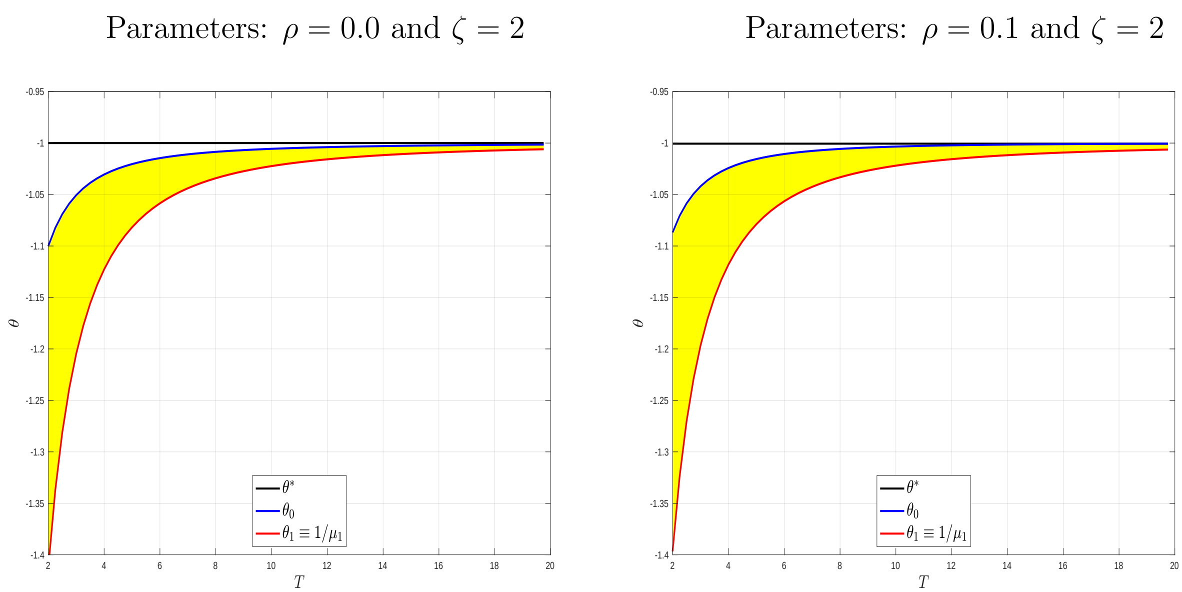
Appendix for Online Publication:

A Range for the Hump-shaped IRF in Calvo

In this section we explore the range of the parameter θ for which the impulse response has a hump shape in the Calvo model. This complements the information in Figure 1. In particular, fixing ρ and T, the impulse response $Y_\theta(t,T)$ is hump-shaped as a function of t if $\theta \in (\theta_1, \theta_0)$. The thresholds $\{\theta_0, \theta_1, \theta^*\}$ are defined in Lemma 2. Note that θ^* does not depend on T, while $\{\theta_0, \theta_1\}$ do. In Figure 6 we plot the thresholds as a function of T, using $\zeta = 2$ and $\rho = 0$ in the left panel while the right panel uses $\rho = 0.10$.

The shaded area contains the values of θ for which a hump shaped impulse response occurs. Two comments are in order. First, the values of T, ρ, ζ scales with the units in which time is measured. We chose $\zeta = 2$, i.e. we interpret time as measured in years (two price changes per year). Second, it is clear that the two regions are almost identical for the two values of ρ considered, which bracket any reasonable yearly interest rate.

Figure 6: Range for hump-shaped IRF in the Calvo model



B Proofs

Proof. (of Proposition 1). Define the markup $m(p/P) \equiv \frac{\eta(p/P)}{\eta(p/P)-1}$. Totally differentiating the first order condition $p^*(P) = m(p^*(P)/P)\,\chi(P)$ with respect to P, completing elasticities and evaluating at $p^* = P$ gives

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=P} = -\frac{m(1)\frac{\chi(P)}{p^*}}{1-m'(1)\frac{\chi(P)}{p^*}}\left(\frac{m'(1)}{m(1)}\right) + \frac{m(1)\frac{\chi(P)}{p^*}}{1-m'(1)\frac{\chi(P)}{p^*}}\left(\frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}\right)$$

and using that $\chi(P)/p^* = 1/m(1)$:

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=P} = \left[\frac{1}{1-\frac{m'(1)}{m(1)}}\right]\left[-\frac{m'(1)}{m(1)} + \frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}\right]$$

To get the expression in equation (1) let $r \equiv p/P$ and note that $m(r) \equiv \frac{\eta(r)}{\eta(r)-1}$ so $m'(r) = \frac{\eta'(r)(\eta(r)-1)-\eta(r)\eta'(r)}{(\eta(r)-1)^2} = -\frac{\eta'(r)}{(\eta(r)-1)^2}$ and hence: $\frac{m'(1)}{m(1)} = -\frac{\eta'(1)}{(\eta(1)-1)^2}\frac{(\eta(1)-1)}{\eta(1)} = -\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}$. That $\eta(1) > 1$ is implied by the first order optimality condition.

The expression in equation (2) is obtained by taking a second order expansion of the profit function at a symmetric equilibrium around the optimal price $p^* = \bar{P}$. Let $x \equiv \frac{p-\bar{P}}{\bar{P}}$ be the firm's price gap, and $X \equiv \frac{P-\bar{P}}{\bar{P}}$ be the aggregate price gap, we have

$$\Pi(p,P) = \Pi(\bar{P},\bar{P}) + \tfrac{1}{2}\Pi_{11}\left(x + \frac{\Pi_{12}}{\Pi_{11}}X\right)^2 + \tfrac{1}{2}\left(\Pi_{22} - \frac{\Pi_{12}^2}{\Pi_{11}}\right)X^2 + o(||x^2, X^2||)$$

Let us normalize the profits by the steady state level of profits $\Pi(\bar{P},\bar{P})$. We can then rewrite the firm's optimal policy as the minimization of the quadratic period loss

$$B\left(x+\theta X\right)^2 \quad \text{where} \quad B \equiv -\tfrac{1}{2}\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P}))}\bar{P}^2 \ \text{ and } \ \theta \equiv \frac{\Pi_{12}(\bar{P},\bar{P})}{\Pi_{11}(\bar{P},\bar{P})}.$$

Finally we show that $1+\frac{\eta'(1)}{\eta(1)(\eta(1)-1)} > 0$. Recall the second order condition for a maximum

$$\Pi_{11}(p^*,P) = D''(p^*/P)(p^*-\chi(P))/P^2 + 2D'(p^*/P)/P < 0$$

Note that $D' < 0$ and that $\chi/p^* = 1/m$ and rewrite the second order condition as

$$\frac{D''(p^*/P)}{D'(p^*/P)}\frac{p^*}{P}\left(1-\frac{1}{m}\right) + 2 > 0 \tag{62}$$

Next differentiate the elasticity $\eta(r) \equiv -\frac{\partial D(r)}{\partial r}\ \frac{r}{D(r)}$ and evaluate it at $r \equiv p^*/P = 1$. We get

$$\eta'(1) = -\frac{D''(1)}{D(1)} + \left(\frac{D'(1)}{D(1)}\right)^2 - \frac{D'(1)}{D(1)} = -\frac{D''(1)}{D(1)} + \eta^2 + \eta$$

where the second equality uses the elasticity definition. We can then write the second order condition equation (62) as $\frac{D''(1)}{D(1)}\ \frac{D(1)}{D'(1)}\ \frac{1}{\eta} + 2 > 0$ or, using the expression for D''/D and the elasticity definition $(\eta' - \eta^2 - \eta)\ \frac{1}{\eta^2} + 2 = \frac{\eta'+\eta(\eta-1)}{\eta^2} > 0$ which establishes that $1 + \frac{\eta'}{\eta(\eta-1)} > 0$, where all η are evaluated at $p^* = P$.

Finally, the expression for $B \equiv -\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P})}\bar{P}^2$, is obtained by direct computation evaluating the objects at $p^* = P = \bar{P}$. We get

$$\frac{\Pi_{11}}{\Pi} = \frac{D''\left(1-\frac{1}{m}\right)\frac{p^*}{P^2} + 2\frac{D'}{P}}{D\,P\left(1-\frac{1}{m}\right)} = \frac{1}{P^2}\left(\frac{D''}{D} + 2\frac{D'}{D}\eta\right) = -\frac{1}{P^2}\left(\eta' + \eta(\eta-1)\right) \ .$$

Proof. (of Lemma 1). Equation (5) is the conventional Hamilton Jacobi equation giving the recursive formulation of the sequence problem in equation (4). As usual the flow value equals the period costs and the expected change in the value function, given by Ito's term and the possibility of a price adjustment. Likewise, the continuation value function for $t > T$ solves the HJB $(\rho+\zeta)\tilde{u}(x) = Bx^2 + \frac{\sigma^2}{2}\tilde{u}_{xx}(x) + \zeta\left(\min_z \tilde{u}(z)\right)$, whose solution is given by $\tilde{u}(x) = \frac{B}{(\rho+\zeta)}\left(\frac{\sigma}{\rho} + x^2\right)$.

Equation (6) follows by taking the first order condition of equation (4) with respect to x and using that the adjustment times are exponentially distributed with parameter ζ to compute the expectation. Equation (7) holds since T is finite, the continuation value function, $e^{-\rho T}\tilde{u}(x)$, is bounded and since $u(x,t)$ is a quadratic function of x (as can be shown by equation (5)).

Proof. (of Lemma 2). Recall that $Y(t) = -X(t)$. The solution in equation (11) is obtained by solving the system of differential equations equation (8) and equation (9) with boundary conditions $x^*(T) = 0$ and $X(0) = -1$. This is a canonical 2 by 2 system whose solution $\dot{y} = Ay$ is readily obtained by a factorization of the matrix $A = S\Lambda S^{-1}$ into a diagonal matrix Λ of eigenvalues $\lambda_j,\ j = 1,2$, given in the proposition, and the matrix of eigenvectors $S \equiv \begin{bmatrix} s_{11} & s_{12} \\ 1 & 1 \end{bmatrix}$ where $s_{11} \equiv \frac{1+\gamma-\Delta}{2\gamma}$, $s_{12} \equiv \frac{1+\gamma+\Delta}{2\gamma}$ and $\Delta \equiv \sqrt{(1+\gamma)^2 + 4\gamma\theta}$ is the discriminant of the characteristic equation. The eigenvalues are real if $\Delta \geq 0$. The rest of the analysis follows from standard computations with the exception of the expression for $\theta_j = \theta^* - \frac{(\Delta_j)^2}{4\gamma}$ which applies when $\Delta < 0$. The latter obtains by setting the denominator of the complex number $\mathfrak{c}(\theta,T)$, given by $d(\theta,T) \equiv (1+\gamma-\Delta(\theta))\,e^{\lambda_1(\theta)T} - (1+\gamma+\Delta(\theta))\,e^{\lambda_2(\theta)T}$, equal to zero, so that θ_j solves $d(\theta,T) = 0$, after replacing the expression for $\Delta, \lambda_1, \lambda_2$. Inspection of the equation reveals that it involves trigonometric function, due to Euler's formula, and thus the equation has countably many zeros. □

Proof. (of Proposition 3). Here we argue that, if $\theta \neq -1$, then the stationary solution displayed above is unique. On the other hand, if $\theta = -1$, then any value X_{ss} corresponds to a steady state. Define $w \equiv x + \theta X_{ss}$. Consider the value function $\hat{u}$ corresponding to the control problem:

$$\hat{u}(w) = \min_{\{\tau_i, \Delta w_i\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} B w^2(t)dt + \sum_{i=1}^{\infty} \psi 1_{\{\tau_i \neq t_i\}} e^{-\rho\tau_i} \,|\, w(0) = w\right]$$

where $dw = \sigma dW$ for $t \in [\tau_i, \tau_{i+1})$ and $w(\tau_i^+) = w(\tau_i^-) + \Delta w_i$ and where t_i are the realizations of the exogenously given times at which the fixed cost is zero, which are exponentially distributed with parameter ζ.

We start making two claims about this problem, and then a third claim about the stationary distribution. First, the value function $\hat{u}$ is symmetric around zero, i.e. $\hat{u}(w) = \hat{u}(-w)$ for all w. This follows because the flow cost Bw^2 is symmetric around zero, and because a standard Brownian motion has, for any collection of times, increments that are normally distributed, and hence symmetric around zero. Second, if the solution of the value function is C^2 then it must satisfy: $(\rho+\zeta)\hat{u}(w) = Bw^2 + \hat{u}_{ww}(w)\frac{\sigma^2}{2} + \zeta\hat{u}(w^*)$ for all $w \in [-\underline{w}, \bar{w}]$ with boundary conditions: $\hat{u}(\bar{w}) = \hat{u}(\underline{w}) = \hat{u}(w^*) + \psi$ and $0 = \hat{u}_w(\bar{w}) = \hat{u}_w(\underline{w}) = \hat{u}_w(w^*)$. Thus, since $\hat{u}$ is symmetric, it must be the case that $\bar{w} = -\underline{w}$ and $w^* = 0$.

Third, and finally, using the symmetry of the thresholds $\{\underline{w}, w^*, \bar{w}\}$, we can find the stationary density $\hat{m}(w)$ which is the unique solution of

$$0 = \hat{m}_{ww}(w)\tfrac{\sigma^2}{2} - \zeta\hat{m}(w) \text{ for all } w \in [\underline{w}, w^*) \cup (w^*, \bar{w}]$$

with boundary conditions: $0 = \hat{m}(\bar{w}) = \hat{m}(\underline{w})$, $\lim_{w\uparrow w^*}\hat{m}(w) = \lim_{w\downarrow w^*}\hat{m}(w)$, and $1 = \int_{\underline{w}}^{\bar{w}}\hat{m}(w)dw$. Importantly, the density $\hat{m}$ must be symmetric, centered at $w^* = 0$.[18] Hence, $\int_{\underline{w}}^{\bar{w}} w\,\hat{m}(w)dw = 0$. Thus, a stationary equilibrium solution of the original problem requires: $x^*_{ss} = w^* - \theta X_{ss}$, $\underline{x}_{ss} = \underline{w} - \theta X_{ss}$, $\bar{x}_{ss} = \bar{w} - \theta X_{ss}$,

$$X_{ss} = \int_{\underline{w}}^{\bar{w}} \hat{m}(w)\left(w - \theta X_{ss}\right)dw = \int_{\underline{w}}^{\bar{w}} \hat{m}(w)w dw - \theta X_{ss}\int_{\underline{w}}^{\bar{w}} \hat{m}(w)dw$$

and thus we can construct a stationary state if and only if: $X_{ss} = -\theta X_{ss}$. Hence if $\theta \neq -1$, then $X_{ss} = 0$ is the only stationary state, and if $\theta = -1$ one can construct a stationary state for any X_{ss}. □

Proof. (of Proposition 4). The proof is constructive. We first we argue that if $X(t) = 0$, then it is optimal for the firm to set $\bar{x}(t) = \bar{x}_{ss} = 1$, $\underline{x}(t) = \underline{x}_{ss}$ and $x^*(t) = x^*_{ss} = 0$. This is immediate since given $X(t) = 0$ the period flow cost for the firm is $B(x + \theta X_{ss})^2 = Bx^2$, which is identical to the one for the stationary problem whose HJB is in equation (25). Hence the optimal policy must be the same as the one for the stationary problem.

Next we prove that $m(x,t)$ is symmetric in x. We will do this by defining a new function $M(x,t) = m(x,t) - m(-x,t)$, and prove that the integral of its square in the Lebesgue measure is zero. This, implies that the only such possible $M(x;t)$ is the zero function, thus establishing that $m(x,t) = m(-x,t)$. We then turn to the existence of a solution to the p.d.e. with the relevant boundary conditions. The argument is based on finding a fixed point for a function $A : [0,T] \to \mathbb{R}_+$ which serves as a Dirichlet boundary at $x = 0$.

Having established that given $\bar{x}(t) = \bar{x}_{ss}, \underline{x}(t) = \underline{x}_{ss}$ and $x^*(t) = x^*_{ss}$, there exists $m(x,t)$ and it is symmetric in x for each t, then $X(t) = \int_{-1}^{1} x\,m(x,t)dx = 0 = X_{ss}$.

That the solution is unique on the class of symmetric m, follows from noting that (i) if m is symmetric, then $X(t) = 0$ and that (ii) the solution to the KFE is unique.

Full details are given in the technical appendix.

□

[18] This can be shown since for $[\underline{w}, 0]$ and $[0, \bar{w}]$, the density is a linear combination of the same two exponentials. Using the boundary conditions at $\underline{w}$ and $\bar{w}$ we express each the density in each segment as function of one constant of integration. Finally by continuity at $w = 0$ we find that the distribution must be symmetric.

Proof. (of Lemma 4). First we show that v is antisymmetric. For that we use that the source $2B\theta xZ(t)$ is antisymmetric as a function of x. To see this, define $w : [0,1] \times [0,T]$ as $w(x,t) = v(x,t) + v(-x,t)$. We will show that $w(x,t)$ is identically zero and solves $0 = w_t(x,t) + kw_{xx}(x,t) - \rho w(x,t)$ with boundary conditions $w(1,t) = v(1,t) + v(-1,t) = 2v(0,t)$ from equation (31) and $w(0,t) = 2v(0,t)$ all t and $w(x,T) = 0$ for all x.

We can use the maximum principle to show that the maximum and minimum of w must occur at the given boundaries, i.e. at either $x \in \{0,1\}$ and any $t \in [0,T)$ or at any $x \in [0,1]$ and $t = T$. To see this, notice that since $w(x,T) = 0$ for all $x \in [0,1]$, then if a minimum will be interior, i.e. if it will occur at $0 < \tilde{x} < 1$ and $0 \le \tilde{t} < T$, then $w(\tilde{x},\tilde{t}) < 0$. Hence, $w_t(\tilde{x},\tilde{t}) = -kw_{xx}(\tilde{x},\tilde{t}) + \rho w(\tilde{x},\tilde{t}) < 0$ since $w_{xx}(\tilde{x},\tilde{t}) \ge 0$ because $(\tilde{x},\tilde{t})$ is an interior minimum and $k > 0$, and since $w(\tilde{x},\tilde{t}) < 0$. Hence $w(\tilde{x},t') < w(\tilde{x},\tilde{t})$ for t' close to $\tilde{t}$, a contradiction with $(\tilde{x},\tilde{t})$ being an interior minimum. A similar argument shows that there cannot be an interior maximum.

Now we show that the maximum and minimum has to occur at $t = T$. For this we use that $w(x,t) = v(x,t) + v(-x,t)$ implies $w_x(0,t) = v_x(0,t) - v_x(0,t) = 0$ for all $t < T$. Thus, suppose that the minimum occurs at $(x,t) = (0,t_1)$ where $t_1 < T$. Then $w(0,t_1) = 2v(0,t_1)$ and $w_t(0,t_1) = 2v_t(0,t_1)$, so $2\rho v(0,t_1) = kw_{xx}(0,t_1) + 2v_t(0,t_1)$. Since $(0,t_1)$ is a minimum, we have $v_t(0,t_1) \ge 0$ and since the minimum occurs at $t_1 < T$, then $v(0,t_1) < 0$, so $w_{xx}(0,t_1) < 0$. But since $w_x(0,t_1) = 0$, then we obtain a contradiction with $(0,t_1)$ being a minimum. A similar argument shows that the maximum cannot occur at $(x,t) = (0,t_2)$ where $t_2 < T$. Thus the minimum and maximum occur at $t = T$, where $w(x,T) = 0$.

So we have shown that $w(x,t) = 0$ for all (x,t), and hence $v(x,t) = -v(-x,t)$ all (x,t). Since v is antisymmetric we have $v(0,t) = -v(-0,t)$ and hence $v(0,t) = 0$.

Second, using smooth pasting at the boundaries ($\tilde{u}_x(-1) = \tilde{u}_x(1) = 0$) and optimality at $x^* = 0$ ($\tilde{u}_x(0) = 0$) in equation (31), we can write the boundary conditions as

$$v(-1,t) = v(0,t) = v(1,t) = 0 \quad \text{all } t \in (0,T)$$

which gives the desired result. □

Proof of Lemma 2. Preliminaries. First we present a lemma that will be used for solving the HJB in equation (30). For notation simplicity we use ρ below to denote the constant parameter $(\rho + \zeta)$ appearing in equation (30).

LEMMA 9. Let f be the solution of the heat equation

$$0 = f_t(x,t) + kf_{xx}(x,t) - \rho f(x,t) + s(x,t) \text{ for all } x \in [-1,1] \text{ and } t \in [0,T) \tag{63}$$

and boundaries

$$f(1,t) = \bar{\phi}(t) \text{ and } f(-1,t) = \underline{\phi}(t) \text{ for all } t \in (0,T) \tag{64}$$

and

$$f(x,T) = \Phi(x) \text{ for all } x \in [-1,1] \tag{65}$$

for functions $\bar{\phi}, \underline{\phi}, \Phi$ and s. Assume that $\rho \ge 0$ and $k > 0$. The solution is unique.

Proof. (of Lemma 9). As a contradiction, assume that there are two solutions f^1 and f^2. Let $F(x,t) \equiv f^2(x,t) - f^1(x,t)$. Note that the p.d.e. in equation (63) is linear, so that F must satisfy

$$0 = F_t(x,t) + kF_{xx}(x,t) - \rho F(x,t) \text{ for all } x \in [-1,1] \text{ and } t \in (0,T) \tag{66}$$

with boundaries:

$$F(1,t) = 0 \text{ and } F(-1,t) = 0 \text{ for all } t \in (0,T) \text{ and} \tag{67}$$

$$F(x,T) = 0 \text{ for all } x \in [-1,1] \tag{68}$$

We use a conservation of energy type of argument. Define $I(t) \equiv \int_{-1}^{1}(F(x,t))^2 dx \geq 0$ for $t \in [0,T]$. Then use the boundary condition $I(T) = 0$ to write $0 = I(T) = I(0) + \int_0^T I'(t)dt$. Next compute:

$$\begin{aligned} I'(t) &= \int_{-1}^{1} \tfrac{d}{dt}(F(x,t))^2 dx = 2\int_{-1}^{1} F(x,t)F_t(x,t)dx = 2\int_{-1}^{1} F(x,t)[\rho F(x,t) - kF_{xx}(x,t)]dx \\ &= 2\rho \int_{-1}^{1} F(x,t)^2 dx + 2k\left(\int_{-1}^{1} F_x(x,t)^2 dx - F(x,t)F_x(x,t)\Big|_{-1}^{1}\right) \end{aligned}$$

where we have substituted the p.d.e. and integrated by parts. Using the boundary conditions in equation (67) we have:

$$I'(t) = 2\rho \int_{-1}^{1} F(x,t)^2 dx + 2k\int_{-1}^{1} F_x(x,t)^2 dx \geq 0$$

Thus $I(T) = 0$ only if I is zero for almost all t, and hence $F(x,t) = 0$ for almost all x, which in turns implies that $f^1 = f^2$ for almost all x, t. □

Proof. (of Lemma 5). Uniqueness follows from the argument given in Lemma 9.

That equation (34) satisfies the zero boundary condition at $t = T$ follows immediately since at $t = T$ equation (34) becomes an integral with zero length. That the Dirichlet boundary condition holds at $x = 1$ and $x = -1$ follows since $\sin(xj\pi) = 0$ for all integers j. Note also that the $v(0,t) = 0$ since $\sin(0) = 0$. It only remains to show that equation (34) satisfies the heat equation with source $CxZ(t)$, where $C \equiv 2B\theta$. Direct computation gives

$$\begin{aligned} v_t(x,t) &= CZ(t)\,2\sum_{j=1}^{\infty}\frac{(-1)^j}{j\pi}\sin(j\pi x) \\ &\quad - 2C\int_t^T \sum_{j=1}^{\infty} e^{(\rho+k(j\pi)^2)(t-\tau)}(\rho + k(j\pi)^2)Z(\tau)\frac{(-1)^j}{j\pi}\sin(j\pi x)d\tau \\ v_{xx}(x,t) &= 2C\int_t^T \sum_{j=1}^{\infty} e^{(\rho+k(j\pi)^2)(t-\tau)}Z(\tau)\frac{(-1)^j}{j\pi}(j\pi)^2\sin(j\pi x)d\tau \end{aligned}$$

and notice that the Fourier series for x in the interval $[0,1]$ is $x = -2\sum_{j=1}^{\infty}\frac{(-1)^j}{j\pi}\sin(j\pi x)$, since $\int_0^1 x\sin(j\pi x)dx/\int_0^1 \sin^2(j\pi x)dx = -2\frac{(-1)^j}{j\pi}$. Replacing these expressions in the equation for $v_t(x,t)$ we can verify that $0 = v_t(x,t) + kv_{xx}(x,t) - \rho v(x,t) + CxZ(t)$ for all $x \in (-1,1)$ and $t \in [0,T)$. $\square$

For use in Proposition 5 we compute the expressions for the second derivative of $\tilde{u}$ when we use the normalization $\bar{x}_{ss} = 1$, i.e. the choice of ψ so that it is attained.

Lemma 10. Fix the parameters σ, B, ζ and ρ and let ψ be such that $\bar{x}_{ss} = 1$. For such case the second derivatives of $\tilde{u}$ evaluated at the thresholds are given by:

$$0 < \tilde{u}_{xx}(0) = \frac{2B}{\rho+\zeta}\left[1 - \eta \operatorname{csch}(\eta)\right], \text{ and } 0 > \tilde{u}_{xx}(1) = \frac{2B}{\rho+\zeta}\left[1 - \eta \coth(\eta)\right] \quad (69)$$

where $\eta \equiv \sqrt{(\rho+\zeta)/k}$. Moreover $|\tilde{u}_{xx}(0)| < |\tilde{u}_{xx}(1)|$.

Proof. (of Lemma 10). The solution for $\tilde{u}$ is given by the sum of the particular solution $a_0 + a_2 x^2$ and of the two homogenous solutions, which given the symmetry can be written as $A\cosh(\eta x)$, so that $\tilde{u}(x) = a_0 + a_2 x^2 + A\cosh(\eta x)$. From the o.d.e. of $\tilde{u}$ we obtain that $\eta = \sqrt{(\rho+\zeta)/k}$. To determine the coefficients a_0, a_2 note the particular solution must satisfy:

$$(\rho+\zeta)(a_0 + a_2 x^2) = Bx^2 + k2a_2 + \zeta(a_0 + a_2(x^*)) = Bx^2 + k2a_2 + \zeta a_0$$

where we use that $x^* = 0$, and hence $a_2 = B/(\rho+\zeta)$ and $a_0 = 2kB/(\rho(\rho+\zeta))$. It remains to find the value of A. For this we use smooth pasting at $\bar{x} = 1$. We have: $\tilde{u}_x(\bar{x}) = 0 = \frac{2B}{\rho+\zeta}\bar{x} + A\eta\sinh(\eta\bar{x})$ and using $\bar{x} = 1$ we get $A = -\frac{2B}{(\rho+\zeta)\eta\sinh(\eta)}$. Since $\tilde{u}_{xx}(x) = \frac{2B}{\rho+\zeta} + A\eta^2\cosh(\eta x)$ then the second derivatives are:

$$\tilde{u}_{xx}(0) = \frac{2B}{\rho+\zeta} + A\eta^2 = \frac{2B}{\rho+\zeta} - \frac{2B\eta^2}{(\rho+\zeta)\eta\sinh(\eta)} = \frac{2B}{\rho+\zeta}\left[1 - \eta\operatorname{csch}(\eta)\right]$$
$$\tilde{u}_{xx}(1) = \frac{2B}{\rho+\zeta} + A\eta^2\cosh(\eta) = \frac{2B}{\rho+\zeta} - \frac{2B\eta^2\cosh(\eta)}{(\rho+\zeta)\eta\sinh(\eta)} = \frac{2B}{\rho+\zeta}\left[1 - \eta\coth(\eta)\right]$$

The inequality is equivalent to: $1 - \frac{\eta}{\sinh(\eta)} < -1 + \frac{\eta\cosh(\eta)}{\sinh(\eta)}$ or $2 < \eta\frac{1+\cosh(\eta)}{\sinh(\eta)}$ or $2\sinh(\eta) < \eta(1+\cosh(\eta))$.
$\square$

Proof. (of Proposition 5). Consider the smooth pasting and optimal return conditions from the original problem, i.e.

$$0 = u_x(\underline{x}(t,\delta),t,\delta)\,, \quad 0 = u_x(\bar{x}(t,\delta),t,\delta), \quad \text{and} \quad 0 = u_x(x^*(t,\delta),t,\delta)$$

Differentiate them w.r.t. δ to find $\bar{z}, \underline{z}$ and z^*:

$$\begin{aligned} \bar{z}(t) &= -\frac{v_x(1,t)}{\tilde{u}_{xx}(1)} \text{ for all } t \in [0,T) \\ \underline{z}(t) &= -\frac{v_x(-1,t)}{\tilde{u}_{xx}(-1)} = \bar{z}(t) \text{ for all } t \in [0,T) \\ z^*(t) &= -\frac{v_x(0,t)}{\tilde{u}_{xx}(0)} \text{ for all } t \in [0,T)\,. \end{aligned}$$

Differentiating equation (34) obtained in Lemma 5 we obtain:

$$\begin{aligned} v_x(1,t) &= -2C\int_t^T \sum_{j=1}^{\infty} e^{-(\rho+k(j\pi)^2)(\tau-t)} Z(\tau)d\tau \\ v_x(0,t) &= -2C\int_t^T \sum_{j=1}^{\infty} e^{-(\rho+k(j\pi)^2)(\tau-t)} Z(\tau)(-1)^j d\tau \end{aligned}$$

The equality of $\bar{z} = \underline{z}$ follows from the antisymmetry of v established in Lemma 4 and from $\bar{z}(t) = -\frac{v_x(1,t)}{\tilde{u}_{xx}(1)}$ and $\underline{z}(t) = -\frac{v_x(-1,t)}{\tilde{u}_{xx}(-1)}$ since $\tilde{u}$ is symmetric, and hence $\tilde{u}_{xx}(-1) = \tilde{u}_{xx}(1)$.

The expressions for $\bar{A}$ and A^* in equation (38) follow from Lemma 10.

That $\bar{H}(s) > 0$ is immediate using that k and s are positive. That $H^*(s) < 0$ follows from grouping each pair of consecutive terms as in

$$H^*(s) = -\sum_{j=1,3,5,\ldots} e^{-(\eta^2+(j\pi)^2)\,k\,s}\left[1 - e^{-(\eta^2+((j+1)^2-j^2)\pi^2)\,k\,s}\right] < 0$$

where the inequality follows because k and s are strictly positive. □

Proof. (of Lemma 6). The proof strategy is to define $N(x,t) = n(x,t) + n(-x,t)$ defined in $(x,t) \in [0,1] \times [0,T]$ satisfying:

$$\begin{aligned} N_t(x,t) &= kN_{xx}(x,t) - \zeta N(x,t) \text{ for } (x,t) \in [0,1] \times [0,T] \\ N(x,0) &= \nu(x) + \nu(-x) = 0 \text{ for all } x \in [0,1] \\ N(1,t) &= n(1,t) + n(-1,t) = 0 \text{ for all } t \in [0,T] \\ N(0,t) &= b(t) + a(t) \equiv C(t) \text{ for all } t \in [0,T] \\ \int_0^1 N(x,t)dx &= \int_{-1}^0 n(x,t)dx + \int_0^1 n(x,t)dx = 0 \text{ for all } t \in [0,T] \end{aligned}$$

for some function $C(t)$. We will show that $C(t) = 0$ for all t and that $N(x,t) = 0$ for all $(x,t) \in [0,1] \times [0,T]$.

The proof proceeds by contradiction. Suppose that $\max_{\{(x,t)\in[0,1]\times[0,T]\}} N(x,t) > 0$ and $\min_{\{(x,t)\in[0,1]\times[0,T]\}} N(x,t) < 0$. The two extremes have different signs since $\int_0^1 N(x,t)dx = 0$ and $N(1,t) = 0$ for all t. We argue that the maximum and the minimum of $N(x,t)$ on the set $[0,1] \times [0,T]$ has to occur on $\{(x,t) : t = 0\} \cup \{(x,t) : x = 0\} \cup \{(x,t) : x = 1\}$. This is based on the strong maximum/minimum principle for the case for $\zeta \geq 0$, see Evans (2010)

Theorem 12, Section 7.1.c. But since $N(1,t) = 0$ for all t, and $N(x,0) = 0$ for all x, then the maximum and the minimum are attained at $x = 0$ for two values $0 \leq \underline{t} < \bar{t} \leq T$. Assume, without loss of generality, that $C(\bar{t}) > 0 > C(\underline{t})$. Since $C(t)$ is non-zero, there must be some $0 < t_0 < T$ for which $C(t)$ does not change and it attains a strictly either positive or negative value. Assume, without loss of generality, that it attains a positive value. Then by redefining the p.d.e. considered above in the range $t \in [0, t_0]$ we have that $C(t) \geq 0$ and $C(t_1) > 0$ for some $t' \in [0, t_0]$. But in this case, using the comparison principle, $N(x,t)$ will be positive everywhere in this domain, which is a contradiction.

□

Proof. (of Lemma 7). In this lemma we use that $m(x,t,\delta)$ is continuous around $x = x^*(t,\delta)$ for all t and δ. Under the assumption that $m(x,t,\delta)$ is right and left differentiable at $x = x^*(t,\delta)$, we have

$$m(x,t,\delta) = \begin{cases} m(0,t,0) + m_x(0^-,t,0)\frac{\partial}{\partial\delta}x^*(0,0)\delta + \frac{\partial}{\partial\delta}m(0^-,t,0)\delta + o(\delta) & \text{if } x < x^*(t,\delta) \\ m(0,t,0) + m_x(0^+,t,0)\frac{\partial}{\partial\delta}x^*(0,0)\delta + \frac{\partial}{\partial\delta}m(0^+,t,0)\delta + o(\delta) & \text{if } x > x^*(t,\delta) \end{cases}$$

We can write these expressions in the notation developed above:

$$m(x,t,\delta) = \begin{cases} \tilde{m}(0) + \tilde{m}_x(0^-)z^*(t)\delta + n(0^-,t)\delta + o(\delta) & \text{if } x < x^*(t,\delta) \\ \tilde{m}(0) + \tilde{m}_x(0^+)z^*(t)\delta + n(0^+,t)\delta + o(\delta) & \text{if } x > x^*(t,\delta) \end{cases}$$

Using the continuity of m, we equate both expansions to obtain:

$$\tilde{m}(0) + \tilde{m}_x(0^-)z^*(t)\delta + n(0^-,t)\delta + o(\delta) = \tilde{m}(0) + \tilde{m}_x(0^+)z^*(t)\delta + n(0^+,t)\delta + o(\delta)$$

using that $\tilde{m}_x(0^-) = -\tilde{m}_x(0^+) > 0$, and the notation $a(t) = n(0^-,t)$ and $b(t) = n(0^+,t)$ we have: $-\tilde{m}_x(0^+)z^*(t) + a(t) + o(\delta)/\delta = z^*(t)\tilde{m}_x(0^+) + b(t) + o(\delta)/\delta$ or taking $\delta \to 0$:

$$z^*(t) = \frac{b(t) - a(t)}{-2\,\tilde{m}_x(0^+)}$$

□

Lemma 11. The solution of the heat equation given by equation (44),(45) and (46) is

$$n(x,t) = r(x,t) + \sum_{j=1}^{\infty} c_j(t)\varphi_j(x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where}$$
$$r(x,t) = w^*(t) + x\left[\bar{w}(t) - w^*(t)\right] \text{ all } x \in [0,1],\ t > 0$$

where $w^*(t) = -\tilde{m}_x(0^+)\, z^*(t)$ and $\bar{w}(t) = -\tilde{m}_x(1)\bar{z}(t)$ and for all $j = 1, 2, \ldots$ we have:

$$\varphi_j(x) = \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h\rangle \equiv \int_0^1 h(x)\varphi_j(x)dx$$

$$c_j(t) = c_j(0)e^{-\lambda_j t} + \int_0^t q_j(\tau)e^{\lambda_j(\tau - t)}d\tau \text{ all } t > 0\,, \text{ where } \lambda_j = (\ell^2 + (j\pi)^2)k\quad,$$

$$q_j(t) = \frac{\langle \varphi_j\,, -r_t(\cdot,t) - \zeta r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} = 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right] w^{\star\prime}(t) + 2\frac{(-1)^j}{j\pi}\left[\bar{w}'(t) - w^{\star\prime}(t)\right]$$

$$+ 2\zeta\left[\frac{\cos(j\pi) - 1}{j\pi}\right] w^\star(t) + 2\zeta\frac{(-1)^j}{j\pi}\left[\bar{w}(t) - w^\star(t)\right] \text{ all } t > 0$$

$$c_j(0) = \frac{\langle \varphi_j\,, \nu - r(\cdot,0)\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\langle \varphi_j\,, \nu\rangle}{\langle \varphi_j, \varphi_j\rangle} + 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right] w^\star(0) + 2\frac{(-1)^j}{j\pi}\left[w(0) - w^\star(0)\right]$$

where for the benchmark case of $\nu = \tilde{m}_x$ we get:

$$\frac{\langle \varphi_j\,, \nu\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\langle \varphi_j\,, \tilde{m}_x\rangle}{\langle \varphi_j, \varphi_j\rangle} = \begin{cases} -\frac{\ell^2 j\pi}{\ell^2 + (j\pi)^2}\left(\frac{1 + e^{\ell}(-1)^{j+1}}{\left(1 - e^{\ell}\right)^2} + \frac{1 + e^{-\ell}(-1)^{j+1}}{\left(1 - e^{-\ell}\right)^2}\right) & \text{if } \zeta > 0 \\ -2\frac{1 + (-1)^{j+1}}{j\pi} & \text{if } \zeta = 0 \end{cases} \tag{70}$$

Proof. (of Lemma 11). This follows from the explicit solution of the heat equation in $\{(x,t) : x \in [0,1], t \in [0,T]\}$ and using $n(x,t) = n(-x,t)$ to extend it to the negative values of x. We use the general solution of the heat equation using Fourier series with two moving boundaries at $x = 0$ and $x = 1$, a given initial condition, and no source. We reproduce this general solution in the technical appendix. In terms of the notation of the general solution we set $w(x,t) = n(x,t)$, no source, i.e. $s(x,t) = 0$, initial conditions given by $f(x) = \nu(x)$, lower and upper space boundaries $A(t) = -\tilde{m}_x(0^+)\, z^*(t)$, $B(t) = -\tilde{m}_x(1)\bar{z}(t)$ and killing rate $\iota = \zeta$. □

Proof. (of Proposition 6). We replace the expression from Lemma 11 for n into the integral for Z obtaining:

$$Z(t) = 2\int_0^1 x n(x,t)dx = w^*(t)\frac{2}{2} + [\bar{w}(t) - w^*(t)]\frac{2}{3} + 2\sum_{j=1}^{\infty} c_j(t)\int_0^1 x\sin(j\pi x)dx$$

$$= w^*(t) + [\bar{w}(t) - w^*(t)]\frac{2}{3} - 2\sum_{j=1}^{\infty} c_j(t)\frac{(-1)^j}{j\pi}$$

Note that using the expression in Lemma 11 we can write the function $c_j(t)$ in closed form as (see the Technical appendix for step by step derivation).

Replacing the $2\frac{(-1)^j}{j\pi}c_j(t)$ back into $Z(t)$ and using that

$$\sum_{j=1}^{\infty}\frac{(-1)^j}{(j\pi)^2} = -\frac{1}{12} \text{ and } \sum_{j=1}^{\infty}\frac{1}{(j\pi)^2} = \frac{1}{6}$$

we get

$$Z(t) = \sum_{j=1}^{\infty} 4k(-1)^{j+1} \int_0^t w^*(\tau) e^{\lambda_j(\tau - t)} d\tau \;\; + \sum_{j=1}^{\infty} 4k \int_0^t \bar{w}(\tau) e^{\lambda_j(\tau-t)} d\tau \;\; - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} \frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j, \varphi_j\rangle} e^{-\lambda_j t}$$

Using the definition of $w^*(t) = -\tilde{m}_x(0^+)z^*(t)$ and $\bar{w}(t) = -\tilde{m}_x(1)\bar{z}(t)$ and exchanging the integral with the sum and replacing $\lambda_j = (\ell^2 + (j\pi)^2)k$ we get:

$$\begin{aligned} Z(t) &= 4k \int_0^t \left(-\tilde{m}_x(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) z^*(\tau) d\tau \\ &\quad + 4k \int_0^t \left(-\tilde{m}_x(1) \sum_{j=1}^{\infty} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) \bar{z}(\tau) d\tau \;\; - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} \frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j, \varphi_j\rangle} e^{-(\ell^2+(j\pi)^2)kt} \end{aligned}$$

Finally computing the projections for ν:

$$\begin{aligned} Z(t) = 4k &\int_0^t \left(-\tilde{m}_x(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) z^*(\tau) d\tau \\ &+ 4k \int_0^t \left(-\tilde{m}_x(1) \sum_{j=1}^{\infty} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) \bar{z}(\tau) d\tau \\ &- 4 \sum_{j=1}^{\infty} (-1)^j \frac{e^{-(\ell^2+(j\pi)^2)kt}}{j\pi} \int_0^1 \sin(j\pi x)\nu(x) dx \end{aligned}$$

which gives the expression for T_Z given the definitions of $\bar{G}, G^*$ and Z_0^η.

That $\bar{G}(s) > 0$ is immediate. That $G^*(s) \geq 0$ follows by noticing that we can write:

$$G^*(s) = \sum_{j=1,3,5,\ldots} e^{-(\ell^2+(j\pi)^2)k\,s} \left[1 - e^{-((j+1)^2 - j^2)\pi^2 k\, s} \right]$$

and each term $\left[1 - e^{-((j+1)^2 - j^2)\pi^2 k\, s} \right] > 0$ since k and s are positive. □

Proof. (of Proposition 7). First we note that we can decompose ν into its symmetric and antisymmetric part. By linearity, the solution is the sum of the solutions for each part. Due to Corollary 1 the solution for the symmetric part is zero, so we can assume without loss of generality that ν is antisymmetric. Given Z, we replace $z^* = T^*(Z)$, given by equation (36), and $\bar{z} = \bar{T}(Z)$, given by equation (35), into $T_Z(z^*, \bar{z})$, given by equation (47), to get $\mathcal{T}(Z) = T_Z\left(T^*(Z), \bar{T}(Z)\right)$. Note that, except for the term with Z_0, each term is a double integral. Changing the order of integration and using that $\bar{G}, \bar{H}$ and G^*, H^* satisfy:

$$-\tilde{m}_x(1)\bar{H}(s) = e^{-\rho\, s}\, \bar{G}(s) \geq 0 \quad \text{and} \quad \tilde{m}_x(0^+)H^*(s) = e^{-\rho\, s}\, G^*(s) \leq 0 \text{ for all } s > 0 \qquad (71)$$

we obtain:

$$Z(t) = Z_0(t) + \theta \int_0^T K(t,s) Z(s) ds$$

where

$$K(t,s) = 4\,k \int_0^{\min\{t,s\}} e^{-\rho\,(s-\tau)} \left[\bar{A}_\ell \, \frac{\bar{G}(s-\tau)}{\tilde{m}_x(1)} \frac{\bar{G}(t-\tau)}{\tilde{m}_x(1)} - A^*_\ell \, \frac{G^*(s-\tau)}{\tilde{m}_x(0^+)} \frac{G^*(t-\tau)}{\tilde{m}_x(0^+)} \right] d\tau \quad (72)$$

Performing the integration of the exponentials we obtain the desired expression.

The expression for Z_0 uses that sin and ν are antisymmetric, hence we have:

$$\int_0^1 \sin(j\pi x)\nu(x)dx = \tfrac{1}{2} \int_{-1}^1 \sin(j\pi x)\nu(x)dx\,.$$

□

Proof. (of Lemma 8). The symmetry of K when $\rho = 0$ in 1 follows directly from its definition in equation (72). That $K \leq 0$ as in 2 uses the expression equation (72) and that $G^* \geq 0$, $A^* > 0$, $\bar{G} \geq 0$, and $\bar{A} < 0$.

For part 1 with $\rho > 0$ and 3 we use the expression for the kernel K derived in the proof of Proposition 7 (see equation (72)).

Part 3 establishes that K is negative definite. To see why this has to hold, we write:

$$\begin{aligned} Q_i &= -\int_0^T \int_0^T \int_0^T e^{-\rho(s-\tau)} G_i(s-\tau) G_i(t-\tau) V(s) V(t) e^{-\rho t}\, d\tau\, ds\, dt \\ &= -\int_0^T e^{\rho\tau} \left(\int_0^T G_i(s-\tau) V(s) e^{-\rho s}\, ds \right)^2 d\tau \leq 0 \end{aligned}$$

with strictly inequality if $V \neq 0$.

Part 4 of the proof establishes the bounds for the integral $\int_0^T |K(t,s)|ds$. This is obtained by direct (but tedious) calculation. See the technical appendix for the full details of the proof. The same calculation gives the bound in 4 for any η and $t \geq 0$. Another direct calculation establishes part 5, a bound for the kernel when $\ell > 0$ in terms of the kernel for $\ell = 0$. The bound uses the expression derived in the proof of Proposition 6, which shows in equation (72).

□

Proof. (of Proposition 8). It is straightforward to compute $K(0,s) = 0$ for all $s \in [0,T]$ using the definition of K in equation (52). Hence $Y_\theta(0) = Y_0(0) + \theta \int_0^T K(0,s) Y_\theta(s) ds = Y_0(0)$. Finally that $Y_0(0) = -Z_0(0) = 1$ follows from evaluation of equation (49) for any $\ell \geq 0$.

□

Proof. (of Proposition 10). That the series in equation (57), whenever it converges, is the solution of equation (56) follows by replacing the series into the integral equation.

To establish that $Y_\theta(t) > 0$ when $\theta < 0$ we note that $\theta K(t,s) > 0$ for all $(t,s) \in (0,T)^2$ and hence $(\theta\mathcal{K})^r(Y_0) > 0$ for $t \in (0,T)$. Note that, for each t, the sequence $S_n(\theta,t) \equiv$

$\sum_{r=0}^{n} \theta^r (\mathcal{K})^r (Y_0)(t)$ is monotone increasing in n and that (by assumption) it converges. Hence, $Y_\theta(t) > 0$.

To establish that $Y_\theta(t)$ is decreasing in θ, we differentiate equation (57) obtaining:

$$\frac{\partial}{\partial \theta} Y_\theta(t) = \sum_{r=1}^{\infty} r\theta^{r-1} \left(\mathcal{K}\right)^r (Y_0)(t)$$

for $t \in (0, T)$. If r is even we have $\theta^{r-1} < 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) > 0$. If r is odd we have $\theta^{r-1} > 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) < 0$, hence all the terms in the sum are strictly negative, and thus $\frac{\partial}{\partial \theta} Y_\theta(t) < 0$.

To establish that $Y_\theta(t)$ is convex in θ, we differentiate twice equation (57) obtaining:

$$\frac{\partial^2}{\partial \theta^2} Y_\theta(t) = \sum_{r=2}^{\infty} r(r-1)\theta^{r-2} \left(\mathcal{K}\right)^r (Y_0)(t)$$

for $t \in (0, T)$. If r is even we have $\theta^{r-2} > 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) > 0$. If r is odd we have $\theta^{r-2} < 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) < 0$, hence all the terms in the sum are strictly positive, and thus $\frac{\partial^2}{\partial \theta^2} Y_\theta(t) > 0$.

□

Proof. (of Proposition 11). We show here a bound for the HS operator norm in terms of the L^2 norm of the kernel. We use that

$$||K||_2^2 \equiv \frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K^2(t,s)\, e^{-\rho(s+t)}\, ds\, dt \tag{73}$$

$$= \sum_{i,j} \left(\frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho(s+t)}\, ds\, dt \right)^2 \tag{74}$$

This equality follows from projecting $K(t,s)$ first as a function of s into $\{f_i(s)\}$. In particular, fix a t:

$$K(t,s) = \sum_{i=1}^{\infty} \langle K(t,\cdot), f_i \rangle f_i(s) = \frac{\rho}{1-e^{-\rho T}} \sum_{i=1}^{\infty} \int_0^T K(t,s') f_i(s') e^{-\rho s'} ds'\, f_i(s)$$

And then project this expression as a function of t into the base $\{f_j(t)\}$

$$K(t,s) = \frac{\rho^2}{(1-e^{-\rho T})^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^T \int_0^T K(t',s') f_j(t') f_i(s') e^{-\rho s'} e^{-\rho t'} ds' dt'\, f_i(s) f_j(t)$$

To simplify we can write this expression as: $K(t,s) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \kappa_{i,j}\, f_i(s) f_j(t)$. Now we can write:

$$(K(t,s))^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \kappa_{i,j} \kappa_{m,n}\, f_i(s) f_j(t) f_m(s) f_n(t)$$

Then integrating with respect to $\rho^2 e^{-\rho(t+s)}/(1-e^{-\rho T})^2$ then:

$$\begin{aligned}
&\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T\int_0^T (K(t,s))^2 e^{-\rho(t+s)} dt ds \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \kappa_{i,j}\kappa_{m,n}\frac{\rho}{1-e^{-\rho T}}\int_0^T f_i(s) f_m(s) e^{-\rho s} ds \frac{\rho}{1-e^{-\rho T}}\int_0^T f_j(t) f_n(t) e^{-\rho t} dt \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \kappa_{i,j}\kappa_{m,n}\delta_{i,m}\delta_{j,n} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\kappa_{i,j})^2
\end{aligned}$$

where we use that $\{f_i\}$ are orthonormal, and $\delta_{\cdot,\cdot}$ is the Kroneker symbol, and thus we obtain equation (74).

Let K_ρ be defined as $K_\rho(t,s) = K(t,s)e^{\rho s}$. Then

$$\begin{aligned}
||K_\rho||_2^2 &= \sum_{i,j}\left(\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T\int_0^T K_\rho(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho(s+t)}\, ds\, dt\right)^2 \\
&= \sum_{i,j}\left(\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T\int_0^T K(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho t}\, ds\, dt\right)^2
\end{aligned}$$

and using Cauchy-Schwarz $||K_\rho||_2^2 \leq ||K||_2^2\, ||e^{\rho s}||_2^2 = ||K||_2^2 \frac{(\rho T)^2}{(1-e^{-\rho T})^2}$ so

$$||\mathcal{K}||_{HS}^2 \leq \frac{(1-e^{-\rho T})^2}{\rho^2}\frac{(\rho T)^2}{(1-e^{-\rho T})^2}||K||_2^2 = T^2||K||_2^2$$

Thus, using this inequality and the results in Lemma 8 we obtain the bound on $||\mathcal{K}||_{HS}$, and thus operator is compact. The rest of the proof is directly from the spectral theorem. □

Proof. (of Proposition 12). Part 1 follows using the compactness and self-adjoint nature of $\mathcal{K}$, as well as the fact that all its eigenvalues are negative, and that μ_1 is the largest, all established in Proposition 11. Under these conditions the uniqueness and existence are a consequence of the Fredholm alternative for the second kind of Fredholm integral equation, such as equation (56). The computation of the projection coefficients is immediate, i.e. using the inner product with respect to the j^{th} eigenfunction ϕ_j with eigenvalue μ_j on the integral equation (56), i.e.:

$$\langle Y_\theta, \phi_j\rangle = \langle Y_0, \phi_j\rangle + \theta\langle \mathcal{K}Y_\theta, \phi_j\rangle = \langle Y_0, \phi_j\rangle + \theta\langle Y_\theta, \mathcal{K}\phi_j\rangle = \langle Y_0, \phi_j\rangle + \theta\mu_j\langle Y_\theta, \phi_j\rangle$$

where we use the self-adjointess of $\mathcal{K}$, and the assumption that $\{\phi_j, \mu_j\}$ are an eigenfunction-eigenvalue pair. From here we obtain that $\langle Y_\theta, \phi_j\rangle = \langle Y_0, \phi_j\rangle/(1-\mu_j\theta)$, i.e. the projection coefficients of the solution Y_θ. Then we use that $\{\phi_j\}_{j=1}^{\infty}$ form an orthonormal base to write Y_θ.

Part 2 follows directly by taking limits, since $\mu_j < 0$ for all j, as shown in Proposition 11.

Part 3 follows directly a consequence of the Fredholm alternative for the second kind of

Fredholm integral equation, such as equation (56). Recall that for the lack of existence it is required to show that $\langle Y_0, \phi_1 \rangle \neq 0$. This, in turns, follows from the Perron Froebenoius theorem, since $-K(t,s) > 0$ for all t, s, and hence the dominant eigenfunction ϕ_1 can be taken to be positive. The two limits follow from direct computation.

Part 4 uses as an intermediate step that $\phi_j(0) = 0$ for all $j = 1, 2, \ldots$. This follows because $K(0,s) = 0$ for all $s \in [0,T]$ and hence $\mu_j \phi_j(0) = \int_0^T K(0,s)\phi_j(s)ds = 0$. Now, using that $\phi_j(0) = 0$ for all $j = 1, 2, \ldots$, and that $Y_0(t) > 0$ for $t \in [0, \epsilon)$ for some $\epsilon > 0$.

See the technical appendix for full details.

□

Proof. (of Proposition 14) We set $T = \infty$. For this value we want to compute

$$\tfrac{d}{d\theta} CIR_\theta|_{\theta=0} = \int_0^\infty \tfrac{d}{d\theta} Y_\theta(t)|_{\theta=0} dt = \int_0^\infty \int_0^\infty K(t,s) Y_0(t)\, ds\, dt$$

which can be written as

$$Q \equiv \int_0^\infty \int_0^\infty K(t,s) Y_0(s) ds\, dt = \sum_{m=1}^{\infty} Q_m \text{ where } Q_m = 4 \int_0^\infty \int_0^\infty K(t,s) \frac{1-\cos(m\pi)}{(m\pi)^2} ds\, dt$$

where we have replaced the expression for Y_0

Replacing the expression for K we get that for each m

$$Q_m = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 16 \left(1 - \cos(m\pi)\right) \left(\bar{A} - A^*(-1)^{i+j}\right) \tilde{\omega}_{i,j,m}$$

where $\tilde{\omega}_{i,j,m}$ is defined as

$$\begin{aligned} \tilde{\omega}_{i,j,m} &= \frac{1}{k^2 \pi^8} \frac{1}{(i^2 + j^2 + r^2) m^2}\, \omega_{i,j,m} \text{ and} \\ \omega_{i,j,m} &= \int_0^\infty \int_0^\infty \left(e^{(j^2+i^2+r^2)s \wedge t} - 1\right) e^{-j^2 t - i^2 s - r^2 s - m^2 s} ds\, dt \end{aligned}$$

were we have used a change on variables for t, and where we use $r \equiv \eta^2/\pi^2$.

Now we compute $\omega_{i,j,m}$ letting $\rho \downarrow 0$, or equivalently $r \to 0$. For this note that we can write the inner integral in $\omega_{i,j,m}$ as follows:

$$\begin{aligned} &\int_0^t e^{-j^2 t} e^{-(m^2 - j^2)s} ds + \int_t^\infty e^{i^2 t} e^{-(i^2+m^2)s} ds - \int_0^\infty e^{-j^2 t} e^{-(i^2+m^2)s} ds \\ &= \frac{e^{-j^2 t} - e^{m^2 t}}{(m^2 - j^2))} + \frac{e^{-m^2 t} - e^{-j^2 t}}{(i^2 + m^2)} \end{aligned}$$

Then, integrating the resulting expression with respect to t between 0 and ∞ we get:

$$\begin{aligned}\omega_{i,j,m} &= \frac{1}{(m^2-j^2)}\left[\frac{1}{j^2}-\frac{1}{m^2}\right]+\frac{1}{(i^2+m^2)}\left[\frac{1}{m^2}-\frac{1}{j^2}\right]=\frac{1}{m^2j^2}+\frac{1}{(i^2+m^2)}\frac{(j^2-m^2)}{m^2j^2}\\ &= \frac{1}{m^2j^2}\left(\frac{i^2+j^2}{i^2+m^2}\right)\end{aligned}$$

Now we replace this expression into $\tilde{\omega}_{i,j,m}$

$$\begin{aligned}\omega_{i,j,m} &= \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{(j^2+i^2)}\omega_{i,j,m}=\frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{(j^2+i^2)}\frac{1}{m^2j^2}\left(\frac{i^2+j^2}{i^2+m^2}\right)\\ &= \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{m^2j^2}\left(\frac{1}{i^2+m^2}\right)=\frac{1}{k^2}\frac{1}{(m\pi)^4}\frac{1}{(j\pi)^2}\frac{1}{(i^2\pi^2+m^2\pi^2)}\end{aligned}$$

Finally we want to compute the infinite sums of the expression for $\omega_{i,j,m}$ over i, j, m. For this we will use that when m is odd:

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2+m^2\pi^2} &= \frac{m\pi\coth(m\pi)-1}{2m^2\pi^2}\\ \sum_{i=1}^{\infty}\frac{(-1)^i}{i^2\pi^2+m^2\pi^2} &= \frac{m\pi\,\mathrm{csch}(m\pi)-1}{2m^2\pi^2}\\ \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i^2\pi^2+m^2\pi^2} &= \frac{1-m\pi\,\mathrm{csch}(m\pi)}{2m^2\pi^2}\end{aligned}$$

and we will also use that

$$\sum_{j=1}^{\infty}\tfrac{1}{(j\pi)^2}=\frac{1}{6}\text{ and }\sum_{j=0}^{\infty}\frac{1}{\pi^2(j+1)^2}=\frac{1}{8}\,.$$

We write $Q=\mathcal{Q}_I-\mathcal{Q}_{II}$:

$$\begin{aligned}\mathcal{Q}_I &= \sum_{m=1,3,5,\ldots}2\times 16\,\bar{A}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\tilde{\omega}_{i,j,m}=\sum_{m=1,3,5,\ldots}32\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^4}\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}\sum_{i=1}^{\infty}\frac{1}{(i^2\pi^2+m^2\pi^2)}\\ &= \sum_{m=1,3,5,\ldots}\frac{32}{12}\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^6}(m\pi\coth(m\pi)-1)\end{aligned}$$

Now we write the second term of Q:

$$\mathcal{Q}_{II} = \frac{32}{k}\frac{A^*}{k}\sum_{1,3,5,\ldots}\frac{1}{(m\pi)^4}\sum_{j=1}^{\infty}\frac{1}{j^2\pi^2}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{\pi^2 i^2+\pi^2 m^2} = \frac{32}{k}\frac{A^*}{k}\sum_{m=1,3,5,\ldots}\frac{1}{(m\pi)^4}\left(\mathcal{O}+\mathcal{E}\right) \text{ where}$$

$$\begin{aligned}
\mathcal{O} &= \sum_{j=1,3,5,\ldots}\frac{1}{(\pi j)^2}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{(i^2\pi^2+m^2\pi^2))} = \sum_{j=0}^{\infty}\frac{1}{\pi^2(j+1)^2}\frac{(1-m\pi\,\mathrm{csch}(m\pi))}{2m^2\pi^2} \\
&= \frac{1}{8}\frac{(1-m\pi\,\mathrm{csch}(m\pi))}{2m^2\pi^2} \text{ and} \\
\mathcal{E} &= \sum_{j=2,4,6,\ldots}\frac{1}{(\pi j)^2}\sum_{i=1}^{\infty}\frac{(-1)^{i}}{(i^2\pi^2+m^2\pi^2))} = \left[\frac{1}{6}-\frac{1}{8}\right]\sum_{i=1}^{\infty}\frac{(-1)^{i}}{(i^2\pi^2+m^2\pi^2))} \\
&= \frac{1}{8}\frac{1}{3}\frac{(m\pi\,\mathrm{csch}(m\pi)-1)}{2m^2\pi^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{Q}_{II} &= \frac{32}{k}\frac{A^*}{k}\sum_{m=1,3,5,\ldots}\frac{1}{(m\pi)^4}\left(\mathcal{O}+\mathcal{E}\right) = \frac{32}{k}\frac{A^*}{k}\frac{1}{8}\left(\frac{1}{3}-1\right)\sum_{m=1,3,5,\ldots}\frac{1}{(m\pi)^4}\frac{(m\pi\,\mathrm{csch}(m\pi)-1)}{2m^2\pi^2} \\
&= \frac{32}{k}\frac{A^*}{k}\frac{1}{8}\frac{1}{3}\sum_{m=1,3,5,\ldots}\frac{1-m\pi\,\mathrm{csch}(m\pi)}{(m\pi)^6}
\end{aligned}$$

Recall that as $\rho \to 0$ then $\bar{A}/k \to -6$ and $A^*/k \to 12$, and thus

$$\begin{aligned}
Q = \mathcal{Q}_I - \mathcal{Q}_{II} &= \sum_{m=1,3,5,\ldots}\frac{32}{12}\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^6}\left(m\pi\coth(m\pi)-1\right) - \frac{32}{k}\frac{A^*}{k}\frac{1}{8}\frac{1}{3}\sum_{m=1,3,5,\ldots}\frac{1-m\pi\,\mathrm{csch}(m\pi)}{(m\pi)^6} \\
&= \frac{16}{k}\sum_{m=1,3,5,\ldots}\frac{\mathrm{csch}(m\pi)-\coth(m\pi)}{(m\pi)^5}
\end{aligned}$$

Finally we have:

$$CIR_0 = \int_0^{\infty} Y_0(t)dt = \sum_{1,3,5,\ldots} 8\int_0^{\infty}\frac{e^{-\pi^2 m^2 kt}}{(m\pi)^2}dt = \frac{8}{k}\sum_{1,3,5,\ldots}\frac{1}{(m\pi)^4} = \frac{8}{k}\frac{1}{96} = \frac{1}{12\,k}$$

Thus

$$\frac{1}{CIR_\theta}\frac{dCIR_\theta}{d\theta}|_{\theta=0} = \frac{Q}{CIR_0} = 16\times 12\sum_{m=1,3,5,\ldots}\frac{\mathrm{csch}(m\pi)-\coth(m\pi)}{(m\pi)^5}$$

and using $16 \times 12 = 192$ we get our final result.

□

C Numerical Computation of Equilibrium

This section develops a simple and accurate algorithm to solve the equilibrium as a finite dimensional linear system, which follows exactly the same equations as the continuous case. This linear system also serves to clarify the expressions of the original continuous case which uses notation and concepts from functional analysis. We prove that our numerical procedure is stable, and that its solution has the same properties as the actual solution. Moreover, in spite of the fact that the kernel K is irregular, i.e. that $K(t,t) = -\infty$ for all $t > 0$, we show that the method is convergent, and analytically characterize its rate of convergence.

Our algorithm uses two assumptions: (A1) replaces the time interval $[0,T]$ by $\{t_r\}_{r=1}^m$, and (A2) replaces the infinite series in the definition of the kernel K in equation (52) and in the definition of Y_0 in equation (48) by finite sums of its first M elements.

We first implement A2, i.e. we define a version of the kernel with sums of M terms. In particular, define the kernel $K_M : [0,T]^2 \to \mathbb{R}$ as

$$K_M(t,s) \equiv 4 \sum_{j=1}^{M} \sum_{i=1}^{M} \left[\bar{A}_\ell - A_\ell^* (-1)^{j+i}\right] \frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2+\ell^2]k(t\wedge s)} - 1\right] e^{-(j\pi)^2kt-\ell^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2+\ell^2}$$

Now we implement A1 by discretizing time as follows. Let $\Delta = T/m$ and $t_r = r\Delta$ for $r = 1, 2, \ldots, m$, and likewise for $s_q = q\Delta$. For finite m we use the kernel K_M to build an $m \times m$ matrix $\mathbf{K}$ with typical element $\mathbf{K}_{r,q}$ where for any pair (r,q) we have:

$$\mathbf{K}_{r,q} \equiv K_M(t_r, s_q) \text{ for } (r,q) \in \{1,2,\ldots,m\}^2 \tag{75}$$

Likewise, we approximate the infinite sums in equation (48) with the first M terms:

$$Y_0^M(t) \equiv 4 \sum_{j=1}^{M} (-1)^j \frac{e^{-(\ell+(j\pi)^2)\,k\,t}}{j\pi} \int_0^1 \sin(j\pi x)\nu(x)dx \text{ for all } t \in [0,T] \tag{76}$$

and in the case of a monetary shock we use equation (49) to write:

$$Y_0^M(t) = -2 \sum_{j=1}^{M} \frac{\ell^2}{\ell^2+(j\pi)^2} \left(\frac{(-1)^j \left(1+e^{2\ell}\right) - 2e^\ell}{\left(1-e^\ell\right)^2}\right) e^{-(\ell^2+(j\pi)^2)k\,t} \text{ for all } t \in [0,T] \tag{77}$$

The equilibrium path $Y_\theta : [0,T] \to \mathbb{R}$ is replaced by a vector $\mathbf{Y}_\theta \in \mathbb{R}^m$, and likewise the equilibrium with no strategic interactions Y_0 is replaced by $\mathbf{Y}_0 \in \mathbb{R}^m$. Thus we define $\mathbf{Y}_0$ as the m dimensional vector: $\mathbf{Y}_0 = \left[Y_0^M(t_1), Y_0^M(t_2), \ldots, Y_0^M(t_m)\right]$. Then the equilibrium vector $\mathbf{Y}_\theta$ solves the following system of m linear equations:

$$\mathbf{Y}_\theta = \mathbf{Y}_0 + \theta \tfrac{T}{m}\, \mathbf{K}\, \mathbf{Y}_\theta \tag{78}$$

Let $\mathbf{R}$ be an $m \times m$ diagonal matrix with typical element $\mathbf{R}_{rr} = e^{-\rho t_r}$ for $r = 1, 2, \ldots, m$.

Proposition 15. Fix a positive integer m and a positive even integer M. The matrix $\mathbf{K}$ has real strictly negative eigenvalues μ_j and real eigenvectors $\phi_j \in \mathbb{R}^m$ satisfying $\mu_j \phi_j = \mathbf{K}\phi_j$.

The matrix $\mathbf{RK}$ is symmetric, and the eigenvectors of $\mathbf{K}$ are orthonormal using the inner product $\phi_j^\top \mathbf{R} \phi_i = 0$ if $i \neq j$ and $\phi_i^\top \mathbf{R} \phi_i = 1$. Letting Φ be the matrix whose columns are the eigenvectors ϕ_j, we have $\Phi^{-1} = \Phi^\top \mathbf{R}$. If $\theta \mu_j \neq 1$ for all $j = 1, \ldots, m$, then the unique solution of equation (78) is given by

$$\mathbf{Y}_\theta = \sum_{j=1}^{m} \frac{\phi_j^\top \mathbf{R}\, \mathbf{Y}_0}{1 - \theta \mu_j}\, \phi_j \equiv \Phi\, \mathbf{D}(\theta)\, \Phi^\top \mathbf{R}\, \mathbf{Y}_0 \tag{79}$$

where $\mathbf{D}(\theta)$ is a diagonal matrix with diagonal element $1/(1 - \theta \mu_j)$.

The previous proposition shows that for a fixed m, M the solution of the discretized system has the same properties as the solution on the original case. Clearly, the expression in equation (79) is the finite dimensional version of equation (59) in Proposition 12. It is also the finite dimensional version of equation (57) in Proposition 10 in the cases where $|\theta\, \mu_1| < 1$. To see this, note that $\Phi\, \mathbf{D}(\theta)\, \Phi^\top \mathbf{R} = \Phi\, \mathbf{D}(\theta)\, \Phi^{-1} = I + \theta \mathbf{K} + (\theta \mathbf{K})^2 + \ldots$.

Its computation is extremely simple, as it only involves finding the eigenvalues and eigenvectors of the well-behaved matrix $\mathbf{K}$. In particular no inverse matrices are needed, the computation of $\{\phi_j, \mu_j\}$ is independent of the value of θ, and the solution is stable even as $\theta \to 1/\mu_j$ while solving equation (78) using a matrix inversion will give rise to a badly behaved problem.

Next we characterize the rate at which the solution of the discrete system converges to the solution of the original case. We use a variation on the Nystrom method. For this we define linear operator and a solution corresponding to each m, M. In particular, let $\mathcal{K}^{m,M}$ be the linear operator defined as $\mathcal{K}_{m,M}(V)(t) = \Delta \sum_{q=1}^{m} K_M(t, s_q) V(s_q)$, and denote by $Y_\theta^{m,M} : [0, T] \to \mathbb{R}$ the function that solves:

$$Y_\theta^{m,M}(t) = Y_0^M(t) + \theta \Delta \sum_{r=1}^{m} K_M(t, s_q) Y_\theta^{m,M}(s_q) \text{ for all } t \in [0, T]$$

Let $\mathbf{Y}_\theta$ be the m dimensional vector solution of equation (79) for the pair (m, M). Clearly $Y_\theta^{m,M}(t_r) = \mathbf{Y}_{\theta,r}$. We first have a preliminary lemma about Y_0^M.

Lemma 12. Assume that ν is absolutely continuous. Then $||Y_0^M||_\infty < \infty$ and there exists a constant $c_0 > 0$ such $||Y_0 - Y_0^M||_\infty \leq \frac{c_0}{M}$ for all M.

The next proposition analyzes the rate of convergence of the extension of the solution of the discretized linear system to the solution of the original system, as a function of the two discretization parameters m and M.

Proposition 16. Assume that $T < \infty$, that ν is absolutely continuous, and that $|\theta| \operatorname{Lip} K < 1$. There exist three positive constants c_1, c_2, c_3 and an integer $\bar{m}$ such that:

$$||Y_\theta^{m,M} - Y_\theta||_\infty \leq c_1 \frac{(\log M)^2}{m} + c_2 \frac{1}{M} + c_3 \frac{1}{M^2} \text{ for all } M \text{ and } m \geq \bar{m} \tag{80}$$

As equation (80) makes clear, the convergence requires that m grows at a faster rate than M. For instance $m = M^2$ is enough. In words, we require a relatively fine discretization of the time interval relative to the number of terms to compute the kernel in the matrix $\mathbf{K}$ and the Fourier series in $\mathbf{Y}_0$. This is required because the kernel K is irregular, i.e. because $K(t,t) = -\infty$ for each $t \in (0,T]$.

C.1 Proofs for Numerical computation

Proof. (of Proposition 15) We first establish that $\mathbf{RK}$ is symmetric and negative definite. The proof is almost identical to the one for the kernel K. To see this note that K_M is defined as K in equation (72), except that we use $\bar{G}_M$ and G^*_M:

$$K_M(t,s) = 4\,k \int_0^{\min\{t,s\}} e^{-\rho\,(s-\tau)} \left[\bar{A}_\ell \,\frac{\bar{G}_M(s-\tau)}{\tilde{m}_x(1)} \frac{\bar{G}_M(t-\tau)}{\tilde{m}_x(1)} - A^*_\ell \,\frac{G^*_M(s-\tau)}{\tilde{m}_x(0^+)} \frac{G^*_M(t-\tau)}{\tilde{m}_x(0^+)} \right] d\tau$$

where $\bar{G}_M$ and G^*_M are defined as G and G^* expect that the infinite sum are replaced by the sums up to M elements, i.e.:

$$\bar{G}_M(s) \equiv -\tilde{m}_x(1) \sum_{j=1}^{M} e^{-(\ell^2+(j\pi)^2)\,k\,s} > 0 \quad \text{and} \quad G^*_M(s) \equiv -\tilde{m}_x(0^+) \sum_{j=1}^{M} (-1)^{j+1} e^{-(\ell^2+(j\pi)^2)k\,s} > 0$$

We want to show that for any two vectors m dimensional vectors V and W:

$$\sum_{r=1}^{m}\sum_{q=1}^{m} K_M(t_r,s_q) V_q\, W_r\, e^{-\rho t_r} = \sum_{r=1}^{m}\sum_{q=1}^{m} K_M(t_r,s_q) W_q\, V_r\, e^{-\rho t_r} \tag{81}$$

and that for any non-zero m dimensional vector V:

$$\sum_{r=1}^{m}\sum_{q=1}^{m} K_M(t_r,s_q) V_q\, V_s\, e^{-\rho t_r} < 0 \tag{82}$$

In both cases we use the expression for K_M and interchange the integral with respect to τ with the sums with respect to q and r. First consider equation (81). Fort this it suffices to show that

$$S_1 \equiv \sum_{r=1}^{m}\sum_{q=1}^{m} \left[\int_0^{\min\{t_r,s_q\}} e^{-\rho\,(s_q-\tau)} \bar{G}_M(s_q-\tau) \bar{G}_M(t_r-\tau) d\tau \right] V_q\, W_r\, e^{-\rho t_r}$$
$$S_2 \equiv \sum_{r=1}^{m}\sum_{q=1}^{m} \left[\int_0^{\min\{t_r,s_q\}} e^{-\rho\,(s_q-\tau)} \bar{G}_M(s_q-\tau) \bar{G}_M(t_r-\tau) d\tau \right] W_q\, V_r\, e^{-\rho t_r}$$

To see why the equality holds, interchange the order of the integrals and rearranging we get that each of these expression are

$$S_1 = \int_0^{\min\{t_r, s_q\}} e^{\rho\tau} \left[\sum_{r=1}^{m} \sum_{q=1}^{m} e^{-\rho s_q} \bar{G}_M(s_q - \tau) e^{-\rho t_r} \bar{G}_M(t_r - \tau) V_q W_r \right] d\tau \text{ and}$$

$$S_2 = \int_0^{\min\{t_r, s_q\}} e^{\rho\tau} \left[\sum_{r=1}^{m} \sum_{q=1}^{m} e^{-\rho s_q} \bar{G}_M(s_q - \tau) e^{-\rho t_r} \bar{G}_M(t_r - \tau) W_q V_r \right] d\tau$$

which established the equality. Now consider equation (82). Again, it suffices to show that:

$$PD \equiv \sum_{r=1}^{m} \sum_{q=1}^{m} \left[\int_0^{\min\{t_r, s_q\}} e^{-\rho(s_q - \tau)} \bar{G}_M(s_q - \tau) \bar{G}_M(t_r - \tau) d\tau \right] V_q V_r e^{-\rho t_r} > 0$$

Rewriting this expression we get:

$$\begin{aligned} PD &= \int_0^{\min\{t_r, s_q\}} e^{\rho\tau} \left[\sum_{r=1}^{m} \sum_{q=1}^{m} e^{-\rho s_q} \bar{G}_M(s_q - \tau) \bar{G}_M(t_r - \tau) V_q V_r e^{-\rho t_r} \right] d\tau \\ &= \int_0^{\min\{t_r, s_q\}} e^{\rho\tau} \left[\sum_{q=1}^{m} e^{-\rho s_q} \bar{G}_M(s_q - \tau) V(s_q) \right]^2 d\tau > 0 \end{aligned}$$

Finally, to show that $\phi_j^\top \mathbf{R} \phi_i = 0$ if $i \neq j$, take two different eigenvalues $\mu_j \neq \mu_i$ and pre-multiply $\mu_j \phi_j = K\phi_j$ by R, and transpose it, to get: $\mu_j \phi_j^\top R = \phi_j^\top (RK)^\top$, use the symmetry of RK, obtaining $\mu_j \phi_j^\top R = \phi_j^\top (RK)$, and post multiplying by ϕ_i, then $\mu_j \phi_j^\top R \phi_i = \phi_j^\top (RK) \phi_i$. Reversing the role of i and j, we obtain $\mu_i \phi_i^\top R \phi_j = \phi_i^\top (RK) \phi_j$. Subtracting one from the other $(\mu_j - \mu_i)\phi_j^\top R \phi_i = 0$. Given this properties the form of the solution is immediate, since we can regard the matrix $\mathbf{K}$ as defining a self-adjoint linear operator, using the inner product defined by $\langle V, W \rangle \equiv V^\top R W$. Thus we reproduce the same argument as for the general case, by using the Fredholm alternative to solve the linear system.

□

Proof. (of Lemma 12) Integrating by part gives

$$\begin{aligned} Y_0^M(t) &= 4 \sum_{j=1}^{M} (-1)^j \frac{e^{-(\ell + (j\pi)^2)\, k\, t}}{j\pi} \int_0^1 \sin(j\pi x)\nu(x) dx \\ &= 4 \sum_{j=1}^{M} (-1)^j \frac{e^{-(\ell + (j\pi)^2)\, k\, t}}{(j\pi)^2} \left[\cos(j\pi)\nu(1) - \nu(0) - \int_0^1 \cos(j\pi x)\nu'(x) dx \right] \end{aligned}$$

Let $\bar{\nu} = |\nu(1)| + \nu(0) + \int |\nu'(x)| dx$, then $|Y_0^M(t)| \leq 4\bar{\nu} \sum_{j=1}^{M} \frac{1}{(j\pi)^2}$. Thus $||Y_0||_\infty \equiv \lim_{M \to \infty} ||Y_0^M|| <$

∞. Moreover:

$$Y_0(t) - Y_0^M(t) = 4 \sum_{j=M}^{\infty} (-1)^j \frac{e^{-(\ell+(j\pi)^2)\,k\,t}}{(j\pi)^2} \left[\cos(j\pi)\nu(1) - \nu(0) - \int_0^1 \cos(j\pi x)\nu'(x)dx \right]$$

and thus

$$|Y_0(t) - Y_0^M(t)| \leq 4\bar{\nu} \sum_{j=M}^{\infty} \frac{1}{(j\pi)^2} \leq \frac{c_0}{M}$$

□

Proof. (of Proposition 16) This proof proceeds in four steps. First, we note that for each M, the Kernel and the solution for Y_θ^M has the same properties as Y_θ. Second, we use some bounds on the approximations of integrals using the trapezoidal rule as applied in this case. We note that for large m our method gives the same limit as the trapezoidal rule. Third we use the first two results, as well as the characterization in Proposition 15 and Lemma 8 to check the sufficient conditions for a bound on $||Y_\theta^{m,M} - Y_\theta^M||$, where $Y_\theta^M = \lim_{m\to\infty} Y_\theta^{m,M}$. The fourth final step uses the hypothesis on θ to obtain a bound on $||Y_\theta^M - Y_\theta||$. Combining the last two steps we obtain the desired result. See the technical appendix for all details.

□