B Online Appendix

B.1 Additional theoretical results for Section 3

B.1.1 Approximate solutions

Proposition 3. *For any $\sigma \in \mathcal{I}_k$, we have*

$$P(X(k), \sigma) \leqslant \sum_x \max_\omega \sum_\theta \left[w(\omega, \theta)\sigma(x, \theta) + \sum_i u_i(\omega, \theta)\tilde{\nabla}_i^+ \sigma(x, \theta) \right].$$

For any $m \in \mathcal{M}_k^0$, we have

$$G(X(k), m) \geqslant \sum_\theta \mu(\theta) \min_x \sum_\omega \left[w(\omega, \theta)m(\omega|x) + \sum_i u_i(\omega, \theta)\nabla_i^+ m(\omega|x) \right].$$

Proof. This is an immediate implication of the proofs of Theorem 1. In particular, the program (UB-P-k) is obtained by taking the dual of the inner maximization over mechanisms and equilibria from (MIN-P), so that any feasible solution to that dual provides an upper bound on the value of the primal, meaning that it provides an upper bound on welfare under any mechanism and equilibrium. Similarly, we obtained (LB-G-k) by taking the dual of the inner minimization program, and any feasible solution to the dual provides a lower bound on welfare in the primal program. □

B.1.2 Robustness to the prior

The following corollary of Proposition 3 generalizes Proposition 7 of Brooks and Du (2021b):

Proposition 4. *Fix a mechanism $m \in \mathcal{M}_k^0$. Then for any $\mu' \in \Delta(\Theta)$, welfare in any information structure I' with prior μ' and equilibrium $b \in \mathcal{E}((X(k), m), I')$ is at least*

$$\sum_\theta \mu'(\theta) \min_x \sum_\omega \left[w(\omega, \theta)m(\omega|x) + \sum_i u_i(\omega, \theta)\nabla_i^+ m(\omega|x) \right]. \tag{41}$$

Proof. From Proposition 3, we replace μ with μ' and conclude that (41) is a lower bound on equilibrium welfare. □

In particular, the bound (41) holds for any mechanism that maximizes the lower bound on the guarantee at a particular prior μ, and at $\mu' = \mu$, the lower bound (41) coincides with the optimum. Thus, while we have treated the prior over θ as a feature of the environment that is ostensibly known by the designer, in fact our theory remains valid and useful even if the designer has only approximate knowledge of the distribution of θ.

B.1.3 Strong maxmin solutions

As discussed in Section 3.3.3, Brooks and Du (2021b) characterize analogues of (LB-G-k) and (UB-P-k) in a setting with infinitely many actions and signals. In this case, equilibrium existence is not guaranteed, and care has to be taken that guarantees and potentials are not vacuous. To finesse this issue, we solved for a *strong maxmin solution*, which is a triple (M, I, b) such that $b \in \mathcal{E}(M, I)$, and $P(I) = G(M)$.

With finite action and signal spaces, an equilibrium always exists, but the optimal guarantee and potential may not be exactly attained. We now formulate an analogous solution concept for the discrete setting and relate its existence to whether or not there is a duality gap.

A pair (M, I) is an *ϵ-strong maxmin solution* if $P(I) - G(M) \leqslant \epsilon$. The following result is immediate from definitions:

Proposition 5. *The min potential is equal to the max guarantee if and only if for every $\epsilon > 0$, there exists an ϵ-strong maxmin solution.*

Proof. If $W(\text{MIN-P}) = W(\text{MAX-G}) = W^*$, then for every $\epsilon > 0$, there exists an M and I such that $G(M) \geqslant W^* - \epsilon/2$ and $P(I) \leqslant W^* + \epsilon/2$, and hence $P(I) - G(M) \leqslant \epsilon$. Conversely, suppose (M, I) is an ϵ-strong maxmin solution. Then $W(\text{MIN-P}) \leqslant P(I)$ and $W(\text{MAX-G}) \geqslant G(M)$. Hence, $W(\text{MIN-P}) - W(\text{MAX-G}) \leqslant P(I) - G(M) \leqslant \epsilon$. Since ϵ is arbitrary, we conclude that $W(\text{MIN-P}) = W(\text{MAX-G})$. □

As a consequence of Proposition 5 and Theorem 2, we obtain the following:

Corollary 1. *In the optimal auctions problem, an ϵ-strong maxmin solution exists for all $\epsilon > 0$.*

Corollary 1 is essentially a "strong minimax theorem" for the zero-sum game in which the designer chooses the mechanism and an adversary chooses the information structure to maximize and minimize welfare, respectively. It is important to note, however, that this is not quite a game, because for a given (M, I), there may be multiple equilibria with different payoffs for the designer, and therefore the standard results on zero-sum games do not apply. In (MAX-G), we have effectively selected the mechanism designer's preferred equilibrium, and in (MIN-P), we selected the worst case for the designer. The theorem therefore implies that the values of these programs would coincide regardless of how an equilibrium is selected. Equivalently, we could consider the collection of zero-sum games that are parameterized by the choice of equilibrium selection rule (as a function of the mechanism and information structure). Then any ϵ-strong maxmin solution is an ϵ-equilibrium of all of the zero-sum games with different equilibrium selection rules.

B.1.4 A sufficient condition for tight bounds and zero duality gap

For a sequence of functions $\{f^k\}_{k=1}^{\infty}$, where $f^k : X(k) \to \mathbb{R}$, and a function $f : \mathbb{R}^N_+ \to \mathbb{R}$, we say that $\{f^k\}$ *converges uniformly* to f if for all $\epsilon > 0$, there exists a K such that if $k \geqslant K$ and $x \in X(k)$, then $|f^k(x) - f(x)| \leqslant \epsilon$. Given a $f : \mathbb{R}^N_+ \to \mathbb{R}$, we denote by $\nabla_i f$ the partial derivative with respect to the ith argument. Also define $X^i(k) = \{x \in X(k) | x_i < k\}$.

Proposition 6. *Suppose that there exist optimal solutions to* (12), $\{(m^k, \lambda^k)\}_{k=1}^{\infty}$, *for which* m^k *converge uniformly to a Lipschitz continuous function* $m : \mathbb{R}_+^N \to \Delta(\Omega)$.[39] *Further suppose that* $\nabla_i m(\omega|x)$ *exists everywhere and is Lipschitz continuous,* $\nabla_i^- m^k(\omega|\cdot)$ *restricted to* $X^i(k)$ *converges uniformly to* $\nabla_i m(\omega|\cdot)$ *for all* i, *and that* $x_i = 0$ *is participation secure under* m *for all* i. *Then*

$$\lim_{k\to\infty} W(\text{UB-P-}k) = W(\text{MIN-P}) = W(\text{MAX-G}) = \lim_{k\to\infty} W(\text{LB-G-}k).$$

Proof. Let C_1 be an upper bound on $|w(\omega,\theta)|$ and $|u_i(\omega,\theta)|$. Since λ^k is optimal,

$$\lambda^k(\theta) = \min_x \sum_\omega \left[w(\omega,\theta) m^k(\omega|x) + \sum_i u_i(\omega,\theta)\nabla_i^- m^k(\omega|x)\right].$$

The program (UB-P-k) is feasible with $\sigma(x,\theta) = \mu(\theta)/(1+k^2)^N$ for all θ and x, which generates an upper bound on the value of (UB-P-k) equal to

$$\sum_x \max_\omega \sum_\theta \frac{\mu(\theta)}{(1+k^2)^N}\left[w(\omega,\theta) + \sum_i u_i(\omega,\theta)\left((1-k)\mathbb{I}_{x_i=k-1/k} - \mathbb{I}_{x_i=k}\right)\right] \leqslant (N+1)C_1.$$

We conclude that for all k, $\sum_\theta \mu(\theta)\lambda^k(\theta) \leqslant (N+1)C_1$.

Now, let $\tilde{m}^k(\omega|\theta)$ be the restriction of m to $X(k)$, and define

$$\tilde{\lambda}^k(\theta) = \min_x \sum_\omega \left[w(\omega,\theta)\tilde{m}^k(\omega|x) + \sum_i u_i(\omega,\theta)\nabla_i^+ \tilde{m}^k(\omega|x)\right].$$

Participation security of $\tilde{m}^k$ follows from that of m, so $(\tilde{m}^k, \tilde{\lambda}^k)$ is feasible for (9), and

$$W(\text{UB-P-}k) - W(\text{LB-G-}k) \leqslant \sum_\theta \mu(\theta)(\lambda^k(\theta) - \tilde{\lambda}^k(\theta)).$$

Let C_2 be a Lipschitz constant for $\nabla_i m$ and m. Fix $\epsilon > 0$. From uniform convergence, there exists a K such that for $k \geqslant K$ and $x \in X(k)$, $|m^k(\omega|x) - m(\omega|x)| \leqslant \epsilon$, and for all i and $x \in X^i(k)$, $|\nabla_i^- m^k(\omega|x) - \nabla_i m(\omega|x)| \leqslant \epsilon$. Without loss, we may also take $K \geqslant C_2/\epsilon$. As a result, if $x \in X^i(k)$ and $k \geqslant K$, then

$$\begin{aligned}\nabla_i^+ \tilde{m}^k(\omega|x) &= (k-1)\int_{y=x_i}^{x_i+1/k} \nabla_i m(\omega|y, x_{-i})dy \\ &= \frac{k-1}{k}\nabla_i m(\omega|x) + (k-1)\int_{z=0}^{1/k} \left(\nabla_i m(\omega|x_i+z, x_{-i}) - \nabla_i m(\omega|x)\right)dz,\end{aligned}$$

and hence,

$$\left|\nabla_i^+ \tilde{m}^k(\omega|x) - \frac{k-1}{k}\nabla_i^- m^k(\omega|x)\right| \leqslant \left|\nabla_i^+ \tilde{m}^k(\omega|x) - \frac{k-1}{k}\nabla_i m(\omega|x)\right| + \epsilon$$

[39] By this, we mean that the sequence of functions $m^k(\omega|\cdot) : X(k) \to \mathbb{R}$ converge uniformly to $m(\omega|\cdot) : \mathbb{R}_+^N \to \mathbb{R}$, for every ω.

$$\leqslant (k-1)\int_{z=0}^{1/k} |\nabla_i m(\omega|x_i+z,x_{-i}) - \nabla_i m(\omega|x)|\, dz + \epsilon$$
$$\leqslant \frac{C_2(k-1)}{k^2} + \epsilon \leqslant \frac{C_2}{K} + \epsilon \leqslant 2\epsilon.$$

In addition, if $x \in X(k)\backslash X^i(k)$ and $k \geqslant K$, then

$$\left|\nabla_i^+ \tilde{m}^k(\omega|k,x_{-i}) - \frac{k-1}{k}\nabla_i^- m^k(\omega|k,x_{-i})\right| = \frac{k-1}{k}\left|m^k(\omega|k,x_{-i}) - m^k(\omega|k-1/k,x_{-i})\right|$$
$$\leqslant |m(\omega|k,x_{-i}) - m(\omega|k-1/k,x_{-i})| + 2\epsilon$$
$$\leqslant \frac{C_2}{k} + 2\epsilon \leqslant 3\epsilon.$$

We conclude that for $k \geqslant K$,

$$\tilde{\lambda}^k(\theta) = \min_{x\in X(k)} \sum_\omega \left[w(\omega,\theta)\tilde{m}^k(\omega|x) + \sum_i u_i(\omega,\theta)\nabla_i^+\tilde{m}^k(\omega|x)\right]$$
$$\geqslant \frac{k-1}{k}\min_{x\in X(k)}\sum_\omega\left[w(\omega,\theta)m^k(\omega|x) + \sum_i u_i(\omega,\theta)\nabla_i^- m^k(\omega|x)\right] - C_1|\Omega|\left(\epsilon\frac{k-1}{k} + \frac{1}{k} + 3N\epsilon\right)$$
$$= \frac{k-1}{k}\lambda^k(\theta) - C_1|\Omega|\left(\epsilon\frac{k-1}{k} + \frac{1}{k} + 3N\epsilon\right).$$

As a result, for $k \geqslant K$,

$$\sum_\theta \mu(\theta)\left(\lambda^k(\theta) - \tilde{\lambda}^k(\theta)\right) \leqslant \frac{1}{k}\sum_\theta \mu(\theta)\lambda^k(\theta) + C_1|\Omega|\left(\epsilon\frac{k-1}{k} + \frac{1}{k} + 3N\epsilon\right)$$
$$\leqslant \frac{1}{k}(N+1)C_1 + C_1|\Omega|\left(\epsilon\frac{k-1}{k} + \frac{1}{k} + 3N\epsilon\right).$$

Since ϵ is arbitrary, we conclude that (UB-P-k) and (LB-G-k) have the same value when $k \to \infty$, as desired. □

The hypotheses of Proposition 6 may be difficult to verify in practice. Their value is in making precise what kind of asymptotic smoothness implies that the bounds are tight. Note that these conditions, Lipschitz continuity of $\nabla_i m(\omega|x)$ in particular, are stronger than needed and are not always satisfied by guarantee maximizers in the optimal auctions problem, such as in Brooks and Du (2021b).

B.2 Bounding programs for the optimal auctions problem

We now derive the bounding programs for the optimal auctions problem.

Theorem 3. *For all $k \in \mathbb{N}$, we have*

$$W(19) \geqslant W(\text{MIN-P}) \geqslant W(\text{MAX-G}) \geqslant W(18).$$

Moreover,

- *If (q,t) solves* (18), *then* $G(X(k),q,t) \geqslant W(\text{LB-G-}k)$.
- *If σ solves* (19), *then* $P(X(k),\sigma) \leqslant W(\text{UB-P-}k)$.

Proof. The proof that W(MIN-P)$\geqslant W$(MAX-G) is the same as for Theorem 1.

For (MAX-G), clearly a subset of participation secure mechanisms are those with the message space $X(k)$ and the action 0 is participation secure, and moreover, the transfer is zero if an agent takes action 0. For any such mechanism, infimum expected revenue across all information structures and equilibria is equal to minimum expected revenue across all Bayes correlated equilibria, i.e.,

$$\begin{aligned}
&\min_{\sigma:X(k)\times\Theta\to\mathbb{R}_+} \sum_{\theta,x,i} t_i(x)\sigma(x,\theta) \\
\text{s.t. } &\sum_{x_{-i},\theta}\left(\sum_l \theta_{i,l}\left[q_{i,l}(x_i,x_{-i}) - q_{i,l}(x_i',x_{-i})\right] - \left[t_i(x_i,x_{-i}) - t_i(x_i',x_{-i})\right]\right)\sigma(x_i,x_{-i},\theta) \geqslant 0 \ \forall i,x_i,x_i' \\
&\sum_x \sigma(x,\theta) = \mu(\theta) \ \forall\theta
\end{aligned}$$

This program is clearly feasible, and because $X(k)$ is finite and t and σ are bounded, the value of this program is also bounded. Hence, by the strong duality theorem, its value is equal to that of its dual:

$$\begin{aligned}
&\max_{\substack{\alpha:X_1(k)\times X_1(k)\to\mathbb{R}_+^N,\\ \lambda:\Theta\to\mathbb{R}}} \sum_\theta \lambda(\theta)\mu(\theta) \\
\text{s.t. } &\lambda(\theta) \leqslant \sum_i t_i(x) \\
&\qquad + \sum_{i,x_i'} \alpha_i(x_i,x_i')\left(\sum_l \theta_{i,l}\left[q_{i,l}(x_i',x_{-i}) - q_{i,l}(x_i,x_{-i})\right] - \left[t_i(x_i',x_{-i}) - t_i(x_i,x_{-i})\right]\right) \ \forall x,\theta.
\end{aligned}$$

(Recall that $X_1(k)$ is one dimension of the action/signal space $X(k)$.) We may further constrain this program (and therefore decrease its value) by fixing

$$\alpha_i(x_i,x_i') = \begin{cases} k-1 & \text{if } x_i < k \text{ and } x_i' = x_i + 1/k; \\ 0 & \text{otherwise,} \end{cases}$$

which results in the program

$$\begin{aligned}
&\max_{\lambda:\Theta\to\mathbb{R}} \sum_\theta \lambda(\theta)\mu(\theta) \\
\text{s.t. } &\lambda(\theta) \leqslant \sum_i t_i(x) + \sum_i \left(\sum_l \theta_{i,l}\nabla_i^+ q_{i,l}(x_i,x_{-i}) - \nabla_i^+ t_i(x_i,x_{-i})\right) \ \forall x,\theta.
\end{aligned} \tag{42}$$

We conclude that (42) is a lower bound on equilibrium expected revenue for any mechanism in $\mathcal{M}_k^0$. Hence, a lower bound on (MAX-G) can be obtained by maximizing this lower bound across all such mechanisms, which is (18).

Similarly, for (MIN-P), an upper bound on the min potential is given by restricting attention to information structures with signals in $X(k)$. Moreover, participation security implies that each agent's interim expected utility is non-negative. Thus, by the revelation principle, maximum expected revenue across participation secure mechanisms and equilibria is bounded above by maximum expected revenue across all incentive compatible and individually rational direct mechanisms, that is

$$\max_{q:X(k)\to\mathbb{R}_+^{NL},t:X(k)\to\mathbb{R}^N} \sum_{x,\theta,i} t_i(x)\sigma(x,\theta)$$

$$\text{s.t. } \sum_i q_{i,l}(x) \leqslant 1 \ \forall l,x$$

$$\sum_{x_{-i},\theta}\left(\sum_l \theta_{i,l}q_{i,l}(x_i,x_{-i}) - t_i(x_i,x_{-i})\right)\sigma(x_i,x_{-i},\theta) \geqslant 0 \ \forall i,x_i$$

$$\sum_{x_{-i},\theta}\left(\sum_l \theta_{i,l}\left[q_{i,l}(x_i,x_{-i}) - q_i(x_i',x_{-i})\right] - \left[t_i(x_i,x_{-i}) - t_i(x_i',x_{-i})\right]\right)\sigma(x_i,x_{-i},\theta) \geqslant 0 \ \forall i,x_i,x_i'.$$

We obtain an even more permissive upper bound on the value of the inner program by dropping the interim individual rationality constraint except for $x_i = 0$ and dropping incentive compatibility except for $x_i' = x_i - 1/k$ and $x_i > 0$:

$$\max_{q:X(k)\to\mathbb{R}_+^{NL},t:X(k)\to\mathbb{R}^N} \sum_{x,\theta,i} t_i(x)\sigma(x,\theta)$$

$$\text{s.t. } \sum_i q_{i,l}(x) \leqslant 1 \ \forall l,x$$

$$\sum_{x_{-i},\theta}\left(\sum_l \theta_{i,l}q_{i,l}(0,x_{-i}) - t_i(0,x_{-i})\right)\sigma(0,x_{-i},\theta) \geqslant 0 \ \forall i$$

$$\sum_{x_{-i},\theta}\Bigg(\sum_l \theta_{i,l}\left[q_{i,l}(x_i,x_{-i}) - q_i(x_i-1/k,x_{-i})\right]$$
$$- \left[t_i(x_i,x_{-i}) - t_i(x_i-1/k,x_{-i})\right]\Bigg)\sigma(x_i,x_{-i},\theta) \geqslant 0 \ \forall i, x_i > 0.$$

Since the constraints are not altered by multiplication by a positive scalar (1 or k), this program is equivalent to

$$\max_{q:X(k)\to\mathbb{R}_+^{NL},t:X(k)\to\mathbb{R}^N} \sum_{x,\theta,i} t_i(x)\sigma(x,\theta)$$

$$\text{s.t. } \sum_i q_{i,l}(x) \leqslant 1 \ \forall l,x$$

$$\sum_{x_{-i},\theta}\left(\sum_l \theta_{i,l}\nabla_i^- q_{i,l}(x_i,x_{-i}) - \nabla_i^- t_i(x_i,x_{-i})\right)\sigma(x_i,x_{-i},\theta) \geqslant 0 \ \forall i,x_i.$$

Note that this program is feasible with $q \equiv 0$ and $t \equiv 0$. However, it may be unbounded. Nonetheless, its value (possibly infinity) is an upper bound on the potential of the information structure $(X(k), \sigma)$. Finally, we can relax the program even further by removing the constraints and adding them to the objective, which gives us a weakly higher value:

$$\max_{\substack{q:X(k)\to\mathbb{R}_+^{NL},\\ t:X(k)\to\mathbb{R}^N}} \sum_{x,\theta,i} \left[t_i(x) + \sum_l \theta_{i,l} \nabla_i^- q_{i,l}(x_i, x_{-i}) - \nabla_i^- t_i(x_i, x_{-i}) \right] \sigma(x,\theta)$$

$$\text{s.t.} \quad \sum_i q_{i,l}(x) \leqslant 1 \ \forall l, x.$$

Using Lemma 2 to sum by parts, the objective in this last program is equivalent to

$$\sum_{x,\theta,i} \left[t_i(x)\sigma(x,\theta) - \left(\sum_l \theta_{i,l} q_{i,l}(x_i, x_{-i}) - t_i(x_i, x_{-i}) \right) \tilde{\nabla}_i^+ \sigma(x,\theta) \right]$$

Hence, minimizing the value of this program across all $\sigma \in \Delta(X(k) \times \Theta)$ is exactly (19). □

B.3 Transfers and the aggregate excess growth

B.3.1 Preliminary observations

The proof of Theorem 2, and Lemma 1 in particular, shows that we can essentially solve out the transfers from the optimal auctions lower bound program (18) in terms of the aggregate excess growth Ξ, to obtain the equivalent program (28). In applications, we have found that it is often more convenient to work with the program (28), and derive the optimal multipliers and allocation (λ, Ξ, q).

Importantly, Lemma 1 tells us that it is possible to go back and forth between solutions of the programs (18) and (28). In particular, given (λ, q, t) that is feasible for (18), we can define Ξ according to (23). Lemma 1 implies that Ξ is balanced, so that (λ, Ξ, q) is feasible for (28) and has the same value as (λ, q, t) in (18). In the other direction, given (λ, Ξ, q) that is feasible for (28), there is another optimal solution $(\lambda + C, \Xi - C, q)$ such that $\Xi - C$ is balanced. Lemma 1 then implies that there exists a transfer rule t with aggregate excess growth $\Xi - C$ and satisfies participation security (24), so that $(\lambda + C, q, t)$ is feasible for (18) and has the same value. This discussion is formalized in the following corollary:

Corollary 2. *The triple* (λ^*, q^*, t^*) *is an optimal solution to* (18) *only if* (λ^*, Ξ^*, q^*) *is an optimal solution to* (28), *where* $\Xi^* = \nabla^+ \cdot t^* - \Sigma t^*$. *The triple* (λ^*, Ξ^*, q^*) *is an optimal solution to* (28) *only if there is a* $C \in \mathbb{R}$ *and a* t^* *where* $\Xi^* - C = \nabla^+ \cdot t^* - \Sigma t^*$ *and* $(\lambda^* + C, q^*, t^*)$ *is an optimal solution to* (18).

The proof of Lemma 1 is non-constructive. But in fact, given a balanced Ξ, we can construct a transfer rule t that is participation secure and has the given aggregate excess growth. To develop this result, let us first denote by $\xi_i(x)$ agent i's *individual excess growth*:

$$\xi_i(x) \equiv \nabla_i^+ t(x) - t_i(x). \tag{43}$$

We can view this as a first-order difference equation in x_i, which we can use to solve for the transfer in terms of the individual excess growth. The solution that satisfies the boundary condition $t_i(k, x_{-i}) = -\xi_i(k, x_{-i})$ (which is just equation (43) when $x_i = k$) is

$$t_i(x) = -\sum_{y_i : y_i \geqslant x_i} \left(\frac{k-1}{k}\right)^{(y_i - x_i)k} \frac{1}{k^{\mathbb{I}_{y_i<k}}} \xi_i(y_i, x_{-i}).$$

Using the definition of ρ_i, we can rewrite this more simply as

$$t_i(x) = -\frac{1}{\rho_i(x_i)} \sum_{y_i : y_i \geqslant x_i} \rho_i(y_i) \xi_i(y_i, x_{-i}). \tag{44}$$

Thus, the transfers will satisfy $t_i(0, x_{-i}) = 0$ if and only if[40]

$$\sum_{y_i \in X_i(k)} \rho_i(y_i) \xi_i(y_i, x_{-i}) = 0. \tag{45}$$

We conclude that there is a correspondence between feasible solutions to (45) and individual excess growths that correspond to participation-secure transfer rules.

Now, for Ξ to be the aggregate excess growth, we must have

$$\Sigma \xi(x) = \Xi(x) \tag{46}$$

for all x. Thus, the task of constructing transfers with a given aggregate excess growth reduces to constructing individual excess growths that satisfy (45) and (46).

Proposition 7. *Fix a transfer rule t and its associated individual excess growth functions ξ defined by* (43)*. Then t is given by the formula* (44)*. Moreover, t is participation secure and has aggregate excess growth Ξ if and only if ξ satisfies* (45) *and* (46)*.*

B.3.2 Construction of transfers for $N = 2$

Given a balanced Ξ, we now explicitly describe the solutions to (45) and (46), which in turn define transfer rules with aggregate excess growth Ξ. For notational simplicity, we specialize to the case of $N = 2$. In the next section, we give a general construction for $N > 2$, which is conceptually the same but more involved in terms of notation.

A *balanced division of* Ξ is a pair of functions $\Xi_i : X(k) \to \mathbb{R}$ for $i = 1, 2$ such that Ξ_i is balanced and $\Xi = \Xi_1 + \Xi_2$. Any balanced division induces individual excess growths that satisfy (45) and (46), as we now explain. We interpret Ξ_i as agent i's "initial" allocation of the aggregate excess growth. We then make "correction" to the initial excess growth to satisfy (45):

$$\xi_i(x) = \Xi_i(x) - \sum_{y_i \in X_i(k)} \Xi_i(y_i, x_j) \rho_i(y_i) + \sum_{y_j \in X_j(k)} \Xi_j(x_i, y_j) \rho_j(y_j). \tag{47}$$

[40] Brooks and Du (2021b) stated and used an analogue of (45) in a continuum model where actions are non-negative real numbers. In particular, the condition (45) is key to showing that truthful reporting is an equilibrium of the strong maxmin solution constructed in that paper. A subtlety arises in the continuum model, in that there is no boundary condition at the top. Instead, the analogue of the condition (45) ensures that transfers remain bounded in the limit as x_i goes to infinity, and the transfer given by (44) converges to $\lim_{x_i \to \infty} \xi_i(x_i, x_{-i})$.

Proposition 8. *Suppose $N = 2$. If (Ξ_1, Ξ_2) is a balanced division of Ξ, then ξ given by* (47) *satisfies* (45) *and* (46). *The corresponding transfers t defined by* (44) *are participation secure and have aggregate excess growth Ξ.*

The proof of Proposition 8 follows immediately from the definitions: Taking the expectation over x_i with respect to $\rho_i(x_i)$, the last term in the right-hand side of (47) vanishes, since Ξ_j is balanced, and the first two terms obviously cancel each other; thus, ξ defined by (47) satisfies (45). Moreover, using the assumption that $\Xi_1(x) + \Xi_2(x) = \Xi(x)$, it is easy to see that $\xi_1(x) + \xi_2(x) = \Xi(x)$ in (47) as well.

The set of solutions to (45) and (46) given by Proposition 8 is complete in the sense that if (ξ_1, ξ_2) satisfies (45) and (46), then $(\Xi_1, \Xi_2) = (\xi_1, \xi_2)$ is clearly a balanced division, and the correction terms in (47) are all zero.

While it is simple to prove, Proposition 8 yields rich possibilities for constructing participation secure transfers with the desired aggregate excess growth. For example, we can set $\Xi_1(x) = c\Xi(x)$ and $\Xi_2 = (1-c)\Xi(x)$, where $c \in [0, 1]$ is a constant, which is a balanced division whenever Ξ is balanced. Moreover, if (Ξ_1, Ξ_2) is a balanced division, then so is $(\Xi_1 + E, \Xi_2 - E)$ for any balanced function E. If E is skew-symmetric—meaning $E(x_1, x_2) = -E(x_2, x_1)$—then E will also be balanced; this follows from the fact that ρ is exchangeable, so the expectation of E over ρ is zero. Simple examples of skew symmetric functions include any odd function of the difference $x_1 - x_2$. In Online Appendix B.3.4 we will illustrate that including a skew-symmetric E in the balanced division can lead to a significant simplification in the functional form of the transfers in the model studied in Section 6.

B.3.3 Construction of transfers for $N > 2$

We now give a general construction of participation secure transfers via balanced divisions of the aggregate excess growth. The idea of the general construction is as follows: When there were two agents, we initially allocated each agent a share of the total aggregate excess growth. Each agent then received an "adjustment" so that their individual excess growth satisfied (45), where the adjustment is essentially the interim expected individual excess growth of the other agent. We now adopt a more general construction where there is an initial allocation of the aggregate excess growth to each agent, and then adjustments (in the form of interim expected excess growths) are passed around from agent to agent, so that (45) is satisfied.

We now proceed formally. For this section, we will suppress k and write $X_i = X_i(k)$. For any subset $N' \subseteq \{1, 2, \ldots, N\}$ of agents, let $X_{N'} = \prod_{i \in N'} X_i$.

Let Z be the set of non-repeating sequences in $\{1, \ldots, N\}$ of length less than or equal to N. We also define $Z(i) \subseteq Z$ to be the set of sequences of length less than or equal to $N - 1$ which do not contain i. Given $z \in Z$, we let $N(z)$ be the set of agents not in z. And for $z \in Z(i)$, we let (z, i) be the sequence that appends i to the end of z.

Fix an aggregate excess growth Ξ that is balanced. We say that a collection $\{\Xi_z\}_{z \in Z}$ is a *balanced division (of Ξ)* if the following conditions are satisfied:

1. $\Xi_\varnothing = \Xi$.

2. For all i, $z \in Z(i)$, $\Xi_{(z,i)} : X_{N(z)} \to \mathbb{R}$.

3. For all i, $z \in Z(i)$, $\sum_{x \in X_{N(z)}} \Xi_{(z,i)}(x) \prod_{j \in N(z)} \rho_j(x_j) = 0$.

4. For all i, $z \in Z(i)$, and $x_{N(z,i)} \in X_{N(z,i)}$,

$$\sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)}) \rho_i(x_i) = \sum_{j \in N(z,i)} \Xi_{(z,i,j)}(x_{N(z,i)}).$$

5. For all $x \in X_{N(\varnothing)} = \prod_{j=1}^{N} X_j$,

$$\Xi_{\varnothing}(x) = \sum_{j \in N(\varnothing)} \Xi_j(x).$$

We interpret $\Xi_{(z,i)}$ as agent i's excess growth (before adjustment) when the agents in z are ahead of i in a queue. Condition 2 says that the excess growth $\Xi_{(z,i)}$ depends on x_i and $x_{N(z,i)}$, but not on the actions of the agents in z. Condition 3 says that $\Xi_{(z,i)}$ is balanced. Condition 4 says that the adjustment one makes to i's excess growth (see equation (48)) is equal to the total unadjusted excess growths of the agents who are behind i in the queue. Finally, Conditions 1 and 5 says that the unadjusted excess growths for agents who are first in the queue adds up to the given aggregate excess growth Ξ.

An example of a balanced division is $\Xi_i(x) = \Xi(x)/N$, and

$$\Xi_{(z,i,j)}(x_{N(z,i)}) = \frac{1}{|N(z,i)|} \sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)}) \rho_i(x_i).$$

Another example is $\Xi_1(x) = \Xi(x)$, and

$$\Xi_{(z,i,j)} = \begin{cases} \sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)}) \rho_i(x_i) & \text{if } j = i+1; \\ 0 & \text{otherwise,} \end{cases}$$

where $i+1$ is defined in modulo arithmetic, i.e., $N+1 = 1$.

Given a balanced division $\{\Xi_z\}_{z \in Z}$, we can define the individual excess growths

$$\xi_i(x) = \sum_{z \in Z(i)} \left(\Xi_{(z,i)}(x_i, x_{N(z,i)}) - \sum_{y_i \in X_i} \Xi_{(z,i)}(y_i, x_{N(z,i)}) \rho_i(y_i) \right). \tag{48}$$

We claim that ξ_i defines participation secure transfers. This can be verified by checking:

$$\begin{aligned} \sum_{x_i \in X_i} \xi_i(x) \rho_i(x_i) &= \sum_{z \in Z(i)} \left(\underbrace{\sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)}) \rho_i(x_i) - \sum_{y_i \in X_i} \Xi_{(z,i)}(y_i, x_{N(z,i)}) \rho_i(y_i)}_{=0} \right) \\ &= 0. \end{aligned}$$

Moreover, $\{\xi_i\}_{1\leqslant i\leqslant N}$ has aggregate excess growth Ξ, since

$$\begin{aligned}\sum_{i=1}^{N}\xi_i(x) = \sum_{i=1}^{N}\Xi_i(x)&\\ &+\sum_{i=1}^{N}\sum_{\{z\in Z(i):|z|<N-1\}}\left[\underbrace{\sum_{j\in N(z,i)}\Xi_{(z,i,j)}(x_{N(z,i)})-\sum_{y_i\in X_i}\Xi_{(z,i)}(y_i,x_{N(z,i)})\rho_i(y_i)}_{=0}\right]\\ &-\sum_{i=1}^{N}\sum_{\{z\in Z(i):|z|=N-1\}}\underbrace{\sum_{y_i\in X_i}\Xi_{(z,i)}(y_i)\rho_i(y_i)}_{=0}\\ =\Xi(x).&\end{aligned}$$

In this last calculation, all we have done is to break up the sum over sequences into those that end in i and those that end in (i,j), then we use the fact that $N(z,i)=\varnothing$ when $|z|=N-1$, so the balanced condition implies that the terms in the third line are all zero.

Given an arbitrary profile of participation secure transfers, we can "recover" their individual excess growths ξ from equation (48) with the balanced division $\{\Xi_z\}_{z\in Z}$ such that $\Xi_{\varnothing}=\Sigma\xi$, $\Xi_i=\xi_i$ and $\Xi_z=0$ for all other z. Thus, the participation secure transfers given by the balanced divisions are complete.

When $N=2$, then Z just consists of $\{\varnothing,1,2,12,21\}$. Thus, for any balanced division $\{\Xi_z\}_{z\in Z}$, it must be that Ξ_1 and Ξ_2 are balanced and satisfy $\Xi=\Xi_1+\Xi_2$; moreover, we have

$$\sum_{x_2\in X_2}\Xi_2(x_1,x_2)\rho_2(x_2)=\Xi_{21}(x_1),$$

and likewise for Ξ_1 and Ξ_{12}. Thus, the expression for the individual excess growths given in (48) reduces to the formula (47) given in the case of two agents.

B.3.4 Transfer rules for constant-sum values

We now apply the results of this section to the construction of transfers for optimal auctions problem with a certain empirical distribution of values from Section 6. We first take $k\to\infty$ and then take $m\to 0$.

Let us first consider the case where the agents initially get half of the aggregate excess growth: $\Xi_1=\Xi_2=\Xi/2$.

Taking $k\to\infty$, the aggregate excess growth from equation (30) becomes

$$\Xi(x)=-\frac{1}{2m}(1-\exp(-m))+\frac{1}{2m}\mathbb{I}_{|x_1-x_2|<m},$$

where we have modified the constant to make Ξ balanced for a fixed $m>0$ (see the paragraph following (30) for a discussion about the distribution of $|x_1-x_2|$).

The individual excess growth (equation (47)) and the transfers (equation (44)) in the limit as $k \to \infty$ are

$$\begin{aligned}\xi_i(x) =& \frac{1}{2}\Xi(x) - \frac{1}{2}\int_{y_i=0}^{\infty} \exp(-y_i)\Xi(y_i, x_j)dy_i + \frac{1}{2}\int_{y_j=0}^{\infty} \exp(-y_j)\Xi(x_i, y_j)dy_j \\ =& -\frac{1}{4m}(1-\exp(-m)) + \frac{1}{4m}\mathbb{I}_{|x_i-x_j|<m} \\ & -\frac{1}{4m}\int_{y_i=0}^{\infty} \exp(-y_i)\mathbb{I}_{|y_i-x_j|<m}dy_i + \frac{1}{4m}\int_{y_j=0}^{\infty} \exp(-y_j)\mathbb{I}_{|x_i-y_j|<m}dy_j\end{aligned} \tag{49}$$

and

$$\begin{aligned}t_i(x) =& \frac{-1}{\exp(-x_i)}\int_{y_i=x_i}^{\infty} \exp(-y_i)\xi_i(y_i, x_j)dy_i \\ =& -\int_{y_i=0}^{\infty} \exp(-y_i)\xi_i(x_i+y_i, x_j)dy_i.\end{aligned}$$

Substituting ξ_i into t_i, we get

$$\begin{aligned}t_i(x) =& \frac{1}{4m}(1-\exp(-m)) - \int_{y_i=0}^{\infty} \frac{1}{4m}\mathbb{I}_{|x_i+y_i-x_j|<m}\exp(-y_i)dy_i \\ & + \frac{1}{4m}\int_{y_i=0}^{\infty} \exp(-y_i)\mathbb{I}_{|y_i-x_j|<m}dy_i - \frac{1}{4m}\int_{y_i=0}^{\infty}\int_{y_j=0}^{\infty} \exp(-y_j-y_i)\mathbb{I}_{|x_i+y_i-y_j|<m}dy_jdy_i \\ =& \underbrace{\frac{1}{4m}(1-\exp(-m))}_{A} - \underbrace{\frac{1}{4m}(\exp(-\max(x_j-x_i-m,0)) - \exp(-\max(x_j-x_i+m,0)))}_{B} \\ & + \underbrace{\frac{1}{4m}(\exp(-\max(x_j-m,0)) - \exp(-(x_j+m)))}_{C} - \underbrace{\frac{1}{4m}(L(-x_i+m) - L(-x_i-m))}_{D},\end{aligned} \tag{50}$$

where L is the CDF of a Laplace distribution:

$$L(z) = \begin{cases}\exp(z)/2 & z < 0; \\ 1-\exp(-z)/2 & z \geqslant 0.\end{cases}$$

Using L'Hôpital's rule to take $m \to 0$, we get

$$\begin{aligned}A &= 1/4; \\ B &= \begin{cases}0 & x_j < x_i; \\ 1/4 & x_j = x_i; \\ \exp(-x_j+x_i)/2 & x_j > x_i;\end{cases} \\ C &= \begin{cases}1/4 & x_j = 0; \\ \exp(-x_j)/2 & x_j > 0;\end{cases} \\ D &= \exp(-x_i)/4.\end{aligned}$$

So in the limit as $m \to 0$, we have

$$t_i(x) = \begin{cases} 1/4 - \exp(-x_j + x_i)/2 + \exp(-x_j)/2 - \exp(-x_i)/4 & x_i < x_j; \\ \exp(-x_j)/2 - \exp(-x_i)/4 & x_j = x_i; \\ 1/4 + \exp(-x_j)/2 - \exp(-x_i)/4 & x_i > x_j, \end{cases} \tag{51}$$

when $x_j > 0$, and

$$t_i(x) = \begin{cases} 0 & x_i = 0; \\ 1/2 - \exp(-x_i)/4 & x_i > 0, \end{cases} \tag{52}$$

when $x_j = 0$.

We now show that an alternative balanced division of the aggregate excess growth yields a simpler transfer rule. To this end suppose initially agent i gets $\Xi_i(x) = \Xi(x)/2 + c\,\mathrm{sign}(x_i - x_j)$, where c is a constant, and $\mathrm{sign}(z)$ is 1 if $z > 0$, -1 if $z < 0$, and zero if $z = 0$.

The individual excess growth (equation (47)) in the limit is now

$$\begin{aligned} \xi_i(x) =& \frac{\Xi(x)}{2} + c\,\mathrm{sign}(x_i - x_j) - \int_{y_i=0}^{\infty} \exp(-y_i)\left(\frac{\Xi(y_i, x_j)}{2} + c\,\mathrm{sign}(y_i - x_j)\right) dy_i \\ &+ \int_{y_j=0}^{\infty} \exp(-y_j)\left(\frac{\Xi(x_i, y_j)}{2} + c\,\mathrm{sign}(y_j - x_i)\right) dy_j \end{aligned}$$

Let us denote the difference between the above equation and equation (49) as

$$v_i(x) \equiv c\,\mathrm{sign}(x_i - x_j) - \int_{y_i=0}^{\infty} \exp(-y_i) c\,\mathrm{sign}(y_i - x_j) dy_i + \int_{y_j=0}^{\infty} \exp(-y_j) c\,\mathrm{sign}(y_j - x_i) dy_j.$$

The transfer given by $\Xi_i(x) = \Xi(x)/2 + c\,\mathrm{sign}(x_i - x_j)$ is equation (50) plus

$$\begin{aligned} \Delta_i(x) \equiv& - \int_{y_i=0}^{\infty} \exp(-y_i) v_i(x_i + y_i, x_j) dy_i \\ =& - \int_{y_i=0}^{\infty} c\,\mathrm{sign}(x_i + y_i - x_j) \exp(-y_i) dy_i + \int_{y_i=0}^{\infty} \exp(-y_i) c\,\mathrm{sign}(y_i - x_j) dy_i \\ &- \int_{y_i=0}^{\infty} \int_{y_j=0}^{\infty} \exp(-y_j - y_i) c\,\mathrm{sign}(y_j - x_i - y_i) dy_j dy_i \\ =& - c(2\exp(-\max(x_j - x_i, 0)) - 1) + c(2\exp(-x_j) - 1) + c(1 - \exp(-x_i)). \end{aligned}$$

Taking $c = -1/4$ and adding $\Delta_i(x)$ to equations (51) and (52), we get as $m \to 0$

$$t_i(x) = \begin{cases} 0 & x_i < x_j; \\ 1/4 & x_j = x_i; \\ 1/2 & x_i > x_j, \end{cases}$$

when $x_j > 0$, and

$$t_i(x) = \begin{cases} 0 & x_i = 0; \\ 1/4 & x_i > 0, \end{cases}$$

when $x_j = 0$. Thus, in the limit as $m \to 0$, the winner simply pays a posted price of 1/2.

B.3.5 Concluding remarks

In summary, given (λ^*, q^*, t^*) that is feasible for (18), there will generally be many transfer rules t such that (λ^*, q^*, t) is also feasible and has the same value. For example, in characterizing guarantee-maximizing transfers in the pure common value model, Brooks and Du (2021b) showed that in addition to the solution given by (47) with $\Xi_i = \Xi/2$, there is a distinct solution with an especially simple form, wherein each agent simply pays a constant price per unit, and that price depends just on the sum of the bids. This multiplicity of optimal transfer rules, not all of which are of practical interest, presents a challenge to the study of guarantee-maximizing auctions, and additional properties may be needed to isolate the most useful transfer rules. Going back to common values, the transfer rule in the proportional auction is characterized by the property that the aggregate transfer depends only on the aggregate action.

B.4 Rate of convergence for optimal auctions

Our next result calculates an upper bound on the rate of convergence of the bounding programs in the optimal auctions problem:

Proposition 9. *For all $k \geqslant 1$,*

$$|W(\text{UB-P-}k) - W(\text{LB-G-}k)| \leqslant O(1/\sqrt{k}).$$

Proof. Theorem 3 shows that $W(\text{UB-P-}k) - W(\text{LB-G-}k) \geqslant 0$. The proof of Theorem 2 shows that $W(\text{UB-P-}k) - W(\text{LB-G-}k)$ is at most

$$\epsilon(k) + \left(1 - \frac{k-1}{k(1+N\tilde{\epsilon}(k))}\right) C_\lambda + N(1-1/k)^{k^2-1}\left(\frac{k-1}{k(1+N\tilde{\epsilon}(k))} kNL\overline{\theta} + (k-1)NL\overline{\theta}\right),$$

where $\epsilon(k)$ is given by (40) and $\tilde{\epsilon}(k) = 2/(h(k)+1)$. The second term can be rewritten as

$$\frac{1+Nk\tilde{\epsilon}(k)}{k(1+N\tilde{\epsilon}(k))} C_\lambda = \frac{1/k + 2N/(h(k)+1)}{1+2N/(h(k)+1)} C_\lambda,$$

which goes to zero at a rate of $1/h(k)$ since $h(k)$ goes to infinity at a slower rate than k. The third term goes to zero at the same rate as $\exp(-k)k$ (cf. Footnote 38). Moreover, considering the formula in (40), we see that the first term in $\epsilon(k)$ goes to zero at a rate equal to that of $1 - \exp(-h(k)/k) = O(h(k)/k)$, while the remaining terms go to zero at a rate of either $1 - \exp(-Nh(k)/k)$ or $\exp(-k)k$. Thus, the rate at which $W(\text{UB-P-}k) - W(\text{LB-G-}k)$ goes to zero is therefore $\max(1/h(k), h(k)/k)$. Clearly the optimal choice is $h(k) = \sqrt{k}$, which gives a rate of $1/\sqrt{k}$. □

B.5 Numerical examples for optimal auctions

B.5.1 Perfectly correlated values

We now present a numerical example with one good and two bidders, $L = 1$, $N = 2$, and $\theta_1 = \theta_2 + c$ for a constant c, i.e., values are perfectly positively correlated. Agent 2's value θ_2 is uniformly distributed on an evenly spaced grid of 10 values between 0 and 1.

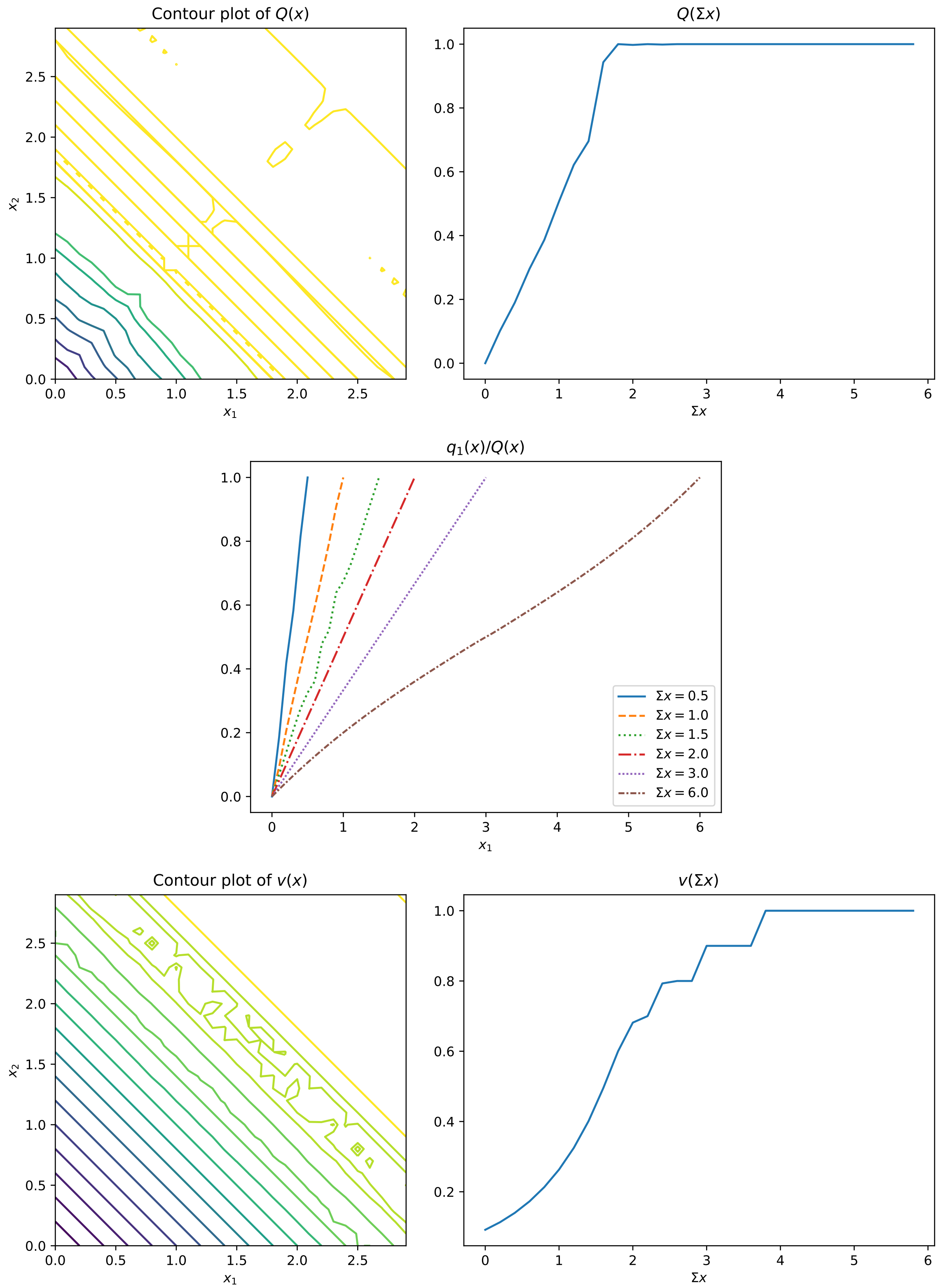

Figure 3: Approximate guarantee-maximizing allocations and potential-minimizing information with pure common values.

We first present numerical results for the case of $c = 0$, that is, there is a pure common value for the good. This case was previously studied in Brooks and Du (2021b), where we solved analytically for optimal mechanisms and information structures in the limit when the action/signal space is all of $\mathbb{R}_+$. The mechanism has the form of a "proportional auction," in which the aggregate allocation and aggregate transfer only depend on the aggregate action, and individual allocations and transfers are proportional to actions. For this example, the aggregate allocation $Q(x) = q_1(x)+q_2(x)$ has the form $Q(\Sigma x) = \min\{\alpha \Sigma x, 1\}$ for a constant α, and $q_i(x) = Q(\Sigma x)\frac{x_i}{\Sigma x}$. Thus, each agent i's allocation on the low rationing region is a simple linear function of their action: $q_i(x) = \alpha x_i$. (This appears to be a general feature of aggregate allocations for solutions to the optimal auctions problem in which the good is rationed at low aggregate actions.)

The top and middle panels of Figure 3 shows the approximate optimal allocation as computed by solving (18) with $k = 10$ (so that each agent has 101 actions). The associated profit guarantee is 0.2620, or 52% of the expected value. The approximate optimal aggregate allocation in the top panels bears a close resemblance to the theoretical solution $Q(\Sigma x) = \min\{\alpha \Sigma x, 1\}$ for $\alpha \approx 1/2$. The middle panel shows that bidder 1's share $q_1(x)/Q(x)$ in the approximate optimal allocation is close to the proportional fraction $x_1/\Sigma x$. Indeed, the solution in Brooks and Du (2021b) was in part motivated by looking at simulations of this sort.

We have omitted the numerical solution for the transfer. As we discussed in Section 5 and Online Appendix B.3, even holding fixed a particular guarantee-maximizing allocation, there may be many transfer rules which could complete a guarantee-maximizing mechanism. Numerical simulations of (18) need not produce the most interesting or tractable solution. In this case, the numerical solution did not suggest the proportional form which is part of the analytical solution in Brooks and Du (2021b). In our subsequent examples, we will similarly focus on optimal allocations.

The bottom panels of Figure 3 shows the approximate potential-minimizing information structure that solves (19). The potential of this information structure is at most 0.2820, so that the gap between $W(18)$ and $W(19)$ is approximately 3.99% of the expected value. The approximate potential-minimizing information very nearly coincides with the theoretical solution with a continuum of signals: The interim expected (common) value $v(x) = v_i(x)$ is an increasing function of the aggregate signal. There is a cutoff (which is around 2 in the approximate solution), below which the interim expected value grows exponentially, and above which the interim expected value is equal to the ex post value. This structure gives rise to the discontinuities in the value function, evident in bottom-right panel, which occur when the interim expected value jumps up to the next higher value in the grid with increments of 0.1. Note that the signals themselves are distributed according to ρ, as per Proposition 1, as they are in all of the simulations reported in this appendix.

Our next simulation has $c = 0.1$, so that there is common knowledge that agent 1's value is higher than agent 2's.[41] In this case, it is socially efficient to always allocate the good to agent 1. In contrast, the guarantee-maximizing mechanism takes into account the cost of incentives, and sometimes allocates to agent 2 so as to reduce information rents, as we

[41] Note that this model does not have pure common values and is not characterized by Brooks and Du (2021b).

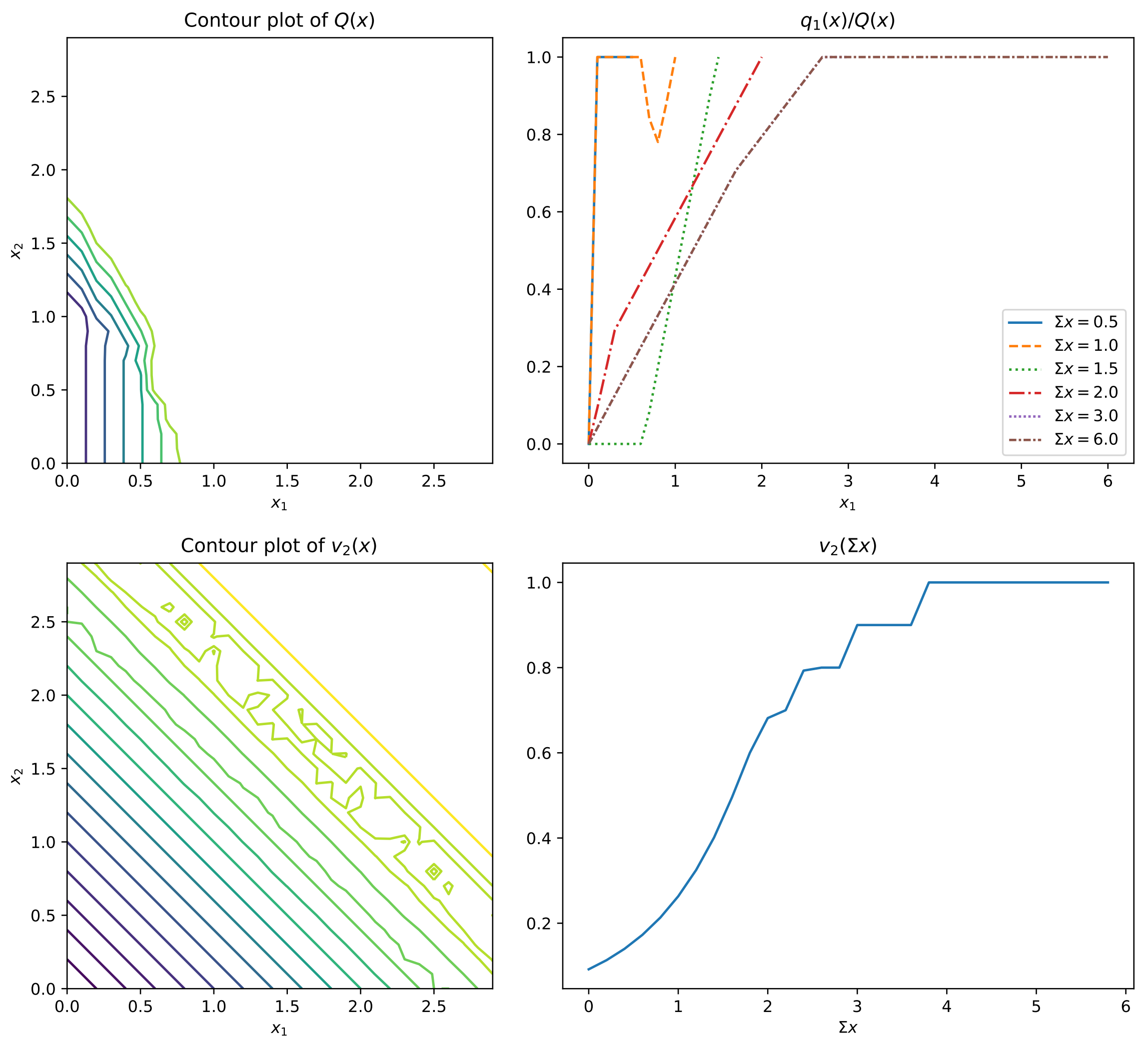

Figure 4: Approximate guarantee-maximizing mechanism and potential-minimizing information with perfectly correlated asymmetric values ($c = 0.1$).

can see in the simulated allocation depicted in the top panels of Figure 4. We also see that the aggregate allocation no longer just depends on the aggregate action, and the contour lines tend to be vertical, i.e., the the aggregate allocation tends to just depend on agent 1's action. Together with the transfer that solves (18), this mechanism guarantees profit at least 0.3045, while the efficient surplus (if the good is always allocated to agent 1) is now 0.6.[42] In the approximate potential-minimizing information structure in the bottom panels of Figure 4, agent 2's interim expected value is identical to that when $c = 0$. (Agent 1's interim expected value is simply $v_1(x) = v_2(x) + 0.1$.) Profit on this information structure is at most 0.3272. Consistent with Theorem 2, we see that the upper and lower bounds on profit are quite close.

[42]Note that the profit guarantee rises by much less than the increase in the efficient surplus, because in order to realize that gain, it would be necessary to allocate the good to agent 1, which in turn would necessitate granting agent 1 a large information rent.

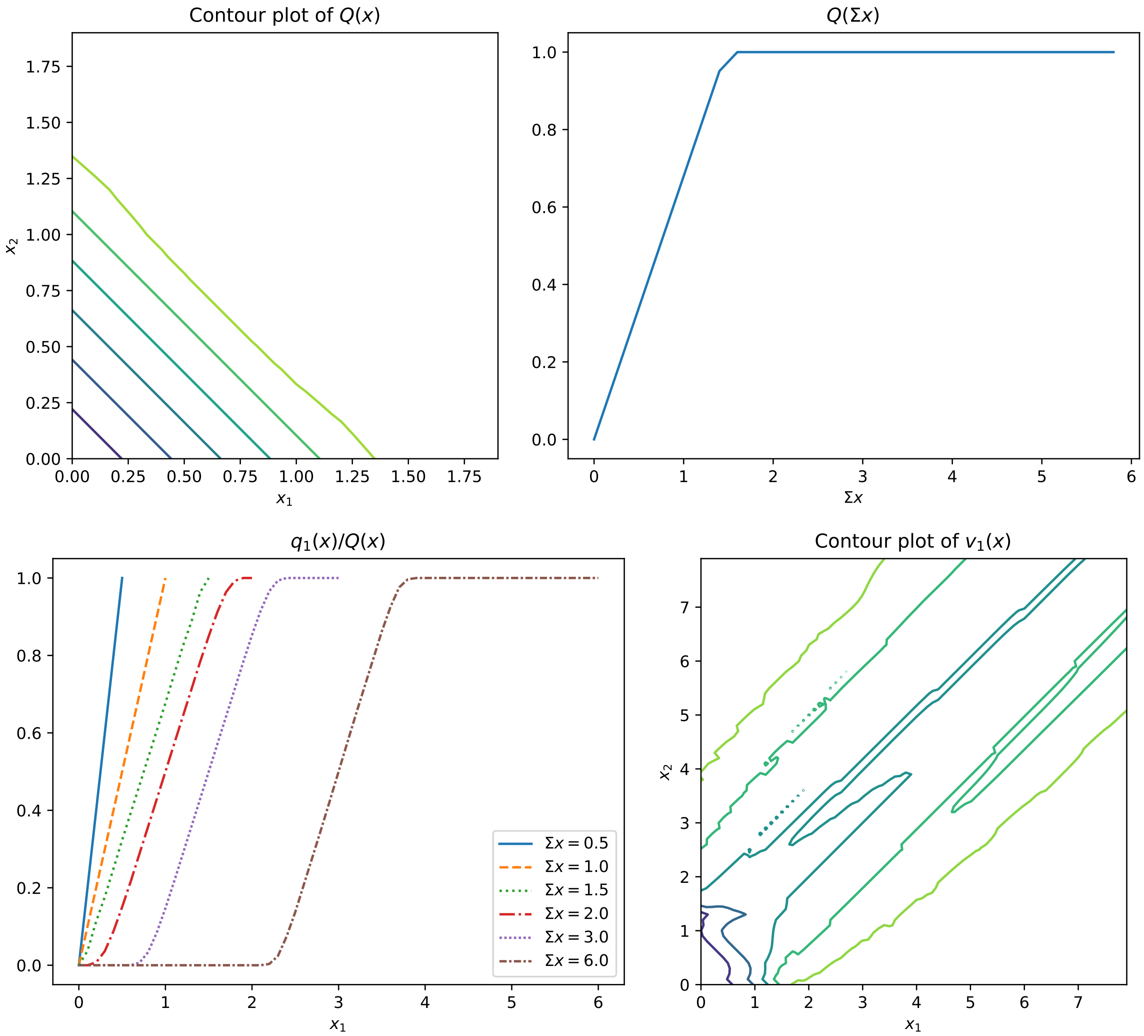

Figure 5: Approximate guarantee-maximizing mechanism and potential-minimizing information for independent values.

As c increases, the region where the aggregate allocation is interior shrinks. When c is sufficiently large, the optimal mechanism always allocates the good to agent 1 at their lowest ex post value.

B.5.2 Independent values

Our next example involves one good and two agents whose values are independently distributed on the same evenly spaced ten-point grid in $[0, 1]$. The simulated allocation and interim values for agent 1 are depicted in Figure 5.

The allocation that solves (18) with $k = 10$ is in the top and bottom-left panels. We again see some striking structure: The aggregate allocation $Q(x)$ has the same functional form as that of the common value, and $Q(x) = \min\{\alpha\Sigma x, 1\}$ for $\alpha \approx 1/1.75$. Agent 1's share of allocation $q_1(x)/Q(x)$ (the bottom-left panel) is $x_1/\Sigma x$ when Σx is below the threshold of 1.75 (just like the case of common value), but $\frac{q_1(x)}{Q(x)} = \min\left\{\frac{\max\{x_1 - f(\Sigma x), 0\}}{\Sigma x - 2f(\Sigma x)}, 1\right\}$ when Σx is above the threshold for some function f of the aggregate action. Moreover, $\Sigma x - 2f(\Sigma x)$ appears to be a constant when Σx is above the threshold of 1.75, which corresponds to the parallel lines when Σx is fixed at 2, 3 and 6 in the bottom-left panel.

The expected highest value in this discrete example is 0.6818, and the lower bound on the max guarantee is 0.2892, or approximately 42% of the efficient surplus.[43]

The interim expected value that solves (19) with $k = 10$ is in the bottom-right panel. While there is more noise than in the allocation, we again see some patterns: In particular, $v_1(x)$ essentially depends only on $|x_1 - x_2|$ when this absolute difference is large enough. While agents' ex post values are independent, their interim expectations are highly correlated, with both agents' interim expected values being higher when the absolute difference in their signals is large. The potential of this information structure is at most 0.3127.

B.5.3 Multiple goods

We now present examples with two agents and two goods. First assume agents have pure common values for each good, so $\theta_{1,l} = \theta_{2,l}$ almost surely for each $l = 1, 2$. The common values are independently distributed across goods and are distributed on the same evenly spaced ten-point grid in $[0, 1]$. The lower and upper bounds from solving (18) and (19) with $k = 10$ are 0.5897 and 0.6306, respectively. The simulated solution in Figure 6 clearly indicates that the interim expected values for the two goods are *exactly the same* and have the same form as that in Figure 3. The approximate guarantee-maximizing allocations for the two goods also have the same form as in Figure 3, and we omit their plots. Thus, the two-good pure common value model reduces to a single-good pure common value model, in which the value for the single good is the sum of the values of the two goods. The logic for this finding was given above in Section 5.3, where we argued that the model should always reduce to a single good problem, where the mechanism only offers the "grand bundle," as long as the ex post value distribution is "exchangeable across goods."

If, however, values are not exchangeable across goods, then the multiple-good problem need not reduce to an auction for the grand bundle, as the following example shows. Agent 2's values $\theta_{2,l}$ are distributed as before, uniform on each good l and independent across goods; agent 1 has the same value for good 2 as agent 2 but assigns more value to good 1 than agent 2: $\theta_{1,2} = \theta_{2,2}$ and $\theta_{1,1} = \theta_{2,1} + 1$. The approximate optimal allocations are depicted in Figure 7 as surface plots, so that it is easier to see levels. As we can see, the two agents receive each good with different probabilities. As we would expect, good 1 is mostly allocated to agent 1, since their value for that good is much higher. Interestingly, agent 1 also tends to get more shares of good 2 than agent 2, even though the two agents have the same value for good 2, because of the endogenous bundling of the two goods in the guarantee-maximizing mechanism.

[43]Thus, while the profit guarantee is higher than with the common value, it does not rise nearly as much as the surplus. The reason, of course, is that the agents also have more private information when their values are independent than when they are common.

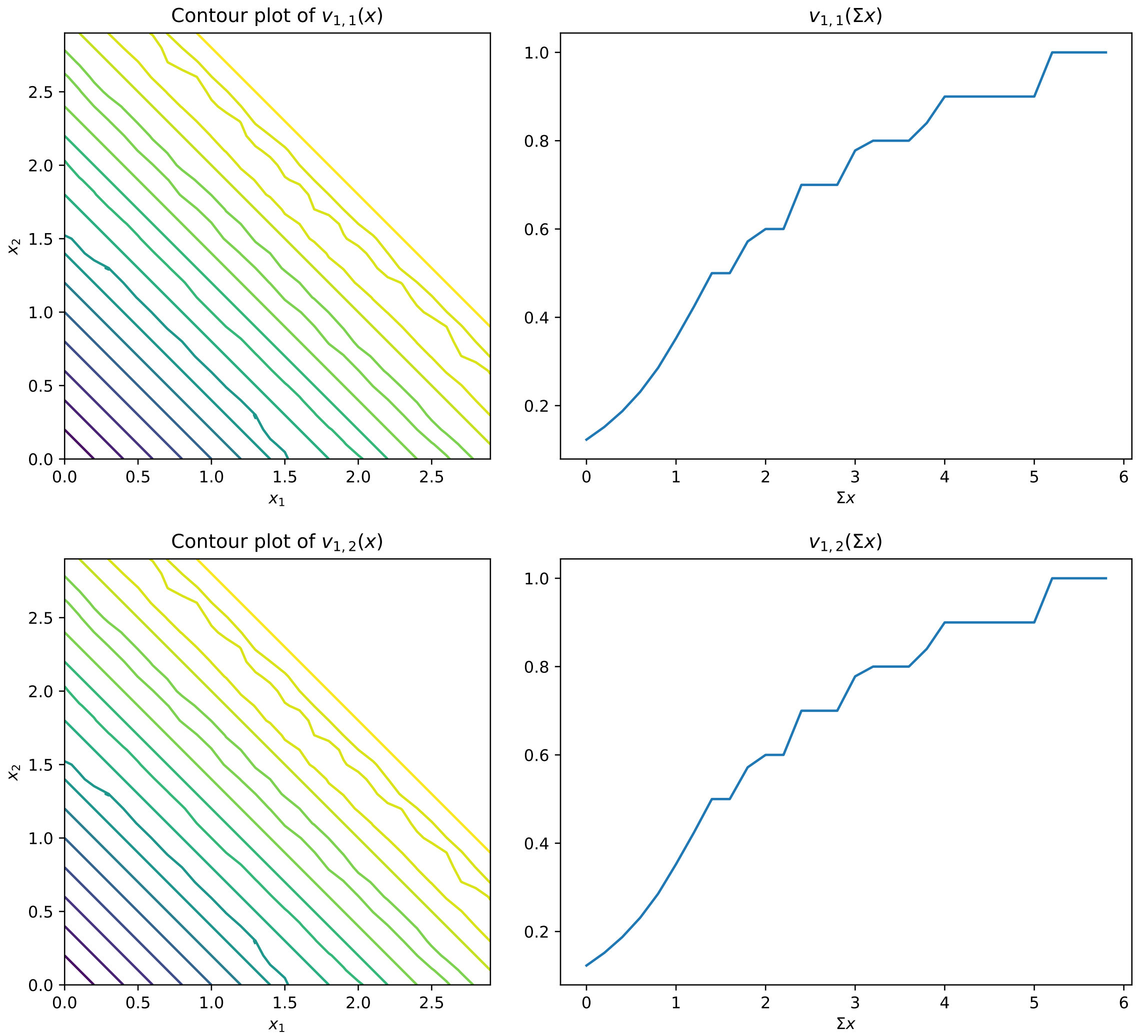

Figure 6: Approximate potential-minimizing information with pure common values and two goods.

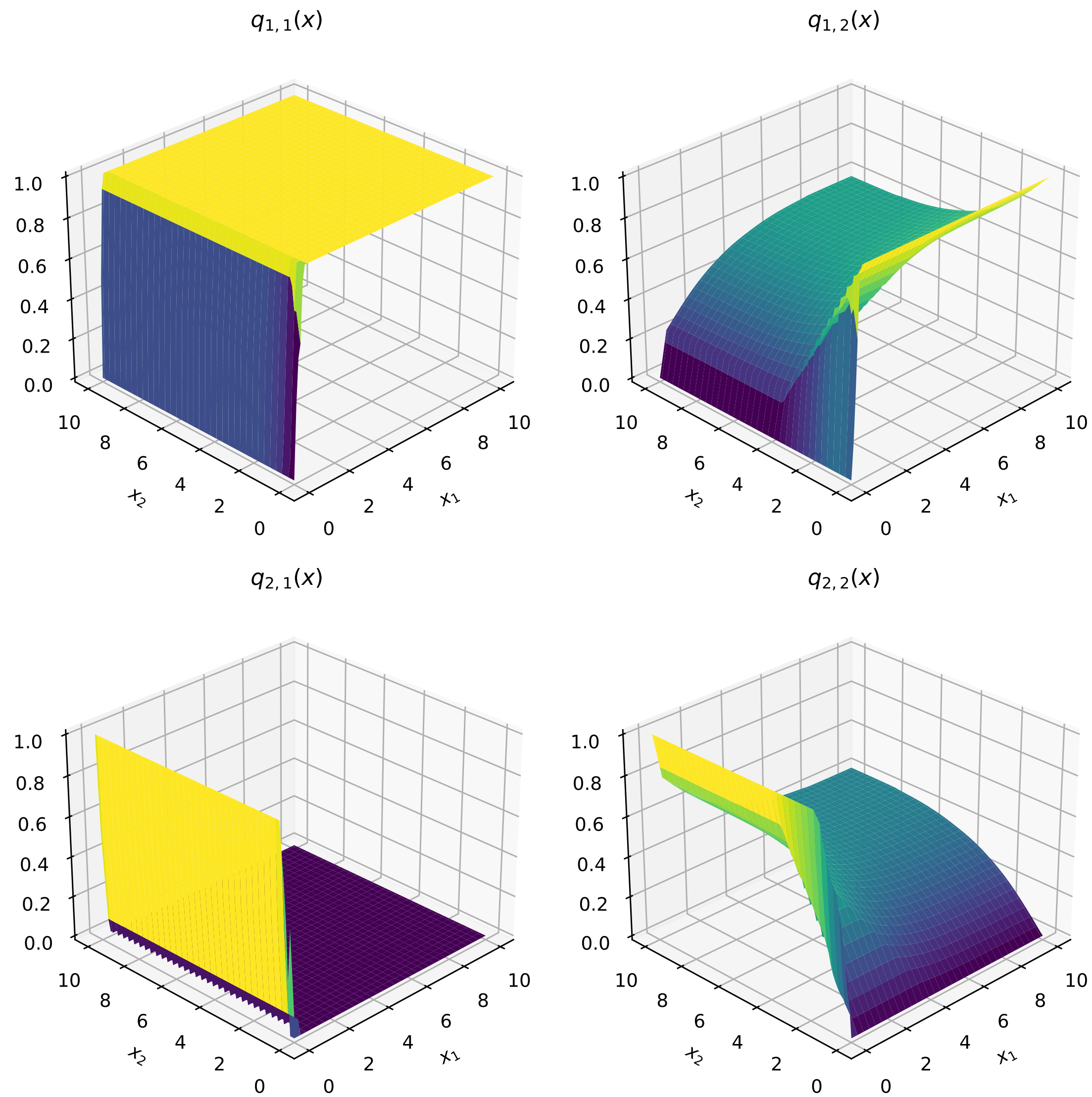

Figure 7: Approximate guarantee-maximizing allocation with two goods with non-exchangeable values.