\hat{\Omega}_n^{\mathrm{D}} = \frac{\hat{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{\hat{\Sigma}_{1n}^{p}}{\phi_n}$.

Next define

$$\hat{\bar{s}}_{1i,n}^{c} = \frac{1}{M}\sum_{j=1}^{M}\hat{s}_{ij,n}$$

$$\hat{\bar{s}}_{1j,n}^{p} = \frac{1}{N}\sum_{i=1}^{N}\hat{s}_{ij,n}.$$

The 'jackknife' estimate of Σ_{1n}^c is

$$\breve{\Sigma}_{1n}^c = \frac{1}{N}\sum_{i=1}^{N} \hat{\bar{s}}_{1i,n}^c \hat{\bar{s}}_{1i,n}^{c\prime}. \tag{59}$$

See, for example, Efron and Stein (1981). Basic manipulation gives

$$\begin{aligned}
\breve{\Sigma}_{1n}^c =& \frac{1}{N}\frac{1}{M^2}\sum_{i=1}^{N}\left[\sum_{j=1}^{M}\hat{s}_{ij,n}\right]\left[\sum_{j=1}^{M}\hat{s}_{ij,n}\right]' \\
=& \frac{1}{N}\frac{1}{M^2}\sum_{i=1}^{N}\left[\sum_{j=1}^{M}\hat{s}_{ij,n}\hat{s}_{ij,n}' + 2\sum_{j=1}^{M-1}\sum_{k=j+1}^{M}\hat{s}_{ij,n}\hat{s}_{ik,n}'\right] \\
=& \frac{1}{M}\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\hat{s}_{ij,n}\hat{s}_{ij,n}' + \frac{1}{N}\frac{2}{M^2}\sum_{i=1}^{N}\sum_{j=1}^{M-1}\sum_{k=j+1}^{M}\hat{s}_{ij,n}\hat{s}_{ik,n}' \\
=& \frac{1}{M}\widehat{\Sigma_{2n} + \Sigma_{3n}} + \frac{M-1}{M}\hat{\Sigma}_{1n}^c
\end{aligned}$$

where I define $\widehat{\Sigma_{2n} + \Sigma_{3n}} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\hat{s}_{ij,n}\hat{s}_{ij,n}'$.

These calculations give the equality

$$\hat{\Sigma}_{1n}^c = \frac{M}{M-1}\left[\breve{\Sigma}_{1n}^c - \frac{1}{M}\widehat{\Sigma_{2n} + \Sigma_{3n}}\right].$$

Analogous calculations yield

$$\breve{\Sigma}_1^p = \frac{1}{M}\sum_{j=1}^{M}\hat{\bar{s}}_{1i,n}^p\hat{\bar{s}}_{1i,n}^{p\prime} = \frac{1}{N}\widehat{\Sigma_{2n} + \Sigma_{3n}} + \frac{N-1}{N}\hat{\Sigma}_{1n}^p$$

and hence that

$$\hat{\Sigma}_{1n}^p = \frac{N}{N-1}\left[\breve{\Sigma}_{1n}^p - \frac{1}{N}\widehat{\Sigma_{2n} + \Sigma_{3n}}\right].$$

The jackknife estimate for $\mathbb{V}\left(n^{1/2}S_n\right)$ in the dense case is thus

$$\begin{aligned}
\hat{\Omega}_n^{\mathrm{JK}} &= \frac{\breve{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{\breve{\Sigma}_{1n}^{p}}{\phi_n} \\
&= \frac{M-1}{M}\frac{\hat{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{N-1}{N}\frac{\hat{\Sigma}_{1n}^{p}}{\phi_n} + \frac{1}{M}\frac{\widehat{\Sigma_{2n}+\Sigma_{3n}}}{N/n} + \frac{1}{N}\frac{\widehat{\Sigma_{2n}+\Sigma_{3n}}}{M/n} \\
&= \frac{\hat{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{\hat{\Sigma}_{1n}^{p}}{\phi_n} + \frac{2n}{NM}\widehat{\Sigma_{2n}+\Sigma_{3n}} - \frac{1}{M}\frac{\hat{\Sigma}_{1n}^{c}}{N/n} - \frac{1}{N}\frac{\hat{\Sigma}_{1n}^{p}}{M/n} \\
&= \frac{\hat{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{\hat{\Sigma}_{1n}^{p}}{\phi_n} + \frac{1}{n}\frac{1}{\phi_n(1-\phi_n)}\left(2\left[\widehat{\Sigma_{2n}+\Sigma_{3n}}\right] - \hat{\Sigma}_{1n}^{c} - \hat{\Sigma}_{1n}^{p}\right).
\end{aligned}$$

This suggests the bias corrected estimate of $\mathbb{V}\left(n^{1/2}S_n\right)$ equal to

$$\begin{aligned}
\hat{\Omega}_n^{\mathrm{JK-BC}} &= \frac{\breve{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{\breve{\Sigma}_{1n}^{p}}{\phi_n} - \frac{1}{n}\frac{\widehat{\Sigma_{2n}+\Sigma_{3n}}}{\phi_n(1-\phi_n)} \\
&= \frac{\hat{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{\hat{\Sigma}_{1n}^{p}}{\phi_n} + \frac{1}{n}\frac{1}{\phi_n(1-\phi_n)}\left(\widehat{\Sigma_{2n}+\Sigma_{3n}} - \hat{\Sigma}_{1n}^{c} - \hat{\Sigma}_{1n}^{p}\right).
\end{aligned}$$

See Cattaneo et al. (2014) for a related estimator in the context of density weighted average derivatives.[14]

To estimate $\mathbb{V}\left(n^{3/2}S_n\right)$, as required for sparse network inference, I use $n^2\hat{\Omega}^{\mathrm{JK-BC}}$ since

$$n^2\hat{\Omega}_n^{\mathrm{JK-BC}} = \frac{n^2\hat{\Sigma}_{1n}^{c}}{1-\phi_n} + \frac{n^2\hat{\Sigma}_{1n}^{p}}{\phi_n} + \frac{n}{\phi_n(1-\phi_n)}\left(\widehat{\Sigma_{2n}+\Sigma_{3n}} - \hat{\Sigma}_{1n}^{c} - \hat{\Sigma}_{1n}^{p}\right)$$

which, under suitable conditions, should be such that

$$n^2\hat{\Omega}_n^{\mathrm{JK-BC}} \to \frac{\tilde{\Sigma}_1^{c}}{1-\phi} + \frac{\tilde{\Sigma}_1^{p}}{\phi} + \frac{\tilde{\Sigma}_3}{\phi(1-\phi)} + O\left(\frac{1}{n}\right).$$

To estimate $\tilde{\Gamma}_0$ I use $-nH_n\left(\hat{\theta}\right)$. To ensure that $\hat{\Omega}_n^{\mathrm{JK-BC}}$ is positive definite I threshold negative eigenvalues as suggested by Cameron and Miller (2014).

The above estimators seem to be obvious places to start based on the prior work on dyadic clustering surveyed in Graham (2020a) and Graham (2020b). However, exploring the strengths and weakness of alternative methods of sparse network inference formally is a topic for future research.

[14]Note that $n^2\hat{\Omega}^{\mathrm{JK}}$ appears to be a conservative estimate of $\mathbb{V}\left(n^{3/2}S_n\right)$ under sparsity (again see Cattaneo et al. (2014) for helpful discussion in a different context).