Supplementary Appendix for: "Near" weighted utilitarian characterizations of Pareto optima

D Other characterizations

Here, we discuss other characterizations of Pareto optima. As will be seen, these characterizations are not only related to our "near"-utilitarian welfare maximizations but they also highlight certain aspects of them and thus help to interpret and understand them. At the same time, we will show that they differ in their axiomatic properties from our "near"-utilitarian welfare characterizations. Our discussion will therefore illustrate that the axiomatic properties of our "near"-utilitarian characterizations are special and not shared by other possible characterizations of Pareto optima.

D.1 Sequential Nash bargaining

The first characterization is motivated by an institutional/behavioral implementation of Pareto optima. As is well known from the second fundamental welfare theorem, a Pareto optimal allocation, say in an exchange economy, can be implemented by a competitive equilibrium under a suitable endowment.[24] In the same spirit, one may ask what institution implements a given Pareto optimum in a more general environment. Our SUWM characterization of Pareto optima allows one to envision sequential negotiations as fulfilling this goal. That is, any Pareto optimal outcome can be seen as emerging from a sequence of negotiations among individuals whose relative bargaining powers in round t are determined by the welfare weights ϕ^t in the corresponding round of SUWM characterization.

To be specific, suppose each agent has a disagreement utility, normalized as zero, that is less than any Pareto optimal utility—i.e., $u \gg 0$ for every $u \in U^P$. Consider a collection of bargaining units $\mathcal{I} = \{I^1, \ldots, I^T\}$ satisfying $I^{t-1} \subsetneq I^t$ for each $t = 2, \ldots, T$ and $I^T = I$. Imagine that the agents engage in a sequence of bargaining: in round 1, agents in I^1 bargain from U to a set $V^1 \subset U$, and in round $t = 2, \ldots, T$, agents in set I^t bargain from V^{t-1} to a set V^t. The bargaining protocol in each round t is a generalized Nash bargaining game (Kalai, 1977) in which each agent $i \in I^t$ has a bargaining power $\psi_i^t > 0$ such that $\sum_{i \in I^t} \psi_i^t = 1$ and a disagreement payoff 0. More specifically, for bargaining units $\mathcal{I} = \{I^1, \ldots, I^T\}$ and bargaining powers $\Psi = (\psi^1, \ldots, \psi^T)$ satisfying the above requirement, we let $V^t := \arg\max_{u \in V^{t-1}} \prod_{i \in I^t} u_i^{\psi_i^t}$ for each $t = 1, \ldots, T$ with $V^0 := U$. Then, we call any

[24] As an aside, in Appendix F, we illustrate how to use some of the techniques established in our proof of Theorem 1 to offer a new proof of the second welfare theorem that allows for weaker assumptions than the standard treatment. We discuss this more in the paper's conclusion section.

$u \in V^T$ a *sequential Nash bargaining solution* (SNBS) over U for $\mathcal{I}$ and Ψ, and call u an SNBS over U if there exist such $\mathcal{I}$ and Ψ.

Observe now that SNBS implements the SUWM procedure for the logarithmic transforms of utilities. Namely, u is an SNBS over $U \subset \mathbb{R}^n_{++}$ if and only if $v := (\ln u_1, ..., \ln u_n)$ is an SUWM solution of $V := \{(\ln u'_1, ..., \ln u'_n) : (u'_1, ..., u'_n) \in U\}$. This connection also makes it clear that SNBS provides another characterization of Pareto optima.

Proposition D.1. A vector $u \in U \cap \mathbb{R}^n_{++}$ is Pareto optimal if and only if u is an SNBS over U.

Proof. For any $u \in \mathbb{R}^n_{++}$, let $\log u := (\log u_i)_i$ and, moreover, for any $u \in \mathbb{R}^n$, let $e^u := (e^{u_i})_i$. Let us also redefine $U := U \cap \mathbb{R}^n_{++}$ for notational simplicity. Now, let $\tilde{U} := \{\log u | u \in U\}$.

Claim 3. Suppose $u \in U$ and let $\tilde{u} = \log u$. Then, u is Pareto optimal with respect to U if and only if $\tilde{u}$ is Pareto optimal with respect to $\text{dc}(\tilde{U})$.

Proof. First, note that $u \in U$ is Pareto optimal with respect to U if and only if $\tilde{u}$ is Pareto optimal with respect to $\tilde{U}$ because $\log(.)$ is a strictly increasing function. Second, note that $\tilde{u} \in \tilde{U}$ is Pareto optimal with respect to $\tilde{U}$ if and only if it is Pareto optimal with respect to $\text{dc}(\tilde{U})$ because Pareto optimality is invariant to adding utility vectors to a set that are smaller than existing utility vectors. These two observations imply the conclusion of this claim. □

Claim 4. Suppose that U is convex. Then $\text{dc}(\tilde{U})$ is convex.

Proof. Suppose $\tilde{u}, \tilde{u}' \in \text{dc}(\tilde{U})$, and $\lambda \in [0, 1]$. By definition of $\text{dc}(.)$, it follows that there exist $\tilde{v}, \tilde{v}' \in \tilde{U}$ such that $\tilde{u} \leq \tilde{v}, \tilde{u}' \leq \tilde{v}'$. Therefore, by definition of $\tilde{U}$, there exist $v, v' \in U$ such that $\tilde{v} = \log v, \tilde{v}' = \log v'$.

Because U is convex, $w := \lambda v + (1 - \lambda)v'$ is in U. This implies that $\tilde{w} := \log w$ is in $\tilde{U}$. Now, because $\log(.)$ is a concave function, we have that

$$\lambda\tilde{v} + (1 - \lambda)\tilde{v}' = \lambda \log v + (1 - \lambda) \log v' \leq \log(\lambda v + (1 - \lambda)v') = \log w = \tilde{w},$$

so $\lambda\tilde{v} + (1 - \lambda)\tilde{v}' \in \text{dc}(\tilde{U})$. Because $\tilde{u} \leq \tilde{v}$ and $\tilde{u}' \leq \tilde{v}'$, it follows that $\lambda\tilde{u} + (1 - \lambda)\tilde{u}' \in \text{dc}(\tilde{U})$, as desired. □

Now we proceed to prove the theorem.

The "if" direction: Suppose that $u \in U$ is an SNBS over U for some bargaining units $\mathcal{I}$ and bargaining powers Ψ (satisfying the requirement). Then, $u \in V^T$ where $V^T = U$ and $V^t := \arg\max_{v \in V^{t-1}} \prod_{i \in I^t} v_i^{\psi_i^t}$ for each $t \geq 1$. This implies that $V^t := \arg\max_{v \in V^{t-1}} \sum_{i \in I^t} \psi_i^t \log v_i$. Setting $\tilde{u} := \log u$ and noting that ψ is a nonnegative and eventually positive sequence, $\tilde{u}$ is a SUWM solution of $\text{dc}(\tilde{U})$ with respect to ψ. Therefore, by Theorem 1, $\tilde{u}$ is Pareto optimal in $\text{dc}(\tilde{U})$. Then, by Claim 3, u is Pareto optimal with respect to U, as desired.

The "only if" direction: Suppose that $u \in U$ is Pareto optimal with respect to U. Then, by Claim 3, $\tilde{u} := \log u$ is Pareto optimal with respect to $\text{dc}(\tilde{U})$. Therefore, by Theorem 1, there exist a sequence $\phi := (\phi^t)_t$ of nonnegative and eventually positive welfare weight vectors such that $\tilde{u} \in \tilde{U}^T$ where $\tilde{U}^0 = \mathrm{dc}(\tilde{U})$ and $\tilde{U}^t := \arg\max_{\tilde{u}' \in \tilde{U}^{t-1}} \sum_{i \in I^t} \phi_i^t \tilde{u}_i'$ for each $t \geq 1$. Then, for $u = e^{\tilde{u}}$, we have $u \in U^T$, where $U^t := \arg\max_{u' \in U^{t-1}} \prod_{i \in I^t} (u_i')^{\phi_i^t} = \arg\max_{u' \in V^{t-1}} \prod_{i \in I^t} (u_i')^{\psi_i^t}$ for each $t \geq 1$, where $V^t := \{e^{\tilde{v}} | \tilde{v} \in U^t\}$ and $\psi_i^t := \frac{\phi_i^t}{\sum_{j \in I^t} \phi_j^t}$, so u is an SNBS, as desired (note that ψ satisfies the condition required of bargaining powers for SNBS). $\square$

This result provides a behavioral interpretation of our near-weighted utilitarian welfare maximization. Despite this close connection, we will see that the SNBS characterization differs in the social welfare ordering it induces from our near-weighted utilitarian characterizations. To see this, we first define the welfare ordering induced by SNBS. We say u *sequentially Nash welfare dominates* v according to bargaining units $\mathcal{I}$ and bargaining powers Ψ if u is an SNBS over $\{u, v\}$ for $\mathcal{I}$ and Ψ.

Since SNBS implements the SUWM procedure for the logarithmic transforms of utilities, Theorem 3 implies that the following axiom would fulfill the same role as Invariance.

- Log Invariance: for any $u, v \in \mathbb{R}^n_{++}$, if $u \succeq v$, then $u' \succeq v'$ for any $u', v' \in \mathbb{R}^n_{++}$ such that, for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}_{++}$, $\ln u_i' = a_i + b \ln u_i$ and $\ln v_i' = a_i + b \ln v_i$ for all $i \in I$.

Combining this axiom with the Pareto Principle and Weak Continuity defined earlier, we obtain the following axiomatization of the welfare ordering based on SNBS.

Corollary D.1. Let $\succeq$ be a social welfare ordering defined on $\mathbb{R}^n_{++}$. Then, the following statements are equivalent.

(i) $\succeq$ satisfies the Pareto Principle, Log Invariance, and Weak Continuity.

(ii) There exist bargaining units $\mathcal{I}$ and bargaining powers Ψ such that for any $u, v \in \mathbb{R}^n_{++}$, $u \succeq v$ if and only if u sequentially Nash welfare dominates v according to $\mathcal{I}$ and Ψ.

Proof. To prove (ii) implies (i), we only check that $\succeq$ satisfies Log Invariance since the other axioms are rather straightforward to check. To do so, suppose that $u \succeq v$ so that for some $t \leq T$, $\prod_{i \in I^s} u_i^{\psi_i^s} = (>) \prod_{i \in I^s} v_i^{\psi_i^s}$ for $s < (=) t+1$, which implies

$$\sum_{i \in I^s} \psi_i^s \ln u_i = (>) \sum_{i \in I^s} \psi_i^s \ln v_i \text{ for } s < (=) t+1. \tag{18}$$

Consider now any u', v' such that for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}_{++}$, $\ln u_i' = a_i + b \ln u_i$ and $\ln v_i' = a_i + b \ln v_i$ for all $i \in I$. By (18), we have

$$\begin{aligned} \sum_{i \in I^s} \psi_i^s \ln u_i' &= \sum_{i \in I^s} \psi_i^s a_i + b \left(\sum_{i \in I^s} \psi_i^s \ln u_i \right) \\ &= (>) \sum_{i \in I^s} \psi_i^s a_i + b \left(\sum_{i \in I^s} \psi_i^s \ln v_i \right) = \sum_{i \in I^s} \psi_i^s \ln v_i' \text{ for } s < (=) t+1, \end{aligned}$$

which implies $\prod_{i \in I^s} (u_i')^{\psi_i^s} = (>) \prod_{i \in I^s} (v_i')^{\psi_i^s}$ for $s < (=) t+1$ or $u' \succeq v'$ as desired.

We now prove that (i) implies (ii). Given any $u \in \mathbb{R}^n$, let e^u denote a vector $(e^{u_i})_{i\in I}$ and $\ln u$ denote a vector $(\ln u_i)_{i\in I}$ for simplicity. Consider any welfare ordering $\succeq$ on $\mathbb{R}^n_{++}$ that satisfies the three axioms. Let us define another ordering $\tilde{\succeq}$ on $\mathbb{R}^n$ as follows: for any $u, v \in \mathbb{R}^n$, $u \tilde{\succeq} v$ if $e^u \succeq e^v$. It is straightforward to check that $\tilde{\succeq}$ satisfies Pareto Principal, Invariace, and Weak Continuity. In particular, Invariance holds for the following reason. Consider any u, v such that $u \tilde{\succeq} v$ or equivalently $\tilde{u} := e^u \succeq e^v =: \tilde{v}$. Invariance requires that for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}_{++}$, $u' := (a+bu) \tilde{\succeq} (a+bv) =: v'$ or equivalently $\tilde{u}' := e^{u'} \succeq e^{v'} =: \tilde{v}'$, which follows from $\tilde{u} \succeq \tilde{v}$ and Log Invariance since $\ln \tilde{u}' = a + b \ln \tilde{u}$ and $\ln \tilde{v}' = a + b \ln \tilde{v}$. Since $\tilde{\succeq}$ satisfies the Pareto Principle, Invariance, and Weak Continuity, Theorem 3 implies that there exists a nonnegative and eventually positive sequence of weight vectors $\Phi = (\phi^1, \phi^2, ..., \phi^T)$ such that for any $u, v \in \mathbb{R}^n$, $u \tilde{\succeq} v$ if and only if u sequentially utilitarian welfare dominates v according to Φ. For each $t = 1, \ldots, T$, let $I^t = \operatorname{supp} \psi^t$ and $\psi^t_i = \frac{\phi^t_i}{\sum_{i\in I^t} \phi^t_i}$ for all $i \in I^t$. Consider any u, v with $u \succeq v$. Then, $\tilde{u} := \ln u \tilde{\succeq} \ln v =: \tilde{v}$ so that $\tilde{u}$ sequentially utilitarian welfare dominates $\tilde{v}$ according to $\Psi = (\psi^1, \ldots, \psi^T)$: that is, for some $t \leq T$, $\sum_{i\in I^s} \psi^s_i \tilde{u}_i = (>) \sum_{i\in I^s} \psi^s_i \tilde{v}_i$ for $s < (=) t+1$, which implies that $\prod_{i\in I^s} u_i^{\psi^s_i} = (>) \prod_{i\in I^s} v_i^{\psi^s_i}$ for $s < (=) t+1$, meaning u sequentially Nash welfare dominates v according to $\mathcal{I}$ and Ψ.

It is straightforward, and thus omitted, to prove that u sequentially Nash welfare dominating v according to $\mathcal{I}$ and Ψ implies $u \succeq v$. □

In particular, this corollary implies that while SNBS characterizes Pareto optimality, the welfare orderings implied by the criterion depart further from utilitarianism than our "near"-utilitarian welfare criteria. While it shares the Pareto Principle and Weak Continuity, it generally fails Invariance.

D.2 Piecewise linear concave welfare function

Some readers may not like the sequentiality of SUWM or the use of hyperreal numbers in LHUWM. This observation leads to the question of whether it is possible to characterize Pareto optima by a one-shot maximization of a real-valued welfare function. For such a characterization, the welfare function cannot be weighted utilitarian. In particular, the function must be nonlinear. Can we achieve the characterization with minimal relaxation of the linearity? This motivates the following approach.

A social welfare function W is a *piecewise linear concave (PLC) welfare function* characterized by $(\psi^1, \psi^2, \ldots, \psi^t)$ if

$$W(v) = \min_{t\in\{1,\ldots,T\}} \langle \psi^t, v \rangle, \tag{19}$$

where $\psi^t \in \mathbb{R}^n_+$ for each t. One candidate for the weight vectors $(\psi^1, \psi^2, \ldots, \psi^T)$ to construct a PLC welfare function are those identified in the SUWM characterization; i.e., eventually positive weights. However, the characterization does *not* hold without an auxiliary condition. For this condition, let us say that a PLC welfare function W *achieves its maximum over U via eventually positive weights* if (i) $(\psi^1, \psi^2, \ldots, \psi^T)$ is nonnegative and eventually positive and (ii) for all $v \in \arg\max_{u'\in U} W(u')$, $W(v) = \langle \psi^T, v \rangle$.

Proposition D.2. Let U be a closed convex subset of $\mathbb{R}^n$. Then, $u \in U \cap \mathbb{R}^n_{++}$ is Pareto optimal if and only if it maximizes a PLC welfare function that achieves its maximum over U via eventually positive weights.[25,26]

Proof. The "only if" direction: By Theorem 1, for any Pareto optimal $u \in U \cap \mathbb{R}^n_{++}$, there are nonnegative and eventually positive weights $(\phi^1, \ldots, \phi^T)$ sequentially maximized by u. Letting U^t be defined as in (3), we have $u \in U^t$ for all $t = 1, \ldots, T$. Consider weights $(\psi^t)_{t=1}^T$ defined as $\psi^1 = \phi^1$ and $\psi^t = \frac{\langle \psi^{t-1}, u \rangle}{\langle \phi^t, u \rangle} \phi^t$ for each $t \geq 2$. First, using the fact that $u \in \mathbb{R}^n_{++}$ and $\phi^t \in \mathbb{R}^n_+, \forall t$, it is straightforward to see that $\langle \psi^t, u \rangle > 0$ and $\langle \phi^t, u \rangle > 0$ for every t. Thus, $\langle \psi^t, v \rangle \geq \langle \psi^t, v' \rangle$ if and only if $\langle \phi^t, v \rangle \geq \langle \phi^t, v' \rangle$ for all $v, v' \in U$. Note also that $\langle \psi^t, u \rangle = \frac{\langle \psi^{t-1}, u \rangle}{\langle \phi^t, u \rangle} \langle \phi^t, u \rangle = \langle \psi^{t-1}, u \rangle$ for each $t \geq 2$. Thus, we have $W(u) = \langle \psi^t, u \rangle$ for all $t = 1, \ldots, T$. Also, for any $v \in U^T$, we have $\langle \psi^t, u \rangle = \langle \psi^t, v \rangle$ for all t, so $W(u) = W(v)$. For any $v \notin U^T$, there is some t such that $v \notin U^t$ so $\langle \psi^t, v \rangle < \langle \psi^t, u \rangle$, implying $W(v) < W(u)$. Thus, u maximizes W, implying that W achieves its maximum over U via eventually positive weights.

The "if" direction: Consider any $u \in U$ maximizing a PLC function W that achieves its maximum via eventually positive weights. Suppose for contradiction that u is not Pareto optimal. Then, there is some $v > u$ so that $\langle \psi^t, v \rangle \geq \langle \psi^t, u \rangle$ for all $t = 1, \ldots, T$. As u maximizes W, so does v. Given this and the fact that W achieves its maximum via eventually positive weights, we must have $\langle \psi^T, v \rangle = W(v) = W(u) = \langle \psi^T, u \rangle$ or $\langle \psi^T, v - u \rangle = 0$, which is a contradiction since $\psi^T \gg 0$ and $v > u$. □

The role of the auxiliary condition is to prevent a Pareto suboptimal point from maximizing the PLC function (so that the "if" direction holds). To see it, observe that for any Pareto suboptimal point u, one can find $v > u$ so that $W(v) \geq W(u)$. If u were a maximizer of W, then the auxiliary condition would require $W(v) = \langle \psi^T, v \rangle = \langle \psi^T, u \rangle = W(u)$ or $\langle \psi^T, v - u \rangle = 0$, which cannot hold since $\psi^T \gg 0$ and $v > u$. While achieving the goal of characterizing Pareto optima, the auxiliary condition also captures the main feature of SUWM that every agent's welfare must count as it requires a PLC function to be maximized via a weight vector that puts a positive weight on every agent's utility.

While our PLC welfare functions successfully characterize Pareto optima, we regard them to be further away from utilitarianism than our "near"-utilitarian welfare criteria. This is because, to our knowledge, no natural axioms characterize PLC welfare functions. In fact, it

[25]We focus on points $u \in \mathbb{R}^n_{++}$ for technical simplicity. This is not a substantive restriction because the economic environment is arguably unchanged when a constant is added to all utility profiles.

[26]This proposition may be reminiscent of construction of a PLC utility function based on an individual's choice data (see Afriat (1967)). The PLC social welfare function reveals the planner's preferences for agents' utilities similarly to how Afriat's PLC utility function reveals an individual's preferences for alternative goods. Note, however, that there are clear differences. The multiple linear components of our PLC welfare function result from multiple welfare weights corresponding to the successive rounds of SUWM. By contrast, the linear components in Afriat's construction reflect different budget lines a consumer faces in different choice scenarios. Moreover, the role played by the auxiliary condition to ensure every agent's welfare counts has no analogue in Afriat's characterization.

is not even obvious how to formulate a PLC function as a social welfare ordering in the face of the auxiliary condition. For instance, define the binary relation $\succeq$ by $u \succeq v$ if $W(u) \geq W(v)$ and $W(u) = \langle \psi^T, u \rangle$. Note that the condition $W(u) = \langle \psi^T, u \rangle$ is an adaptation of the auxiliary condition to the context of social welfare ordering. Then, $\succeq$ is not necessarily a complete binary relation, as the following example shows.

Example 1. Let there be two agents 1 and 2, $T = 2$, $\psi^1 = (1, 0)$, $\psi^2 = (1, 1)$, $u = (1, 1)$ and $v = (0, 0)$. Then, we have $W(u) = 1 > 0 = W(v)$ while $W(u) = 1 < 2 = \langle \psi^2, u \rangle$, so neither $u \succeq v$ nor $v \succeq u$ holds. Hence, $\succeq$ is not complete.

E Pareto optimality (U^P) and positive utilitarianism (U^{++})

This section aims to discover natural conditions for U^P to coincide with U^{++}. The following lemma, which follows easily from the proof of Theorem 1, is the key to our investigation.

Lemma E.1. If u is a maximal element of U that lies in the relative interior of an exposed face of $\mathrm{dc}(U)$ then u maximizes a positive weight vector over U.

Proof. In the proof of the "only if" part of Theorem 1 in Appendix A.3.2, if u is a maximal element of U that lies in the relative interior of an exposed face of $\mathrm{dc}(U)$, then $T = 1$ in Step 3 and by Step 4 we know ϕ^1 is positive. Hence, $\Phi = (\phi^1)$ and so by Step 5, we conclude that u maximizes the positive weight vector ϕ^1 over U. □

To characterize when $U^P = U^{++}$, we need to introduce a few notions and establish their properties. First, the *normal cone of U at a point $u \in U$* is the set

$$N_U(u) = \{\phi \in \mathbb{R}^n \mid \langle \phi, u \rangle \geq \langle \phi, v \rangle \text{ for all } v \in U\}.$$

If $\phi \in N_U(u)$ then u is a maximizer of the linear function $\langle \phi, u \rangle$ over the set U. Then, *the normal cone of a face $F \subset U$*, denoted $N_U(F)$, as the normal cone of each of its relative interior points. Next, the *relative boundary of F* is defined as $F \setminus \mathrm{ri}(F)$.

The next two lemmas give us some properties of these notions.

Lemma E.2. Let F be a face of a convex set U. Then, every point in the relative interior of F has the same normal cone.

Proof. Let u, u' be distinct in the relative interior of F and suppose $N_U(u)$ contains an element ϕ not in $N_U(u')$. This implies $\langle \phi, u \rangle > \langle \phi, u' \rangle$. Since u is the relative interior, the point $v = u + \lambda(u - u')$ lies in F for a sufficiently small positive λ. However, $\langle \phi, v \rangle = \langle \phi, u \rangle + \lambda \langle \phi, u - u' \rangle > \langle \phi, u \rangle$, violating the assumption that ϕ is in $N_U(u)$. □

Lemma E.3. Let F be a face of a convex set U. Then, every point u in the relative boundary of F has $N_U(u) \supset N_U(F)$.

Proof. Let u be in the relative boundary of F. Suppose there is a weight vector ϕ in $N_U(v)$ (where v is any relative interior element of F) that is not in $N_U(u)$. That is,

$$\langle \phi, u \rangle \neq \langle \phi, v \rangle. \tag{20}$$

By the definition of the relative interior, we can get an element of the relative interior of F arbitrarily close to u, which yields a contradiction of the continuity of $\langle \phi, \cdot \rangle$ because of (20). □

We are now ready to provide the condition that characterizes when $U^P = U^{++}$:

Proposition E.1. Let U be a closed convex set. Then $U^P = U^{++}$ if and only if every maximal element of U belongs to some exposed maximal face of $\mathrm{dc}(U)$.

Proof. The "if" direction. Observe that $U^{++} \subset U^P$ is immediate from Proposition 3.23 in Bewley (2009). It remains to show that $U^P \subset U^{++}$. Let $u \in U^P$. If u lies in the relative interior of an exposed face of $\mathrm{dc}(U)$, then $u \in U^{++}$ from Lemma E.1. The remaining case is where u lies on the relative boundary of a maximal exposed face F of $\mathrm{dc}(U)$. Since F is a maximal exposed face, then an element v in its relative interior maximizes a positive weight vector ϕ, again by Lemma E.1. By Lemma E.2, this implies that the normal cone $N_U(F)$ of face F contains ϕ and so, by Lemma E.3, the normal cone $N_U(u)$ of the point u contains ϕ. In other words, u maximizes the positive weight vector ϕ. This completes the proof.

The "only if" direction. Let u be a maximal element of U. By the equivalence of U^P and U^{++}, u maximizes a positive weight vector ϕ. Let $F = \arg\max_{v \in U} \langle \phi, v \rangle$. We claim that F is a maximal exposed face of $\mathrm{dc}(U)$, which contains u. The fact that F is maximal in $\mathrm{dc}(U)$ follows since Proposition 3.23 in Bewley (2009) (along with Lemma A.2) implies F is maximal in U and thus maximal in $\mathrm{dc}(U)$ by Lemma A.4. Suppose to the contrary that F is not exposed in $\mathrm{dc}(U)$. Then, there must exist an element $u' \in \mathrm{dc}(U) \setminus U$ that maximizes ϕ but is not in F. However, since $u' \in \mathrm{dc}(U) \setminus U$, there must exist $u'' \in U$ such that $u' \leq u''$ and $u'_i < u''_i$ for some index i. But this implies that $\langle \phi, u \rangle \geq \langle \phi, u'' \rangle > \langle \phi, u' \rangle$, where the weak inequality holds by the definition of F and the strict inequality holds since ϕ is positive. This yields a contradiction and so we conclude that F is an exposed face of $\mathrm{dc}(U)$. □

We now discuss a few of the nuances in the statement of Proposition E.1. First, the condition cannot be weakened so that every maximal element of U simply lies in a (potentially nonmaximal) exposed face of $\mathrm{dc}(U)$. Consider our canonical example in Figure 1. The point u lies on an exposed face of $\mathrm{dc}(U)$, but this face is not a maximal face of $\mathrm{dc}(U)$.

Figure 1 also demonstrates that it is not sufficient for a point to lie on a maximal exposed face of U (as opposed to $\mathrm{dc}(U)$) to guarantee it maximizes a positive weight vector. Consider the point u'', which is a maximal exposed extreme point of U but does not maximize any positive weight vector over U. However, u'' does not lie on a maximal exposed face of $\mathrm{dc}(U)$ and so does not contradict the theorem.

Given the above nuance, a simpler sufficient condition may be useful. Consider the setting where all maximal faces of $\mathrm{dc}(U)$ are exposed.

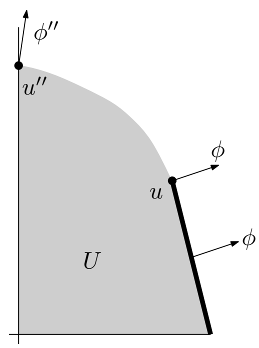

Figure 6: The maximal extreme point u is not exposed while $U^P = U^{++}$.

Corollary E.1. If U is a closed convex set such that all maximal faces of $\mathrm{dc}(U)$ are exposed, then $U^P = U^{++}$.[27]

Proof. Note that every maximal element of U lies in a maximal face of $\mathrm{dc}(U)$ by Lemma A.4. This and the hypothesis imply that every maximal element of U belongs to some exposed maximal face of $\mathrm{dc}(U)$. Applying Proposition E.1, we obtain the desired conclusion. □

However, the converse of Corollary E.1 is false, as illustrated by the example in Figure 6. One sufficient condition for the hypothesis of Corollary E.1 to hold is that U is a polyhedron. In that case, all faces of U are exposed (Theorem 13.21 of Soltan (2015)); moreover, its downward closure of a polyhedron is also a polyhedron (Theorem 13.20 of Soltan (2015)), so all of its faces are exposed.

Let X be a polyhedral subset of $\mathbb{R}^m_+$ (possibly $\mathbb{R}^m_+$ itself). The utility function $u_i : X \to \mathbb{R}$ is *piecewise-linear concave (PLC)* if there exist finite index set K_i and affine functions $u_{i,k} : \mathbb{R}^m_+ \to \mathbb{R}$ for each $k \in K_i$ such that $u_i(x) = \min_{k \in K_i} u_{i,k}(x)$ for all $x \in X$. The lemma uses some of the following facts.

Lemma E.4. The following properties on polyhedra hold:

(i) Let $P_1, P_2, \ldots, P_n$ be a finite collection of polyhedra in $\mathbb{R}^m$. The Cartesian product $P_1 \times P_2 \times \cdots \times P_n$ is a polyhedron in $\mathbb{R}^{mn}$.
(ii) Let $\pi : \mathbb{R}^d \to \mathbb{R}^m$ be an affine map and let P be a polyhedron in $\mathbb{R}^d$. Then $\pi(P)$ is a polyhedron.
(iii) All faces of a polyhedron are exposed.
(iv) The downward closure of a polyhedron is also a polyhedron.

Proof. (i) Consider two polyhedra in $\mathbb{R}^m$, P_1 and P_2. Letting $Q_1 := P_1 \times \mathbb{R}^m$ and $Q_2 := \mathbb{R}^m \times P_2$, each Q_k is a polyhedron in $\mathbb{R}^{2m}$, so $P_1 \times P_2 = \cap_{k=1,2} Q_k$ is a polyhedron in $\mathbb{R}^{2m}$. The

[27] This cannot be derived easily from Arrow, Barankin, and Blackwell (1953). To see this, recall that they establish $U^{++} \subset U^P \subset \mathrm{cl}(U^{++})$. This implies that if U^{++} is closed then $U^P = U^{++}$. However, in the "tilted cone" in Figure 2, U^{++} is not closed since the point K does not lie in U^{++} but is the limit point of elements in U^{++} (indicated by the line in the figure). However, it is straightforward to check that U^P and U^{++} coincide. One can also check that all maximal faces of $\mathrm{dc}(U)$ for U in Figure 2 are exposed, the condition of Corollary E.1.

result follows from applying this argument repeatedly. (ii) This is Theorem 13.21 in Soltan (2015). (iii) This is Corollary 13.12 in Soltan (2015). (iv) This follows by nothing that since $\mathrm{dc}(P) = P + \mathbb{R}^n_-$ where $\mathbb{R}^n_-$ is the nonpositive orthant of $\mathbb{R}^n$ and by applying Theorem 13.20 of Soltan (2015). □

Lemma E.5. If each agent has a PLC utility function defined on a polyhedron X and U is defined according to (1), then $\mathrm{dc}(U)$ is a polyhedron.

Proof. For each $k \in K_i$, let $X_{i,k} = \{x \in X \mid u_{i,k}(x) \le u_{i,k'}(x), \forall k' \in K_i\}$. Since X is a polyhedron and all functions $(u_{i,k})_{k \in K_i}$ are affine, $X_{i,k}$ is an intersection of finitely many polyhedra and thus a polyhedron.

Now let $\mathcal{K} = \{\mathbf{k} = (k_i)_{i \in I} \mid k_i \in K_i \text{ for all } i\}$. For each $\mathbf{k} \in \mathcal{K}$, let $X_{\mathbf{k}} = \cap_{i \in I} X_{i,k_i}$ and observe that $X_{\mathbf{k}}$ is a polyhedron. Also, all functions $u_1(\cdot), \ldots, u_I(\cdot)$ are affine on $X_{\mathbf{k}}$ since for each $i \in I$, $u_i(x) = u_{i,k_i}(x), \forall x \in X_{\mathbf{k}}$. Then, by Lemma E.4(ii), the set $U_{\mathbf{k}} = \{(u_i(x))_{i \in I} \mid x \in X_{\mathbf{k}}\}$ is a polyhedron. Observe that $U = \{(u_i(x))_{i \in I} \mid x \in X\} = \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}}$. While we do not know whether the set U, which is a union of polyhedra, is a polyhedron, Theorem 13.19 of Soltan (2015) shows that $\overline{U} := \mathrm{cl}(\mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}})$ is a polyhedron, where cl and conv denote the closure and convex hull, respectively.

Next, we show that $\mathrm{dc}(U) = \mathrm{dc}(\overline{U})$. By definition of $\overline{U}$, $\mathrm{dc}(U) \subset \mathrm{dc}(\overline{U})$ is clear. To show $\mathrm{dc}(\overline{U}) \subset \mathrm{dc}(U)$, consider any $\tilde{u} \in \mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}}$ so that $\tilde{u} = \sum_{\mathbf{k} \in \mathcal{K}} \lambda_{\mathbf{k}} \tilde{u}_{\mathbf{k}}$ for some weight $(\lambda_{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}}$ and $\tilde{u}_{\mathbf{k}} \in \cup_{\mathbf{k}' \in \mathcal{K}} U_{\mathbf{k}'}$. Also, for each $\tilde{u}_{\mathbf{k}}$, we can find $\tilde{x}_{\mathbf{k}} \in X_{\mathbf{k}}$ such that $(u_i(\tilde{x}_{\mathbf{k}}))_{i \in I} = \tilde{u}_{\mathbf{k}}$. Letting $x = \sum_{\mathbf{k} \in \mathcal{K}} \lambda_{\mathbf{k}} \tilde{x}_{\mathbf{k}}$, observe that $x \in X$ by the convexity of X and that for all $i \in I$, $u_i(x) \ge \sum_{\mathbf{k} \in \mathcal{K}} \lambda_{\mathbf{k}} u_i(\tilde{x}_{\mathbf{k}}) = \tilde{u}_i$ by the concavity of $u_i(\cdot)$, which means that $\tilde{u} \in \mathrm{dc}(U)$. Thus, $\mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}} \subset \mathrm{dc}(U)$, implying that $\mathrm{cl}(\mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}}) \subset \mathrm{dc}(U)$ since $\mathrm{dc}(U)$ is closed, from which $\mathrm{dc}(\overline{U}) \subset \mathrm{dc}(U)$ follows, as desired.

Lastly, observe that $\mathrm{dc}(\overline{U}) = \overline{U} + \mathbb{R}^n_-$ and that both $\overline{U}$ and $\mathbb{R}^n_-$ are polyhedra, which implies (by Lemma E.4(iv)) that $\mathrm{dc}(\overline{U}) = \mathrm{dc}(U)$ is a polyhedron. □

The following is obtained immediately from Corollary E.1 and Lemma E.5, and the fact that all faces of polyhedra are exposed. It is a clean economic setting where U^P and U^{++} coincide.

Proposition E.2. If each agent has a PLC utility function defined on a polyhedron X and U is defined according to (1), then $U^P = U^{++}$.[28]

F Second welfare theorem with piecewise-linear concave utility functions

In the paper, we showed that the notions of exposed faces and normal vectors play crucial roles for our characterization of a Pareto optimal utility profile as a welfare-maximizing

[28]It is worth noting that the ABB theorem provides an alternative proof of this result. Recall that it suffices to argue U^{++} is closed in order to conclude $U^P = U^{++}$. Clearly, the elements of U^{++} come in faces, and a polyhedron has finitely many faces. Since the faces of a polyhedron are closed, and a finite union of closed sets is closed, this implies that U^{++} is closed.

point. Recall that the normal vector also plays an important role in the second theorem of welfare economics in identifying a price vector that supports a Pareto optimal allocation as a competitive equilibrium outcome. Unlike in our characterization, the idea of a normal vector in the second welfare theorem applies to the space of goods, not the space of utility profiles. However, the fact that the two spaces are closely connected hints at the possibility of establishing the second welfare theorem using the machinery we have developed so far. We do so in the current section under a set of assumptions on the agent preferences and endowments that generalize the existing welfare theorem in a certain direction.

To begin, consider an exchange economy with m types of goods with some integer $m > 0$. For each $k \in \{1, \ldots, m\}$, let $\bar{e}^k > 0$ be the total supply of type-k goods in the environment. Let $\bar{e}$ denote the vector $(\bar{e}^k)_{k=1}^m$. Each alternative $x = (x_i)_{i \in I}$, $x_i = (x_i^k)_{k=1}^m \in \mathbb{R}_+^m$, specifies consumption bundle x_i for each $i \in I$. A profile of consumption bundles x is said to be feasible if and only if $\sum_{i \in I} x_i \leq \bar{e}$. In this context, the choice set X is defined as the set of all feasible profiles of consumption bundles. Each individual $i \in I$ is endowed with a utility function $u_i : \mathbb{R}_+^m \to \mathbb{R}$. Suppose that each agent i is endowed with a vector of goods $e_i \in \mathbb{R}_+^m \backslash \{0\}$ and let $\bar{e} = \sum_{i \in I} e_i$. A vector $p \in \mathbb{R}^m$ is referred to as a price profile. A pair (p, x) of a price profile p and a profile $x = (x_i)_{i \in I}$ of consumption bundles is a *Walrasian equilibrium* if

1. $\sum_{i \in I} x_i = \bar{e}$, and
2. $x_i \in \arg\max_{y_i \in B_i(p)} u_i(y_i)$ for each $i \in I$, where $B_i(p) := \{y_i \in \mathbb{R}_+^m \mid \langle p, y_i \rangle \leq \langle p, e_i \rangle\}$ is the budget set of i.

We consider a case where utility functions of all players are piecewise-linear concave (PLC), as defined in Appendix E. PLC utility functions may appear somewhat restrictive, but any concave function can be approximated arbitrarily closely by a PLC utility function (Bronshtein and Ivanov, 1975). Meanwhile, we make a weaker assumption in another dimension—preference monotonicity. The existing second welfare theorem assumes agents' utility functions to be strictly monotonic. We invoke a weaker form of monotonicity. Say that an allocation $(x_i)_{i \in I}$ is *strictly feasible for good k* if it is feasible and satisfies $\sum_{i \in I} x_i^k < \bar{e}^k$. We assume that the agent preferences are *monotonic under limited resources* in the following sense: for any allocation $(x_i)_{i \in I}$ that is strictly feasible for good k, there exist an agent j and $\tilde{x}_j \in \mathbb{R}_+^m$ such that $u_j(\tilde{x}_j) > u_j(x_j)$ while $\tilde{x}_j^{k'} = x_j^{k'}, \forall k' \neq k$, $\tilde{x}_j^k > x_j^k$, and $\tilde{x}_j^k + \sum_{i \neq j} x_j^k \leq \bar{e}^k$. That is, given any allocation that does not exhaust the endowment of good k, there exists an agent who gets better off by consuming more of that good within its endowment. This condition is fairly weak. For instance, it allows for agents to consider a certain good indifferently, or even as bads (rather than goods), as long as there is at least one agent who likes to consume that good. We are now ready to prove the second welfare theorem under the above assumptions.

Proposition F.1. Consider the exchange economy described above. If $(u_i(e_i))_{i \in I}$ is Pareto optimal, then there exists a positive price vector $p \gg 0$ such that $(p, (e_i)_{i \in I})$ is a Walrasian equilibrium.

Proof of Proposition F.1. Let $A_i := \{x \in \mathbb{R}_+^m \mid u_i(x) \geq u_i(e_i)\}$ for each agent i. Observe that each A_i is a polyhedron since it is an intersection of two polyhedra, $\{x \in \mathbb{R}^m \mid x \geq 0\}$ and $\{x \in \mathbb{R}^m \mid u_i(x) \geq u_i(e_i)\} = \cap_{k\in K_i}\{x \in \mathbb{R}^m \mid u_{i,k}(x) \geq u_i(e_i)\}$.

Consider the set $A = \left\{x \in \mathbb{R}^m_+ \mid \exists x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n \text{ s.t. } x = \sum_{i\in I} x_i\right\}$. Observe that A is the image of the set $A_1 \times A_2 \times \cdots \times A_n$ under the affine mapping π that maps $(x_i)_{i\in I}$ to $\sum_{i\in I} x_i$. By Lemma E.4(i) and (ii), A itself is a polyhedron.

Next, we argue that $\bar{e}$ is a minimal element of the set A. Suppose for contradiction that there exists an element $x \in A$ where $x < \bar{e}$ where $x^k < \bar{e}^k$ for some good k. Since $x \in A$, there exists an allocation $(y_i)_{i\in I}$ where $y_i \in A_i$ such that $x = \sum_{i\in I} y_i$. Since this allocation is strictly feasible for the good k, the monotone preference under limited resources implies that there are some agent j and $\tilde{y}_j \in \mathbb{R}^m_+$ such that $u_j(y_j) < u_j(\tilde{y}_j)$ while $\tilde{y}_j^{k'} = y_j^{k'}, \forall k' \neq k$, $\tilde{y}_j^k > y_j^k$, and $\tilde{y}_j^k + \sum_{i\neq j} y_i^k \leq \bar{e}^k$. Now consider an alternative allocation $(z_i)_{i\in I}$, which is identical to $(y_i)_{i\in I}$ except that $z_j = \tilde{y}_j$. Note that this allocation is feasible under the endowment $\bar{e}$ and that $u_j(z_j) > u_j(y_j) \geq u_j(e_j)$ while $u_i(z_i) = u_i(y_i) \geq u_i(e_i), \forall i \neq j$, which contradicts the Pareto optimality of $(e_i)_{i\in I}$.

That $\bar{e}$ is a minimal element of A implies that $-\bar{e}$ is a maximal element of $-A$. By Lemma A.4, this implies that $-\bar{e}$ is a maximal element of $\mathrm{dc}(-A)$. Moreover, by Lemma E.4(iv) $\mathrm{dc}(-A)$ is a polyhedron and so by Lemma E.4(iii) all of its faces are exposed. Thus, by Lemma E.1, there exists a supporting hyperplane of $-A$ through the point $-\bar{e}$ with a positive normal ϕ. The same normal $p := \phi$ can define a supporting hyperplane to A through the point $\bar{e}$; that is,

$$\langle p, y\rangle \geq \langle p, \bar{e}\rangle, \forall y \in A,$$

where p is a positive vector of prices.

It remains to show that the positive price vector p just constructed supports the allocation $(e_i)_{i\in I}$ as a Walrasian equilibrium. For this, it suffices to show that each e_i maximizes $u_i(\cdot)$ under the prices p and the budget $\langle p, e_i\rangle$. To do so, we take any x_i with $u_i(x_i) > u_i(e_i)$ and show that agent i cannot afford x_i.

By continuity of u_i, the inequality $u_i(x_i) > u_i(e_i)$ implies that for some $\lambda < 1$ but sufficiently close to 1, we have $u_i(\lambda x_i) > u_i(e_i)$, so by definition we have $\lambda x_i \in A_i$. This implies that $\lambda x_i + \sum_{j\neq i} e_j \in A$. Since $\langle p, \lambda x_i + \sum_{j\neq i} e_j\rangle \geq \langle p, \sum_{i\in I} e_i\rangle$, we must also have $\langle p, \lambda x_i\rangle \geq \langle p, e_i\rangle$. Dividing through by λ gives $\langle p, x_i\rangle \geq \langle \frac{1}{\lambda}p, e_i\rangle > \langle p, e_i\rangle$ where the strict inequality holds since e_i is nonnegative and nonzero while p is positive. □

In addition to the weakening of preference monotonicity, we also dispense with the typical assumption required by the existing second welfare theorem that every consumer has a positive endowment for every type of good (i.e., $e_i \gg 0, \forall i \in I$). The positive endowment assumption can be quite restrictive, excluding many realistic situations. Relaxing the same assumption was an important motivation behind Arrow's generalization of the first welfare theorem.[29] At the same time, the theorem assumes PLC utility functions. This

[29] "While listening to a talk about housing by Franko (sic) Modigliani, Arrow realized that most people consume nothing of most goods (for example, living in just one particular kind of house), and thus that the prevailing efficiency proofs assumed away all the realistic cases," according to Geanakoplos in https://www.econometricsociety.org/sites/default/files/inmemoriam/arrow_geanakoplos.pdf.

assumption guarantees that the "upper contour set" of the target allocation—or the set of goods weakly preferred to $(e_i)_{i \in I}$—is a polyhedron. Meanwhile, preference monotonicity and Pareto-optimality of $(u_i(e_i))_{i \in I}$ ensure that the vector $\bar{e}$ is a (minimal) face of this set. Invoking Proposition E.2, $\bar{e}$ is then exposed by a positive normal (or price vector) that supports $(e_i)_{i \in I}$ as a competitive equilibrium allocation.