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# On (constrained) efficiency of strategy-proof random assignment

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We study random assignment of indivisible objects among a set of agents, when each agent is to receive one object and has strict preferences over the objects. Random Serial Dictatorship (RSD) satisfies equal treatment of equals, ex-post efficiency and strategy-proofness. Answering a longstanding open question, we show that RSD is not characterized by those properties – there are other mechanisms satisfying equal treatment of equals, ex-post efficiency and strategy-proofness which are not welfare-equivalent to RSD. On the other hand, we show that RSD is not Pareto-dominated by any mechanism that is (i) strategy-proof and (ii) boundedly invariant. Moreover, the same holds for all mechanisms that are ex-post efficient, strategy-proof and boundedly invariant: no such mechanism is dominated by any other mechanism that is strategy-proof and boundedly invariant.

KEYWORDS: random assignment, strategy-proofness, ex-post efficiency, bounded invariance.

## 1. INTRODUCTION

Consider the problem of assigning indivisible objects among a set of agents – each agent is to receive one object and has strict preferences over the set of objects. Further, while objects' characteristics may include a fixed monetary payment, there are no additional transfers. Problems like this arise in many real-life situations such as the assignment of

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on-campus housing (where rents are fixed), organ allocation, school choice with ties in applicants' priorities, etc.. Whenever several agents prefer the same object over any other, the indivisible nature of objects, together with the absence of compensating transfers, will render any deterministic assignment unfair. For that reason, both theorists and policy makers have turned to random assignments in such contexts.

To implement random assignments, a mechanism will have to elicit agents' preferences to then determine a probability distribution over deterministic assignments. Since eliciting preferences over all possible lotteries is often impractical, agents are typically only asked to report their preference ranking over objects – for example, school choice programs will typically ask applicants to provide a list of schools, ranked from most- to least-preferred. Crucially, given that preferences are private information, the design of random assignment mechanisms has to take into account agents' incentives to reveal their preferences.

Strategy-proofness makes truthful reporting a dominant strategy and thus should ensure that agents truthfully reveal their ordinal preferences over objects for any underlying utility representation of preferences. Unfortunately, the literature on random assignment mechanisms contains numerous impossibility results as soon as strategy-proofness and equal-treatment-of-equals, as a minimal fairness requirement, are married with different ex-ante notions of efficiency.[1] Hence we will focus on ex-post efficiency and analyse *constrained* ex-ante efficiency in a class of mechanisms satisfying certain properties. Furthermore, we will consider situations where each agent needs to be assigned exactly one object. We will refer to this as the acceptable domain, as agents either desire all objects (but cannot consume more than one) or cannot unilaterally reject an assignment.

One of the most prominent procedures, frequently used in real life, is random serial dictatorship (RSD): After ordering agents uniformly at random, the first agent gets to pick their most preferred object and each subsequent agent gets to pick their most preferred among all remaining objects. Besides being easily implementable, RSD satisfies many desirable properties: (1) equal treatment of equals – any two agents with the same preferences obtain identical random assignments ex-ante, (2) ex-post efficiency – for any realized ordering of agents, the associated deterministic assignment is Pareto efficient, and (3) strategy-

[1] Throughout 'ex-ante' is to be understood as before realizing the final deterministic assignment; this corresponds to the term 'interim' used in mechanism design outside of the literature on random assignments.

proofness – no agent has an incentive to pick a less preferred object when it is their turn to choose. Moreover, the procedure can be readily implemented as a direct mechanism where agents report their ordering over objects and the procedure picks optimally on their behalf. In that case strategy-proofness is satisfied in that it is a dominant strategy to report preferences truthfully.

It has been an open question for more than two decades whether RSD is characterized by these properties in terms of welfare, i.e., whether any mechanism satisfying (1)-(3) necessarily yields the same individual random assignments as RSD.[2] Our first main result invalidates this conjecture—there exist mechanisms which satisfy equal treatment of equals, ex-post efficiency and strategy-proofness, and for which for some preference profile and some agent, her random assignment does not coincide with the one of RSD. In fact for some preference profiles our constructed mechanism yields random assignments that Pareto dominate, in a stochastic dominance sense, the ones arising under RSD. Hence, as preferences over objects are strict, for any extension of preferences from objects to random assignments, all agents prefer this random assignment (with strict preference holding for some agents). Thus, the mechanism is not welfare equivalent to RSD.

In the mechanism constructed for our first main result, the random assignment of a given object may depend on agents' preferences over less preferred objects. This is disturbing as strategy-proofness implies that an individual agent's probability share for a given object *does not* depend on their own preferences over less preferred objects. In contrast, RSD satisfies (4) bounded invariance according to which the random assignment of any object $x$ depends only on agents' preferences over objects which are preferred to $x$ – changing the reported ordering of less preferred objects does not affect the probability with which other agents are assigned object $x$. Hence, for strategy-proof mechanisms, bounded invariance may be viewed as a weak, object-wise, non-bossiness condition.[3]

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[2] Bogomolnaia and Moulin [2001] were able to prove this for the case of three agents and three objects.

[3] Many mechanisms considered in the literature as well as real-life mechanisms used in practice satisfy bounded invariance. For example, Probabilistic Serial, Immediate Acceptance and the Top-Trading-Cycles (TTC) mechanism are all boundedly invariant. In Basteck and Ehlers [2024] we show bounded invariance to unify two key properties in the literature: (i) DA satisfies bounded invariance if and only if the underlying deterministic priority structure ensures Pareto efficiency, adding to the main theorem by Ergin [2002], and (ii) in the characterization of hierarchical exchange rules by Pápai [2000], reallocation-proofness may be replaced by bounded invariance. The latter also follows from Pycia and Ünver [2017, Theorem 2] without using the term bounded invariance:

Our second main result is that no mechanism satisfying properties (2)-(4) is Pareto dominated (in terms of first-order stochastic dominance) by a strategy-proof and boundedly invariant mechanism. As an immediate corollary we find that RSD is not Pareto dominated by any mechanism satisfying strategy-proofness and bounded invariance. It is important to stress that our second main result applies to any mechanism and is not exclusive to RSD. For instance, in applications one might take into account affirmative action constraints with respect to minorities or disadvantaged groups by not choosing certain orders of agents (where majorities or advantaged groups come first in the order) and apply a weighted version of RSD.[4] Any such mechanism satisfies (2)-(4) and is therefore not Pareto dominated by any strategy-proof and boundedly invariant mechanism. This addresses a question whether RSD is constrained efficient in the class of strategy-proof mechanisms (where we impose bounded invariance in addition), and provides a positive partial answer on the acceptable domain to the long-standing open question by Zhou [1990] whether RSD is undominated in the class of mechanisms satisfying (1)-(3) – our result does not impose (1) but instead imposes (4). This is the first affirmative result for RSD in connection with ex-post efficiency and strategy-proofness.

We connect our main results to the previous literature. In the cardinal framework, Zhou [1990] showed that no mechanism satisfies equal treatment of equals, strategy-proofness and ex-ante efficiency (where the latter postulates always to choose a random assignment which is not Pareto dominated in terms of expected utilities by any other one). In the ordinal framework, Bogomolnaia and Moulin [2001] establish an analogous impossibility result with the notion of ordinal efficiency (which postulates that no other random assignment Pareto dominates the chosen one for all underlying utility representations of preferences). Pycia and Troyan [2021] recently showed that RSD is characterized by anonymity, ex-post efficiency, and obvious strategy-proofness.[5]

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their result shows that every mechanism satisfying Pareto efficiency, strategy-proofness and non-bossiness is a hierarchical exchange rule or includes a "broker" (where brokerage implies a specific type of failure of bounded invariance).

[4]For example, if there are several subgroups of agents, we may wish to randomize over orders in which members of the groups alternate when 'picking' objects.

[5]This notion implies, roughly speaking, that the mechanism can be modelled as a dynamic perfect information game form where at any decision node an agent can either clinch an object or pass, and passing never results in

Another strand of the literature allows the possibility for agents to rank objects unacceptable and possibly prefer to receive no object instead of an unacceptable one. Notions of efficiency then have to take into account the set of (un)assigned objects: a deterministic assignment is non-wasteful if no agent prefers an unassigned object to her assignment. As a stronger requirement, ex-ante non-wastefulness demands that if an agent prefers an object over another and is assigned the less-preferred with positive probability, then the more-preferred object must be assigned with probability one. Erdil [2014] established that there are mechanisms Pareto-dominating RSD which are less ex-ante wasteful, which is a negative answer on the full domain to a question first raised by Zhou [1990]. Notably, the mechanism constructed in Erdil [2014, Proposition 3] coincides with RSD on the acceptable domain, i.e., it does not Pareto-dominate RSD for the domain where all objects are acceptable. His constructed mechanism satisfies equal treatment of equals and strategy-proofness but violates bounded invariance. Our second main result implies that any strategy-proof and boundedly invariant mechanism, which dominates RSD on the full domain, must coincide with RSD on the acceptable domain. In other words, Pareto improvements over RSD are only possible for profiles where objects are classified unacceptable in a "certain" way.

The paper is organized as follows. Section 2 introduces random assignments, their properties and several prominent mechanisms. Section 3 constructs a mechanism which satisfies equal treatment of equals, ex-post efficiency and strategy-proofness, and which is not welfare-equivalent to RSD. Section 4 states our second main result pertaining to the constrained efficiency of any mechanism satisfying ex-post efficiency, strategy-proofness and bounded invariance. Section 5 concludes. The Appendix contains the proofs of our main results.

## 2. MODEL

Let $N = \{1, \ldots, n\}$ denote the set of agents and $O = \{o_1, \ldots, o_n\}$ denote the finite set of objects. Throughout the main text we suppose $|N| = |O| \geq 3$ and allow for unequal numbers of agents and objects in the Appendix. Each agent $i$ has strict preferences over $O \cup \{i\}$ where $i$ stands for being unassigned; let $R_i$ denote the corresponding linear order[6]

an object worse than the one she could have clinched. The characterization has been cut from the original version and is now available as Pycia and Troyan [2024].

[6] Thus, $R_i$ is (i) complete, (ii) transitive and (iii) antisymmetric ($xR_iy$ and $yR_ix$ implies $x = y$).

and write $P_i$ for its asymmetric part (where $xP_iy$ is defined by $xR_iy$ and $x \neq y$). Let $\mathcal{R}^i$ denote the set of all strict preferences of agent $i$ over $O \cup \{i\}$ such that $oR_ii$ for all $o \in O$, i.e., where all objects are acceptable. Let $\mathcal{R}^N = \times_{i \in N} \mathcal{R}^i$ denote the set of all preference profiles $R = (R_1, \ldots, R_n)$, which we call the acceptable domain.

An assignment is a mapping $\mu : N \to O \cup N$ such that[7] $\mu_i \in O \cup \{i\}$ for all $i \in N$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Let $\mathcal{M}$ denote the set of all assignments.

An assignment $\mu$ is efficient under $R$ if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_j P_j \mu_j$ for some $j \in N$. As all objects are acceptable and $|O| = |N|$, this implies that no agent is unassigned under $\mu$. Let $\mathcal{PO}(R)$ denote the set of all efficient assignments under $R$. An assignment $\mu$ is weakly efficient under $R$ if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i P_i \mu_i$ for all $i \in N$. Let $\mathcal{WPO}(R)$ denote the set of all weakly efficient assignments under $R$.

Let $\Delta(\mathcal{M})$ denote the set of all probability distributions over $\mathcal{M}$. Given $p \in \Delta(\mathcal{M})$, let $p_{ia}$ denote the associated probability of $i$ being assigned $a$ and refer to $p_i = (p_{ia})_{a \in O \cup \{i\}}$ as agent $i$'s (individual) random assignment. Let $supp(p)$ denote the support of $p$. Then (i) $p$ is ex-post efficient under $R$ if $supp(p) \subseteq \mathcal{PO}(R)$, and (ii) $p$ is ex-post weakly efficient under $R$ if $supp(p) \subseteq \mathcal{WPO}(R)$.

For all $i \in N$, all $R_i \in \mathcal{R}^i$ and all $x \in O \cup \{i\}$, let $B(x, R_i) = \{y \in O \cup \{i\} : yR_ix\}$. Then given any $p, q \in \Delta(\mathcal{M})$, $p_i$ stochastically $R_i$-dominates $q_i$ if for all $x \in O \cup \{i\}$,

$$\sum_{y \in B(x,R_i)} p_{iy} \geq \sum_{y \in B(x,R_i)} q_{iy}.$$

A random assignment $p$ stochastically $R$-dominates (or sd-dominates) another random assignment $q$ if $p_i$ $R_i$-dominates $q_i$ for all $i \in N$. A random assignment is stochastic dominance (sd)-efficient if there is no random assignment $q \neq p$ that stochastically $R$-dominates

[7] We will use throughout the convention to write $\mu_i$ instead of $\mu(i)$ for any $i \in N$.

it.[8] Given two random assignments $p$ and $q$, we say that $p$ and $q$ are *welfare-equivalent* if $p_i = q_i$ for all $i \in N$.[9]

A mechanism (or rule) is a mapping $f : \mathcal{R}^N \to \Delta(\mathcal{M})$. Then $f(R)$ denotes the random assignment chosen for $R$, and $f_{ia}(R)$ denotes the probability of agent $i$ being assigned object $a$. For $i \in N$, $f_i(R)$ denotes the tuple of assignment probabilities $(f_{ia}(R))_{a\in O}$, and for $a \in O$, $f_a(R)$ is defined accordingly as the tuple of probability shares with which $a$ is assigned to the various agents. A mechanism $f$ *sd-dominates* another mechanism $g$, denoted as $f \rhd^{sd} g$, if for any profile $R$ the random assignment $f(R)$ stochastically $R$-dominates the random assignment $g(R)$, and for some profile $\bar{R}$ and $i \in N$ we have $f_i(\bar{R}) \neq g_i(\bar{R})$. Further $f$ is *sd-efficient* if for all $R \in \mathcal{R}^N$, $f(R)$ is sd-efficient under $R$. Similarly, we define ex-post (weak) efficiency for a mechanism. A mechanism $f$ is *deterministic* if for any profile $R$, $|supp(f(R))| = 1$, i.e., the mechanism chooses one assignment with probability one.

Then $f$ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$ and all $R'_i \in \mathcal{R}^i$, $f_i(R)$ stochastically $R_i$-dominates $f_i(R'_i, R_{-i})$. Strategy-proofness is equivalent to the requirement that for any von Neumann-Morgenstern utility presentation compatible with a given ordinal ranking of objects, submitting the true ordinal ranking maximizes an agent's expected utility. Most real-life mechanisms only elicit this ordinal information (instead of von Neumann-Morgenstern utilities).

Furthermore, $f$ is *envy-free* if for all $R \in \mathcal{R}^N$ and all $i \in N$, $f_i(R)$ stochastically $R_i$-dominates $f_j(R)$ (where in $f_j(R)$ the outside option $j$ is replaced by $i$). If $f(R)$ attaches probability one to assignment $\mu$, then this is equivalent to $\mu_i R_i \mu_j$ for all $i, j \in N$. Finally, $f$ satisfies *symmetry* (or more descriptively, *equal treatment of equals*) if for all $R \in \mathcal{R}^N$ and all $i, j \in N$, $R_i = R_j$ implies $f_{io}(R) = f_{jo}(R)$ for all $o \in O$.

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[8] Bogomolnaia and Moulin [2001] refer to this as "ordinal efficiency". It implies Pareto-efficiency with respect to expected utilities for some von Neumann-Morgenstern-representations of agents' ordinal preferences over objects [McLennan, 2002].

[9] Some papers directly define a bistochastic matrix $(p_{ia})_{i\in N, a\in O}$ rather than a random assignment per se, i.e., a convex combination of deterministic assignments. Nonetheless, corresponding random assignments exist as any bistochastic matrix $(p_{ia})_{i\in N, a\in O}$ can be decomposed as a convex combination of deterministic assignments by the Birkhoff-von Neumann Theorem [Birkhoff, 1946]. Abdulkadiroğlu and Sönmez [2003] observe that an ex-post efficient random assignment may be welfare-equivalent to a random assignment with support contained in the set of inefficient assignments so that $p_i = q_i$ for all $i \in N$ does not imply $p = q$.

Most properties are defined in terms of an agents' random assignments. For a given set of properties, we say that a mechanism $f$ is *unique in terms of probability shares*, if for any other mechanism $\phi$ satisfying this set of properties, $f(R)$ and $\phi(R)$ are welfare-equivalent for any profile $R$, i.e., if individual random assignments coincide. Below we introduce two well-known mechanisms.

Let $\succ$ denote a strict priority ranking over $N$ and let $\Pi$ denote the set of all strict priority orders. Given $\succ \in \Pi$, let $f^{\succ}$ denote the (deterministic) serial dictatorship (SD) mechanism where agents are assigned their most-preferred among all available objects in order of their priority.[10] Then the random serial dictatorship (RSD) mechanism is defined by $RSD(R) = \frac{1}{n!}\sum_{\succ \in \Pi} f^{\succ}(R)$ for all $R \in \mathcal{R}^N$.

We omit the formal definition of the probabilistic serial (PS) mechanism[11] and provide an intuitive formulation instead: agents start eating, with uniform speed, from their most-preferred object; once an object is exhausted, each agent eats with uniform speed from his most-preferred among the remaining objects, and so on until all objects are exhausted. The assignment probabilities of any agent in PS are simply the shares of objects the agent has eaten over the course of this process.[12]

The literature widely discusses the trade-off among these two mechanisms: on the one hand RSD satisfies ex-post efficiency, symmetry and strategy-proofness but violates sd-efficiency and envy-freeness while on the other hand PS satisfies sd-efficiency and envy-freeness but violates strategy-proofness.

## 3. EX-POST EFFICIENCY, SYMMETRY AND STRATEGY-PROOFNESS

It has long been an open question, at least since Bogomolnaia and Moulin [2001] were able to prove the statement for $|N| = |O| = 3$, whether random serial dictatorship is char-

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[10] For any $R \in \mathcal{R}^N$ and $i_1 \succ i_2 \succ \cdots \succ i_n$, $i_1$ receives his most $R_{i_1}$-preferred object in $O$ (denoted by $f^{\succ}_{i_1}(R)$), and for $l = 2, \ldots, n$, $i_l$ receives his most $R_{i_l}$-preferred object in $O \backslash \{f^{\succ}_{i_l}(R), \ldots, f^{\succ}_{i_{l-l}}(R)\}$ (denoted by $f^{\succ}_{i_l}(R)$).

[11] For that, we refer the reader to Bogomolnaia and Moulin [2001] who introduced the probabilistic serial (PS) mechanism and showed that it is envy-free and ex-ante efficient (hence necessarily violates strategy-proofness). Bogomolnaia [2015] offers an alternative definition of PS, and Katta and Sethuraman [2006] extend PS to the domain where indifferences are allowed.

[12] The PS-mechanism pins down individuals' object assignment probabilities directly but can be decomposed as a convex combination of deterministic assignments by the Birkhoff-von Neumann Theorem [Birkhoff, 1946].

acterized (in terms of welfare) by ex-post efficiency, symmetry and strategy-proofness. As we show, this is not the case for five agents or more.

THEOREM 1: *For five agents or more, there exist mechanisms satisfying ex-post efficiency, symmetry and strategy-proofness, which are not welfare-equivalent to random serial dictatorship.*

We give an informal description of the main steps of the construction of such a mechanism for five agents and five objects below. The starting point of the construction is inspired by Erdil [2014, Proposition 3], and adapted to the acceptable domain in an inventive way.[13] The detailed demonstration is relegated to the Appendix.

First, we describe an alternative formulation of RSD. Namely, define the mechanism $f^{\underline{i}}$ where the four agents in $N \backslash \{i\}$ get to choose in random order as under RSD, while $i$ is assigned the residual object. Now randomizing over all such mechanisms $f^{\underline{i}}$, $i \in N$, with probability $1/5$ gives us back RSD, i.e., $RSD = \frac{1}{5} \sum_{i \in N} f^{\underline{i}}$.

Second, we construct an ex-post efficient and strategy-proof mechanism $g^{1-\underline{5}}$ which weakly sd-dominates $f^{\underline{5}}$ for agents 1 to 4 and where, as under $f^{\underline{5}}$, agent 5 receives the residual assignment.

Thus, intuitively, if we consider all preference profiles to be equally likely ex-ante, the new mechanism $g^{1-\underline{5}}$ improves upon $f^{\underline{5}}$ on average, i.e., in terms of the average expected rank of assigned objects.[14] Permuting the roles of agents in $f^{\underline{5}}$ and $g^{1-\underline{5}}$ allows us to recover equal-treatment-of equals. Moreover, as $g^{1-\underline{5}}$ constitutes an improvement over $f^{\underline{5}}$ on average, its symmetrized version constitutes an improvement over the symmetrized version of $f^{\underline{5}}$, i.e., over RSD, establishing welfare-non-equivalence.[15]We construct $g^{1-\underline{5}}$ so that it strictly sd-dominates $f^{\underline{5}}$ for agent 1 and to yield the same random assignments for agents 2, 3 and 4.

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[13]Erdil [2014] considers random assignment with *unacceptable* objects where RSD may leave some objects unassigned and shows that it is possible to assign them with higher probability without violating strategy-proofness. Note that on the acceptable domain, his constructed mechanism coincides with RSD.

[14]Given $R_i$, let $rank(x, R_i)$ denote the rank of $x$ in $R_i$, where $rank(x, R_i) = 5$ means $x$ is the most preferred object and $rank(x, R_i) = 1$ means $x$ is the least preferred object. Then for random assignment $p$ agent $i$'s expected rank under $R_i$ is given by $\sum_{x \in O} p_{ix} rank(x, R_i)$.

[15]The formal proof in the Appendix establishes welfare-non-equivalence by reference to a particular preference profile for which the constructed mechanism yields an sd-improvement over RSD.

The constructed mechanism $g^{1-\underline{5}}$ differs from $f^{\underline{5}}$ on a subdomain of preference profiles where preferences of agents 2, 3, and 4 are as follows: [16]

| $R_2$ | $R_3$ | $R_4$ |
|---|---|---|
| $c$ | $c$ | $c$ |
| $a$ | $b$ | $e$ |
| $d$ | $d$ | $d$ |
| $e$ | $e$ | $a$ |
| $b$ | $a$ | $b$ |

Now for $f^{\underline{5}}$, agent 1 does not get $a$, $b$ or $c$ for the orders $4-2-3-1-5$ and $4-3-2-1-5$ (where any order has probability $1/24$), i.e., at least with probability $1/12$.

Now suppose $R_1 : ab\ldots$.[17] Note that under $f^{\underline{5}}$ agent 5 gets object $b$ for the orders $3-1-2-4-5$ and $3-1-4-2-5$, i.e., at least with probability $1/12$. For $g^{1-\underline{5}}$, we will increase for agent 1 the share of $b$ by $1/12$ (while keeping her share of $a$ unchanged) and reduce for agent 5 the share of $b$ by $1/12$, i.e., agent 1 will receive $(18a+6b)/24$ in the new mechanism $g^{1-\underline{5}}$. Agents 2, 3 and 4 always get the same random assignment under $g^{1-\underline{5}}$ and $f^{\underline{5}}$, and agent 5's random assignment is the residual. In the Appendix, we show that the same shift in probability shares of $b$ from agent 5 to agent 1 is feasible no matter where $c$ is ranked in $R_1$. If $R_1 : ba\ldots$, then analogously we increase for agent 1 the share of $a$ by $1/12$ (while keeping her share of $b$ unchanged), and reduce it for agent 5.

We verify that ex-post efficiency for $g^{1-\underline{5}}$ is preserved. This can be done by replacing the efficient assignments on the left below with the efficient assignments on the right with probability share $1/12$ each. Let $\{d,e\}=\{x,y\}$ and $xP_1y$:

$$\begin{pmatrix} 1\,2\,3\,4\,5 \\ x\,a\,b\,c\,y \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4\,5 \\ a\,d\,c\,e\,b \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,4\,5 \\ b\,a\,c\,e\,d \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4\,5 \\ a\,d\,b\,c\,e \end{pmatrix}.$$

[16] Our construction will allow agent 1 to sometimes receive a higher probability share of $a$ or $b$ and a lower share of $e$ or $d$ than under $f^{\underline{5}}$. We verify that this can be done without violating ex-post efficiency and strategy-proofness. For this it will be crucial for agents 2 and 3 to rank the same object $d$ above $e$ and for agent 4 to rank $e$ above $d$.

[17] We use this notation to write $aP_1bP_1o$ for all $o\in O\backslash\{a,b\}$.

Note that under $f^{\underline{5}}$ the first assignment is obtained for the orders $4-2-3-1-5$ and $4-3-2-1-5$, the second one for the orders $3-1-2-4-5$ and $3-1-4-2-5$ (since agents 2 and 4 have opposite preferences over $d$ and $e$), the third one for the orders $3-2-1-4-5$ and $3-2-4-1-5$, and the fourth one for the orders $4-1-2-3-5$ and $4-1-3-2-5$.

Indeed, we thus improve agent 1's share of $b$ by $1/12$ whenever agent 1 prefers $a$ over $b$ and prefers both over $d$ and $e$. In an analogous way, we improve agent 1's share of $a$ by $1/12$ whenever agent 1 prefers $b$ over $a$ and prefers both over $d$ and $e$. Otherwise, whenever the preferences of agents 2, 3 and 4 are not as above or agent 1 prefers $d$ or $e$ over $a$ or $b$, let $g^{1-\underline{5}}(R)$ and $f^{\underline{5}}(R)$ coincide.

We verify strategy-proofness of $g^{1-\underline{5}}$. It is obvious that agents 2, 3, 4 and 5 cannot gain from manipulation as agents 2, 3 and 4 always obtain the same random assignment as under $f^{\underline{5}}$ and agent 5 gets the residual random assignment (which is independent of her preferences). For agent 1, the probability with which she is assigned her most preferred object remains identical under $g^{1-\underline{5}}$ and $f^{\underline{5}}$. Now if 2, 3 and 4 report $(R_2, R_3, R_4)$, then agent 1's increase for $b$ (relative to $f^{\underline{5}}$) under $R_1 : ab\ldots$ is identical to the increase of $a$ (relative to $f^{\underline{5}}$) under $R'_1 : ba\ldots$. If agent 1 ranks object $d$ or $e$ above either $a$ or $b$, then she obtains the same random assignment as under $f^{\underline{5}}$, which avoids the increase of the probability share of the less preferred object $a$ or $b$. In the Appendix we list for agent 1 all possible deviations and random assignments.

If we compare both mechanisms across all possible preference profiles, we find that, on average, agent $5$ receives her third ranked object[18] under both $f^{\underline{5}}$ and $g^{1-\underline{5}}$. In contrast, the average expected rank of objects received by agent $1$ is strictly higher under $g^{1-\underline{5}}$ than under $f^{\underline{5}}$ while the average expected rank for agents $2$ to $4$ is the same. Averaging over all agents, we thus find $g^{1-\underline{5}}$ to strictly improve the average expected rank of agents' assigned objects.

Obviously, both mechanisms treat agents differently. However, permuting the roles of agents in each mechanism and taking their convex combination allows us to recover equal-treatment-of equals (in fact even anonymity) for both. Since symmetrizing mechanisms in this way preserves the improvement in terms of average expected ranks, the symmetrized

[18]Averaging over all possible preferences of agent $5$, any given object is received with (average) rank $3$.

version of $g^{1-\underline{5}}$ is not welfare equivalent to the symmetrized version of $f^{\underline{5}}$, i.e., to RSD. In fact, in the Appendix we show that our symmetrized, constructed mechanism sd-dominates RSD at some profiles.

To generalize the result to more than five agents and objects, not necessarily with the same number of objects and agents, the first four agents play the same role as above and we let the remaining agents choose in a fixed order, and obtain the same conclusions.

One can even recover neutrality by permuting the names of the objects. Hence, Theorem 1 remains true when neutrality is added. We also show that for this conclusion it suffices to have at least five agents and at least five objects (but possibly with unequal numbers of agents and objects). We establish all this in the Appendix.

We finish with the observation that in the above construction, we may ignore agent 5 and consider assigning five objects among four agents. Then the constructed mechanism sd-dominates RSD, i.e., even though under RSD any agent is always assigned an object (or RSD has size one), RSD might be sd-dominated when there are more objects than agents. Erdil [2014] has studied in detail random assignment with outside options, i.e., where agents may be unassigned and may prefer this to certain objects. He showed that RSD may be sd-dominated where his constructed mechanism coincides with RSD when agents find all objects acceptable, i.e., on the acceptable domain.

## 4. NON-DOMINATION

In the constructed mechanism above, the random assignment of a certain object may depend on preferences over less preferred objects.[19] Hence, even though strategy-proofness ensures that an agent's probability share for a particular object is unaffected by changes to the order in which they rank less preferred objects, such changes may still affect the probability shares of this object for *other* agents. The following invariance condition, called bounded invariance, rules out such effects and may therefore be interpreted as a weak object-wise non-bossiness condition for strategy-proof mechanisms. It was first proposed

[19] For instance, for the above profile $R$ agent 1 receives $1/12$ more of object $b$ under $g_1^{1-\underline{5}}(R)$ compared to $f_1^{\underline{5}}(R)$, i.e., $g_{1b}^{1-\underline{5}}(R) \neq f_{1b}^{\underline{5}}(R)$. Now if we change $R_3$ to $R_3'$ : *cba*..., then for $R' = (R_3', R_{-3})$, we have $g_1^{1-\underline{5}}(R') = f_1^{\underline{5}}(R')$.

by Bogomolnaia and Heo [2012] who used it in conjunction with ex-ante efficiency and envy-freeness to characterize the probabilistic serial mechanism.[20]

Suppose the mechanism chooses a random assignment for given preference profile $R$. Pick an object $x$ and consider a profile $R'$ that differs from $R$ only in how objects are ranked below $x$. That is, for each agent, their ranking of all the objects until $x$ remains the same, and the rankings are altered arbitrarily after $x$. In particular, if an agent ranks $x$ as his last choice, nothing changes in this agent's ranking, and an agent ranking $x$ as their second choice should have the same first and second choice object, but can alter their rankings of the other objects. If a mechanism satisfies bounded invariance, then the random assignment of $x$ under $R$ and $R'$ should be identical.

DEFINITION 1: *Given $i \in N$, $R_i \in \mathcal{R}^i$ and $x \in O$, let $R_i(x) = R_i|B(x, R_i)$ denote the restriction of $R_i$ to the weak upper contour set of $x$. Now a mechanism $f$ satisfies bounded invariance (BI) if for all $R \in \mathcal{R}^N$, all $i \in N$, all $R'_i \in \mathcal{R}^i$ and all $x \in O$, if $R'_i(x) = R_i(x)$, then $f_x(R) = f_x(R'_i, R_{-i})$.*

Recall that a mechanism $f$ sd-dominates another mechanism $g$ if for any profile $R$ the random assignment $f(R)$ stochastically $R$-dominates the random assignment $g(R)$, and for some profile $\bar{R}$ and $i \in N$ we have $f_i(\bar{R}) \neq g_i(\bar{R})$.

THEOREM 2: *On the acceptable domain, if a mechanism $g$ satisfies ex-post efficiency, bounded invariance and strategy-proofness, then no boundedly invariant and strategy-proof mechanism sd-dominates $g$.*

RSD satisfies ex-post efficiency, bounded invariance and strategy-proofness – hence, by Theorem 2, RSD is not sd-dominated by any mechanism satisfying bounded invariance and strategy-proofness. The same is true for weighted versions of RSD, i.e., where we attach different weights to different orders of agents and apply SD. Such weights could take into

[20] Bogomolnaia and Heo [2012] weakened and unified stronger invariance conditions of two previous papers characterizing the probabilistic serial mechanism, which were merged to Hashimoto et al. [2014]. In the final version the latter article further weakened bounded invariance to weak invariance in this characterization, a property which is satisfied by any strategy-proof mechanism. The set of all ex-ante efficient, strategy-proof, non-bossy, neutral, and boundedly invariant mechanisms has recently been characterized by Alva et al. [2024].

account minorities/majorities and (dis)advantaged groups.[21] Furthermore, in Theorem 2 ex-post efficiency cannot be weakened to ex-post weak efficiency. For instance, the Random-Dictatorship-cum-Equal-Division[22] by Basteck and Ehlers [2023] satisfies ex-post weak efficiency, bounded invariance and strategy-proofness, but is sd-dominated by RSD.

Several questions remain. First, does Theorem 2 remain true when we drop bounded invariance as a requirement on the second mechanism, i.e., keep bounded invariance only for the first mechanism, thus strengthening the implication? Second, could we drop bounded invariance as a requirement on the first mechanism, weakening Theorem 2's premise? Third, is RSD characterized by ex-post efficiency, bounded invariance, strategy-proofness and symmetry?

We provide an outline of the proof of Theorem 2. As a basic step we show that for any efficient deterministic assignment, any agent must rank his allocated object weakly above some non-top ranked object. Then for a fixed object, say $z$, we count for any profile and for any agent the number of non-top ranked objects below $z$, and consider lexicographic minimization with respect to those numbers. If $g$ sd-dominates $f$, then the set of profiles where $f$ and $g$ differ is non-empty. Now in this set we choose a profile where object $z$ is ranked as low as possible with respect to the minimization outlined above and show that the random assignment of $z$ must coincide for $f$ and $g$. Remaining in the set of profiles where $f$ and $g$ differ and $z$ is ranked as low as possible, we take another object, say $y$, choose a profile where $y$ is ranked as low as possible and show that the random assignment of $y$ (and $z$) is identical for $f$ and $g$. Iterating we eventually exhaust the set of objects and obtain that $f$ and $g$ coincide, which implies that the set of profiles where $f$ and $g$ differ was empty yielding the final contradiction.

REMARK 1: When agents may rank objects as unacceptable, several contributions have considered Pareto domination among strategy-proof and individually rational mechanisms, and showed that then the size of a mechanism matters, i.e., in the context of random assignment the aggregate probability of any agent being assigned a real object. Erdil [2014]

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[21] E.g., if objects are to be assigned to a set of agents comprised of two groups, one might consider orders that alternate between members of the two subgroups and randomize within each.

[22] We omit the formal definition and refer to Basteck and Ehlers [2023]. Informally, the mechanism works as follows: any agent $i$ is chosen with probability $\frac{1}{n}$, then agent $i$ picks his most preferred object and the remaining objects are assigned uniformly among the other agents.

showed for the random assignment model that when a strategy-proof mechanism Pareto dominates another strategy-proof and individually rational mechanism, then the former mechanism has to be of greater size than the latter one. Alva and Manjunath [2019] considered the same question for deterministic mechanisms in a general model and Zhang [2023] considered its random variant. The main difference here is that all objects are acceptable and the size of any ex-post efficient mechanism is always identical (as all objects are always assigned), and these results do not apply to our context.

## 5. CONCLUSION

Numerous contributions establish the impossibility of strategy-proofness, envy-freeness and ex-ante efficiency. In the ordinal framework, Bogomolnaia and Moulin [2001] establish the impossibility result where envy-freeness is weakened to equal-treatment-of-equals. Nesterov [2017] shows that the impossibility persists when ex-ante efficiency is weakened to ex-post efficiency (while maintaining envy-freeness).[23] Shende and Purohit [2023] show that strategy-proofness and envy-freeness are incompatible with unanimity[24] (which they refer to as contention-free efficiency), a significant weakening of ex-post efficiency. Further, Basteck and Ehlers [2023] show that a strategy-proof and envy-free mechanism is ex-post unanimous with probability of at most $\frac{2}{n}$ (where $n$ is the number of agents). In other words, for any strategy-proof and envy-free mechanism there exist preference profiles where the unique ex-post efficient assignment is chosen with probability of at most $\frac{2}{n}$ (and inefficient assignments are chosen with probability of at least $1 - \frac{2}{n}$). This finding strengthens significantly the incompatibility of strategy-proofness, envy-freeness and ex-post efficiency and provides an exact upper bound for ex-post unanimity.[25]

Instead of reporting ordinal preferences, one might ask agents to report cardinal utility functions, assuming that they evaluate random assignments according to their expected utilities. We implicitly assume ordinality of mechanisms, i.e., constrain random assignments

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[23] Zhang [2019] proves a strong group-manipulability result, imposing ex-post efficiency and auxiliary fairness axioms that are by themselves weaker than envy-freeness.

[24] Unanimity requires that whenever all agents consider a different object most-preferred, each should receive their most-preferred object. In other words, whenever there is a unique Pareto efficient assignment, it is chosen with probability one.

[25] For the assignment of one object, Ehlers [2002] characterized the uniform random dictatorship mechanism by ex-post efficiency, envy-freeness and strategy-proofness.

to be the same across cardinal utility profiles which induce identical ordinal preferences. For applications ordinality is a natural requirement as it facilitates reporting, when agents are unable to determine their exact utilities but are able to compare individual objects. Of course, allowing cardinal reports but imposing ordinality yields the same result as imposing ordinal preference reports. In particular, in such contexts RSD is not dominated by any mechanism satisfying ordinality, strategy-proofness and bounded invariance. This is a positive answer on the acceptable domain and addresses a question raised by Zhou [1990], who showed that in the cardinal framework no mechanism satisfies equal treatment of equals, strategy-proofness and ex-ante efficiency. The latter postulates always to choose a random assignment which is not Pareto dominated in terms of expected utility by any other. It is clear that in the cardinal context the properties of ordinality, equal treatment of equals and ex-ante efficiency are incompatible: as a simple example, let $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, $u_1 = (u_{1a}, u_{1b}, u_{1c}) = (1, \epsilon, 0) = u_2$ and $u_3 = (1, 1 - \epsilon, 0)$ where $\epsilon > 0$ is small; when all agents have utility function $u_3$ equal treatment of equals requires each agent to obtain $a$ with probability one third, and similarly, when all agents have utility function $u_1$ equal treatment of equals requires each agent to obtain $c$ with probability one third; now ordinality requires for the profile $(u_1, u_2, u_3)$ that each agent obtains any object with probability $\frac{1}{3}$, which is dominated in terms of expected utility by assigning agent 3 object $b$ with probability one, and assigning agents 1 and 2 objects $a$ and $c$ each with probability one half.

The last example shows the disrelation of Zhou's result and the impossibility results in the ordinal framework with respect to efficiency, equity and strategy-proofness. Ordinality, sd-efficiency and envy-freeness are compatible as PS satisfies all those properties. As soon as strategy-proofness is added, we obtain an incompatibility, which is robust when weakening sd-efficiency to ex-post efficiency, or envy-freeness to equal treatment of equals.

Theorem 1 invalidated the conjecture that RSD was characterized by ex-post efficiency, equal treatment of equals and strategy-proofness.[26][27] Another strand of the literature studies large markets. In particular one may enlarge markets in two different ways: either by

[26] Pycia and Troyan [2023] provided an earlier weaker counterexample whereby an ex-post efficient random assignment may be welfare-equivalent but not identical to the RSD random assignment. For instance, for three agents and three objects, if all agents have identical preferences, then we may choose each of the allocations $\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$ with probability $\frac{1}{3}$ whereas RSD chooses any allocation with probability $\frac{1}{6}$.

[27] Brandt et al. [2024] analyze characterizing RSD via linear algebra and computational methods.

keeping the set of object types fixed and adding copies to match an increasing number of agents, or by considering a sequence of economies where the number of distinct agents and the number of distinct objects grow at the same rate. First, when we add object copies, Liu and Pycia [2016, Theorem 2] have shown that any two symmetric and "regular"[28] mechanisms, which are asymptotically strategy-proof and asymptotically efficient, coincide asymptotically, i.e., they choose the same allocations in the limit. For instance, this implies asymptotic coincidence of RSD[29] and PS (which was first shown by Che and Kojima [2010]), and that RSD and, respectively, PS satisfy ex-post efficiency and asymptotically both strategy-proofness and envy-freeness. In some sense, then, it does not matter in the large whether we choose RSD or PS (or any other mechanism satisfying the above three properties). One of the earliest contributions to identify a link between RSD and PS is Kesten [2009] who establishes that PS is the average of RSD in the large where both any object and any priority order are replicated sufficiently many times.[30] However when we consider economies with a large number of distinct agents and distinct objects,[31] Manea [2009] has shown that RSD is sd-efficient with probability approaching zero, and hence RSD and PS diverge with probability one. Thus, continued discussions in real-life markets show the importance of the choice of the random assignment mechanism to be implemented. As we have shown, RSD cannot be improved in an unambiguous way while maintaining our two basic properties.

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[28] Loosely speaking, this means that agents cannot change to "too much" the random assignments of other agents (in terms of probability shares) as the market becomes large.

[29] RSD is regular, provided the number of copies for each object type grows at the same rate as the number of agents, e.g., in replica economies.

[30] More precisely, for any $k$ and any priority order of agents Kesten [2009] considers the economy with $k$ copies of each object with agents choosing objects according to the priority order repeated $k$ times. Dividing by $k$ afterwards and averaging over all possible priority orders we obtain a random allocation for each agent in the original economy. If $k$ becomes sufficiently large, then this 'random repeated serial dictatorship' is asymptotically equivalent to PS.

[31] E.g., in a school choice context, this would describe a scenario where the number of applicants and schools grows, but the capacity of individual schools is bounded.

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# APPENDIX.

## APPENDIX A: GENERAL VERSION OF THEOREM 1

Below we allow for the possibility of unequal numbers of agents and objects. With fewer objects than agents, some agents may remain unassigned. All our definitions then extend in a straightforward way. Theorem 1 is a corollary from this more general result.

We further strengthen symmetry to anonymity where agents' names are treated equally: a mechanism $f$ satisfies *anonymity* if for any permutation $\pi : N \to N$ of agents and for any profile $R$, we have $f_i(R) = f_{\pi(i)}((R_{\pi(i)})_{i\in N})$.

In addition, the constructed mechanism is immune to renaming objects: a mechanism $f$ satisfies *neutrality* if for any permutation $\tau : O \to O$ of objects and for any profile $R$, we define $\tau(R) = (\tau(R_i)_{i\in N}) \in \mathcal{R}^N$ such that for $o, o' \in O$, $\tau(o)\tau(R_i)\tau(o')$ iff $oR_io'$, and we have for all $i \in N$ and all $o \in O$, $f_{io}(R) = f_{i\tau(o)}(\tau(R))$.

THEOREM 3: *For $|N| \geq 5$ and $|O| \geq 5$, there exist mechanisms satisfying ex-post efficiency, anonymity, neutrality and strategy-proofness, which are not welfare equivalent to random serial dictatorship.*

**Proof.** We begin with five agents and five objects, i.e., let $N = \{1,\ldots,5\}$ and $O = \{a,b,c,d,e\}$.

First, we define the following mechanism $f^{\underline{5}}$ whereby agents 1, 2, 3 and 4 are ranked arbitrarily and choose in that order (as in RSD for four agents) and afterwards agent 5 receives the remaining object. Making it symmetric for agents (by choosing $f^{\underline{i}}$ with probability $\frac{1}{5}$) we get back RSD (as then any order is chosen with equal probability $\frac{1}{5!}$), i.e., $RSD = \frac{1}{5}\sum_{i\in N} f^{\underline{i}}$.

Second, let $R_2, R_3, R_4$ be as follows

| $R_2$ | $R_3$ | $R_4$ |
|---|---|---|
| $c$ | $c$ | $c$ |
| $a$ | $b$ | $e$ |
| $d$ | $d$ | $d$ |
| $e$ | $e$ | $a$ |
| $b$ | $a$ | $b$ |

It will turn out to be crucial that the same object $d$ is ranked below $c$ and $a$ for agent 2 and below $c$ and $b$ for agent 3, and that agent 4 ranks the different object $e$ below $c$. We define the mechanism $g^{1-\underline{5}}$ whereby agent 1 is improved over $f^{\underline{5}}$, agents 2, 3 and 4 receive identical random assignments under $g^{1-\underline{5}}$ and $f^{\underline{5}}$, and agent 5 is worse off or better off under $g^{1-\underline{5}}$ compared to $f^{\underline{5}}$.[32]

We decompose the preference domain for agent 1 as the disjoint union of the following three sets:

$$\mathcal{R}^1_a = \{R_1 : aP_1bP_1x \text{ for all } x \in O\backslash\{a,b,c\}\}$$
$$\mathcal{R}^1_b = \{R_1 : bP_1aP_1x \text{ for all } x \in O\backslash\{a,b,c\}\}$$
$$\hat{\mathcal{R}}^1 = \{R_1 : \text{ there exists } x \in O\backslash\{a,b,c\} \text{ such that } xP_1a \text{ or } xP_1b\}.$$

[32] Strictly speaking, $g^{1-\underline{5}}$ shall also make reference to the preferences $(R_2,R_3,R_4)$, for instance by denoting it $g^{1-(R_2,R_3,R_4)-\underline{5}}$. For ease of notation we write $g^{1-\underline{5}}$ while fixing $(R_2,R_3,R_4)$.

For all $Q \in \mathcal{R}^N$, let

$$g_i^{1-\underline{5}}(Q) = f_i^{\underline{5}}(Q) \text{ for } i = 2, 3, 4. \quad (1)$$

Moreover, for any profile $Q$, if $Q_{-15} \neq (R_2, R_3, R_4)$ or $Q_1 \in \hat{\mathcal{R}}^1$, then $g^{1-\underline{5}}(Q) = f^{\underline{5}}(Q)$.

Otherwise, suppose $Q_1 \in \mathcal{R}_a^1 \cup \mathcal{R}_b^1$ and $Q_{-15} = (R_2, R_3, R_4)$. Note that once we have defined $g_1^{1-\underline{5}}(Q)$, then $g_5^{1-\underline{5}}(Q)$ is the residual given (1).

If $Q_1 \in \mathcal{R}_a^1$, then under $f^{\underline{5}}(Q)$ agent 1 receives her most preferred object from $O\backslash\{a, b, c\}$ for the orders 4-2-3-1-5 and 4-3-2-1-5 (where any such order is chosen with probability $1/24$), i.e., with probability $1/12$. Similarly, object $b$ is assigned to agent 5 for the orders $3-1-4-2-5$ and $3-1-2-4-5$, i.e., at least with probability $1/12$.

Then let $g_1^{1-\underline{5}}(Q) = f_1^{\underline{5}}(Q) + 1/12b - 1/12x$ where $x$ is 1's most preferred object from $O\backslash\{a, b, c\}$. Note that then $g_1^{1-\underline{5}}(Q)$ strictly sd-improves agent 1 over $f_1^{\underline{5}}(Q)$ (and if $Q_5 : be\ldots$, then $f_5^{\underline{5}}(Q)$ strictly sd-dominates $g_5^{1-\underline{5}}(Q)$, i.e., agent 5 is unambiguously worse off, and if $Q_5 : \ldots b$, then $g_5^{1-\underline{5}}(Q)$ strictly sd-dominates $f_5^{\underline{5}}(Q)$, i.e., agent 5 is unambiguously better off).

Analogously, if $Q_1 \in \mathcal{R}_b^1$, then let $g_1^{1-\underline{5}}(Q) = f_1^{\underline{5}}(Q) + 1/12a - 1/12x$ where $x$ is 1's most preferred object from $O\backslash\{a, b, c\}$.

It remains to show that $g^{1-\underline{5}}$ is ex-post efficient and strategy-proof.

For ex-post efficiency of $g^{1-\underline{5}}$, it is crucial that the same object $d$ is ranked below $c$ and $a$ for agent 2 and below $c$ and $b$ for agent 3, and that agent 4 ranks the different object $e$ below $c$. Improving 1's assignment when $R_1 \in \mathcal{R}_a^1$ involves increasing her share of $b$ by $1/12$, while holding unchanged the assignment of agents 2, 3, and 4. This can be done by replacing the assignments on the left below with the assignments on the right with probability share $1/12$ each. Each assignment on the left is realized with probability $1/12$ or more, therefore this improvement is indeed feasible. As the assignments on the right are efficient, ex-post efficiency is preserved (where $\{d, e\} = \{x, y\}$ and $xP_1y$).

$$\begin{pmatrix} 1\,2\,3\,4\,5 \\ x\,a\,b\,c\,y \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4\,5 \\ a\,d\,c\,e\,b \end{pmatrix} \rightarrow \begin{pmatrix} 1\,2\,3\,4\,5 \\ b\,a\,c\,e\,d \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4\,5 \\ a\,d\,b\,c\,e \end{pmatrix}$$

Note that the first assignment is obtained for the orders $4-2-3-1-5$ and $4-3-2-1-5$, the second one for the orders $3-1-2-4-5$ and $3-1-4-2-5$, the third one for the orders $3-2-1-4-5$ and $3-2-4-1-5$, and the fourth one for the orders $4-1-2-3-5$ and $4-1-3-2-5$.

The argument for the case when $R_1 \in \mathcal{R}_b^1$ is analogous but for completeness we verify it below. Improving 1's assignment when $R_1 \in \mathcal{R}_b^1$ involves increasing her share of $a$ by $1/12$, while holding unchanged the assignment of agents 2, 3, and 4. This can be done by replacing the assignments on the left below with the assignments on the right with probability share $1/12$ each. Each assignment on the left is realized with probability $1/12$ or more, therefore this improvement is indeed feasible. As the assignments on the right are efficient, ex-post efficiency is preserved (where $\{d,e\}=\{x,y\}$ and $xP_1y$).

$$\begin{pmatrix} 1\,2\,3\,4\,5 \\ x\,a\,b\,c\,y \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4\,5 \\ b\,c\,d\,e\,a \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,4\,5 \\ a\,c\,b\,e\,d \end{pmatrix} + \begin{pmatrix} 1\,2\,3\,4\,5 \\ b\,a\,d\,c\,e \end{pmatrix}$$

The first assignment is obtained for the orders $4-2-3-1-5$ and $4-3-2-1-5$, the second one for the orders $2-1-3-4-5$ and $2-1-4-3-5$, the third one for the orders $2-3-1-4-5$ and $2-3-4-1-5$, and the fourth one for the orders $4-1-2-3-5$ and $4-1-3-2-5$.

Finally, we verify strategy-proofness of $g^{1-\underline{5}}$. It is obvious that agents 2, 3, 4 and 5 cannot gain from manipulation – it remains to verify that $g_1^{1-\underline{5}}(Q)$ stochastically $Q_1$-dominates $g_1^{1-\underline{5}}(R_1,Q_{-1})$ for all $Q \in \mathcal{R}^N$ and all $R_1 \in \mathcal{R}^1$. By Lemma 2 of Gibbard [1977] it suffices to consider pairwise switches of objects ranked adjacently, i.e., compare the random assignment of agent 1 when reporting $Q_1$ and when reporting $R_1 = Q_1^{y\leftrightarrow z}$ for two objects $y,z \in O$ ranked adjacent in $Q_1$.[33]

If $Q_{-15} \neq (R_2,R_3,R_4)$, then agent 1 cannot gain from manipulation as $f^{\underline{5}}$ is strategy-proof while $g_1^{1-\underline{5}}(Q) = f_1^{\underline{5}}(Q)$ and $g_1^{1-\underline{5}}(R_1,Q_{-1}) = f_1^{\underline{5}}(R_1,Q_{-1})$ for all $R_1$. Thus, let $Q_{-15} = (R_2,R_3,R_4)$.

Suppose that $Q_1 \in \mathcal{R}_a^1$ and $R_1 \in \mathcal{R}_a^1 \cup \mathcal{R}_b^1$. Note that by ex-post efficiency and as agent 5 chooses last, agent 1 receives object $c$ with probability zero when $c$ is not ranked first in $Q_1$.

[33] If $R_1 : uvwyzx$, then $Q_1 : uvwzyx$.

Suppose $c$ is not ranked first in $Q_1$. Then by construction $g_1^{1-\underline{5}}(Q) = (18a + 6b)/24$. If $c$ is not ranked first under $R_1$ as well, then 1's random assignment is either unchanged when reporting $R_1$ instead of $Q_1$ or changed to $(18b + 6a)/24$. If instead $c$ is now ranked first under $R_1$, then $g_1^{1-\underline{5}}(R_1, Q_{-1}) = (6c + 12a + 6b)/24$. The case where $c$ is ranked first in $Q_1 \in \mathcal{R}_a^1$ is analysed analogously.

Suppose that $Q_1 \in \mathcal{R}_a^1$ and $R_1 \in \hat{\mathcal{R}}^1$. Then $g_1^{1-\underline{5}}(R_1, Q_{-1}) = f_1^{\underline{5}}(R_1, Q_{-1})$. But then $g_1^{1-\underline{5}}(Q)$ by construction stochastically $Q_1$-dominates $f_1^{\underline{5}}(Q)$, which, by strategy-proofness of $f^{\underline{5}}$, stochastically $Q_1$-dominates $f_1^{\underline{5}}(R_1, Q_{-1}) = g_1^{1-\underline{5}}(R_1, Q_{-1})$. Thus, reporting $R_1$ leads to a $Q_1$-dominated random assignment.

This completes the analysis for $Q_1 \in \mathcal{R}_a^1$. The case $Q_1 \in \mathcal{R}_b^1$ is analysed analogously.

Suppose $Q_1 \in \hat{\mathcal{R}}^1$. If $R_1 \in \hat{\mathcal{R}}^1$ then $g_1^{1-\underline{5}}$ and $f_1^{\underline{5}}$ coincide both at $Q$ and at $(R_1, Q_{-1})$ – hence strategy-proofness of $f^{\underline{5}}$ implies that $g_1^{1-\underline{5}}(Q)$ stochastically $Q_1$-dominates $f_1^{\underline{5}}(R_1, Q_{-1})$. If $R_1 \in \mathcal{R}_a^1$, then the pairwise swap of two objects must have involved $b$ (ranked below $a$ under both $Q_1$ and $R_1$) and the preferred object in $\{d, e\}$, i.e., object $x \in \{d, e\}$ such that $xQ_1d$, $xQ_1e$, and $R_1 = Q_1^{b \leftrightarrow x}$. By strategy-proofness of $f^{\underline{5}}$ and the construction of $g^{1-\underline{5}}$, when reporting $R_1$ instead of $Q_1$, this only moves probability mass from $x$ to $b$ – hence $g_1^{1-\underline{5}}(Q)$ stochastically $Q_1$-dominates $f_1^{\underline{5}}(R_1, Q_{-1})$. Again, the case $R_1 \in \mathcal{R}_b^1$ is analysed analogously.

Below, for completeness, we list all possible misreports $R_1$, derived from $Q_1$ by a pairwise swap given that $Q_1 \in \mathcal{R}_a^1 \cup \mathcal{R}_b^1$. For this, we list the first three objects in $R_1$ where $x$ denotes the preferred object among $\{d, e\}$, i.e., $x \in \{d, e\}$, $xR_1d$, and $xR_1e$. Note that $x$ cannot be ranked at the top of $R_1$ (as otherwise $x$ would have been ranked above $a$ or $b$ under $Q_1$, a contradiction to $Q_1 \in \mathcal{R}_a^1 \cup \mathcal{R}_b^1$), and similarly, $x$ cannot be ranked second while $c$ is ranked third in $R_1$:

If $aP_1bP_1x$ or $aP_1cP_1b$, then her assignment is $(18a + 6b)/24$.

If $bP_1aP_1x$ or $bP_1cP_1a$, then her assignment is $(18b + 6a)/24$.

If $cP_1aP_1b$, then her assignment is $(6c + 12a + 6b)/24$.

If $cP_1bP_1a$, then her assignment is $(6c + 12b + 6a)/24$.

If $aP_1xP_1b$, then her assignment is $(18a + 4x + 2b)/24$.

If $aP_1cP_1x$, then her assignment is $(18a + 6x)/24$.

If $bP_1cP_1x$, then her assignment is $(18b + 6x)/24$.

If $bP_1xP_1a$, then her assignment is $(18b + 4x + 2a)/24$.

If $cP_1aP_1x$, then her assignment is $(6c + 12a + 6x)/24$.

If $cP_1bP_1x$, then her assignment is $(6c + 12b + 6x)/24$.

Under the first four announcements agent 1 receives objects $d$ and $e$ with probability zero, and at each announcement agent 1 receives with probability one his first three objects. A straightforward pairwise comparison of these ten outcomes verifies that at each preference ranking $Q_1 \in \mathcal{R}^1_a \cup \mathcal{R}^1_b \cup \hat{\mathcal{R}}^1$, truthful revelation (weakly or strongly) first-order stochastically dominates untruthful revelation.

Third, the mechanism $g^{1-\underline{5}}$ treats agent 1 differently in comparison to agents 2, 3 and 4. In order to recover equal treatment of equals among agents 1, 2, 3, and 4, we again appeal to randomization.

Let $\pi$ be a permutation of the agents with agent 5 staying put, i.e., let $\pi : N \to N$ be a bijection such that $\pi(5) = 5$. Then $\pi(g^{1-\underline{5}})$ is defined via changing the roles of the agents in mechanism $g^{1-\underline{5}}$ according to the permutation $\pi$. Denoting with $\Pi^5$ the set of all permutations of $N$ where agent 5 stays put, we define $h^{\underline{5}}$ as

$$h^{\underline{5}} = \frac{1}{4!} \sum_{\pi \in \Pi^5} \pi(g^{1-\underline{5}}).$$

Then $h^{\underline{5}}$ inherits ex-post efficiency and strategy-proofness from $g^{1-\underline{5}}$, and agents 1, 2, 3, and 4 are treated symmetrically. Note also that $\sum_{\pi \in \Pi^5} \pi(f^{\underline{5}}) = f^{\underline{5}}$, and hence agents 1, 2, 3 and 4 are better off under $h^{\underline{5}}$ and agent 5 is worse off or better off under $h^{\underline{5}}$ (when compared to $f^{\underline{5}}$).

Fourth, in order to recover anonymity (and, respectively, symmetry) completely, let $h = \frac{1}{5} \sum_{i \in N} h^{\underline{i}}$. Again, $h$ inherits ex-post efficiency and strategy-proofness from $h^{\underline{i}}$, and satisfies anonymity and symmetry.

Fifth, we show that $h$ does not coincide with RSD, i.e., there exists a profile $Q$ such that $h_i(Q) \neq RSD_i(Q)$ for some $i \in N$. Thus, there exist other random mechanisms (in terms of probability shares) satisfying ex-post efficiency, symmetry and strategy-proofness. Let $Q \in \mathcal{R}^N$ be such that $Q_1 : abecd$, $Q_{-15} = (R_2, R_3, R_4)$ and $Q_5 : edcab$.

Then $g_1^{1-\underline{5}}(Q) = (18a + 6b)/24 \neq (18a + 4b + 2e)/24 = f_1^{\underline{5}}(Q)$ (and correspondingly $g_5^{1-\underline{5}}(Q) = f_5^{\underline{5}}(Q) + (2e - 2b)/24$). Now let us consider $g^{i-\underline{j}}(Q)$ and $f^{\underline{j}}(Q)$, i.e., where we permute the roles of agents. For $i = 5$ and $j = 1$ the two coincide, since agent $5$ does not rank $a$ and $b$ over $d$ and $e$. Finally, if $\{i, j\} \neq \{1, 5\}$ then again $g^{i-\underline{j}}(Q) = f^{\underline{j}}(Q)$ since either $1$ or $5$ is now in the role of agents $2, 3$, or $4$ in $g^{1-\underline{5}}$ but neither $1$ nor $5$ ranks $c$ first.

But now it follows that $h_{1e}(Q) \neq RSD_{1e}(Q)$, and $h(Q)$ stochastically $Q$-dominates $RSD(Q)$ (as agents 1 and 5 are better off and agents 2, 3 and 4 receive identical random assignments).[34]

Finally, we show that neutrality can be recovered from the above mechanism. Let $\tau : O \to O$ be a renaming of the objects and denote by $\Gamma$ the set of all such bijections. Let $\tau(h)$ denote the permuted mechanism where the names of the objects in $h$ are changed according to $\tau$, and let $\overline{h} = \frac{1}{5!}\sum_{\tau\in\Gamma}\tau(h)$. But then $\overline{h}$ inherits all the properties from $h$ and satisfies neutrality. Furthermore, for the above profile, we continue to have $\overline{h}_{1e}(Q) \neq RSD_{1e}(Q)$.

Lastly, suppose that there are at least five agents and at least five objects, i.e., $N = \{1, \ldots, n\}$ with $n \geq 5$ and $O = \{a, b, c, d, e, o_1, \ldots, o_{|O|-5}\}$ with $|O| \geq 5$ where possibly $|N| \neq |O|$. We then define the mechanism $f^{\underline{56\cdots n}}$ by letting choose agents $1 - 4$ in a random order and then the remaining agents in the order $5 - 6 - \cdots - n$. Again permuting gives us back RSD. For the mechanism $g^{1-\underline{56\cdots n}}$ we add to the preferences $R_2$, $R_3$ and $R_4$ the other objects in the same order $o_1 - \cdots - o_{|O|-5}$ at the bottom. Now for agent 1 an improvement is applied under the same conditions (where no $x \in O\backslash\{a, b, c\}$ shall be ranked above $a$ or $b$). One can check again strategy-proofness and ex-post efficiency, and make the mechanism symmetric. In showing that the new mechanism does not coincide with RSD, let the preferences of agents 1-5 be as above in profile $Q$ with the other objects being ranked in the same order $o_1 - \cdots - o_{|O|-5}$ at the bottom and let agent $i$ (with $i \geq 5$) have the same preference as agent 5. Note that then the above improvement is applied for agent 1 in the mechanism $f^{\underline{56\cdots n}}$ but when at least one agent $i$ with $i \geq 5$ plays the role

[34] Note that under profile $Q$, agent 5 ranks the (leftover) objects $d$ and $e$ first and second, and after permuting the names of objects agent 5 continues to rank at least one (leftover) object first or second, i.e., no Pareto improvement is applied when agent 1 is last, independently whether 5 is first or not and 1 continues to get the same share of $e$ under both mechanisms. When agent 1 is first and a Pareto improvement is applied, either $e$ is not a leftover object anymore and her share of $e$ weakly increases whereas when $e$ is a (leftover) object, then agent 1 receives zero probability share of $e$. If no Pareto improvement is applied or agent 1 is neither last nor first, then agent 1 continues to get the same share of $e$ under both mechanisms.

of agent 2, 3 or 4, then no improvement can be applied as $i$ ranks $e$ at the top. Finally, neutrality can be recovered as above. □

## APPENDIX B: PROOF OF THEOREM 2

We begin by introducing some additional notation. Given $R_i \in \mathcal{R}^i$, let $top(R_i) \in O$ denote the top-ranked object in $O$ according to $R_i$, i.e., $top(R_i)R_i x$ for all $x \in O$. For a subset $I' \subseteq N$, let $top(R_{I'}) = \cup_{i \in I'}\{top(R_i)\}$ and denote the set of objects top-ranked by some $i \in N$ by $top(R) = \cup_{i \in N}\{top(R_i)\}$. Conversely, let $\overline{top}(R) = O \backslash top(R)$ denote the set of objects which are not top-ranked by any $i \in N$.

If all agents rank a different object at the top, i.e., if $top(R) = O$, Pareto efficiency requires that each agent receives their top-ranked object. Our first lemma concerns an implication of efficiency when preferences are at least partially in conflict, i.e., if $top(R) \neq O$ – top-ranked objects will not be assigned to agents who rank them at the bottom, i.e., below non-top ranked objects.

LEMMA 1: *Consider any ex-post efficient mechanism $g$ and any preference profile $R \in \mathcal{R}^N$ such that $top(R) \neq O$. Then for all $i \in N$ and all $y \in top(R)$ such that $xP_iy$ for all $x \in \overline{top}(R)$, we have $g_{iy}(R) = 0$. Moreover, for any mechanism $f$ such that $f \rhd^{sd} g$, we also have $f_{iy}(R) = 0$.*

**Proof.** Towards a contradiction, assume there exists an $i \in N$ and $y \in top(R) \subsetneq O$, such that $g_{iy}(R) > 0$ while for all $x \in \overline{top}(R)$ we have $xP_iy$. Since $g$ is ex-post efficient, there exists $\mu \in \mathcal{PO}(R)$ such that $\mu_i = y$.

Since $top(R_i)P_ixP_i\mu_i$ for all $x \in \overline{top}(R)$, we have $top(R_i) \neq \mu_i$ – and since $\mu_i \in top(R)$, there must be another agent, $j \in N\backslash\{i\}$, for whom $top(R_j) = \mu_i$.

But then $top(R_j) \neq \mu_j$. By efficiency, $\mu_j \in top(R)$ – as otherwise $\mu_jP_i\mu_i$ and $\mu_i = top(R_j)P_j\mu_j$, creating a possible trading cycle where all included agents become strictly better off. Thus, there must be another agent, $k \in N\backslash\{i,j\}$, for whom $top(R_k) = \mu_j$.

But then $top(R_k) \neq \mu_k$. By efficiency, $\mu_k \in top(R)$ – as otherwise $\mu_kP_i\mu_i$, $\mu_iP_j\mu_j$, and $\mu_jP_k\mu_k$, creating a possible trading cycle where all included agents become strictly better off. Thus there must be another agent, $l \in N\backslash\{i,j,k\}$ for whom $top(R_l) = \mu_j$.

Continue in this way. Since $N$ is finite, we eventually arrive at a contradiction once we have exhausted $N$. This establishes the first part of the lemma: no agent receives objects ranked below their least preferred object in $\overline{top}(R)$ under $g$. A fortiori, the same needs to hold for any mechanism $f$ which stochastically dominates $g$. □

Suppose now that mechanism $g$ satisfies ex-post efficiency, bounded invariance and strategy-proofness. Towards a contradiction, assume there exists a bounded invariant and strategy-proof mechanism $f$ such that $f \rhd^{sd} g$. In particular, this implies that there is a non-empty set of preference profiles where $f$ and $g$ are not welfare-equivalent. Let $\mathcal{R}_0^{\neq}$ denote this set, i.e.,

$$\mathcal{R}_0^{\neq} = \{R \in \mathcal{R}^N : f_i(R) \neq g_i(R) \text{ for some } i \in N\}.$$

To prove Theorem 2 by contradiction, we will show that $\mathcal{R}_0^{\neq} = \emptyset$. For this, we will consider an arbitrary sequence of objects $z_1, z_2, \ldots, z_n \in O$ along with a decreasing sequence of subsets of preference profiles

$$\mathcal{R}_0^{\neq} \supseteq \mathcal{R}_1^{\neq} \supseteq \mathcal{R}_2^{\neq} ... \supseteq \mathcal{R}_n^{\neq}$$

where (i) for each $k = 1, ..., n$, $\mathcal{R}_{k-1}^{\neq} \neq \emptyset$ implies $\mathcal{R}_k^{\neq} \neq \emptyset$ while (ii) $f_{z_l}(R) = g_{z_l}(R)$ for all $l \leq k$ and all $R \in \mathcal{R}_k^{\neq}$. This way, (i) implies $\mathcal{R}_n^{\neq} \neq \emptyset$ (given $\mathcal{R}_0^{\neq} \neq \emptyset$), while (ii) implies that for $R \in \mathcal{R}_n^{\neq}$ all objects $z_1, ..., z_n$ have to be assigned with the same assignment probabilities under $f$ and $g$ – i.e., $\mathcal{R}_n^{\neq} = \emptyset$.

Intuitively, each $\mathcal{R}_k^{\neq}$ is the set of preference profiles where $z_k$ is ranked as low as possible by all agents (relative to objects in $\overline{top}(R)$), subject to the constraint that $\mathcal{R}_k^{\neq} \subseteq \mathcal{R}_{k-1}^{\neq} \subseteq ... \subseteq \mathcal{R}_0^{\neq}$, i.e., subject to preserving a difference between $f$ and $g$ and subject to ranking the preceding objects $z_{k-1}, z_{k-2} ..., z_1$ as low as possible. Moreover, note that for all preference profiles in $\mathcal{R}_0^{\neq}$, and hence also for all profiles in $\mathcal{R}_k^{\neq}$, we have $top(R) \neq O$ as otherwise ex-post efficiency requires that all agents receive their top ranked object with probability one so that $f(R) = g(R)$.

To make this precise and define the sets $\mathcal{R}_k^{\neq}$ formally, let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of natural numbers including zero. Let $\mathbb{N}_{\geq}^{|N|}$ denote the set of all vectors $v \in \mathbb{N}^{|N|}$ such that $v_1 \geq v_2 \geq \cdots \geq v_{|N|}$, i.e., the coordinates of $v$ are arranged in non-increasing order. Let $\precsim$

denote the lexicographical ordering on $\mathbb{N}_{\geq}^{|N|}$: for all $v,w \in \mathbb{N}_{\geq}^{|N|}$, $v \precsim w$ means either $v = w$ or there is $1 \leq t \leq n$, such that $v_t < w_t$ and $v_i = w_i$ for every $i < t$ and . We write $v \prec w$ if $v \precsim w$ and $w \neq v$.

Furthermore, for any $z \in O$ and $R_i$, let

$$L(z, R_i) = \{y \in O : zP_iy\}$$

denote the strict lower contour set of $z$ at $R_i$. Note that this set excludes $z$.

Next, let $O = \{z_1, \ldots, z_n\}$, define $O_{\geq t} = \{z_t, \ldots, z_n\}$, for any $1 \leq t < n$, and for any $i \in N$ let

$$\rho_i(z_t, R) = |L(z_t, R_i) \cap O_{\geq t} \cap \overline{top}(R)|$$

be the rank that $z_t$ occupies in agents' preferences, where the rank of $z_t$ is the number of non-top-ranked objects below $z_t$, ignoring objects $z_l$ with $l < t$. Further, let $\theta(z_t, R) \in \mathbb{N}_{\geq}^{n}$ be the vector of ranks, ordered in non-increasing fashion, i.e., $\theta_i(z_t, R) = \rho_{\tau(i)}(z_t, R)$ for an appropriate permutation $\tau : N \to N$. For any $t \geq 1$ define

$$\mathcal{R}_t^{\neq} = \{R \in \mathcal{R}_{t-1}^{\neq} : \text{there exists no } \bar{R} \in \mathcal{R}_{t-1}^{\neq} \text{ such that } \theta(z_t, \bar{R}) \prec \theta(z_t, R)\},$$

where $\theta(z_t, \bar{R})$ and $\theta(z_t, R)$ are ordered by lexicographic minimization. Hence, $\mathcal{R}_t^{\neq}$ contains all profiles where $z_t$ is ranked as low as possible, provided that (i) $f$ and $g$ still differ in the assignment probability shares of *some* object, and that (ii) all objects $z_l \in O$, $l < t$, are likewise ranked as low as possible (with rank-minimization of $z_m$ taking precedence over the rank-minimization of $z_{m'}$ for any $m < m' < t$).

We first show in two lemmas that for any preference profile in $\mathcal{R}_1^{\neq}$, the assignment probabilities of $z_1$ coincide under $f$ and $g$. We then proceed by induction to show that the same holds for any $\mathcal{R}_k^{\neq}$, and objects $z_l$, $l \leq k$. This implies that $\mathcal{R}_n^{\neq} = \emptyset$ and thus establishes the desired contradiction.

LEMMA 2: *Consider $z_1 \in O$ and $R \in \mathcal{R}_1^{\neq}$, and partition $N$ as follows: $N = I_1 \cup I_2$ with $I_1 = \{i \in N : L(z_1, R_i) \cap \overline{top}(R) = \emptyset\}$ and $I_2 = N \backslash I_1$ (i.e., $I_1$ consists of those agents for which $z_1$ is ranked least relative to $\overline{top}(R)$ while agents in $I_2$ rank some object from $\overline{top}(R)$ below $z_1$). If there is some $j \in N$ such that $f_{jz_1}(R) > g_{jz_1}(R)$, then $j \in I_2$ and for* *all $i \in I_2\backslash\{j\}$ we have $(L(z_1, R_i) \cap \overline{top}(R)) \supseteq (L(z_1, R_j) \cap \overline{top}(R)) \neq \emptyset$ (i.e., $i$'s lower contour set of $z_1$ at $R_i$ contains all objects in $\overline{top}(R)$ which are contained in $j$'s lower contour set).*

**Proof.** First, given that $f \triangleright^{sd} g$, $f_{jz_1}(R) > g_{jz_1}(R)$ implies that there is some object $x$ ranked below $z_1$ by $j$, for which $f_{jx}(R) < g_{jx}(R)$. If $x \in \overline{top}(R)$, then we have $L(z_1, R_j) \cap \overline{top}(R) \neq \emptyset$; and if $x \in top(R)$, then Lemma 1 implies $L(z_1, R_j) \cap \overline{top}(R) \subseteq L(x, R_j) \cap \overline{top}(R) \neq \emptyset$. Hence, in either case we have $j \in I_2$. Now take any $a \in L(z_1, R_j) \cap \overline{top}(R)$ and move it up to just below $z_1$, arriving at $R'$. Note that $top(R) = top(R')$ so that $\theta(z_1, R) = \theta(z_1, R')$ and hence $R' \in \mathcal{R}_1^{\neq}$. By strategy-proofness we have $f_{jz_1}(R') > g_{jz_1}(R')$ and since $R' \in \mathcal{R}_1^{\neq}$ we have $f_{ja}(R') < g_{ja}(R')$ – otherwise, we could swap $a$ and $z_1$, arriving at $R''$ where $f(R'') \neq g(R'')$, yet $top(R'')\backslash\{z_1\} \supseteq top(R')\backslash\{z_1\}$ so that $z_1$ is ranked lower relative to non-top-ranked objects in $R''$ than in $R'$, contradicting $R' \in \mathcal{R}_1^{\neq}$. Any agent $i \neq j$ who does not rank $z_1$ least relative to $\overline{top}(R') = \overline{top}(R)$, i.e., for whom $L(z_1, R_j) \cap \overline{top}(R') \neq \emptyset$, must also rank $a$ below $z_1$ in $R_i = R_i'$: otherwise, they could move $z_1$ to the bottom of their preferences in $R'$ – call the new profile $R'''$. By BI, we still have $f_{ja}(R''') < g_{ja}(R''')$. Again, this would contradict $R' \in \mathcal{R}_1^{\neq}$, i.e., that $z_1$ is ranked as low as possible in $R'$.

Since $a \in L(z_1, R_j) \cap \overline{top}(R)$ was chosen arbitrarily, this proves the Lemma 2. □

LEMMA 3: *Let $R \in \mathcal{R}_1^{\neq}$. Then $f_{iz_1}(R) = g_{iz_1}(R)$ for all $i \in N$.*

**Proof.** Towards a contradiction, assume there exists $j \in N$ with $f_{jz_1}(R) > g_{jz_1}(R)$ and consider the partition $\{I_1, I_2\}$ as in Lemma 2. By Lemma 2 we know that $j \in I_2$, i.e., $L(z_1, R_j) \cap \overline{top}(R) \neq \emptyset$, and $0 = \rho_h(z_1, R) < \rho_j(z_1, R) \leq \rho_l(z_1, R)$ for all $h \in I_1$ and $l \in I_2$. We will construct a new profile $\tilde{R}^*$ such that $f(\tilde{R}^*) \neq g(\tilde{R}^*)$ but where the number of non-top ranked objects below $z_1$ for agents in $I_2$ is lower than in $R$ – strictly so for $j \in I_2$ – and where for all agents in $i \in I_1$ we have $\rho_i(z_1, \tilde{R}^*) < \rho_j(z_1, R)$. Thereby we will find that $\theta(z_1, \tilde{R}^*) \prec \theta(z_1, R)$, contradicting $R \in \mathcal{R}_1^{\neq}$ .

First we will rule out $z_1 \in top(R)$. For that, note that since $R \in \mathcal{R}_1^{\neq}$, we need to have $f_{iz_1}(R) \geq g_{iz_1}(R)$ for all $i \in I_2$ – otherwise, for $i \in I_2$ such that $f_{iz_1}(R) < g_{iz_1}(R)$, $f \triangleright^{sd} g$ would imply there to be a higher-ranked object, $xP_iz_1$, such that $f_{ix}(R) > g_{ix}(R)$ and we could move $z_1$ to the bottom of $i$ preference order. For the new profile, denoted $\hat{R}$, strategy-proofness would imply $f_{ix}(\hat{R}) > g_{ix}(\hat{R})$, i.e., $f(\hat{R}) \neq g(\hat{R})$. Since $top(R) = top(\hat{R})$ and $z_1$ is now ranked lower for $i$ but unchanged for all $k \neq i$, i.e., $0 = |L(z_t, \hat{R}_i) \cap \overline{top}(\hat{R})| < |L(z_t, R_i) \cap \overline{top}(R)|$ and $|L(z_t, \hat{R}_k) \cap \overline{top}(\hat{R})| = |L(z_t, R_k) \cap \overline{top}(R)|$, this contradicts $R \in \mathcal{R}_1^{\neq}$. We conclude that $f_{iz_1}(R) \geq g_{iz_1}(R)$, for all $i \in I_2$.

But then it cannot be the case that $z_1 \in top(R)$, since Lemma 1 would imply $f_{iz_1}(R) = 0 = g_{iz_1}(R)$, for all $i \in I_1$, which, together with $f_{iz_1}(R) \geq g_{iz_1}(R)$, for all $i \in I_2$, as well as $f_{jz_1}(R) > g_{jz_1}(R)$, would contradict the fact that $z_1$ is assigned with probability one in both $f$ and $g$.

Now, if $top(R_{I_1}) \cap L(z_1, R_j) \neq \emptyset$, take any $x \in top(R_{I_1}) \cap L(z_1, R_j)$ and move up $x$ in $R_j$ just below $z_1$ to arrive at $R_j^x$. Note that $top(R_j^x, R_{-j}) = top(R)$ and $L(z_1, R_j^x) = L(z_1, R_j)$. By strategy-proofness, we still have $f_{jz_1}(R_j^x, R_{-j}) > g_{jz_1}(R_j^x, R_{-j})$. We have either $f_{jx}(R_j^x, R_{-j}) < g_{jx}(R_j^x, R_{-j})$ or $f_{jx}(R_j^x, R_{-j}) \geq g_{jx}(R_j^x, R_{-j})$. We show that for both cases we obtain a new profile $R'$ where $f_{jz_1}(R') > g_{jz_1}(R')$, where $\rho_i(z_1, R') = \rho_i(z_1, R)$ for all $i \in I_2$ and where $\rho_i(z_1, R') \leq \rho_j(z_1, R)$ for all $i \in I_1$. Let $I_1^x$ denote the set of agents in $I_1$ who rank $x$ at the top.

*Case (1.x):* if $f_{jx}(R_j^x, R_{-j}) < g_{jx}(R_j^x, R_{-j})$, let all $i \in I_1^x$ push $\{z_1\} \cup (L(z_1, R_j) \cap \overline{top}(R))$ to the bottom of their preference order, in the same order as they are ranked in $R_j$, to arrive at $R_i'$. For $j$, relabel $R_j' = R_j^x$ and for all other $i \in N \backslash (I_1^x \cup \{j\})$, relabel $R_i' = R_i$ to arrive at $R' = (R_k')_{k \in N}$. By BI, we still have $f_{jx}(R') < g_{jx}(R')$. Towards a contradiction, assume $f_{jz_1}(R') \leq g_{jz_1}(R')$. Then there would be some object $yP_j' z_1$ such that $f_{jy}(R') > g_{jy}(R')$. Moreover, $yP_i' z_1$ for all $i \in I_1^x$. Hence we could push $z_1$ to the bottom of the preference order for all agents $i \in I_1^x$ as well as for $j$ and, by BI, arrive at a profile $\hat{R}$ where $f$ and $g$ differ in the assignment probabilities of $y$. Since in $\hat{R}$, $z_1$ is ranked lower relative to objects $\overline{top}(\hat{R}) = \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}_1^{\neq}$ – and we conclude that $f_{jz_1}(R') > g_{jz_1}(R')$.

*Case (2.x):* if instead we have $f_{jx}(R_j^x, R_{-j}) \geq g_{jx}(R_j^x, R_{-j})$, swap $x$ and $z_1$ in the ranking of $j$ – let us denote this new preference order as $R_j'$ and the new preference profile $(R_j', R_{-j})$ simply as $R'$. Note that since $z_1 \neq top(R_j^x) = top(R_j)$ we have $top(R') = top(R)$. Towards a contradiction, assume $f_{jx}(R') > g_{jx}(R')$. Then we could push down $z_1$ to the bottom of $j$'s preference order, below all other $\overline{top}(R')$, and do the same for all $i \in I_1^x$ – call the new preference profile $\hat{R}$. By BI this preserves $f_{jx}(\hat{R}) > g_{jx}(\hat{R})$. Since in $\hat{R}$ object $z_1$ is ranked lower relative to objects $\overline{top}(\hat{R}) = \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}_1^{\neq}$. Therefore, after having swapped $x$ and $z_1$, we must have $f_{jx}(R') \leq g_{jx}(R')$ and thus $f_{jz_1}(R') > g_{jz_1}(R')$.

Thus, independently of whether Case $(1.x)$ or Case $(2.x)$ applies, we arrive at a new profile $R'$ where $f_{jz_1}(R') > g_{jz_1}(R')$ and where $\rho_i(z_1, R') = \rho_i(z_1, R)$ for all $i \in I_2$, i.e., $z_1$ is ranked as low as before for all agents in $I_2$. While $z_1$ might be ranked higher than in $R$ for agents in $I_1^x$, we still have $\rho_k(z_1, R') \leq \rho_j(z_1, R') \leq \rho_l(z_1, R')$ for all $k \in I_1$ and $l \in I_2$.

Next, if there is any other $x' \in (top(R_{I_1}) \cap L(z_1, R_j)) \backslash \{x\} \subseteq top(R'_{I_1}) \cap L(z_1, R'_j)$, we proceed as before and move up $x'$ in $R'_j$ just below $z_1$. Refer to this preference order as $R_j^{x'}$. By strategy-proofness, we still have $f_{jz_1}(R_j^{x'}, R'_{-j}) > g_{jz_1}(R_j^{x'}, R'_{-j})$. We proceed as above and obtain profile $R''$ where $f_{jz_1}(R'') > g_{jz_1}(R'')$ and the rank of $z_1$ relative to non-top-ranked objects remains unchanged for agents in $I_2$.

*Case (1.x'):* if $f_{jx'}(R_j^{x'}, R'_{-j}) < g_{jx'}(R_j^{x'}, R'_{-j})$ we proceed as in Case $(1.x)$ – the only difference is that we now need to take into account the possible changes made to preferences of agents in $I_1^x$ in Case $(1.x)$. Let all $i \in I_1^{x'}$ push $\{z_1\} \cup (L(z_1, R_j) \cap \overline{top}(R))$ to the bottom of their preference order, in the same order as they are ranked in $R_j$, to arrive at $R''_i$. For $j$, relabel $R''_j = R_j^{x'}$ and for all other $i \in N \backslash (I_1^{x'} \cup \{j\})$, relabel $R''_i = R'_i$ to arrive at $R'' = (R''_k)_{k \in N}$. By BI, we still have $f_{jx'}(R'') < g_{jx'}(R'')$. Towards a contradiction, assume $f_{jz_1}(R'') \leq g_{jz_1}(R'')$. Then there would be some object $y P''_j z_1$ such that $f_{jy}(R'') > g_{jy}(R'')$. Moreover, $y P''_i z_1$ for all $i \in I_1^{x'}$ as well as for all $i \in I_1^x$ if we arrived at $R'$ via Case $(1.x)$. Hence we could push $z_1$ to the bottom of the preference order for all agents in $I_1$ for whom we have so far constructed new preferences[35] as well as for $j$ and, by BI, arrive at a profile $\hat{R}$ where $f$ and $g$ differ in the assignment probabilities of $y$. Since in $\hat{R}$, $z_1$ is ranked lower relative to objects $\overline{top}(\hat{R}) = \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}_1^{\neq}$ – and we conclude that $f_{jz_1}(R'') > g_{jz_1}(R'')$.

*Case (2.x'):* if instead we have $f_{jx'}(R_j^{x'}, R'_{-j}) \geq g_{jx'}(R_j^{x'}, R'_{-j})$, swap $x'$ and $z_1$ in the ranking of $j$ – let us denote this new preference order as $R''_j$ and the new preference profile $(R''_j, R'_{-j})$ simply as $R''$. Note that since $z_1 \neq top(R_j^x) = top(R_j)$ we have $top(R'') = top(R)$. Towards a contradiction, assume $f_{jx}(R'') > g_{jx}(R'')$. Then we could push down

[35] I.e., for $i \in I_1^{x'} \cup I_1^x$ if we arrived at $R'$ via Case $(1.x)$, and for $i \in I_1^{x'}$ if we arrived at $R$ via Case $(2.x)$.

$z_1$ to the bottom of $j$'s preference order, below all other $\overline{top}(R'')$, and do the same for all $i \in I_1^{x'}$, as well as for all other $i \in I_1$ for whom we may have so far constructed new preferences – call the new preference profile $\hat{R}$. By BI this preserves $f_{jx}(\hat{R}) > g_{jx}(\hat{R})$. Since in $\hat{R}$ object $z_1$ is ranked lower relative to objects $\overline{top}(\hat{R}) = \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}_1^{\neq}$. Therefore, after having swapped $x$ and $z$, we must have $f_{jx}(R'') \le g_{jx}(R'')$ and thus $f_{jz_1}(R'') > g_{jz_1}(R'')$.

Repeat these steps for all $x^* \in top(R_{I_1}) \cap L(z_1, R_j)$, i.e., move up $x^*$ in the preference order of $j$ to just below $z_1$ and then proceed as in Case $(1.x')$ or $(2.x')$. This way, we arrive at a profile, refer to it as $R^\dagger$, where $top(R_i^\dagger) = top(R_i)$ for all $i \in N$, $f_{jz_1}(R^\dagger) > g_{jz_1}(R^\dagger)$, and $I_1$ has been partitioned into two subsets: $I_1'$ includes all agents $i \in I_1$ for whom $R_i^\dagger = R_i$ and hence $L(z_1, R_i^\dagger) \cap \overline{top}(R) = \emptyset$, and whose top ranked objects are ranked above $z_1$ by $j$ – in $R$ but also in $R^\dagger$ since $j$'s lower contour set has only gotten weakly smaller as we moved away from $R$ to $R^\dagger$ (strictly smaller whenever Case 2 applied). Second, $I_1''$ includes all agents $i \in I_1$ whose lower contour set $L(z_1, R_i^\dagger)$ consists of all objects $L(z_1, R_j) \cap \overline{top}(R)$. Third, compared to $R$, $j$'s lower contour set at $z_1$ has gotten weakly smaller in that some objects from $top(R_{I_1})$ may now be ranked above $z_1$ – however, no object in $\overline{top}(R)$ has been raised above $z_1$ as we moved to $R_j^\dagger$, i.e., $L(z_1, R_j) \cap \overline{top}(R) = L(z_1, R_j^\dagger) \cap \overline{top}(R^\dagger)$. Last the ranking of other agents $i \in I_2 \backslash \{j\}$ is unchanged, i.e., $R_i^\dagger = R_i$.

By Lemma 2 as well as the preceding construction, we have for all $h \in I_1''$ and all $l \in I_2$,

$$L(z_1, R_h^\dagger) \cap \overline{top}(R^\dagger) \subseteq L(z_{k+1}, R_j^\dagger) \cap \overline{top}(R^\dagger) \subseteq L(z_1, R_l^\dagger) \cap \overline{top}(R^\dagger),$$
$$\rho_h(z_1, R^\dagger) \le \rho_j(z_1, R^\dagger) \le \rho_l(z_1, R^\dagger) \text{ and } \quad \rho_l(z_1, R^\dagger) = \rho_l(z_1, R).$$

Now, for all $i \in I_1'' \cup I_2$ (including $j$) change the order of objects in the lower contour set $L(z_1, R_i^\dagger)$ as follows: (i) objects that are in $L(z_1, R_i^\dagger) \backslash L(z_1, R_j^\dagger)$ are ranked immediately below $z_1$ (beyond that, their order does not matter), (ii) objects that are also in $L(z_1, R_j^\dagger) \cap \overline{top}(R^\dagger)$ are ranked next, in the same order as by $R_j^\dagger$, (iii) last, all objects in $L(z_1, R_i^\dagger) \cap L(z_1, R_j^\dagger) \cap top(R^\dagger)$ are ranked below (beyond that, their order does not matter). Call this new (and penultimate) profile $\tilde{R}$. By BI, we still have $f_{jz_1}(\tilde{R}) > g_{jz_1}(\tilde{R})$. By Lemma 1 and $f \rhd^{sd} g$, we have $f_{ix}(\tilde{R}) = 0 = g_{ix}(R)$ for all $i \in I_2$ and all $x \in L(z_1, \tilde{R}_i) \cap L(z_1, \tilde{R}_j) \cap top(\tilde{R})$.

Hence, we now have all agents in $I_1'' \cup I_2$ ranking objects $L(z_1, \tilde{R}_j) \cap \overline{top}(\tilde{R})$ adjacent and in the same order as $\tilde{R}_j$, and below that only objects in $top(\tilde{R}) = top(R)$ for which the assignment probabilities are equal to zero under $f$ and $g$ by Lemma 1. Since $f_{jz_1}(\tilde{R}) > g_{jz_1}(\tilde{R})$, there is some $y$, ranked below $z_1$ by $\tilde{R}_j$, such that $f_{jy}(\tilde{R}) < g_{jy}(\tilde{R})$ – and thus some $i \in N$ with $f_{iy}(\tilde{R}) > g_{iy}(\tilde{R})$. Moreover, by Lemma 1, we have $y \in L(z_1, \tilde{R}_j) \cap \overline{top}(\tilde{R})$.

If $i \in I_1'' \cup I_2$, then there is $y'$ with $y\tilde{R}_i y'$, such that $f_{iy'}(\tilde{R}) < g_{iy'}(\tilde{R})$ – and thus some $i' \in N$ with $f_{i'y'}(\tilde{R}) > g_{i'y'}(\tilde{R})$. Hence, by Lemma 1, it must be that $y \in \overline{top}(\tilde{R})$, so that $y' \in L(y, \tilde{R}_j) \cap \overline{top}(\tilde{R})$. Thus, $y'$ is ranked lower than $y$ according to $\tilde{R}_j$.

If $i' \in I_1'' \cup I_2$, then there is $y''$ with $y'\tilde{R}_{i'} y''$, such that $f_{i'y''}(\tilde{R}) < g_{i'y''}(\tilde{R})$ – and thus some $i'' \in N$ with $f_{i''y''}(\tilde{R}) > g_{i''y''}(\tilde{R})$, and so on.

Since $L(z, \tilde{R}_j) \cap \overline{top}(\tilde{R})$ is finite and we move down (according to $\tilde{R}_j$) in each iteration, eventually there is some $y^* \in L(z, \tilde{R}_j) \cap \overline{top}(\tilde{R})$ and $i^* \in I_1' = N \backslash (I_1'' \cup I_2)$ such that $f_{i^*y^*}(\tilde{R}) > g_{i^*y^*}(\tilde{R})$.

Note that $\tilde{R}_i = R_i$, and thus, $y^* \tilde{P}_i z_1$ for any $i \in I_1'$. If $y^* P_i top(\tilde{R}_{i^*})$, then change $\tilde{R}_i$ to $\tilde{R}_i'$ as follows: (i) objects in $B(y^*, R_i)$ are ranked first according to $R_i$, (ii) then $top(\tilde{R}_{i^*})$ and (iii) then objects in $L(y^*, R_i) \backslash \{top(\tilde{R}_{i^*})\}$ according to $R_i$. After having done this for all such $i \in I_1'$ and denoting the obtained profile by $\tilde{R}'$, by BI we continue to have $f_{i^*y^*}(\tilde{R}') > g_{i^*y^*}(\tilde{R}')$. But then let $i^*$ exchange the positions of $y^*$ and $top(\tilde{R}_{i^*})$ in $\tilde{R}_{i^*}'$ and call this final profile $\tilde{R}^*$. This strictly decreases the number of non-top objects ranked below $z_1$ for $j$, as well as all $i \in I_1''$, and weakly decreases it for all $i \in I_1'$ (as either $top(\tilde{R}_{i^*})\tilde{P}_i' y^* \tilde{P}_i' z_1$ or $top(\tilde{R}_{i^*})$ is ranked immediately below $y^*$ in $\tilde{R}_i'$) and for all $i \in I_2 \backslash \{j\}$ (only weakly if $i \in I_2 \backslash \{j\}$ ranked both $top(\tilde{R}_{i^*})$ and $top(\tilde{R}_{i^*}^*)$ below $z_1$). Hence, $\rho_i(z_1, \tilde{R}^*) \leq \rho_i(z_1, R)$ for $i \in I_2 \backslash \{j\}$, $\rho_j(z_1, \tilde{R}^*) < \rho_j(z_1, R)$, and $\rho_i(z_1, \tilde{R}^*) \leq \rho_j(z_1, \tilde{R}^*)$ for $i \in I_1$ contradicting $R \in \mathcal{R}_1^{\neq}$. □

The following two lemmas extend Lemma 2 and Lemma 3 to $\mathcal{R}_t^{\neq}$, $t = 1, ..., n$, thereby completing the proof. Recall that $O_{\geq t} = \{z_t, ..., z_n\}$, for any $1 \leq t < n$. Let $Z_t = \{z_1, \ldots, z_t\}$ for any $1 \leq t < n$.

LEMMA 4: *Consider $1 \leq t < n$, $z_t \in O$ and $R \in \mathcal{R}_t^{\neq}$, and partition $N$ as follows: $N = I_1 \cup I_2$ with $I_1 = \{i \in N : L(z_t, R_i) \cap O_{\geq t} \cap \overline{top}(R) = \emptyset\}$ and $I_2 = N \backslash I_1$ (i.e., $I_1$ consists* *of those agents for which $z_t$ is ranked least relative to $O_{\geq t} \cap \overline{top}(R)$ while agents in $I_2$ rank some object from $O_{\geq t} \cap \overline{top}(R)$ below $z_t$). If there is some $j \in N$ such that $f_{jz_t}(R) > g_{jz_t}(R)$ then $j \in I_2$ and for all $i \in I_2 \backslash \{j\}$ we have $L(z_t, R_i) \cap O_{\geq t} \cap \overline{top}(R) \supseteq L(z_t, R_j) \cap O_{\geq t} \cap \overline{top}(R) \neq \emptyset$ (i.e., $i$'s lower contour set of $z_t$ at $R_i$ contains all objects in $O_{\geq t} \cap \overline{top}(R)$ which are contained in $j$'s lower contour set).*

LEMMA 5: *Let $R \in \mathcal{R}_t^{\neq}$. Then $f_{iz_t}(R) = g_{iz_t}(R)$ for all $i \in N$.*

**Proof of Lemma 4 and 5.** For $t = 1$, this is established by Lemma 2 and 3, which serve as the basis for the following induction. For the induction step, assume we have established both statements for all $1 \leq t \leq k < n$ (the induction hypothesis). It remains to show that both hold for $t = k+1$. For this, the following observation will be useful.

CLAIM 4: *If there is some $j \in I$ such that $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ for $R \in \mathcal{R}_{k+1}^{\neq}$, then $z_{k+1} \neq top(R_j)$.*

*Proof of Claim 4:* Suppose $top(R_j) = z_{k+1}$. Since $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$, there exists $i$ with $f_{iz_{k+1}}(R) < g_{iz_{k+1}}(R)$. We have either $L(z_{k+1}, R_i) \cap O_{\geq k+1} \cap \overline{top}(R) \neq \emptyset$ or $L(z_{k+1}, R_i) \cap O_{\geq k+1} \cap \overline{top}(R) = \emptyset$.

Suppose $L(z_{k+1}, R_i) \cap O_{\geq k+1} \cap \overline{top}(R) \neq \emptyset$. By $f \rhd^{sd} g$ there exists $x$ with $xP_iz_{k+1}$ and $f_{ix}(R) > g_{ix}(R)$. But then we could move $Z_{k+1} \cap L(z_{k+1}, R_i)$ to the bottom of $R_i$ in unchanged order, arriving at a contradiction to $R \in \mathcal{R}_{k+1}^{\neq}$.

Suppose instead $L(z_{k+1}, R_i) \cap O_{\geq k+1} \cap \overline{top}(R) = \emptyset$. By Lemma 1 and $z_{k+1} \in top(R)$, there exists $y \in \overline{top}(R) \cap L(z_{k+1}, R_i)$. Hence we must have $y \in Z_k$. Furthermore, by $f \rhd^{sd} g$ and $f_{iz_{k+1}}(R) < g_{iz_{k+1}}(R)$, we have $\sum_{o \in O: oP_iz_{k+1}} f_{io}(R) > \sum_{o \in O: oP_iz_{k+1}} g_{io}(R)$. Now reorder the objects in $R_i$ as follows: first rank $O \backslash (L(z_{k+1}, R_i) \cup \{z_{k+1}\})$, then $\overline{top}(R) \cap L(z_{k+1}, R_i)$, and then, at the bottom $top(R) \cap L(z_{k+1}, R_i) \cup \{z_{k+1}\}$. Call the new preference profile $R'$. Since objects $O \backslash (L(z_{k+1}, R_i) \cup \{z_{k+1}\})$ are still ranked above objects in $(L(z_{k+1}, R_i) \cup \{z_{k+1}\})$, strategy-proofness implies $\sum_{o \in O \backslash (L(z_{k+1}, R_i) \cup \{z_{k+1}\})} f_{io}(R') > \sum_{o \in O \backslash (L(z_{k+1}, R_i) \cup \{z_{k+1}\})} g_{io}(R')$. Yet by the induction hypothesis and Lemma 1, we have $f_{iy}(R') = g_{iy}(R')$ for all $y \in (L(z_{k+1}, R_i) \cup \{z_{k+1}\})$. Hence $\sum_{o \in O} f_{io}(R') > \sum_{o \in O} g_{io}(R')$ – contradicting feasibility and thus establishing Claim 4.

*Induction step for Lemma 4*. First, given that $f \rhd^{sd} g$, $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ implies that there is some object $x$ ranked below $z_{k+1}$ for which $f_{jx}(R) < g_{jx}(R)$. Thus, by Lemma 1 and the induction hypothesis, we have $L(z_{k+1}, R_j) \cap \overline{top}(R) \cap O_{\geq k+1} \neq \emptyset$, i.e., $j \in I_2$: otherwise all objects in $L(z_{k+1}, R_j)$ would either be top-ranked objects or from $Z_k$. But then we could move to $R'_j$ by reordering $j'$s lower contour set, pushing all objects in $top(R)$ to the bottom. Since this leaves the rank of objects $z \in Z_{k+1}$ unaffected, we would still have $(R'_j, R_{-j}) \in \mathcal{R}^{\neq}_{k+1}$. But now Lemma 1 implies that $f_{jx}(R'_j, R_{-j}) = g_{jx}(R'_j, R_{-j})$ for all $x \in top(R) = top(R'_j, R_{-j})$ while the induction hypothesis implies that $f_{jx}(R'_j, R_{-j}) = g_{jx}(R'_j, R_{-j})$ for all $x \in Z_k$. Since strategy-proofness ensures $f_{jz_{k+1}}(R'_j, R_{-j}) > g_{jz_{k+1}}(R'_j, R_{-j})$, this would contradict $f \rhd^{sd} g$.

Now take any $a \in L(z_{k+1}, R_j) \cap O_{\geq k+1} \cap \overline{top}(R)$ and move it up to just below $z_{k+1}$, arriving at $R'$. Note that $top(R) = top(R')$ so that $\rho(z_{k+1}, R) = \rho(z_{k+1}, R')$ and hence $R' \in \mathcal{R}^{\neq}_{k+1}$. By strategy-proofness we have $f_{jz_{k+1}}(R') > g_{jz_{k+1}}(R')$ and since $R' \in \mathcal{R}^{\neq}_{k+1}$ we have $f_{ja}(R') < g_{ja}(R')$ – otherwise, we could swap $a$ and $z_{k+1}$ in $R'_j$, arriving at $R''$ where $f_l(R'') \neq g_l(R'')$ for some $l \in N$, yet, by Claim 4, $top(R'') = top(R')$, so that $z_{k+1}$ is ranked lower relative to non-top-ranked objects from $O_{\geq k+1}$ in $R''$ than in $R'$, contradicting $R' \in \mathcal{R}^{\neq}_{k+1}$. Any agent $i \neq j$ who does not rank $z_{k+1}$ least relative to $\overline{top}(R') = \overline{top}(R)$ and $O_{\geq k+1}$, i.e., for whom $L(z_{k+1}, R_j) \cap O_{\geq k+1} \cap \overline{top}(R') \neq \emptyset$, must also rank $a$ below $z_{k+1}$ in $R_i = R'_i$: otherwise, they could move $z_{k+1}$ to the bottom of their preferences in $R'$ – call the new profile $R'''$. By BI, we still have $f_{ja}(R''') < g_{ja}(R''')$. Again, this would contradict $R' \in \mathcal{R}^{\neq}_{k+1}$, i.e., that $z_{k+1}$ is ranked as low as possible in $R'$. Since $a \in L(z_{k+1}, R_j) \cap O_{\geq k+1} \cap \overline{top}(R)$ was chosen arbitrarily, this completes the induction step for Lemma 4.

*Induction step for Lemma 5*. Suppose the statement is not true for $t = k+1$. Then there exist $R \in \mathcal{R}^{\neq}_{k+1}$ and $j \in N$ with $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$. By Claim 4, $z_{k+1} \in \overline{top}(R)$. Moreover, without loss of generality, we may assume that all objects in $Z_k$ are ranked at the bottom of $R_j$ such that $z_{m'} R_j z_m$ for $m < m' \leq k$: otherwise we can begin by moving $z_1$ to the bottom of $j$'s preference list in single, pairwise swaps. Since these transformations keep the profile in $\mathcal{R}^{\neq}_k \subseteq \mathcal{R}^{\neq}_1$ we have $f_{jz_1}(\hat{R}) = g_{jz_1}(\hat{R})$ both before and after the swap and hence, by SP, $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ (where $\hat{R}$ denotes an arbitrary profile in the sequence starting at $R$). Repeating this for each $m$ with $1 < m \leq k$ establishes the claim.

Consider the partition $\{I_1, I_2\}$ as in Lemma 4 – by the induction hypothesis and the induction step for Lemma 4 above, this exists for $t = k+1$. By Lemma 4 we know that $j \in I_2$, i.e., $L(z_{k+1}, R_j) \cap O_{\geq k+1} \cap \overline{top}(R) \neq \emptyset$, and $0 = \rho_k(z_{k+1}, R) < \rho_j(z_{k+1}, R) \leq \rho_l(z_{k+1}, R)$ for all $k \in I_1$ and $l \in I_2$. As in Lemma 2, we will construct a new profile $\tilde{R}^*$ in which $z_{k+1}$ is ranked lower to contradict $R \in \mathcal{R}^{\neq}_{k+1}$.

Now, if $top(R_{I_1}) \cap L(z_{k+1}, R_j) \cap O_{\geq k+1} \neq \emptyset$, take any $x \in top(R_{I_1}) \cap L(z_{k+1}, R_j) \cap O_{\geq k+1}$ and move up $x$ in $R_j$ just below $z_{k+1}$ to arrive at $R^x_j$. Note that $top(R^x_j, R_{-j}) = top(R)$ and $L(z_{k+1}, R^x_j) = L(z_{k+1}, R_j)$. By strategy-proofness, we still have $f_{jz_{k+1}}(R^x_j, R_{-j}) > g_{jz_{k+1}}(R^x_j, R_{-j})$. We have either $f_{jx}(R^x_j, R_{-j}) < g_{jx}(R^x_j, R_{-j})$ or $f_{jx}(R^x_j, R_{-j}) \geq g_{jx}(R^x_j, R_{-j})$. We show that for both cases we obtain a new profile $R'$ where $f_{jz_{k+1}}(R') > g_{jz_1}(R')$, where $\rho_i(z_{k+1}, R') = \rho_i(z_{k+1}, R)$ for all $i \in I_2$ and where $\rho_i(z_{k+1}, R') \leq \rho_j(z_{k+1}, R)$ for all $i \in I_1$. Let $I^x_1$ denote the set of agents in $I_1$ who rank $x$ at the top.

*Case (1.x):* if $f_{jx}(R^x_j, R_{-j}) < g_{jx}(R^x_j, R_{-j})$, let all $i \in I^x_1$ push $\{z_{k+1}\} \cup (L(z_{k+1}, R_j) \cap \overline{top}(R)) \cup Z_k$ to the bottom of their preference order, in the same order as they are ranked in $R_j$, to arrive at $R'_i$. For $j$, relabel $R'_j = R^x_j$ and for all other $i \in N \backslash (I^x_1 \cup \{j\})$, relabel $R'_i = R_i$ to arrive at $R' = (R'_l)_{l \in N}$. By BI, we still have $f_{jx}(R') < g_{jx}(R')$. Towards a contradiction, assume $f_{jz_{k+1}}(R') \leq g_{jz_{k+1}}(R')$. Then there would be some object $yP'_j z_{k+1}$ such that $f_{jy}(R') > g_{jy}(R')$. Moreover, $yP'_i z_{k+1}$ for all $i \in I^x_1$. Hence we could push $z_{k+1}$ down in the preference order, ranking just above $Z_k$, for all agents $i \in I^x_1$ as well as for $j$ and, by BI, arrive at a profile $\hat{R}$ where $f$ and $g$ differ in the assignment probabilities of $y$. Since in $\hat{R}$, $z_{k+1}$ is ranked lower relative to objects $O_{\geq k+1} \cap \overline{top}(\hat{R}) = O_{\geq k+1} \cap \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}^{\neq}_{k+1}$ – and we conclude that $f_{jz_{k+1}}(R') > g_{jz_{k+1}}(R')$.

*Case (2.x):* if instead we have $f_{jx}(R^x_j, R_{-j}) \geq g_{jx}(R^x_j, R_{-j})$, swap $x$ and $z_{k+1}$ in the ranking of $j$ – let us denote this new preference order as $R'_j$ and the new preference profile $(R'_j, R_{-j})$ simply as $R'$. Since $z_{k+1} \in \overline{top}(R)$, we have $top(R') = top(R)$ – thus the set of (non-)top ranked objects relevant to determine the ranks of $z_l$, $l \leq k+1$ in agents preferences is unchanged. Towards a contradiction, assume $f_{jx}(R') > g_{jx}(R')$. Then we could push down $z_{k+1}$ in $j$'s preference order, ranking just above $Z_k$ and hence below all other

$O_{\geq k+1} \cap \overline{top}(R')$, and do the same for all $i \in I_1^x$, i.e., push down $\{z_{k+1}\} \cup Z_k$ to the bottom of $i$'s preferences. Call the new preference profile $\hat{R}$. By BI the transformation from $R'$ to $\hat{R}$ preserves $f_{jx}(\hat{R}) > g_{jx}(\hat{R})$. Since in $\hat{R}$ object $z_{k+1}$ is ranked lower relative to objects $O_{\geq k+1} \cap \overline{top}(\hat{R}) = O_{\geq k+1} \cap \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}^{\neq}_{k+1}$. Therefore, after having swapped $x$ and $z_{k+1}$, we must have $f_{jx}(R') \leq g_{jx}(R')$ and thus $f_{jz_{k+1}}(R') > g_{jz_{k+1}}(R')$.

Thus, independently of whether Case $(1.x)$ or Case $(2.x)$ applies, we arrive at a new profile $R'$ where $f_{jz_{k+1}}(R') > g_{jz_{k+1}}(R')$ and where $\rho_i(z_{k+1}, R') = \rho_i(z_{k+1}, R)$ for all $i \in I_2$, i.e., $z_{k+1}$ is ranked as low as before for all agents in $I_2$. While $z_{k+1}$ might be ranked higher than in $R$ for agents in $I_1^x$, we still have $\rho_h(z_{k+1}, R') \leq \rho_j(z_{k+1}, R') \leq \rho_l(z_{k+1}, R')$ for all $h \in I_1$ and $l \in I_2$.

Next, if there is any other $x' \in (top(R_{I_1}) \cap L(z_{k+1}, R_j) \cap O_{\geq k+1}) \setminus \{x\} \subseteq top(R'_{I_1}) \cap L(z_{k+1}, R'_j) \cap O_{\geq k+1}$, we proceed as before and move up $x'$ in $R'_j$ just below $z_{k+1}$. Refer to this preference order as $R_j^{x'}$. By strategy-proofness, we still have $f_{jz_{k+1}}(R_j^{x'}, R'_{-j}) > g_{jz_{k+1}}(R_j^{x'}, R'_{-j})$. We proceed as above and obtain profile $R''$ where $f_{jz_{k+1}}(R'') > g_{jz_{k+1}}(R'')$ and the rank of $z_{k+1}$ relative to non-top-ranked objects in $O_{\geq k+1}$ remains unchanged for agents in $I_2$.

*Case (1.x'):* if $f_{jx'}(R_j^{x'}, R'_{-j}) < g_{jx'}(R_j^{x'}, R'_{-j})$ we proceed as in Case $(1.x)$ – the only difference is that we now need to take into account the possible changes made to preferences of agents in $I_1^x$ in Case $(1.x)$. Let all $i \in I_1^{x'}$ push $\{z_{k+1}\} \cup (L(z_{k+1}, R_j) \cap \overline{top}(R)) \cup Z_k$ to the bottom of their preference order, in the same order as they are ranked in $R_j$, to arrive at $R''_i$. For $j$, relabel $R''_j = R_j^{x'}$ and for all other $i \in N \setminus (I_1^{x'} \cup \{j\})$, relabel $R''_i = R'_i$ to arrive at $R'' = (R''_l)_{l \in N}$. By BI, we still have $f_{jx'}(R'') < g_{jx'}(R'')$. Towards a contradiction, assume $f_{jz_{k+1}}(R'') \leq g_{jz_1}(R'')$. Then there would be some object $yP''_j z_{k+1}$ such that $f_{jy}(R'') > g_{jy}(R'')$. Moreover, $yP''_i z_{k+1}$ for all $i \in I_1^{x'}$ as well as for all $i \in I_1^x$ if we arrived at $R'$ via Case $(1.x)$. Hence we could push $z_{k+1}$ down in the preference order, ranking just above $Z_k$, for all agents in $I_1$ for whom we have so far constructed new preferences[36] as well as for $j$ and, by BI, arrive at a profile $\hat{R}$ where $f$ and $g$ differ in the assignment probabilities of $y$. Since in $\hat{R}$, $z_{k+1}$ is ranked lower relative to objects

[36] I.e., for $i \in I_1^{x'} \cup I_1^x$ if we arrived at $R'$ via Case $(1.x)$, and for $i \in I_1^{x'}$ if we arrived at $R$ via Case $(2.x)$.

$O_{\geq k+1} \cap \overline{top}(\hat{R}) = O_{\geq k+1} \cap \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}^{\neq}_{k+1}$ – and we conclude that $f_{jz_{k+1}}(R'') > g_{jz_{k+1}}(R'')$.

*Case (2.x'):* if instead we have $f_{jx'}(R^{x'}_j, R'_{-j}) \geq g_{jx'}(R^{x'}_j, R'_{-j})$, swap $x'$ and $z_{k+1}$ in the ranking of $j$ – let us denote this new preference order as $R''_j$ and the new preference profile $(R''_j, R'_{-j})$ simply as $R''$. Since $z_{k+1} \in \overline{top}(R)$, we have $top(R') = top(R)$ – thus the set of (non-)top ranked objects relevant to determine the ranks of $z_l$, $l \leq k+1$ in agents preferences is unchanged. Towards a contradiction, assume $f_{jx}(R'') > g_{jx}(R'')$. Then we could push down $z_{k+1}$ in $j$'s preference order, ranking just above $Z_k$ and hence below all other $O_{\geq k+1} \cap \overline{top}(R')$, and do the same for all $i \in I^x_1$, , i.e., push down $\{z_{k+1}\} \cup Z_k$ to the bottom of $i$'s preferences. Moreover, do the same for all other $i \in I_1$ for whom we may have so far constructed new preferences. Call the new preference profile $\hat{R}$. By BI this preserves $f_{jx}(\hat{R}) > g_{jx}(\hat{R})$. Since in $\hat{R}$ object $z_{k+1}$ is ranked lower relative to objects $O_{\geq k+1} \cap \overline{top}(\hat{R}) = O_{\geq k+1} \cap \overline{top}(R)$ than at our initial profile $R$, this contradicts $R \in \mathcal{R}^{\neq}_{k+1}$. Therefore, after having swapped $x$ and $z$, we must have $f_{jx}(R'') \leq g_{jx}(R'')$ and thus $f_{jz_{k+1}}(R'') > g_{jz_{k+1}}(R'')$.

Repeat these steps for all $x^* \in top(R_{I_1}) \cap L(z_{k+1}, R_j) \cap O_{\geq k+1}$, i.e., move up $x^*$ in the preference order of $j$ to just below $z_{k+1}$ and then proceed as in Case $(1.x')$ or $(2.x')$. This way, we arrive at a profile, refer to it as $R^\dagger$, where $top(R^\dagger_i) = top(R_i)$ for all $i \in N$, $f_{jz_{k+1}}(R^\dagger) > g_{jz_{k+1}}(R^\dagger)$, and $I_1$ has been partitioned into two subsets: $I'_1$ includes all agents $i \in I_1$ for whom $R^\dagger_i = R_i$ and hence $L(z_{k+1}, R^\dagger_i) \cap O_{\geq k+1} \cap \overline{top}(R) = \emptyset$, and whose top ranked objects are in $Z_k$ or ranked above $z_{k+1}$ by $j$ – in $R$ but also in $R^\dagger$ since $j$'s lower contour set has only gotten weakly smaller as we moved away from $R$ to $R^\dagger$ (strictly smaller whenever Case 2 applied). Second, $I''_1$ includes all agents $i \in I_1$ whose lower contour set $L(z_{k+1}, R^\dagger_i)$ consists of all objects $(L(z_{k+1}, R_j) \cap \overline{top}(R)) \cup Z_k$. Third, compared to $R$, $j$'s lower contour set at $z_{k+1}$ has gotten weakly smaller in that some objects from $top(R_{I_1})$ may now be ranked above $z_{k+1}$ – however, no object in $O_{\geq k+1} \cap \overline{top}(R)$ has been raised above $z_{k+1}$ as we moved to $R^\dagger_j$, i.e., $L(z_{k+1}, R_j) \cap O_{\geq k+1} \cap \overline{top}(R) = L(z_{k+1}, R^\dagger_j) \cap O_{\geq k+1} \cap \overline{top}(R^\dagger)$. Last the ranking of other agents $i \in I_2 \backslash \{j\}$ is unchanged, i.e., $R^\dagger_i = R_i$.

By the induction step for Lemma 4 as well as the preceding construction, we have for all $h \in I_1''$ and all $l \in I_2$,

$$L(z_{k+1}, R_h^\dagger) \cap O_{\geq k+1} \cap \overline{top}(R^\dagger) \subseteq L(z_{k+1}, R_j^\dagger) \cap O_{\geq k+1} \cap \overline{top}(R^\dagger) \subseteq L(z_{k+1}, R_l^\dagger) \cap O_{\geq k+1} \cap \overline{top}(R^\dagger),$$

$$\rho_h(z_{k+1}, R^\dagger) \leq \rho_j(z_{k+1}, R^\dagger) \leq \rho_l(z_{k+1}, R^\dagger) \text{ and } \qquad \rho_l(z_{k+1}, R^\dagger) = \rho_l(z_{k+1}, R).$$

Now, for all $i \in I_1'' \cup I_2$ (including $j$) change the order of objects in the lower contour set $L(z_{k+1}, R_i^\dagger)$ as follows: (i) objects that are in $L(z_{k+1}, R_i^\dagger) \backslash L(z_{k+1}, R_j^\dagger)$ are ranked immediately below $z_{k+1}$ (beyond that, their order does not matter), (ii) objects that are also in $L(z_{k+1}, R_j^\dagger) \cap \overline{top}(R^\dagger)$ are ranked next, in the same order as by $R_j^\dagger$, (iii) last, all objects in $L(z_{k+1}, R_i^\dagger) \cap L(z_{k+1}, R_j^\dagger) \cap top(R^\dagger)$ are ranked below (beyond that, their order does not matter). Call this new (and penultimate) profile $\tilde{R}$. By BI, we still have $f_{jz_{k+1}}(\tilde{R}) > g_{jz_{k+1}}(\tilde{R})$. By Lemma 1 and $f \rhd^{sd} g$, we have $f_{ix}(\tilde{R}) = 0 = g_{ix}(R)$ for all $i \in I_1'' \cup I_2$ and all $x \in L(z_{k+1}, \tilde{R}_i) \cap L(z_{k+1}, \tilde{R}_j) \cap top(\tilde{R})$. Lastly, since in moving from $R$ to $R^\dagger$ and on to $\tilde{R}$ objects in $Z_k$ were only moved down relative to non-top ranked objects, we still have $\tilde{R} \in \mathcal{R}_k^{\neq}$.

Hence, we now have all agents in $I_1'' \cup I_2$ ranking objects $L(z_{k+1}, \tilde{R}_j) \cap \overline{top}(\tilde{R})$ adjacent and in the same order as $\tilde{R}_j$, and below that only objects in $top(\tilde{R}) = top(R)$ for which the assignment probabilities are equal to zero under $f$ and $g$ by Lemma 1. Since $f_{jz_{k+1}}(\tilde{R}) > g_{jz_{k+1}}(\tilde{R})$, there is some $y$, ranked below $z_{k+1}$ by $\tilde{R}_j$, such that $f_{jy}(\tilde{R}) < g_{jy}(\tilde{R})$ – and thus some $i \in N$ with $f_{iy}(\tilde{R}) > g_{iy}(\tilde{R})$. Moreover, by Lemma 1 and the induction hypothesis, we have $y \in L(z_{k+1}, \tilde{R}_j) \cap O_{\geq k+1} \cap \overline{top}(\tilde{R})$.

If $i \in I_1'' \cup I_2$, then there is $y' \neq y$ with $y \tilde{R}_i y'$, such that $f_{iy'}(\tilde{R}) < g_{iy'}(\tilde{R})$ – and thus some $i' \in N$ with $f_{i'y'}(\tilde{R}) > g_{i'y'}(\tilde{R})$. Moreover, given that $y \in L(z_{k+1}, \tilde{R}_j)$, our construction of $\tilde{R}_i$ implies that $y'$ is ranked lower than $y$ according to $\tilde{R}_j$, while by Lemma 1 and the induction hypothesis, it must be that $y' \in L(y, \tilde{R}_j) \cap O_{\geq k+1} \cap \overline{top}(\tilde{R})$.

If $i' \in I_1'' \cup I_2$, then there is $y'' \neq y'$ with $y' \tilde{R}_{i'} y''$, such that $f_{i'y''}(\tilde{R}) < g_{i'y''}(\tilde{R})$ – and thus some $i'' \in N$ with $f_{i''y''}(\tilde{R}) > g_{i''y''}(\tilde{R})$, and so on.

Since $L(z_{k+1}, \tilde{R}_j) \cap O_{\geq k+1} \cap \overline{top}(\tilde{R})$ is finite and we move down (according to $\tilde{R}_j$) in each iteration, eventually there is some $y^* \in L(z, \tilde{R}_j) \cap O_{\geq k+1} \cap \overline{top}(\tilde{R})$ and $i^* \in I_1' = N \backslash (I_1'' \cup I_2)$ such that $f_{i^*y^*}(\tilde{R}) > g_{i^*y^*}(\tilde{R})$.

Suppose $top(\tilde{R}_{i^*}) \in O_{\geq k+1}$. Note that $\tilde{R}_i = R_i$, and thus, $y^* \tilde{P}_i z_1$ for any $i \in I_1'$. For any $i \in I_1'$ where $y^* P_i top(\tilde{R}_{i^*})$, change $\tilde{R}_i$ to $\tilde{R}_i'$ as follows: (i) objects in $B(y^*, R_i)$ are ranked first according to $R_i$, (ii) then $top(\tilde{R}_{i^*})$ and (iii) then objects in $L(y^*, R_i) \backslash \{top(\tilde{R}_{i^*})\}$ according to $R_i$. After having done this for all such $i \in I_1'$ and denoting the obtained profile by $\tilde{R}'$, by BI we continue to have $f_{i^* y^*}(\tilde{R}') > g_{i^* y^*}(\tilde{R}')$. As only $top(\tilde{R}_{i^*})$ was moved up and $\tilde{R} \in \mathcal{R}_k^{\neq}$, we still have $\tilde{R}' \in \mathcal{R}_k^{\neq}$. But then let $i^*$ exchange the positions of $y^*$ and $top(\tilde{R}_{i^*})$ in $\tilde{R}_{i^*}'$ and call this final profile $\tilde{R}^*$. This strictly decreases the number of non-top objects ranked below $z_{k+1}$ for $j$, as well as all $i \in I_1''$, and weakly decreases it for all $i \in I_1'$ (as either $top(\tilde{R}_{i^*}) \tilde{P}_i' y^* \tilde{P}_i' z_{k+1}$ or $top(\tilde{R}_{i^*})$ is ranked immediately below $y^*$ in $\tilde{R}_i'$) and for all $i \in I_2 \backslash \{j\}$ (only weakly if $i \in I_2 \backslash \{j\}$ ranked both $top(\tilde{R}_{i^*})$ and $top(\tilde{R}_{i^*}^*)$ below $z_{k+1}$). Note that this also weakly decreases the number of non-top objects ranked below any object in $Z_k$. Hence, $\rho_i(z_{k+1}, \tilde{R}^*) \leq \rho_i(z_{k+1}, R)$ for $i \in I_2 \backslash \{j\}$, $\rho_j(z_{k+1}, \tilde{R}^*) < \rho_j(z_{k+1}, R)$, and $\rho_i(z_{k+1}, \tilde{R}^*) \leq \rho_j(z_{k+1}, \tilde{R}^*)$ for $i \in I_1$ contradicting $R \in \mathcal{R}_{k+1}^{\neq}$.

Finally, consider $top(\tilde{R}_{i^*}) = z_m \in Z_k$ (i.e., $m \leq k$). Since $f_{i^* y^*}(\tilde{R}) > g_{i^* y^*}(\tilde{R})$ there must be some lower ranked $\hat{y}$ such that $f_{i^* \hat{y}}(\tilde{R}) < g_{i^* \hat{y}}(\tilde{R})$. But then, consider the strict upper contour set of $\hat{y}$, i.e., $U(\hat{y}, \tilde{R}_i^*) = \{o \in O : o \tilde{P}_i^* \hat{y}\}$. Push all elements in $U(\hat{y}, \tilde{R}_{i^*}) \cap Z_k$ to just above $\hat{y}$ to arrive at $\tilde{R}^*$. This preserves $f_{i^* \hat{y}}(\tilde{R}^*) < g_{i^* \hat{y}}(\tilde{R}^*)$ (by SP). Moreover, since we have pushed these objects below $y^*$ and $y^* \in O_{\geq k+1} \cap \overline{top}(\tilde{R})$, we have reduced their rank. But that contradicts $\tilde{R} \in \mathcal{R}_k^{\neq}$ – which concludes the proof. □