"Near" weighted utilitarian characterizations of Pareto optima[1]

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Abstract

We give two characterizations of Pareto optimality via "near" weighted utilitarian welfare maximization. One characterization sequentially maximizes utilitarian welfare functions using a finite sequence of nonnegative and eventually positive welfare weights. The other maximizes a utilitarian welfare function with a certain class of positive hyperreal weights. The social welfare ordering represented by these "near" weighted

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utilitarian welfare criteria is characterized by the standard axioms for weighted utilitarianism under a suitable weakening of the continuity axiom.

Keywords: Pareto optima, weighted utilitarian welfare maximization, sequential utilitarian welfare maximization, lexicographic hyperreal utilitarian welfare maximization, weak continuity

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1 Introduction

Pareto optimality is a central concept in economics for its normative appeal. Also central is weighted utilitarian welfare maximization; e.g., Harsanyi (1955) famously defended it as a social welfare function based on several normative axioms. Moreover, weighted utilitarianism is widely invoked in practice, including applied research and policy debates. Given the prominent roles played by these two concepts, attempts have been made to establish a connection between Pareto optima and weighted utilitarianism—or more precisely, a characterization of Pareto optima via weighted utilitarian welfare maximization. Yet, such a characterization has so far been elusive.

It is well known that, given a closed and convex utility possibility set, which we assume throughout, every Pareto optimal utility vector maximizes some *nonnegatively* weighted sum of utilities of agents (see Proposition 3.45 in Bewley (2009)). But the converse is false: not every such maximizer is Pareto optimal. To see this, suppose a society consists of two agents, 1 and 2, and the utility possibility set is given by U in Figure 1. All points on

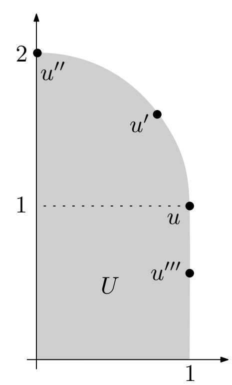

Figure 1: Weighted utilitarian welfare maximization need not yield a Pareto optimum.

the "outer" boundary, including the vertical segment, maximize suitably weighted sums of agents' utilities within U, but not all of them are Pareto optimal. In particular, the points on the vertical segment strictly below u, such as u''', all maximize the utility sum with weights $\phi = (1, 0)$—i.e., only 1's utility. Yet, none of these points is Pareto optimal. The reason is that the welfare of the agent receiving zero weight is *not* counted.

By contrast, if weights are restricted to be (strictly) positive for all agents, weighted utilitarian welfare maximization does always yield a Pareto optimum (Proposition 3.23 of Bewley (2009)). But the converse is false: not every Pareto optimal outcome can be obtained in this way. In Figure 1, u' is Pareto optimal and obtained by weighted utilitarian welfare maximization with positive weights, but u and u'', which are also Pareto optimal, cannot be obtained.

While positive welfare weights do not yield points like u in Figure 1, one may conjecture that they may "in the limit"; for instance, u is a limit of welfare-maximizing utility vectors with positive weights $(1, 1/n)$, as $n \to \infty$. Indeed, Arrow, Barankin, and Blackwell (1953) show that every Pareto optimal vector is a limit of a sequence of utility vectors that maximize some positively weighted sum of utilities—a result known as the ABB theorem.[1] Unfortunately, this too does not lead to a characterization when there are more than two agents:[2] again its converse is false—namely, a limit point of such a sequence may not be Pareto optimal. To see this, suppose there are three agents, 1, 2, and 3, with possible utility vectors depicted in Figure 2. The point K is a limit of the sequence of points maximizing a positively weighted sum of utilities (see the arrow) but is Pareto dominated, say, by the point V. The relationship between Pareto optima and the alternative notions of weighted utilitarianism is depicted in Figure 3, where U^P is the set of Pareto optimal utility vectors while U^+ and U^{++} are the sets of utility vectors that maximize nonnegatively weighted and (strict) positively weighted utilitarian welfare, respectively, with $\text{cl}(U^{++})$ being the closure of U^{++}.

This paper provides exact characterizations of Pareto optima by close variants of weighted utilitarian welfare maximization. To ease language, we will refer to weighted utilitarian welfare maximization simply as *utilitarianism*.[3]

We show that a utility vector u is Pareto optimal if and only if there exists a finite sequence of nonnegative and "eventually positive" welfare weights such that in each round t, u maximizes the round-t weighted sum of utilities out of those surviving from round $t-1$. Here, "eventually positive" means that the support of the weight vector strictly grows over the rounds with the weight vector in the final round having full support. We call this

[1] This theorem has spawned a series of extensions to spaces more general than Euclidean space. See Daniilidis (2000) for a survey of ABB theorems.

[2] When there are two agents, the limit $u \in U$ of any sequence $\{u^k\}$ of utilities $u^k \in U$ maximizing a positively weighted sum of utilities is Pareto optimal, where U is the utility possibility set, assumed to be closed and convex. To see it, let $\{\phi^k\}$ be the sequence of positive weights, normalized to be in the simplex, such that $u^k \in \arg\max_{(u'_1,u'_2)\in U} \sum_{i=1}^2 \phi_i^k u'_i$, and let ϕ denote its limit (say of a convergent subsequence). Clearly, $u \in \arg\max_{(u'_1,u'_2)\in U} \sum_{i=1}^2 \phi_i u'_i$. If ϕ_1 and ϕ_2 are both positive, then u is Pareto optimal, so assume $\phi_1 = 1$ and $\phi_2 = 0$ without loss. Suppose for contradiction u is not Pareto optimal. Then, there must exist $v \in U$ such that $v_1 = u_1$ and $v_2 > u_2$, where the equality holds since $u \in \arg\max_{(u'_1,u'_2)\in U} \sum_{i=1}^2 \phi_i u'_i = \arg\max_{(u'_1,u'_2)\in U} u'_1$. Since u^k's are all Pareto optimal, we have $u_1^k \leq v_1 = u_1$ and $u_2^k \geq v_2 > u_2$ for all k, so u^k never converges to u, a contradiction.

[3] In particular, note that we drop the qualifier "weighted" in our usage of the concept of utilitarianism while keeping in mind that utilitarianism is always used in the weighted sense. Indeed, we have no occasion to discuss unweighted utilitarianism. It is only for emphasis or to provide further clarification that we use the qualifier "weighted" in connection to utilitarianism.

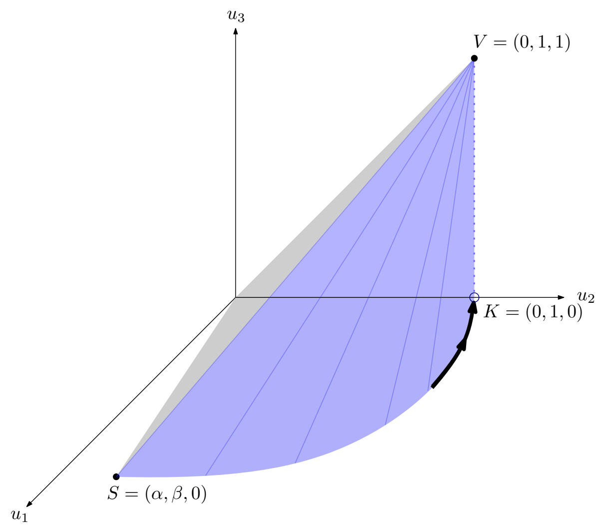

Figure 2: The "tilted cone" adapted from Arrow, Barankin, and Blackwell (1953) and Bitran and Magnanti (1979). The set is the convex hull of the portion of the unit disk centered at the origin in the u_1-u_2 plane from point K to point S (where $\alpha^2 + \beta^2 = 1$ with $\alpha \in (0, 1)$) and the apex point $V = (0, 1, 1)$. The blue surface, including all of its boundaries except for the dotted line, is the set of Pareto optimal utility vectors.

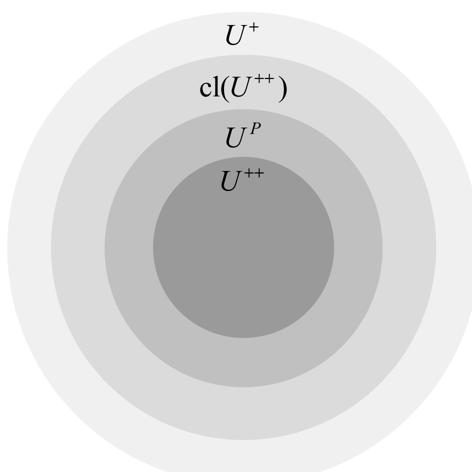

Figure 3: Alternative notions of utilitarian welfare maximization in relationship with Pareto optimality. The containment $U^{++} \subset U^P \subset U^+$ follows from Propositions 3.23 and 3.45 in Bewley (2009). The containment $U^P \subset \mathrm{cl}(U^{++})$ is from Arrow, Barankin, and Blackwell (1953). The containment $\mathrm{cl}(U^{++}) \subset U^+$ is straightforward.

approach *sequential utilitarian welfare maximization* (SUWM).

To illustrate why our SUWM successfully characterizes Pareto optima, let us revisit why neither the "nonnegative utilitarianism" captured by U^+ nor the "positive utilitarianism" captured by U^{++} in Figure 3 works. Nonnegative utilitarianism can include Pareto suboptimal outcomes because some individual's utility may not "count" at all. Positive utilitarianism avoids this problem by requiring that every individual's utility carry positive weights. However, as can be seen from Figure 1, it excludes Pareto optimal outcomes that can be achieved only by assigning some individuals "infinitely smaller" weights than others. SUWM resolves this seeming conflict by assigning positive weights to individuals so that "every agent's welfare counts" but in different rounds: Individuals with strictly positive weights only in later rounds can be regarded as carrying infinitely smaller weights than those with positive weights in earlier rounds.

The preceding observation gives rise to our second characterization of Pareto optimality, via one-shot maximization of utilitarian welfare with *hyperreal* weights. Hyperreal numbers include not only standard real numbers but also infinitesimal "numbers." The space of hyperreal numbers is very large, which may limit the usefulness of the characterization. By contrast, our characterization places an added discipline and structure on such social welfare functions. The resulting criterion, called *lexicographic hyperreal utilitarian welfare maximization* (LHUWM), employs a finite sequence of nonnegative and eventually positive real weights in lexicographically infinitesimal orders (for a precise definition of LHUWM see Definition 2).

Both of these characterizations of Pareto optimality capture the essential feature of standard weighted utilitarian welfare maximization. First, SUWM and LHUWM reduce to the standard weighted utilitarianism in many situations in which the former involves one-round maximization and the latter involves no infinitesimal weights. Second, the welfare functions used in these characterizations are inherently *linear* (based on weighted sums of agent utilities), albeit with SUWM having several rounds of linear optimizations and LHUWM involving hyperreal weights. Third, a consequence of this linearity is that the utilities of individuals are aggregated by weights that do not depend on the particular utility profile under consideration, a property we refer to as having "constant weights." This is in contrast to other social welfare functions, such as Rawlsian and leximin whose weighting of an agent's utility depends on her relative position in a given utility vector.

The sense in which our characterizations constitute "near" utilitarianism is further clarified by the social welfare orderings that underpin our characterizations. d'Aspremont and Gevers (2002) show that for social welfare orderings to be represented by a utilitarian welfare function, they must not only satisfy the Pareto Principle—namely, they must preserve Pareto domination order—but they must also satisfy two additional axioms: Invariance and Continuity. Invariance requires the orderings to be robust to translation and/or scaling of the utility profiles of individuals. Continuity requires the orderings to be robust to perturbations of utility profiles. Continuity effectively forces the welfare weights of agents to be in the same order of magnitude, thus making it impossible for the weight of an agent to be infinitesimally smaller than that of another agent. Since the latter feature is crucial for characterizing

Pareto optima, Continuity must be relaxed.

Indeed, we show that the welfare orderings associated with SUWM and LHUWM can be obtained by the same set of axioms under a suitable weakening of Continuity—more precisely, by the Pareto Principle, Invariance, and Weak Continuity. The last axiom weakens Continuity by requiring welfare orderings to be robust to perturbations of utilities of *some, but not necessarily all*, individuals, which is in line with our characterization of Pareto optima that allows some individuals to be assigned infinitely larger weights than others. That our welfare notions preserve a version of continuity, albeit weakened, is a nontrivial marker of the sense in which SUWM and LHUWM closely resemble utilitarianism. In particular, the same marker is not shared with other reasonable characterizations. For instance, as we show in a subsequent section, an (unrestricted) hyperreal-weighted utilitarian welfare function does not satisfy Weak Continuity.

Our characterizations of Pareto optimality fulfill a long-standing intellectual pursuit of providing a weighted utilitarian foundation for Pareto optimality. In addition, our characterizations of Pareto optimality serve other useful purposes.

First, the SUWM characterization could provide a tractable method for computing Pareto optimal allocations, which may be useful in the market design context. In fact, SUWM can be viewed as a generalization of the serial dictatorship mechanism in which each agent acts sequentially according to serial order to maximize her utility. Serial dictatorship is used widely for Pareto optimally allocating indivisible resources when monetary transfers cannot be used. For instance, serial dictatorship with a randomized serial order—known as random serial dictatorship—is used for assigning public school seats, public and campus housing, and human organs. One could imagine that SUWM can serve a similar practical purpose, but in a much more general setting that goes beyond a one-to-one assignment. In each round, one can let a group of agents negotiate over feasible allocations at that round, as is made precise in the supplementary appendix (Appendix D.1). Indeed, a procedure like this is used in the assignment of campus housing.[4] Alternatively, a central clearinghouse may compute an optimal choice for the group in each round.[5]

Second, the SUWM characterization could serve as a useful analytical tool for analyzing the behavior of Pareto optima as a set. For instance, one may study the comparative statics of Pareto optima—i.e., how they change as the primitives change—utilizing monotone comparative statics methods developed for optimization (e.g., Topkis (1998) and Milgrom and Shannon (1994)). The "round-wise" linear structure of SUWM admits a convenient aggregation property that is crucial for such an analysis. Indeed, Che, Kim, and Kojima

[4] For example, the campus housing assignment at Columbia university uses a cohort-based serial dictatorship, in which a group of students chooses a suite collectively in each round of the serial dictatorship procedure; presumably, the students then negotiate among themselves to allocate rooms within the assigned suite.

[5] In both scenarios, we are implicitly assuming complete information. In case agents' preferences are unobserved, the designer must rely on their preference reports, in which case agents' incentives become an important aspect of the market design. While this issue is beyond the scope of the current paper, it can be addressed in some specific settings such as cohort-based serial dictatorship mentioned in Footnote 4, where the standard strategy-proofness property would extend to a group of students as long as they know their preferences.

(2019) use this property to develop a theory of monotone comparative statics of Pareto optima: they show that when agents' utility functions shift in a way that leads to higher *individual* choices of decisions (e.g., Milgrom and Shannon (1994)), the Pareto optima shift to a higher set of actions in a suitable sense.[6]

The remainder of the paper is organized as follows. Section 2 describes our setting and establishes a few preliminaries used in our main results. Section 3 states our characterization of Pareto optimality. Here, we discuss the tools used to prove the result. Section 4 establishes the axiomatization of SUWM and LHUWM. A supplementary appendix (Appendix D) looks at other reasonable characterizations of Pareto optimality that fail at least one of the "near" utilitarian axioms set out in the previous section. Section 5 concludes with some suggestions for future work. The appendix provides proofs of our main characterization and axiomatization results. The supplementary appendix contains statements and proofs of additional results.

2 Setting and preliminaries

In this section, we introduce our basic setting and introduce some elementary concepts needed for stating our main results.

Let $I = \{1, 2, \dots, n\}$ denote a finite set of agents and the *utility possibility set* $U \subset \mathbb{R}^n$ be the set of possible utility vectors the agents may attain. We assume that U is closed and convex. If U stems from an underlying choice space X via utility functions $(u_i)_{i\in I} : X \to \mathbb{R}^n$, then we let

$$U = \{u \in \mathbb{R}^n \mid u \leq (u_i(x))_{i\in I} \text{ for some } x \in X\}.^7 \tag{1}$$

That U is closed and convex is arguably a mild assumption that is satisfied if, for instance, U is induced by utility functions $(u_i)_{i\in I}$ that are upper semicontinuous and concave on a choice set X that is compact and convex.[8]

For any $u, v \in \mathbb{R}^n$, we write $v \geq u$ if $v_i \geq u_i$ for all $i \in I$, $v > u$ if $v \geq u$ and $v \neq u$, and $v \gg u$ if $v_i > u_i$ for all $i \in I$. We say a point u in U is *Pareto optimal* with respect to U if there exists no $v \in U$ with $v > u$. Let $U^P \subset U$ denote the set of all Pareto optimal points (or, more simply, Pareto optima).

For any $\phi \in \mathbb{R}^n$, consider the optimization problem:

$$\max_{u\in U} \langle \phi, u \rangle, \tag{2}$$

[6] In particular, properties such as supermodularity and increasing differences, which are important for the monotone comparative statics analysis, are preserved under this aggregation. The same proof would not have been possible with nonlinear welfare functions.

[7] To be precise, the utility possibility set is often defined as $\{u \in \mathbb{R}^n \mid u = (u_i(x))_{i\in I} \text{ for some } x \in X\}$, which differs from (1). However, the two sets share the same set of Pareto optima since those points are on the common outer boundary of the sets. Thus, formulating the set U either way makes no difference for our results while the current formation facilitates our analysis.

[8] Note that compactness and convexity of the choice set X are satisfied if, for instance, all lotteries of social outcomes, which are in turn finite, or more generally compact, are feasible.

where $\langle \phi, u \rangle := \sum_{i=1}^n \phi_i u_i$. We call ϕ a *weight vector*. Throughout the paper, we only consider nonzero weight vectors (i.e., $\phi \neq 0$). We say a point $u \in U$ *maximizes* the weight vector ϕ over U (or simply *maximizes* ϕ) if u is a solution to (2). We call a weight vector ϕ *nonnegative* if $\phi > 0$ and *positive* if $\phi \gg 0$. For any vector $v \in \mathbb{R}^n$, the *support* of v is the set of indices where v is nonzero; i.e., $\operatorname{supp} v := \{i \in I \mid v_i \neq 0\}$. A positive ϕ has full support; i.e., $\operatorname{supp} \phi = I$.

One of our characterizations uses the language of hyperreal numbers. We introduce the basics here. The set of *hyperreal numbers* ${}^*\mathbb{R}$ consists of real numbers as well as "infinite" and "infinitesimal" numbers. Infinite numbers are larger than any real number. Infinitesimal numbers (or simply infinitesimals) are closer to 0 than any real number. A formal definition of ${}^*\mathbb{R}$ is somewhat tedious and we will not reproduce it here. Instead, we refer the reader to Goldblatt (2012). Although the use of hyperreal numbers (in what is termed *nonstandard analysis*) is not completely standard in economics, it has been used in a variety of settings including choice under uncertainty (Blume, Brandenburger, and Dekel, 1991), game theory (Dilmé, 2022), and exchange economies (Brown and Robinson, 1975). See Anderson (1991) for a survey of applications of nonstandard analysis to economics.

Important properties of the set of hyperreal numbers for our purposes are that (i) ${}^*\mathbb{R}$ contains a (positive) infinitesimal number, i.e., an element $\epsilon \in {}^*\mathbb{R}$ such that $\epsilon < r$ for every positive real number r while $\epsilon > 0$, and that (ii) arithmetic operations such as addition and multiplication, as well as order relations, are well defined and extended from $\mathbb{R}$ to ${}^*\mathbb{R}$ in expected ways.

3 Characterizations of Pareto optimality

This section presents our main result, Theorem 1, that provides two alternative "near" (weighted) utilitarian characterizations of the set U^P of Pareto optimal points of a given closed convex set U. To state these characterizations, we first introduce some additional terminology and definitions. These definitions will be interpreted after the statement of Theorem 1.

A sequence $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ of weight vectors is *nonnegative* if ϕ^t is nonnegative for every $t \in \{1, \ldots, T\}$. We say that a sequence Φ of weight vectors is *eventually positive* if $\operatorname{supp} \phi^{t-1} \subsetneq \operatorname{supp} \phi^t$ for all $t = 2, \ldots, T$ and $\operatorname{supp} \phi^T = I$. Note that eventual positivity implies $T \leq n$ since the support *strictly* grows along the sequence.

Definition 1 (Sequential utilitarian welfare maximization (SUWM)). We say $u \in U$ *sequentially maximizes* a sequence $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ of weight vectors over U if

$$u \in U^t := \arg \max_{u' \in U^{t-1}} \langle \phi^t, u' \rangle, \text{ for each } t = 1, \ldots, T, \tag{3}$$

where $U^0 = U$. We say $u \in U$ *sequentially maximizes utilitarian welfare* over U—or, more simply, u is an *SUWM solution* of U—if there exists a sequence Φ of nonnegative and eventually positive weight vectors such that u sequentially maximizes Φ.

The following definition uses the concept of hyperreals introduced in the preliminaries section. We call a vector $\phi \in ({}^*\mathbb{R})^n$ of hyperreal weights *lexicographic* if there exists a positive infinitesimal number ϵ and a sequence $\Phi = (\phi^1, \phi^2, \dots, \phi^T)$ of nonnegative and eventually positive weight vectors in $\mathbb{R}^n$ such that

$$\phi = \sum_{t \in \{1,\dots,T\}} \epsilon^{t-1} \phi^t. \tag{4}$$

The name "lexicographic" indicates that the terms corresponding to weight vectors ϕ^t become progressively infinitesimal in a lexicographic manner. An example with two individuals can illustrate the restriction associated with a lexicographic hyperreal vector. Consider the hyperreal weight vector $(1+\epsilon, 1)$, with ϵ being a positive infinitesimal number. This vector is not lexicographic. To see this, note that the only way to express the vector $(1+\epsilon, 1)$ in the form $\phi = \sum_{t \in \{1,\dots,T\}} \epsilon^{t-1} \phi^t$ is to set $T = 2$, $\phi^1 = (1, 1)$, and $\phi^2 = (1, 0)$. The sequence (ϕ^1, ϕ^2) violates the eventual positivity requirement. By contrast, the weight vector $(1 + \epsilon, \epsilon) = (1, 0) + \epsilon(1, 1)$ is lexicographic. The relevance of the distinction between lexicographic and non-lexicographic hyperreal vectors, as well as the role played by the former, will become clear in Section 4.

Definition 2 (Lexicographic hyperreal utilitarian welfare maximization (LHUWM))**.** A social welfare function W is a *lexicographic hyperreal utilitarian welfare function* if

$$W(u) = \langle \phi, u \rangle = \sum_{t \in \{1,\dots,T\}} \epsilon^{t-1} \langle \phi^t, u \rangle, \tag{5}$$

where ϕ is a lexicographic hyperreal weight vector. We say $u \in U$ maximizes a lexicographic hyperreal utilitarian welfare function over U—or, more simply, u is a *LHUWM solution* of U—if there exists a lexicographic hyperreal utilitarian welfare function W such that $W(u) \geq W(v)$ for all $v \in U$.

We can now state the first main result of the paper.

Theorem 1. Let U be a closed convex subset of $\mathbb{R}^n$ and let u be a vector in U. Then, the following are equivalent:

(i) u is Pareto optimal with respect to U.
(ii) u is a SUWM solution of U.
(iii) u is a LHUWM solution of U.

Proof. See Appendix A. □

In the remainder of the section, we will offer interpretations of this result and insights into its proof.

We first start with the SUWM characterization of Pareto optimality implied by the equivalence between (i) and (ii). In SUWM, utilitarian welfare is maximized over multiple rounds for growing sets of agents until all agents are considered. From the social choice perspective, one can imagine a utilitarian social planner who prioritizes some agents—that is, those considered in earlier rounds of SUWM—and maximizes their (weighted) welfare before others. To achieve Pareto optimality, the social planner must assign *some* weights to

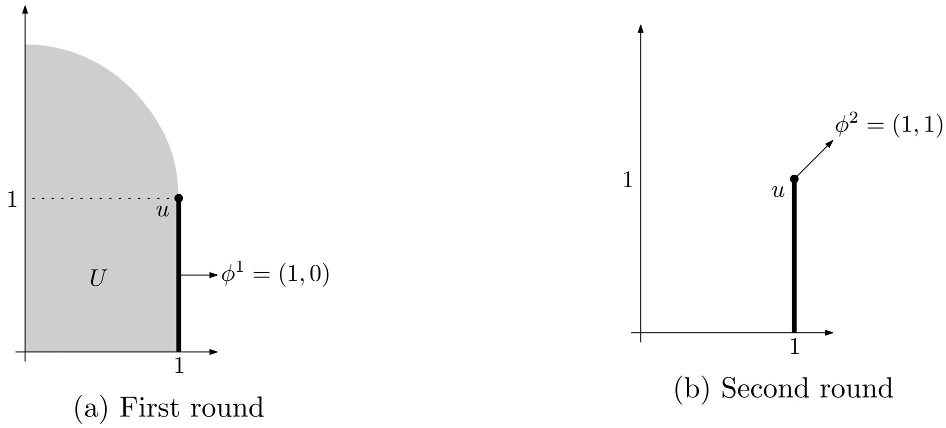

Figure 4: Determining a Pareto optimal point in two rounds of sequential utilitarian welfare maximization.

all agents, but the welfare weights for some individuals (those who receive positive weights in later rounds) may need to be infinitely smaller than those for others (those who receive positive weights in the earlier rounds). SUWM allows such flexibility by placing positive weights on individuals in different rounds. The eventual positivity condition encodes the requirement of Pareto optimality that "every agent's welfare counts" since the utility of each agent i has a positive weight in some round of welfare maximization.

The equivalence between (i) and (ii) is easy to visualize with the example in Figure 1, reproduced in Figure 4(a). In the first round, utilities are maximized within U with weights ϕ^1, which is maximized by the thick vertical segment containing u. One can interpret this as the social planner first maximizing the utility of agent 1 while disregarding the welfare of the other individual completely. Since agent 1 is indifferent among all of these points, the social planner seeks to engage in further optimization. In the second (and last) round, utilities are again maximized but only within the vertical segment, now with an (arbitrary) nonnegative weight vector ϕ^2 that places a positive weight on agent 2. Hence, $\Phi = (\phi^1, \phi^2)$ is eventually positive. The weights ϕ^2 determine u as the unique maximizer, as illustrated in Figure 4(b). The theorem shows that the flexibility in assigning the weights in different rounds in SUWM enables an exact characterization.

The equivalence of (ii) and (iii) in the theorem shows that the sequential optimization involved in SUWM can be encoded in a one-shot weighted utilitarian welfare maximization with the introduction of lexicographic hyperreal weights. This introduction of hyperreals allows for some agents to be prioritized over others in the sense of being assigned positive weights in earlier rounds of SUWM. One can then interpret the former agents as carrying infinitely larger weights than the latter agents to constitute social welfare. The characterization in (iii) formalizes this idea by constructing the lexicographic hyperreal weight vector $\phi = \sum_{t \in \{1,\ldots,T\}} \epsilon^{t-1} \phi^t$. Since ϵ^s is infinitely larger than ϵ^t for any $t > s \geq 0$, the hyperreal vector ϕ assigns infinitely larger weights to the agents with higher priority than those with lower priority. For example, in Figure 4 the vector $u = (1, 1)$ maximizes the lexicographic hyperreal utilitarian welfare function with hyperreal weights $\phi^1 + \epsilon\phi^2 = (1+\epsilon, \epsilon)$. While serving as a useful step toward our proof, this result lacks an important element that is fundamental in the economics context—that the weights be nonnegative and eventually positive. A nontrivial and crucial part of our proof lies in showing that nonnegative and eventually positive weights can be found if and only if the face consists of Pareto optimal points. The proof of Theorem 1 in Appendix A provides additional details and discussion.

Let us now explore some of the insights behind the proof of Theorem 1. The argument showing that (i) implies (ii) exploits a remarkable parallel between our problem and the question in convex geometry pertaining to *extreme faces* of a closed convex set. An extreme face, or simply a *face,* F of U is its convex subset whose elements cannot be expressed as convex combinations of points outside that set. (An extreme point is a special case of a face comprised of a singleton.) Geometrically, Pareto optimal points of U are made up of such faces (a result we establish). We say a hyperplane of U "exposes" a face F if it intersects U precisely at F, namely when F constitutes the set of points that maximize a linear function. A standard utilitarian welfare characterization of Pareto optima implies that the corresponding faces are "exposed" by hyperplanes with nonnegative weight vectors. From this perspective, the failure of standard weighted utilitarianism can be traced to the fact known in convex geometry that extreme faces may not always be exposed. However, an important finding in that literature is that an extreme face is "eventually exposed," that is, the face can be represented by the set of points that sequentially maximize *possibly negatively*-weighted sum of utilities.[9] While serving as a useful step toward our proof, this result lacks an element that is important for us and fundamental in the economics context—that the weights be nonnegative and eventually positive. A nontrivial and crucial part of our proof lies in showing that nonnegative and eventually positive weights can be found if and only if the face consists of Pareto optimal points.

Next, the fact that (ii) implies (iii) follows since any SUMW solution constitutes a LHUWM solution with the lexicographic hyperreal weights constructed using a sequence of the SUWM weights as in (4). Finally, we establish that (iii) implies (i), by observing that any LHUWM solution must be Pareto optimal, given the positivity of the lexicographic hyperreal weights.

4 Axiomatic foundation for "near" utilitarianism

In the previous section, we showed that "near" utilitarian welfare maximization—in the form of either SUWM or LHUWM—characterizes Pareto optima. Here we provide an axiomatic foundation for these welfare criteria. That is, we identify axioms of welfare orderings represented by these social welfare criteria.

This exercise serves at least two purposes. First, one can view the preceding characterization (Theorem 1) as providing a foundation for *some* version of utilitarianism. It is important to ask exactly what social welfare ordering corresponds to that version of utilitarianism. Second, our version of utilitarianism relaxes standard utilitarianism by allowing for a

[9] See Theorem 12.7 in Soltan (2015), reproduced as Lemma A.3 in the appendix.

sequence of welfare weights or for hyperreal welfare weights in utilitarian welfare maximization. Identifying the social welfare orderings that justify such procedures will lay bare the precise nature of departure from those generating standard utilitarianism. This difference will in turn make precise, and flesh out, the sense in which our utilitarianism is "near" the standard one.

We begin with a state-of-the-art axiomatization of (weighted) utilitarianism. Let the social welfare ordering $\succeq$ be a complete and transitive binary relation defined over $\mathbb{R}^n$, the set of utility profiles of agents I, and let $\succ$ and $\sim$ denote the strict and indifferent parts of $\succeq$, respectively. For any $u \in \mathbb{R}^n$ and any real number $\delta > 0$, let $B_\delta(u) := \{v \in \mathbb{R}^n : ||v - u|| < \delta\}$ be the δ-ball centered at u. Utilitarianism (with positive welfare weights) satisfies the following three axioms:

- Pareto Principle: for any $u > v$, we have $u \succ v$.
- Invariance: for any $u, v \in \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}_{++}$, if $u \succeq v$, then $(a + bu) \succeq (a + bv)$.
- Continuity: If $u \succ v$, then there exists $\delta > 0$ such that $u' \succ v$ for all $u' \in B_\delta(u)$.

Pareto Principle requires the welfare ordering to preserve the Pareto domination order. Invariance means that rescaling utility profiles by adding the same constant vector or by multiplying with the same positive coefficient does not alter their social welfare ordering. This property permits just the right scope of interpersonal utility comparison that yields linear social welfare evaluation. Continuity means that perturbing the utilities of possibly all agents slightly does not alter social welfare ordering. Continuity forces welfare weights on alternative individuals to be of the same order of magnitude at the margin, meaning that no individual is treated infinitely better or worse compared with the others. Theorem 4.2-(2) of d'Aspremont and Gevers (2002) shows that utilitarianism is the only social welfare ordering that satisfies the three axioms:

Theorem 2. [d'Aspremont-Gevers] Let $\succeq$ be a social welfare ordering. The following statements are equivalent:[10]

(i) $\succeq$ satisfies the Pareto Principle, Invariance, and Continuity.
(ii) There exists $\phi \in \mathbb{R}^n_{++}$ such that $u \succeq v$ if and only if $\sum_{i \in I} \phi_i u_i \geq \sum_{i \in I} \phi_i v_i$.

It is easy to see that Continuity fails in our lexicographic hyperreal utilitarian welfare function. Recall that in Figure 4, the Pareto optimum $u = (1, 1)$ maximizes the lexicographic hyperreal utilitarian welfare function $W(\cdot)$ with weights $(1 + \epsilon, \epsilon)$, where $\epsilon > 0$ is an infinitesimal. Hence, $W(1, 1) > W(1, 1/2)$, for example. Yet, for any real number $\delta > 0$, $W(1 - \delta, 1 - \delta) < W(1, 1/2)$, so W fails Continuity. Indeed, it is well-known that lexicographic preference orderings cannot be represented by a continuous utility function (see, for instance, pages 46-7 of Mas-Colell, Whinston, and Green (1995)).

While continuity in its general form cannot be satisfied, the additional structure of our *near* utilitarianism may accommodate some weaker version of continuity. Indeed, we identify

[10] Theorem 4.2-(1) of d'Aspremont and Gevers (2002) gives the characterization with nonnegative welfare weights when Pareto is replaced with a weaker Pareto-like condition.

the precise form of weakening of Continuity compatible with our near utilitarianism. For each agent $i \in I$ and a real number $\delta > 0$, let $B^i_\delta(u) := \{v \in \mathbb{R}^n : |v_i - u_i| < \delta, v_j = u_j, \forall j \neq i\}$ be the δ-ball around u but *only in the i-th coordinate*. This notion allows us to define:

- Weak Continuity: for any $u \succ v$, there exist $i \in I$ and $\delta > 0$ such that $u' \succ v$ for all $u' \in B^i_\delta(u)$.

Weak Continuity requires the social welfare ordering to be robust to perturbations of only *some* individual agent's utility, and not necessarily to *all* possible perturbations of the utility profile, as required by Continuity. We next present the desired axiomatization of our "near" weighted utilitarian welfare functions. To this end, we adapt SUWM to welfare orderings in a natural way.

Definition 3. We say u *sequentially utilitarian welfare dominates v according to Φ* if u sequentially maximizes utilitarian welfare over $\{u, v\}$ according to Φ.[11,12]

Theorem 3. Let $\succeq$ be a social welfare ordering. The following statements are equivalent.

(i) $\succeq$ satisfies the Pareto Principle, Invariance, and Weak Continuity.

(ii) There exists a nonnegative and eventually positive sequence of weight vectors $\Phi = (\phi^1, \phi^2, ..., \phi^T)$ such that for any $u, v \in \mathbb{R}^n$, $u \succeq v$ if and only if u sequentially utilitarian welfare dominates v according to Φ.

(iii) There exists a lexicographic hyperreal weight vector $\psi \in ({}^*\mathbb{R}_{++})^n$ such that for any $u, v \in \mathbb{R}^n$, $u \succeq v$ if and only if $\sum_{i\in I} \psi_i u_i \geq \sum_{i\in I} \psi_i v_i$.[13]

Proof. See Appendix B. □

For (iii), the restriction to *lexicographic* hyperreal utilitarian welfare functions is crucial. Recall that simplicity captures the eventual positivity of the weight vectors required in our SUWM, and this feature is essential for a hyperreal utilitarian welfare function to retain the weak continuity property. To see this, recall the non-lexicographic weight vector $\psi := (1+\epsilon, 1)$ with an infinitesimal $\epsilon > 0$ discussed in Section 3. The welfare function associated with this weight vector fails Weak Continuity. To see this, consider utility profiles $u := (1, 0)$ and $v := (0, 1)$. We have $u \succ v$ because $\langle \psi, u\rangle = 1 + \epsilon > 1 = \langle \psi, v\rangle$. However, for any $i \in I$, real number $\delta > 0$, and $u' \in B^i_\delta(u)$ with $u' < u$, we have $\langle \psi, u'\rangle < 1 = \langle \psi, v\rangle$,

[11]In words, u sequentially utilitarian welfare dominates v, if there exists a sequence of eventually positive weight vectors $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ satisfying: either $\phi^t u = \phi^t v$ for all t or there exists $\tau \geq 1$ such that $\langle \phi^t, u\rangle = \langle \phi^t, v\rangle$ for all $t < \tau$ and $\langle \phi^\tau, u\rangle > \langle \phi^\tau, v\rangle$.

[12]Note that this ranking leads to a *rational* order, i.e., a binary relation that is reflexive, complete, and transitive. To see the transitivity (since the other properties are obvious), consider profiles u, v, and w such that u and v sequentially utilitarian welfare dominate v and w, respectively: that is, $\langle \phi^\tau, u\rangle > \langle \phi^\tau, v\rangle$ for some τ and $\langle \phi^t, u\rangle = \langle \phi^t, v\rangle$ for all $t < \tau$ while $\langle \phi^{\tau'}, v\rangle > \langle \phi^{\tau'}, w\rangle$ for some τ' and $\langle \phi^t, v\rangle = \langle \phi^t, w\rangle$ for all $t < \tau$. Then, letting $\tau'' = \min\{\tau, \tau'\}$, we have $\langle \phi^{\tau''}, u\rangle > \langle \phi^{\tau''}, w\rangle$ and $\langle \phi^t, u\rangle = \langle \phi^t, w\rangle$ for all $t < \tau''$, implying u sequentially utilitarian welfare dominates w.

[13]As with the order based on sequential utilitarian welfare domination, an order based on this ranking is also rational (as hyperreal numbers follow the same ordering system as real numbers).

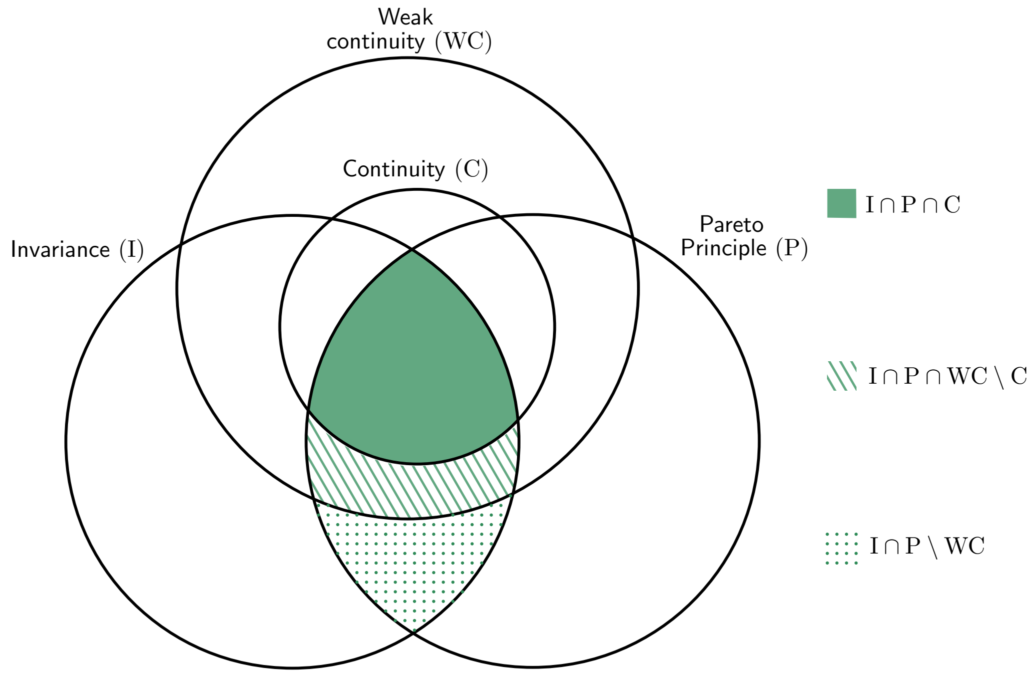

Figure 5: Illustrating the axiomatizations of different notions of utilitarianism. The universe is the set of all social welfare orderings. The shaded region is the set of weighted utilitarian social welfare orderings (via Theorem 2). The union of the shaded and hatched regions is the set of sequential weighted utilitarian (or lexicographic hyperreal weighted utilitarian) social welfare orderings (via Theorem 3). The union of all three highlighted regions (shaded, hatched, and dotted) is the set of general hyperreal weighted utilitarian social welfare orderings (via Proposition 2).

so $u' \succ v$ does not hold, a violation of Weak Continuity. The reason for this difference is that this non-lexicographic hyperreal function cannot be supported by a nonnegative and eventually-positive sequence of weight vectors required by SUWM.

By contrast, consider the lexicographic hyperreal utilitarian welfare function $W(\cdot)$ with weights $(1+\epsilon, \epsilon)$ that exposes u in Figure 1. While W fails to be continuous, it is weakly continuous. Although W fails Continuity, it satisfies Weak Continuity. Recall $W(u) > W(v)$, for $u = (1,1)$ and $v = (1, 1/2)$. And, $W(u') > W(v)$ for any $u' \in B^2_\delta(u)$ if $\delta \in (0, 1/2)$.

To visualize some of this discussion, Figure 5 illustrates the relationship between the axiomatizations of different notions of utilitarianism described in Theorems 2 and 3 and Proposition 2 (the last result is discussed in the next section).

Weighted utilitarianism with general hyperreal weights. As we discussed, the restriction to *lexicographic* hyperreal utilitarian welfare functions disciplines them to resemble utilitarianism. At the same time, hyperreal utilitarian welfare maximization, with no restriction, also characterizes Pareto optimality.

Proposition 1. Let U be a closed convex subset of $\mathbb{R}^n$. Then, $u \in U$ is Pareto optimal if and only if

$$u \in \arg\max_{u' \in U} \langle \psi, u' \rangle,$$

for some weight vector $\psi = (\psi_i)_{i \in I} \in ({}^*\mathbb{R}_{++})^n$.

Proof. See Appendix C.1 in the appendix. □

This proposition highlights the ability to assign an infinitely larger weight to one agent relative to another as a crucial feature that enabled LHUWM to characterize Pareto optimality. Compared with lexicographic hyperreal utilitarian welfare functions, however, the class of general hyperreal utilitarian welfare functions is too large to be declared near-utilitarian. As we already saw, the class includes non-lexicographic hyperreal functions that do not satisfy Weak Continuity, let alone Continuity.

Theorem 1 tells us that such non-lexicographic functions are not needed for characterizing Pareto optimality. To illustrate their superfluity, recall the non-lexicographic hyperreal vector $\psi = (1 + \epsilon, 1)$, where ϵ is a positive infinitesimal number. We can see that any such vector can be replaced by a lexicographic hyperreal vector (which does satisfy Weak Continuity), in this particular case, a real vector, with no loss on the ability to characterize Pareto optima.[14] We showed that the welfare function associated with $\psi = (1 + \epsilon, 1)$ fails Weak Continuity. Indeed, the next proposition shows that the class of hyperreal utilitarian welfare functions in Proposition 1 entails no restriction on social welfare orderings beyond the Pareto Principle and Invariance.

Proposition 2. Let $\succeq$ be a social welfare ordering. The following statements are equivalent.

(i) $\succeq$ satisfies the Pareto Principle and Invariance.

(ii) There exists a hyperreal weight vector $\psi \in ({}^*\mathbb{R}_{++})^n$ such that $u \succeq v$ if and only if $\sum_{i \in I} \psi_i u_i \geq \sum_{i \in I} \psi_i v_i$.

Proof. See Appendix C.2 in the appendix. □

Figure 5 illustrates the differences in how general hyperreal utilitarian and other utilitarian welfare functions are axiomatized.

5 Conclusion

We have provided two characterizations of Pareto optimal solutions of a closed convex set that are "near" to weighted utilitarian maximization in an axiomatic sense. They arise from relaxing the Continuity axiom that defines weighted utilitarian to a Weak Continuity axiom. While one can imagine other characterizations of Pareto optimality, they are more "distant" from weighted utilitarianism. We already saw that utilitarian welfare maximization with

[14] In this case, the real weight vector $(1, 1)$ can be used in place of ψ in the sense that every Pareto optimal point that maximizes the hyperreal weight vector ψ also maximizes the real weight vector $(1, 1)$.

general hyperreal welfare weights characterizes Pareto optima but the associated welfare orderings do not satisfy Weak Continuity. In the supplementary appendix (Appendix D), we provide two additional characterizations of Pareto optima inspired by SUWM. First, one can, in the same spirit as the fundamental theorems of welfare economics, characterize Pareto optima as emerging from sets of agents negotiating sequentially according to a generalized Nash bargaining solution. Second, Pareto optimality can be also characterized by the maximization of piecewise-linear concave welfare functions satisfying a certain restriction. These characterizations are closely connected with SUWM, so they help interpret some features of SUWM. Nevertheless, the associated welfare orderings may not be well defined or are more distant from weighted utilitarianism than our main characterizations. These results constitute significant progress in clarifying the connection between Paretian and utilitarian notions that are foundational to welfare economics.

Although our paper directly worked with the space of utility profiles U, our results drive implications for problems stated in the choice space X. Indeed, examining the structure of what points in the choice set give rise to Pareto optima has been a major focus in the multiobjective optimization literature. An early contribution in that literature is Charnes and Cooper (1967), who showed an equivalence between the problem of finding Pareto optimal solutions (in the choice set X) and that of solving a constrained nonlinear programming problem. Following their contribution, techniques in nonlinear programming were utilized to characterize Pareto optima under various conditions (Ben-Israel, Ben-Tal, and Charnes, 1977; Van Rooyen, Zhou, and Zlobec, 1994; Glover, Jeyakumar, and Rubinov, 1999; Ben-Tal, 1980) all of which require some form of differentiability of the utility functions. We believe further investigation into our approach may have the potential to add to this literature in at least two aspects. First, our characterization does not assume any form of differentiability. Indeed, the subtlety of non-exposure of Pareto optimal faces can also arise when utility functions are not smooth, as is often the case. Our methods may suggest ways to handle Pareto optimality when differentiability fails. Second, our methods may suggest a bridge between existing results in the choice space and results in the utility possibility space, where notions of (sequential) welfare maximization are salient and allow for more natural economic interpretations. Indeed, none of the characterizations in the above references speak to notions of welfare maximization.

A second area of future work would be to examine how the notion of exposure can be used to enhance separating hyperplane arguments that may arise in other economic settings. For instance, the second welfare theorem relies on the existence of a strictly positive weight vector for a supporting hyperplane (which constitutes equilibrium prices). One can prove this with a weaker assumption than in the existing proof of the theorem by leveraging the idea of exposing a Pareto optimal point—which is a target Pareto efficient allocation—, as we show in the supplementary appendix (Appendix F). We believe there is scope to explore other economic settings where separating hyperplane arguments are used and similarly relax the conditions needed to ensure strict positivity when Pareto optimality (in combination with notions of exposure) may be used to assure the existence of a separating hyperplane with a positive weight vector.

A third area of future work is to extend the characterization presented in this paper to the case of infinite-dimensional economies. This is not a straightforward extension. Our argument in the finite-dimensional case depends on a termination condition that counts dimension. In the infinite-dimensional case, this termination condition is not accessible to us. Generalization would likely require a set convergence argument (for instance, using Hausdorff or Kuratowksi set-based metrics) that avoids discussion of dimension.

A Appendix: Proof of Theorem 1

A.1 Proof of (ii) ⇒ (iii)

Given the sequence $\Phi = (\phi^1, \dots, \phi^T)$ that is sequentially maximized by u, let us construct a lexicographic hyperreal weighted welfare function $W(\cdot)$ as in (5). Letting $U^0, U^1, \dots, U^T$ be the notation used in Definition 1, observe first that

$$W(v) = \langle \phi, v \rangle = \langle \phi, u \rangle = W(u) \text{ for every } v \in U^T, \tag{6}$$

by construction of ϕ and U^T. Next, consider any $v \notin U^T$. Then, there exists $\tau \in \{1, \dots, T\}$ such that $\langle \phi^t, v \rangle = \langle \phi^t, u \rangle$ for all $t < \tau$ and $\langle \phi^\tau, v \rangle < \langle \phi^\tau, u \rangle$. Therefore,

$$\begin{aligned}
\langle \phi, u \rangle - \langle \phi, v \rangle &= \sum_{t=1}^{T} \epsilon^{t-1} \langle \phi^t, u \rangle - \sum_{t=1}^{T} \epsilon^{t-1} \langle \phi^t, v \rangle \\
&= \epsilon^{\tau-1} \left(\langle \phi^\tau, u \rangle - \langle \phi^\tau, v \rangle \right) + \sum_{t \in \{\tau+1, \dots, T\}} \epsilon^{t-1} \left(\langle \phi^\tau, u \rangle - \langle \phi^\tau, v \rangle \right) \\
&= \epsilon^{\tau-1} \left[\left(\langle \phi^\tau, u \rangle - \langle \phi^\tau, v \rangle \right) + \sum_{t \in \{\tau+1, \dots, T\}} \epsilon^{t-\tau} \left(\langle \phi^\tau, u \rangle - \langle \phi^\tau, v \rangle \right) \right]
\end{aligned} \tag{7}$$

where all arithmetic operations are valid because $^*\mathbb{R}$ is an ordered field (Theorem 3.6.1 of Goldblatt (2012)). Because ϵ is an infinitesimal number strictly larger than zero, and $\langle \phi^\tau, u \rangle - \langle \phi^\tau, v \rangle$ is a positive real number, the expression inside the square bracket in (7) is positive, so $\langle \phi, u \rangle - \langle \phi, v \rangle > 0$ (see pages 50 and 51 of Goldblatt (2012)). Thus, we have shown that

$$W(v) = \langle \phi, v \rangle < \langle \psi, u \rangle = W(u) \text{ for every } v \notin U^T. \tag{8}$$

By equations (6) and (8), we have established that $u \in \arg\max_{v \in U} W(v)$.

A.2 Proof of (iii) ⇒ (i)

Suppose for contradiction that u maximizes the lexicographic hyperreal weighted welfare function W as in (5) but is not Pareto optimal. Then, there exists $v \in U$ such that $v > u$. Since the sequence $(\phi^1, \dots, \phi^T)$ is nonnegative and eventually positive, we have $\phi_i = \sum_{t \in \{1, \dots, T\}} \epsilon^t \phi_i^t > 0$ for each $i \in I$. Thus, $W(v) - W(u) = \langle \phi, v \rangle - \langle \phi, u \rangle = \sum_{i \in I} \phi_i (v_i - u_i) > 0$, a contradiction.

A.3 Proof of (i) $\Rightarrow$ (ii)

We begin with some preliminaries before providing the proof in Appendix A.3.2.

A.3.1 Preliminaries

Let us first introduce a few concepts that are crucial for our analysis. A *face* of U is a nonempty convex subset F of U with the property that if $u \in F$ and $u = \alpha v + (1-\alpha)w$ for some $0 < \alpha < 1$ and $v, w \in U$ then it must be that $v, w \in F$. That is, F is a face of a convex set if none of its elements are convex combinations of elements that lie outside of F. A *proper face* of U is a face of U that is a proper subset of U. A face F is an *exposed face* of U if there is a weight vector $\phi \in \mathbb{R}^n$ such that $F = \arg\max_{u \in U} \langle \phi, u \rangle$. In this case, we say that ϕ exposes F out of U. A face need not be exposed, as can be seen in Figure 1, where u is a singleton face that is not exposed.

For any convex subset G of U, its *relative interior* $\text{ri}(G)$ is the set of all $u \in G$ such that for every $u' \in G$ there exists $\lambda > 0$ such that $u + \lambda(u - u') \in G$.

The following lemma shows a face structure of a convex set that is interesting in itself and useful for our analysis.

Lemma A.1 (Corollary 11.11(a) in Soltan (2015))**.** For a convex set $U \subseteq \mathbb{R}^n$, the collection of relative interiors of faces—that is, $\{\text{ri}(F) : F \text{ is a face of } U\}$—forms a partition of U.

The next lemma shows that Pareto optimal points "come in faces." It is standard in the convex analytic literature to refer to Pareto optimal points as *maximal* points, so we use that language here.

Lemma A.2. Suppose a maximal point u of a closed convex set U lies in the relative interior of a face F of U. Then, every point in F is maximal.

Proof. The stated result is immediate in the case F is a singleton, so we may assume that F is not a singleton. Suppose for contradiction that F contains a nonmaximal element u'. Thus, there exists a $v \in U$ such that $v > u'$. Since $u \in \text{ri}(F)$, there exists $\lambda > 0$ such that $w' = u + \lambda(u - u') \in F$. Now let $z = \alpha w' + (1-\alpha)v$, where $\alpha = \frac{1}{1+\lambda}$ or $\alpha(1+\lambda) = 1$. Note that $z \in U$ since U is convex. Moreover,

$$z = \alpha\left(u + \lambda(u - u')\right) + (1-\alpha)v = u - \alpha\lambda u' + (1-\alpha)v = u + (1-\alpha)(v - u') > u,$$

contradicting the maximality of u. $\square$

According to Lemma A.2, we say a face is *maximal* if all of its elements are maximal. Importantly for our purpose, Lemmas A.1 and A.2 imply that every maximal point of U belongs to a relative interior of a unique maximal face of U (possibly U itself).

The next result provides a key step of our argument: every face, possibly non-exposed, is *eventually* exposed.[15]

[15] Theorem 5 of Lopomo, Rigotti, and Shannon (2022) proves the same result for singleton faces F, i.e., extreme points.

Lemma A.3 (Theorem 12.7 in Soltan (2015))**.** Let $U \subset \mathbb{R}^n$ be a convex set and F be a nonempty proper face of U. There is a sequence of convex sets $(G^t)_{t=0}^T$ such that

$$F = G^T \subset G^{T-1} \subset \cdots \subset G^1 \subset G^0 = U,$$

where G^t is a nonempty proper exposed face of G^{t-1} for each $t = 1, \ldots, T$.

This lemma is already illustrated in the Introduction. In Figure 4, the singleton face u is exposed in two rounds: the vertical segment is exposed first by a weight vector $(1, 0)$, and then u is exposed by weight vector $(1, 1)$ (among many others) out of that vertical segment. This lemma is not enough for our result, however, as it is silent about any additional properties on the weight vectors that expose the sequence of faces. Crucially, our characterization requires the weight vectors to be nonnegative and eventually positive.

For these additional features, we need to introduce a set of analytical tools. Let J be any subset of the index set I and let χ^J denote the vector whose i-th coordinate is equal to 1 for every $i \in J$ and equal to 0 for every $i \notin J$. When J is the singleton $\{i\}$ we simplify $\chi^{\{i\}}$ to χ^i. A convex set U is *downward closed in coordinates* $J \subset I$ if, for all $u \in U$ and all $\tau \geq 0$, $u - \tau\chi^K \in U$ for any subset K of J. A convex set that is downward closed in all coordinates I is simply called *downward closed*. The *downward closure* of a closed convex set U is the downward closed set $\mathrm{dc}(U) := \bigcup_{u \in U}(u - \mathbb{R}^n_+)$. It is straightforward to see that $\mathrm{dc}(U)$ is closed and convex if U is closed and convex.

One useful feature of downward closure is that it preserves maximal elements and thus maximal faces.

Lemma A.4. The set of maximal elements of a closed convex set coincides with that of its downward closure. If F is a maximal face of U then F is a maximal face of $\mathrm{dc}(U)$.

Proof. Let U be a closed convex set and $\mathrm{dc}(U)$ its downward closure. Let u be a maximal element of $\mathrm{dc}(U)$; that is, $(u + \mathbb{R}^n_+) \cap \mathrm{dc}(U) = \{u\}$. If $u \in U$ then this implies $(u + \mathbb{R}^n_+) \cap U = \{u\}$ since $U \subset \mathrm{dc}(U)$ and so u is a maximal element of U. Note that if $u \in \mathrm{dc}(U) \setminus U$ then it cannot be maximal. Indeed, this implies that $u = v - w$ for some $v \in U$ and nonzero $w \in \mathbb{R}^n_+$ and so $v > u$ and so u is not maximal.

Conversely, we prove the contrapositive. Suppose $u \in \mathrm{dc}(U)$ is not a maximal element. This implies that there exists a $w \neq u$ with $w \in \mathrm{dc}(U)$ and $w \geq u$. However, then we can find a $v \geq w \geq u$ and $v \neq u$ and $v \in U$. This implies that u is not a maximal element of U. We next prove the second statement. To see that F is a face of $\mathrm{dc}(U)$, consider any $x, y \in \mathrm{dc}(U)$ and $\lambda \in (0, 1)$ such that $z = \lambda x + (1 - \lambda)y \in F$. We need to show that both x and y belong to F. We first show that x and y are both maximal. Suppose for contradiction that x is not maximal. Then, we must have some $x' \in \mathrm{dc}(U)$ such that $x' > x$. Let $z' = \lambda x' + (1-\lambda)y$ and observe that $z' \in \mathrm{dc}(U)$, $z' \geq z$, and $z' \neq z$, which contradicts the maximality of z. Given that x and y are both maximal, we must have $x, y \in U$ since there is no maximal point in $\mathrm{dc}(U) \setminus U$. That F is a face of U then implies $x, y \in F$ as desired. □

Crucially for our arguments, halfspaces of the form $\{u : \langle \phi, u \rangle \leq W_\phi\}$ that contain downward-closed sets must have nonnegative weight vectors.

Lemma A.5. Let U be a set that is downward closed in coordinates $J \subset I$. If U is contained in the halfspace $\{u \in \mathbb{R}^n : \langle \phi, u \rangle \leq W_\phi\}$, then $\phi_j \geq 0, \forall j \in J$.

Proof. Suppose for contradiction that $\phi_j < 0$ for some $j \in J$. Let v be an arbitrary element of U. Since U is downward closed in coordinates J, we also have $v - \lambda \chi^j \in U$ for any $\lambda \geq 0$, where χ^j is the unit vector with 1 in component j. However, observe that $\langle \phi, v - \lambda \chi^j \rangle = \langle \phi, v \rangle - \lambda \langle \phi, \chi^j \rangle = \langle \phi, v \rangle - \lambda \phi_j$. But $\langle \phi, v \rangle - \lambda \phi_j \to \infty$ as $\lambda \to \infty$ since $\phi_j < 0$. This contradicts the fact that is contained $\{u \in \mathbb{R}^n : \langle \phi, u \rangle \leq W_\phi\}$. ☐

Lemma A.6. Let F be a face of a closed convex set U that is downward closed in coordinates $J \subset I$. If ϕ exposes F out of U, then F is downward closed in coordinates $J \setminus \operatorname{supp} \phi$.

Proof. Take any $j \in K := J \setminus \operatorname{supp} \phi$ and set $u' = u - \epsilon \chi^j$ for some $u \in F$ and $\epsilon > 0$. Since U is downward closed in coordinates J and $j \in J$, we have $u' \in U$. Moreover, $\langle \phi, u' \rangle = \langle \phi, u - \epsilon \chi^j \rangle = \langle \phi, u \rangle - \epsilon \langle \phi, \chi^j \rangle = \langle \phi, u \rangle - \epsilon \phi_j = \langle \phi, u \rangle$ since $\phi_j = 0$ when $j \in K$ since no element of K lies in $\operatorname{supp} \phi$. However, then $u' \in F$ since $\langle \phi, u' \rangle = \langle \phi, u \rangle = \max_{v \in U} \langle \phi, v \rangle$ and $F = \arg\max_{v \in U} \langle \phi, v \rangle$ since F is exposed by ϕ. ☐

A.3.2 Proof of (i) ⇒ (ii)

Fix any maximal point u of U. We wish to show that u sequentially maximizes utilitarian welfare over U. The proof consists of several steps.

Step 1. There exists a unique face F of $\mathrm{dc}(U)$ such that $u \in \mathrm{ri}(F)$. All points of F are maximal in $\mathrm{dc}(U)$.

Proof. By Lemma A.4, u is a maximal point of $\mathrm{dc}(U)$. By Lemma A.1 there is a unique face F of $\mathrm{dc}(U)$ which contains u in $\mathrm{ri}(F)$. By Lemma A.2, every point of F is maximal in $dc(U)$, as desired. ☐

Step 2. The face F (containing u) is a proper face of $\mathrm{dc}(U)$.

Proof. If not, we must have $F = \mathrm{dc}(U)$. Pick any $u' \in \mathrm{dc}(U)$. Then, for any $\epsilon > 0$, $u'' = u' - \epsilon \chi^I$ is also in $\mathrm{dc}(U)$ by the downward closure property. Clearly, u'' is not a maximal point of $\mathrm{dc}(U)$ and cannot belong to F by Step 1, a contradiction. ☐

Step 3. There exists a sequence of convex sets $(G^t)_{t=0}^T$ of $\mathrm{dc}(U)$ such that G^t is a proper exposed face of G^{t-1} for $t = 1, \ldots, T$, where $G^0 = \mathrm{dc}(U)$, $G^T = F$, and $T \leq n$.

Proof. Since F is a proper face of $\mathrm{dc}(U)$ by Step 2, the result follows from Lemma A.3. For any set V, let $\dim(V)$ denote its dimension.[16] If V' is a proper face of convex set V, then $\dim(V') < \dim(V)$ by Theorem 11.4 in Soltan (2015). Thus, we have $T \leq n$ since $\dim(G^t) < \dim(G^{t-1})$ and since $\dim(G^0) = \dim(\mathrm{dc}(U)) = n$. ☐

[16]The dimension $\dim(V)$ of a convex subset V of U, including one of U's faces, is defined by the dimension of its affine hull: $\mathrm{aff}(V) := \{\sum_{j=1}^k \alpha_j v^j \mid k \in \mathbb{N}, v^j \in V, \alpha_j \in \mathbb{R}, \sum_{j=1}^k \alpha_j = 1\}$.

Step 4. There exists a sequence $\Phi = (\phi^1, \ldots, \phi^T)$ such that for each $t = 1, \ldots, T$,

$$G^t = \arg\max_{x \in G^{t-1}} \langle \phi^t, x \rangle,$$

where $\phi^t > 0$ and $\operatorname{supp} \phi^1 \subset \operatorname{supp} \phi^2 \subset \ldots \subset \operatorname{supp} \phi^T = I$.[17]

Proof. By Step 3, there exists a sequence of weight vectors $\Psi = (\psi^1, \ldots, \psi^T)$ such that, for each $t = 1, \ldots, T$, ψ^t exposes G^t out of G^{t-1}. We construct $\Phi = (\phi^1, \ldots, \phi^T)$ with the stated properties.

The construction is recursive. First, since $G^0 = \mathrm{dc}(U)$, by Lemma A.5, $\phi^1 := \psi^1$ is nonnegative. For an inductive hypothesis, suppose that there are ϕ^k, $k = 1, \ldots, t-1$, with the stated properties and that for each $k = 1, \ldots, t-1$, G^k is downward-closed in coordinates $J^k := \{i \in I \mid \phi^k_i = 0\} = I \setminus \operatorname{supp} \phi^k$. Note that $J^{t-1} \subset J^{t-2} \subset \cdots \subset J^0 := I$. We will now construct ϕ^t and show that G^t is downward-closed in coordinates $J^t = \{i \in I \mid \phi^t_i = 0\}$.

First, observe G^{t-1} is contained in $\{u : \langle \psi^t, u \rangle \leq \max_{u' \in G^{t-1}} \langle \psi^t, u' \rangle\}$ and G^{t-1} is downward-closed in coordinates J^{t-1}. Hence, Lemma A.5 implies that $\psi^t_j \geq 0$ on coordinates $j \in J^{t-1}$. Consider next $i \in \operatorname{supp} \phi^{t-1} = I \setminus J^{t-1}$. For such i, it is indeed possible for ψ^t_i to be negative. But noting $\phi^{t-1}_i > 0$ for such i, we define

$$\phi^t = \lambda^t \phi^{t-1} + \psi^t,$$

where $\lambda^t > \max_{i \in \operatorname{supp} \phi^{t-1}} |\psi^t_i| / \phi^{t-1}_i$ is a (sufficiently large) positive scalar. Given this construction, $\phi^t_i \geq 0$ for all $i \in I$ and $\phi^t_i > 0$ for all $i \in \operatorname{supp} \phi^{t-1}$; i.e., $\operatorname{supp} \phi^t \supset \operatorname{supp} \phi^{t-1}$.

Let us show that ϕ^t exposes G^t out of G^{t-1}. To this end, let $M^t := \max_{x \in G^{t-2}} \langle \phi^{t-1}, x \rangle$. For all $x \in G^{t-1}$, we have

$$\langle \phi^t, x \rangle = \lambda^t \langle \phi^{t-1}, x \rangle + \langle \psi^t, x \rangle = \lambda^t M^t + \langle \psi^t, x \rangle,$$

since $\langle \phi^{t-1}, x \rangle = M^t$ for all $x \in G^{t-1}$. Henceforth,

$$\arg\max_{x \in G^{t-1}} \langle \phi^t, x \rangle = \arg\max_{x \in G^{t-1}} \langle \psi^t, x \rangle = G^t.$$

Since G^{t-1} is downward-closed in coordinates J^{t-1} and ϕ^t exposes G^t out of G^{t-1}, Lemma A.6 implies that G^t is downward-closed in coordinates $J^{t-1} \setminus \operatorname{supp} \phi^t = (I \setminus \operatorname{supp} \phi^{t-1}) \setminus \operatorname{supp} \phi^t = I \setminus \operatorname{supp} \phi^t = J^t$, where the penultimate equality holds since $\operatorname{supp} \phi^{t-1} \subset \operatorname{supp} \phi^t$.

It remains to show that for each $i \in I$, there exists $t \in \{1, \ldots, T\}$ such that $\phi^t_i > 0$. To show this, it suffices to show that $\phi^T \gg 0$. Supposing not, there must be some $i \in I$ such that $\phi^t_i = 0$ for all $t = 1, \ldots, T$, so $i \in J^t$ for all $t = 1, \ldots, T$. Then, Lemma A.6 implies that for all $t = 1, \ldots, T$, G^t is downward-closed in coordinate i, which contradicts the fact that $G^T = F$ is maximal. □

We have so far shown that u sequentially maximizes welfare over $\mathrm{dc}(U)$. We now prove the main result: u sequentially maximizes welfare over U. To this end, the following last step suffices.

Step 5. u sequentially maximizes utilitarian welfare over U.

[17] Note that Φ, with these properties, is eventually positive since $\phi^T \gg 0$.

Proof. Recall a sequence of weight vectors Φ from Step 4. Let $U^0, U^1, \ldots, U^T$ be convex subsets of U such that, for each $t = 1, \ldots, T$, U^t is the face of U^{t-1} exposed by weight vector ϕ^t; i.e.,

$$U^t = \arg\max_{x \in U^{t-1}} \langle \phi^t, x \rangle,$$

where $U^0 := U$. It suffices to prove that $U^T = F$, as this will prove that u sequentially maximizes utilitarian welfare over U.

To this end, it suffices to prove that $F \subset U^t \subset G^t$ for each $t = 0, \ldots, T$. We proceed inductively for the proof. First, note that the claim is trivially true for $t = 0$ because $U^0 := U \subset dc(U) := G^0$ and $F \subset U = U^0$ by definition. Now, suppose that the claim holds for t. We show (i) $F \subset U^{t+1}$ and (ii) $U^{t+1} \subset G^{t+1}$ as follows.

For (i), fix any point v in F. Then, since $F \subset G^{t+1}$ and ϕ^{t+1} exposes G^{t+1} out of G^t, we have $\langle \phi^{t+1}, v \rangle \geq \langle \phi^{t+1}, w \rangle$ for every $w \in G^t$. Because $U^t \subset G^t$ by the inductive assumption,

$$\langle \phi^{t+1}, v \rangle \geq \langle \phi^{t+1}, w \rangle \tag{9}$$

for every $w \in U^t$. Moreover, $v \in U^t$ by the assumption that $F \subset U^t$. This fact, combined with (9), implies that ϕ^{t+1} is maximized by v over U^t and so $v \in U^{t+1}$, since ϕ^{t+1} exposes U^{t+1} out of U^t. This holds for every $v \in F$ and so $F \subset U^{t+1}$, implying (i) holds for $t + 1$.

As for (ii), fix any point v in U^{t+1}. By (i), we know that

$$\langle \phi^{t+1}, v \rangle = \langle \phi^{t+1}, w \rangle \tag{10}$$

for any $w \in F$, since U^{t+1} is exposed by ϕ^{t+1} and F is a subset of U^{t+1}. Moreover, by the definition of G^{t+1} and the fact that $F \subset G^{t+1}$ by construction, we know that

$$\langle \phi^{t+1}, w \rangle \geq \langle \phi^{t+1}, z \rangle \tag{11}$$

for any $w \in F$ and $z \in G^t$. Combining (10) and (11) implies that $\langle \phi^{t+1}, v \rangle \geq \langle \phi^{t+1}, z \rangle$ for any $z \in G^t$. This, and the fact that $v \in G^t$ (which immediately follows from $v \in U^{t+1} \subset U^t \subset G^t$), means that $v \in G^{t+1}$. Since this holds for any $v \in U^{t+1}$, we can conclude that $U^{t+1} \subset G^{t+1}$, so (ii) holds for $t + 1$.

This completes the induction and establishes the result. □

B Appendix: Proof of Theorem 3

B.1 Proof of (i) ⇒ (ii)

We first show that if $\succeq$ satisfies the Pareto Principle and Invariance, then there exists a sequence $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ of nonnegative and eventually positive weight vectors such that for any $u, v \in \mathbb{R}^n$, $u \succ v$ if u sequentially utilitarian welfare dominates v, that is,

$$\langle \phi^t, u \rangle > \langle \phi^t, v \rangle \text{ for some } t \text{ and } \langle \phi^s, u \rangle = \langle \phi^s, v \rangle \text{ for all } s < t. \tag{12}$$

We will then show if $\succeq$ satisfies Weak Continuity in addition, then (12) implies $u \succ v$. Thus, $u \succ v$ if and only if u sequentially utilitarian welfare dominates v.

To do so, define $S := \{s \in \mathbb{R}^n : 0 \succeq s\}$ and $Q := \{s + p : s \in S, p \in \mathbb{R}^n, \text{ and } p \ll 0\}$. We now observe that Q is convex. To this end, let $q, q' \in Q$ and $q'' = tq + (1-t)q'$ for any $t \in (0,1)$. Then, $q = s + p$ and $q' = s' + p'$ for some $s, s' \in S$ and $p, p' \ll 0$, and $q'' = ts + (1-t)s' + tp + (1-t)p'$. Then, by Invariance, we have $0 \succeq ts$, $0 \succeq (1-t)s'$, and $(1-t)s' \succeq ts + (1-t)s'$. Thus, by transitivity, we have $0 \succeq ts + (1-t)s'$, i.e., $ts + (1-t)s' \in S$. Since $tp + (1-t)p' \ll 0$, we have $q'' \in Q$.

Letting $\bar{Q}$ denote the closure of Q, $\bar{Q}$ is also convex. Note that $0 \in S \subset \bar{Q}$ and that by the Pareto Principle, 0 is a maximal point of both S and $\bar{Q}$. Also, there is a maximal face $F \subset S$ with $0 \in F$. Letting $G^0 := \bar{Q}$, the same proof as Step 4 in the proof of Theorem 1 can be used to show there exists a sequence $\Phi = (\phi^1, \ldots, \phi^T)$ such that for each $t = 1, \ldots, T$,

$$G^t = \arg\max_{x \in G^{t-1}} \langle \phi^t, x \rangle, \tag{13}$$

where $\phi^t > 0$, $\operatorname{supp} \phi^t \supsetneq \operatorname{supp} \phi^{t-1}$, $\phi^T \gg 0$, and $G^T = F$.

Claim 1. (12) implies $u \succ v$

Proof. Suppose for contradiction that there are some u, v for which (12) holds but $v \succeq u$. Let $w := u - v$. Then, by Invariance, $0 \succeq w$, so $w \in S \subset \bar{Q} = G^0$. By the hypothesis, we have $\langle \phi^s, w \rangle = 0, \forall s < t$. Since $0 \in F = G^T$, (13) implies $w \in G^s, \forall s < t$, which in turn implies $\langle \phi^t, w \rangle \leq \langle \phi^t, 0 \rangle = 0$, or $\langle \phi^t, u \rangle \leq \langle \phi^t, v \rangle$. We thus have a contradiction. □

Claim 2. $u \succ v$ implies (12)

Proof. Let us first prove that $\langle \phi^s, u \rangle = \langle \phi^s, v \rangle, \forall s$ implies $u \sim v$. Suppose for a contradiction that $u \succ v$. By Weak Continuity, there are i and $\delta > 0$ such that $u' \succ v$ for all $u' \in B^i_\delta(u)$. We can then find a round t in which $i \in \operatorname{supp} \phi^t \setminus \operatorname{supp} \phi^{t-1}$. Since $\langle \phi^s, \chi^i \rangle = 0, \forall s < t$ and $\langle \phi^t, \chi^i \rangle > 0$ for the unit vector χ^i whose i-th component is equal to 1, we have $\langle \phi^s, u - \delta' \chi^i \rangle = \langle \phi^s, u \rangle = \langle \phi^s, v \rangle, \forall s < t$ and $\langle \phi^t, u - \delta' \chi^i \rangle < \langle \phi^t, u \rangle = \langle \phi^t, v \rangle$ for any $\delta' > 0$, which implies by the former statement that $v \succ (u - \delta' \chi^i)$, contradicting that $u' \succ v$ for all $u' \in B^i_\delta(u)$.

Since $\langle \phi^s, u \rangle = \langle \phi^s, u \rangle, \forall s$ implies $u \sim v$, $u \succ v$ implies that there must be some t such that $\langle \phi^s, u \rangle = \langle \phi^s, v \rangle, \forall s < t$ and $\langle \phi^t, u \rangle \neq \langle \phi^t, v \rangle$. Since $\langle \phi^t, u \rangle < \langle \phi^t, v \rangle$ would imply $v \succ u$ by Claim 1, we must have $\langle \phi^t, u \rangle > \langle \phi^t, v \rangle$ as desired. □

B.2 Proof of (ii) ⇒ (iii)

Consider the sequence $(\phi^1, \phi^2, \ldots, \phi^T)$ in (ii). Then, $u \succ v$ is equivalent to (12), which implies

$$\sum_{i \in I} \psi_i u_i - \sum_{i \in I} \psi_i v_i = \epsilon^{t-1} \Big[\langle \phi^t, u - v \rangle + \sum_{s > t} \epsilon^{s-t} \langle \phi^s, u - v \rangle \Big] > 0, \tag{14}$$

where the inequality holds since the first term in the square bracket is a positive real and the second term is infinitesimal. Conversely, if $\sum_{i \in I} \psi_i u_i > \sum_{i \in I} \psi_i v_i$, then there must be some s such that $\langle \phi^s, u \rangle > \langle \phi^s, v \rangle$. Letting t be the smallest such s, we must have $\langle \phi^r, u \rangle = \langle \phi^r, v \rangle, \forall r < t$: else if $\langle \phi^r, u \rangle < \langle \phi^r, v \rangle$ for some $r < t$, then one can use a similar argument to (14) to obtain $\sum_{i \in I} \psi_i v_i > \sum_{i \in I} \psi_i u_i$, a contradiction.

B.3 Proof of (iii) ⇒ (i)

That the welfare function in (5)—or a social welfare ordering it represents—satisfies the Pareto Principle and Invariance is straightforward to check. To check that it satisfies Weak Continuity, consider any $u \succ v$ so that $W(u) > W(v)$. As argued before, there must be some t such that $\langle \phi^t, u \rangle > \langle \phi^t, v \rangle$ and $\langle \phi^s, u \rangle = \langle \phi^s, v \rangle, \forall s < t$. Pick any $i \in \operatorname{supp} \phi^t \setminus \operatorname{supp} \phi^{t-1}$. For sufficiently small $\delta > 0$ and all $u' \in B^i_\delta(u)$, we have $\langle \phi^t, u' \rangle > \langle \phi^t, v \rangle$ while $\langle \phi^s, u' \rangle = \langle \phi^s, v \rangle, \forall s < t$. This implies as desired that for all $u' \in B^i_\delta(u)$,

$$W(u') - W(v) = \epsilon^{t-1} \Big[\langle \phi^t, u' - v \rangle + \sum_{s>t} \epsilon^{s-t} \langle \phi^s, u' - v \rangle \Big] > 0,$$

where the inequality holds for the same reason as (14) holds.

C Proof of Proposition 1 and Proposition 2

C.1 Proof of Proposition 1

The "only if" direction. Suppose $u \in U$ is Pareto optimal. Then, by Theorem 1, there is a sequence $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ of $T \le n$ nonnegative and eventually positive weight vectors such that u sequentially maximizes Φ. Defining $\psi := \sum_{t \in \{1,\ldots,T\}} \epsilon^t \phi^t$, we have $\psi_i > 0$ for each $i \in I$ since the vectors $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ are nonnegative and eventually positive. Also, we have $u \in \arg\max_{v \in U} \langle \psi, v \rangle$ by Theorem 1.

The "if" direction. To show the contrapositive, assume that u is not Pareto optimal. Then there exists $v \in U$ such that $v > u$. Then, for any weight vector $\psi = (\psi_i)_{i \in I}$ with $\psi_i \in {}^*\mathbb{R}$ and $\psi_i > 0$ for each $i \in I$, $\langle \psi, v \rangle - \langle \psi, u \rangle = \sum_{i \in I} \psi_i (v_i - u_i) > 0$. This means that $u \notin \arg\max_{u' \in U} \langle \psi, u' \rangle$, as desired.

C.2 Proof of Proposition 2

The (ii) ⇒ (i) direction is obvious. To prove (i) ⇒ (ii), we adopt the proof approach of Theorem 1' of an unpublished work by Blume (1986) who studies an individual's decision under uncertainty.[18] Suppose that the social welfare ordering $\succeq$ satisfies the Pareto Principle and Invariance.

Lemma C.1. Let $U' = \{u^1, u^2, \ldots, u^m\} \subset \mathbb{R}^n$ be a finite subset of utility profiles such that $u \succ \tilde{u}$ for some $u, \tilde{u} \in U'$. Then, there exists a nonnegative and non-zero weight vector $\phi^{U'} \in \mathbb{R}^n_+$ such that, for any $u, \tilde{u} \in U'$, $u \succeq \tilde{u}$ if and only if $\langle \phi^{U'}, u \rangle \ge \langle \phi^{U'}, \tilde{u} \rangle$.

Proof. We utilize the following fact:

Lemma C.2 (Lemma 7 of Blume (1986))**.** Let $v^1, \ldots, v^K$ and $w^{K+1}, \ldots, w^L$ be vectors in $\mathbb{R}^n$. Then, one of the following two statements holds.

[18] Blume (1986) is superseded by the published version, Blume, Brandenburger, and Dekel (1991), although our proof is more closely related to the former.

1. There exists $x \in \mathbb{R}^n_+ \setminus \{0\}$ such that

$$\langle x, v^k \rangle > 0 \text{ for all } k \in \{1, \ldots, K\}, \text{ and}$$
$$\langle x, w^\ell \rangle = 0 \text{ for all } \ell \in \{K+1, \ldots, L\}.$$

2. There exist $y \in \mathbb{R}^K_+, z \in \mathbb{R}^{L-K}$ such that

$$\sum_{k=1}^{K} y_k v^k + \sum_{\ell=K+1}^{L} z_\ell w^\ell \leq 0.$$

   Moreover, if $y = 0$, then $\sum_{\ell=K+1}^{L} z_\ell w^\ell \neq 0$.

Let $v^k, k = 1, \ldots, K$ be the vectors of the form $v^k = u^k - \tilde{u}^k$ where $u^k, \tilde{u}^k \in U'$ and $u^k \succ \tilde{u}^k$, and $w^\ell, \ell = K+1, \ldots, L$ be the vectors of the form $w^\ell = u^\ell - \tilde{u}^\ell$ where $u^\ell, \tilde{u}^\ell \in U'$ and $u^\ell \sim \tilde{u}^\ell$. It suffices to show that the case 1 of Lemma C.2 holds, as then the conclusion of Lemma C.1 holds when we set the solution x for the case 1 of Lemma C.2 as $\phi^{U'}$. To show this, we will show that the case 2 of Lemma C.2 does not hold. Suppose to the contrary that the case 2 of Lemma C.2 holds, with solution (y, z). We will obtain a contradiction.

Let $\tilde{z}_\ell := |z_\ell|$ and $\tilde{w}^\ell := sgn(z_\ell)w^\ell$. Then $(y, \tilde{z}) > 0$ and $\sum_{k=1}^{K} y_k v^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{w}^\ell \leq 0$.[19] Because v^k for each $k = 1, \ldots, K$ is of the form $u^k - \tilde{u}^k$ with $u^k \succ \tilde{u}^k$ while $\tilde{w}^\ell$ for each $\ell = K+1, \ldots, L$ is of the form $u^\ell - \tilde{u}^\ell$ with $u^\ell \sim \tilde{u}^\ell$, it follows that

$$\sum_{k=1}^{K} y_k \tilde{u}^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{u}^\ell \geq \sum_{k=1}^{K} y_k u^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell u^\ell. \tag{15}$$

Since $\succeq$ satisfies the Pareto Principle, this implies that

$$\left(\sum_{k=1}^{K} y_k \tilde{u}^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{u}^\ell \right) \succeq \left(\sum_{k=1}^{K} y_k u^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell u^\ell \right). \tag{16}$$

Meanwhile, since $u^k \succ \tilde{u}^k$ for each k and $u^\ell \sim \tilde{u}^\ell$ for each ℓ by assumption, by repeated applications of Invariance,[20] it follows that

$$\left(\sum_{k=1}^{K} y_k u^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell u^\ell \right) \succ \left(\sum_{k=1}^{K} y_k \tilde{u}^k + \sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{u}^\ell \right),$$

if there exists k with $y_k > 0$, a contradiction to (16). If $y_k = 0$ for all k, then $\sum_{\ell=K+1}^{L} z_\ell w^\ell \neq 0$ by assumption, so we have by (15),

$$\sum_{\ell=K+1}^{L} \tilde{z}_\ell u^\ell < \sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{u}^\ell,$$

[19] To see why $(y, \tilde{z}) > 0$, note that if $(y, \tilde{z}) = 0$, then $y = 0$ and $\sum_{\ell=K+1}^{L} z_\ell w^\ell = 0$, a contradiction to case 2 of Lemma C.2.

[20] Specifically, for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}^n$ with $u \succeq v$ and $\tilde{u} \succeq \tilde{v}$ as well as $\alpha, \beta \in \mathbb{R}_+$, we have $(\alpha u + \beta \tilde{u}) \succeq (\alpha v + \beta \tilde{v})$, with $\succeq$ replaced with $\succ$ if $u \succ v$ and $\alpha > 0$. This is because $(\alpha u + \beta \tilde{u}) \succeq (\alpha v + \beta \tilde{u})$ and $(\alpha v + \beta \tilde{u}) \succeq (\alpha v + \beta \tilde{v})$ by Invariance, and hence by transitivity of $\succeq$, the desired relationship follows (and the relation being strict if $u \succ v$ and $\alpha > 0$).

holds.[21] Since $\succeq$ satisfies the Pareto Principle, this implies that

$$\left(\sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{u}^\ell\right) \succ \left(\sum_{\ell=K+1}^{L} \tilde{z}_\ell u^\ell\right). \tag{17}$$

Meanwhile, recalling again $u^\ell \sim \tilde{u}^\ell$ for each $\ell = K+1, ..., L$, and applying Invariance repeatedly, we have that

$$\left(\sum_{\ell=K+1}^{L} \tilde{z}_\ell u^\ell\right) \sim \left(\sum_{\ell=K+1}^{L} \tilde{z}_\ell \tilde{u}^\ell\right),$$

a contradiction to (17). This completes the proof. □

Now we proceed to complete the theorem. To do so, we define

$$\mathcal{U} := \{U' \subset \mathbb{R}^n : |U'| < \infty, \exists u, v \in U', u \succ v\},$$

and, for each $u \in \mathbb{R}^n$, define the collection $U^u \subset \mathcal{U}$ by

$$\mathcal{U}^u := \{U' \in \mathcal{U} : u \in U'\}.$$

Then, we consider a family V of collections defined by

$$V := \{\mathcal{U}^u : u \in \mathbb{R}^n\}.$$

Now, let $u^1, u^2, \ldots, u^m \in \mathbb{R}^n$ and consider

$$\bigcap_{k=1}^{m} \mathcal{U}^{u^k}.$$

Note that $\bigcap_{k=1}^m \mathcal{U}^{u^k} \neq \emptyset$ since $\{u^1, u^2, \ldots, u^m\} \in U^{u^k}$ for all $k \in \{1, \ldots, m\}$, that is, the family V has the finite intersection property.

Now, we invoke the following fact:

Lemma C.3 (Proposition 3.6 of Joshi (1983))**.** A collection of sets has the finite intersection property if and only if there is a filter that is a superset of that collection.

The preceding argument and the claim imply that there exists a filter that is a superset of V. By Zorn's lemma, there exists an ultrafilter Ω that is a superset of the above filter, and hence a superset of V. This ultrafilter is clearly free, that is, the intersection of all sets in the collection Ω is empty: This is because all sets of the form $\mathcal{U}^u$ is an element of Ω, and $\cap_{u \in \mathbb{R}^n} \mathcal{U}^u = \emptyset$.[22]

Now, consider the set of functions from $\mathcal{U}$ to $\mathbb{R}$. We say that two functions r and s are equivalent if $\{U' \in \mathcal{U} : r(U') = s(U')\}$ is in Ω. It is straightforward to show that this is an equivalence relation and the set ${}^*\mathbb{R}$ of those equivalence classes is an ordered field which extends $\mathbb{R}$.[23] We call ${}^*\mathbb{R}$ the set of hyperreal numbers. It is well known that addition and

[21]To see this, suppose for contradiction that (15) holds as equality. Then it would imply $\sum_\ell \tilde{z}_\ell \tilde{w}^\ell = \sum_\ell z_\ell w^\ell = 0$, where the first equality follows from the definitions of $\tilde{z}_\ell$ and $\tilde{w}^\ell$.

[22]To show $\cap_{u \in \mathbb{R}^n} \mathcal{U}^u = \emptyset$, suppose for contradiction that $\cap_{u \in \mathbb{R}^n} \mathcal{U}^u$ is nonempty, so there exists $U' \in \cap_{u \in \mathbb{R}^n} \mathcal{U}^u$. Then, by definition U' is a finite subset of $\mathbb{R}^n$. So there exists $v \in \mathbb{R}^n$ such that $v \notin U'$. This implies $U' \notin U^v$, so $U' \notin \cap_{u \in \mathbb{R}^n} \mathcal{U}^u$, a contradiction.

[23]See Blume (1986) for proofs of this property as well as others in this paragraph.

multiplication defined on $\mathbb{R}$ extend readily to ${}^*\mathbb{R}$ by pointwise operations, while the orders $\geq$ and $>$ also extend in a similar manner. It is also standard to show that there exists an infinitesimal number in ${}^*\mathbb{R}$.

Now, let $\psi \in ({}^*\mathbb{R})^n$ be such that, for each $i \in I$, ψ_i is the equivalence class that contains the element r_i such that $r_i(U') = \phi_i^{U'}$ for each $U' \in \mathcal{U}$ and $\phi^{U'}$ given in Lemma C.1. Consider any $u, v \in \mathbb{R}^n$. We know that $\mathcal{U}^u \cap \mathcal{U}^v \in \Omega$. For any $U' \in \mathcal{U}^u \cap \mathcal{U}^v$, $u, v \in U'$, so if $u \succeq v$, then $\langle \phi^{U'}, u\rangle \geq \langle \phi^{U'}, v\rangle$. By construction of ψ, this implies that $\langle \psi, u\rangle \geq \langle \psi, v\rangle$. A similar argument shows that $u \succ v$ implies $\langle \psi, u\rangle > \langle \psi, v\rangle$. Finally, for each $i \in I$, note that $\chi^i \succ 0$ as $\succeq$ satisfies the Pareto Principle. Therefore, it follows that $\psi_i = \langle \psi, \chi^i\rangle > \langle \psi, 0\rangle = 0$, showing that $\psi \in ({}^*\mathbb{R}_{++})^n$. This completes the proof.

References

Afriat, S. N. (1967): "The construction of utility functions from expenditure data," *International Economic Review*, 8(1), 67–77. SA.5

Anderson, R. M. (1991): "Non-standard analysis with applications to economics," *Handbook of Mathematical Economics*, 4, 2145–2208. 8

Arrow, K., E. Barankin, and D. Blackwell (1953): "Admissible points of convex sets," in *Contributions to the Theory of Games*, ed. by Kuhn, and Tucker, pp. 87–91. Princeton University Press. 3, 4, SA.8

Ben-Israel, A., A. Ben-Tal, and A. Charnes (1977): "Necessary and sufficient conditions for a Pareto optimum in convex programming," *Econometrica*, pp. 811–820. 16

Ben-Tal, A. (1980): "Characterization of Pareto and lexicographic optimal solutions," in *Multiple Criteria Decision Making Theory and Application*, pp. 1–11. Springer. 16

Bewley, T. F. (2009): *General Equilibrium, Overlapping Generations Models, and Optimal Growth Theory*. Harvard University Press. 2, 3, 4, SA.7

Bitran, G. R., and T. L. Magnanti (1979): "The structure of admissible points with respect to cone dominance," *Journal of Optimization Theory and Applications*, 29(4), 573–614. 4

Blume, L. (1986): "Lexicographic refinements of Nash equilibrium," mimeo. 24, 26

Blume, L., A. Brandenburger, and E. Dekel (1991): "Lexicographic probabilities and choice under uncertainty," *Econometrica*, pp. 61–79. 8, 24

Bronshtein, E., and L. Ivanov (1975): "The approximation of convex sets by polyhedra," *Siberian Mathematical Journal*, 16(5), 852–853. SA.10

Brown, D. J., and A. Robinson (1975): "Nonstandard exchange economies," *Econometrica*, pp. 41–55. 8

Charnes, A., and W. W. Cooper (1967): *Management Models and Industrial Applications of Linear Programming*. Wiley. 16

Che, Y.-K., J. Kim, and F. Kojima (2019): "Weak monotone comparative statics," arXiv:1911.06442 [econ.TH]. 6

Daniilidis, A. (2000): "Arrow-Barankin-Blackwell theorems and related results in cone duality: A survey," in *Optimization*, pp. 119–131. Springer. 3

d'Aspremont, C., and L. Gevers (2002): "Social welfare functionals and interpersonal comparability," *Handbook of Social Choice and Welfare*, 1, 459–541. 5, 12

Dilmé, F. (2022): "Lexicographic Numbers and Stability in Extensive Form Games," . 8

Glover, B. M., V. Jeyakumar, and A. M. Rubinov (1999): "Dual conditions characterizing optimality for convex multi-objective programs," *Mathematical Programming*, 84(1), 201. 16

Goldblatt, R. (2012): *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*. Springer. 8, 17

Harsanyi, J. C. (1955): "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility," *Journal of Political Economy*, 63(4), 309–321. 2

Joshi, K. D. (1983): *Introduction to General Topology*. New Age International. 26

Kalai, E. (1977): "Nonsymmetric Nash solutions and replications of 2-person bargaining," *International Journal of Game Theory*, 6, 129–133. SA.1

Lopomo, G., L. Rigotti, and C. Shannon (2022): "Detectability, duality, and surplus extraction," *Journal of Economic Theory*, 204, 1–37. 18

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press. 12

Milgrom, P., and C. Shannon (1994): "Monotone comparative statics," *Econometrica*, 62, 157–180. 6, 7

Soltan, V. (2015): *Lectures on Convex Sets*. World Scientific. 11, 18, 19, 20, SA.8, SA.9

Topkis, D. M. (1998): *Supermodularity and Complementarity*. Princeton University Press. 6

Van Rooyen, M., X. Zhou, and S. Zlobec (1994): "A saddle-point characterization of Pareto optima," *Mathematical Programming*, 67(1-3), 77–88. 16