Infinite Debt Rollover in Stochastic Economies

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Abstract

This paper shows that there is *more* scope for a borrower to engage in a *sustainable infinite debt rollover* (a "Ponzi scheme") when interest/growth rates are stochastic. In this context, I prove that the relevant "r vs. g" comparison uses the yield r_{long} to an infinite-maturity zero-coupon bond. I show that r_{long} is lower than the risk-neutral expectation of the short-term yield when it is variable, and that r_{long} is close to the minimal realization of the short-term yield when it is highly persistent. The paper applies these results to illustrative heterogeneous agent dynamic stochastic general equilibrium models to obtain similarly weakened sufficient conditions for the existence of public debt bubbles.

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1 Introduction

Suppose that a borrower faces a constant interest rate $\bar{r}$ and a supply of available loanable funds that is growing deterministically at rate $\bar{g}$. If $\bar{g} \geq \bar{r}$, it is possible for the borrower to engage in what I will term a *sustainable infinite debt rollover.* In such a (Ponzi) scheme, the borrower issues debt in the current period, repays the principal and interest by issuing new debt next period, and then so on ad infinitum (literally). Of course, this infinite rollover plan is not sustainable if $\bar{g} < \bar{r}$ because the requisite repayments necessarily must eventually exceed the funds accessible to the borrower.

But what are the analogs of these conditions on interest rates and/or growth rates if they are stochastic? This paper tackles this question in a general Markovian setting. Its main finding is that there is *more* scope for a sustainable infinite debt rollover (based on risky and/or long-term debt)[1] when interest rates and growth rates exhibit persistent fluctuations. I then use this *partial equilibrium* result to show, in the context of two prototypical macroeconomic models, that there is broader scope for public debt bubbles in *general equilibrium* if non-bubbly interest rates/growth rates exhibit persistent fluctuations.

The specifics are as follows. One-period interest rates and one-period growth rates are governed by a discrete-time time-homogeneous Markov process (with respect to risk-neutral probabilities, which are treated as exogenous until the last section). The key variable is the yield r_{long} on a zero-coupon bond with arbitrarily long maturity (the far right end of the yield curve). Consistent with much earlier work (notably, Hansen and Scheinkman (2009) and Alvarez and Jermann (2005)), the paper assumes that r_{long} is really a parameter, in the sense that it is constant across dates and states. Section 3 shows that this assumption of the constancy of r_{long} is satisfied whenever the driving Markov process is in fact a finite-state Markov chain with a primitive transition matrix.

The paper establishes its two main sets of results in Section 4.

[1] The resulting liabilities are required to have positive value in all dates and states (ensuring the rollover is truly infinite in nature).

1. Suppose the growth rate g_t of available loanable funds is a deterministic constant $\bar{g}$. There is a sustainable infinite debt rollover if and only if the long-term yield r_{long} is less than or equal to $\bar{g}$. **Thus, when interest rates are stochastic, the relevant "r vs. g" comparison is "r_{long} vs. g".**

2. The long-term yield r_{long} is less than the (risk-neutral) expectation of one-period bond yields if they are stochastic. If they are highly persistent, then r_{long} is well-approximated by the *lowest* possible realization of the one-period yield. **It is in this sense that allowing for volatility and persistence of short-term interest rates makes sustainable infinite debt rollover *more* possible.**

The following two findings, developed in Sections 5-6, extend the scope of the two main results described above.

- If the one-period growth rate g_t is stochastic, construct a *detrended* economy with deterministically zero growth by subtracting the realized growth rate in each date and state from the one-period riskfree bond yield in that date and state. Then, there is a sustainable infinite debt rollover in the original (undetrended) economy if and only if the long-term yield in the detrended economy ($\hat{r}_{long}$) is non-positive. It is possible to measure $\hat{r}_{long}$ using an appropriate normalization of the *price* of a very long-dated *growth asset* in the original (undetrended) economy. A growth asset makes only a single risky payment, indexed to the cumulative growth of loanable funds, at maturity (and so is entirely distinct from a Lucas tree based on the borrower's own income stream).

- Section 6 translates the above partial equilibrium results into general equilibrium conditions for the existence of public debt bubbles. In particular, it considers two leading example heterogeneous agent dynamic stochastic general equilibrium models (overlapping generations and incomplete financial markets). In those settings, it shows that the partial equilibrium conditions for a sustainable infinite debt rollover, assessed using autarkic (shadow) interest rates and growth rates (as in Samuelson (1958)), are also sufficient to ensure the existence of a public debt bubble.

Why is r_{long} the appropriate benchmark interest rate, as is shown in Result 1? Pick any horizon T. A borrower who rolls over debt for T years, and then stops doing so, is receiving resources today but has to give up resources in T years. A T-year debt rollover is hence equivalent in a cash flow sense to issuing a T-year discount (zero coupon) bond. Taking limits, an infinite debt rollover is equivalent to issuing an infinite horizon zero coupon bond. Accordingly, the yield on an infinite debt rollover is the interest rate at the far right end of the yield curve - that is, r_{long}.

The intuition behind the first part of Result 2 (r_{long} is low when short term yields are volatile) arises in the economics of fixed income securities[2] and also plays a key role in environmental economics.[3] Consider a situation in which the current riskfree one-year yield is zero. A fair coin is flipped in 20 periods to determine whether the riskfree one-period yield is 0% or 10% thereafter. A risk-neutral investor in period 1 wants to use the available bonds to generate an expected payoff of $1000 in 40 years. How much does the investor need to have available in period 1?

It may seem like the answer to this question should at least be well-approximated by $1000/1.05^{20}$ - that is, the present value of $1000 dollars calculated using the average yield of 5% from periods 20 to 40. But this averaging ignores the fundamental *convexity* of the compounding of interest, which makes it very costly to generate $1000 if the yields don't jump up in period 20. That convexity means that the investor actually needs:

$$1000(\frac{0.5}{1.1^{20}} + \frac{0.5}{1}) = 574 > 377 = 1000/1.05^{20}.$$

This kind of consideration implies that the yield on a long-term investment can be much lower than the average of the expected one-year yields over the investment's life. In this example, the implied average annual yield from periods 20 to 40 is only 2.8%, rather than

[2] See, among others, Litterman, Scheinkman and Weiss (1991) and Gilles (1996) for discussions of what is often termed the *convexity* factor in the determination of longer-term yields.

[3] See Newell and Pizer (2003).

5%.

The second part of Result 2 (that r_{long} is near the minimal short-term yield when the latter is highly persistent) echoes the logic of Weitzman (1998). Intuitively, suppose that, instead of a coin flip in 20 years and a bond maturity of 40 years, we use 2000 and 4000 years in the above calculation. Then, the yield on the long-term investment falls to only 0.3%. This calculation illustrates the second part of Result 2: if short-term interest rates are highly persistent, then the long-term yield is close to the *minimal* short-term yield.

Throughout the paper, the relevant (one-period) growth rate (whether stochastic or deterministic) is that of the supply of loanable funds. The whole point of a sustainable infinite debt rollover is that the borrower's position is self-financing. Hence, the growth of the borrower's own income is irrelevant in determining the viability of the rollover. It may seem that in general equilibrium, the two growth rates (of the borrower's income and of the available loanable funds) should be aligned, and also should determine the risk-neutral probabilities. However, as the examples in Section 6 demonstrate, these various connections are typically disrupted in bubbly economies.

This paper is related to two main strands of literature. The first is the pricing theory for long-term assets, as developed by Alvarez and Jermann (2005), Hansen and Scheinkman (2009), and Martin (2012). This research is particularly salient because, in much of the paper, I follow W. Jiang, et al. (2022) and treat asset prices as exogenous.

The second relevant line of research dates back to the seminal work of Samuelson (1958). It studies what Brunnermeier, Merkel, and Sannikov (2022) term public debt bubbles, in which government liabilities have positive value even though primary surpluses are known to be non-positive with probability one.[4] There has been a recent revival of interest in this phenomenon, spurred in no little part by Blanchard (2019)'s provocative address.

However, the research on public debt bubbles in the presence of stochastic interest/growth rates is (surprisingly) limited. Peled (1982) and Manuelli (1990) provide sufficient conditions

[4]See Z. Jiang, et al. (2022) for a countervailing perspective on the valuation of US government debt that abstracts from the possibility of public debt bubbles..

for the existence of a monetary equilibrium (that is, a public debt bubble) in classes of stochastic overlapping generations (OG) economies without growth. Their conditions are both strictly stronger than the "$r_{long} \leq 0(= g)$" restriction developed in this paper.

Chattopadhyay and Gottardi (1999) (CG) also study a class of non-growing stochastic OG models. Their Theorem 4 provides a simple condition on autarkic contingent claims prices under which there exists an allocation that is (conditionally) Pareto superior to autarky. Their condition is mathematically equivalent to requiring r_{long} to be non-positive (as is derived in this paper). However, CG do not make any connection between their characterization and the yield curve (or to public debt bubbles).[5]

One of the many challenges with obtaining general conditions for the existence of (public debt) bubbles is that they are typically studied in a wide range of distinct models. In this paper, I obtain generality by focusing instead on infinite debt rollover with exogenous asset prices in stochastic economies. But rolling over debt in partial equilibrium is similar to rolling over a bubble in general equilibrium. Section 6 illustrates this intuitive connection in two classes of macroeconomic models by demonstrating that the partial equilibrium results derived in the first part of the paper can be translated directly into similarly weak conditions for the existence of bubbles in dynamic stochastic general equilibrium.

All proofs are in an Appendix.

2 Illustrative Two-State Example

In this section, I use an example to illustrate the main results in the paper. As in Sections 3-5 of the paper, the analysis is deliberately posed in a partial equilibrium setting, in which the stochastic evolution of the one-period interest rate is treated as exogenous. I defer the treatment of general equilibrium issues until Section 6.

[5] See also Aiyagari and Peled (1991). Of course, the connections to the yield curve are now much clearer thanks to research done in the decade after CG was published.

2.1 Description of the Example

At each date, investors trade a one-period risk-free bond with a payoff of one unit (which could be either nominal or real). The bond's price evolves over time according to a two-state Markov chain, with a state space given by (q_1^1, q_2^1), where $q_1^1 < 1 < q_2^1$. The bond price in state 1 corresponds to a one-period yield $y_1^1 = -ln(q_1^1) > 0$ and the bond price in state 2 corresponds to a one-period yield $y_2^1 = -ln(q_2^1) < 0$. The positive 2×2 transition matrix is denoted by P, where $P_{11}, P_{22} \geq 0.5$.

A borrower faces a fixed supply of loanable capacity L_0 from a pool of risk-neutral lenders.[6] The borrower wants to engage in an ***infinite debt rollover***. In such a financing strategy, she borrows B today, which she repays only by issuing new debt. That new debt is repaid by borrowing again, and so on, ad infinitum. The debt used in the rollover is not restricted in terms of maturity or risk, except that the value of the resulting outstanding liabilities must have positive value in all dates and states. (This last restriction is to ensure that the debt rollover is, in fact, infinite.) The main question in this paper is: Under what conditions is such an infinite debt rollover sustainable without exhausting the lending capacity L_0?

To answer this question, it is helpful to first consider a zero coupon bond that makes a single risk-free payment of one unit in T periods. In general, the per-period (continuously compounded) yield y_j^T on this bond at any date depends (only) on the state j - that is, the current one-period bond price q_j^1. But Proposition 1 (in section 3) shows that in this two-state example, the *limiting* yield, as the bond maturity T converges to infinity, is a time and state-invariant constant:

$$r_{long} = lim_{T \to \infty} y_j^T, j = 1, 2.$$

Indeed, as discussed in the introduction, this constancy of r_{long} is a property of a much broader class of commonly-used asset pricing models.

[6] In the body of the paper, I allow lenders to be risk averse, and exploit *risk-neutral* probabilities as the mathematical basis of the results. Section 5 also allows L_0 to exhibit stochastic growth.

The next two subsections show (in the context of the two-state example) that:

- a sustainable infinite debt rollover exists iff $r_{long} \leq 0$.
- r_{long} tends to be low and is especially so when P is close to the identity matrix (one-period bond yields are highly persistent).

The analysis makes use of a key formula (established in Section 3 in Proposition 1) that connects r_{long} to P and q. Define the matrix Q by:

$$Q = [\begin{array}{cc} q_1^1 P_{11} & q_1^1 P_{12} \\ q_2^1 P_{12} & q_2^1 P_{22} \end{array}] \quad (1)$$

Then, r_{long} satisfies the condition:

$$r_{long} = -ln(max(eig(Q)), \quad (2)$$

where $max(eig(Q))$ represents the maximal eigenvalue of Q.

2.2 The Role of the Very Long-Term Yield

This subsection illustrates why the magnitude of r_{long} determines whether an infinite debt rollover is sustainable.

2.2.1 Negative r_{long}

Suppose that the long-term yield r_{long} is negative (that is, less than the zero growth rate of lending capacity). Then, there is some bond maturity T^* such that the T-period bond yields y_j^T are negative in either state $j \in \{1, 2\}$ and for any maturity $T \geq T^*$. A borrower in any period t can borrow B today in exchange for a repayment of $Bexp(y_{s_t}^T)$ in T periods, where $s_t \in \{1, 2\}$ is the state in period t. She repays that obligation by borrowing $Bexp(y_{s_t}^T)$ in period $(t + T)$ in exchange for $Bexp(y_{s_t}^T + y_{s_{T+t}}^T)$ in period $(t + 2T)$ periods and so on ad infinitum. This infinite debt rollover is sustainable: since $y_j^T < 0$ for both values of j, the repayment amounts are falling over time.

This financing strategy relies on T-period bonds with risk-free repayments. However, these long-term bonds are not risk-free in terms of their values. Thus, in period $(t+1)$, the value of the (now) $(T-1)$-period bonds is given by:

$$\frac{Bexp(y_{s_t}^T)}{exp(y_{s_{t+1}}^{T-1})}$$

which is low if $s_{t+1} = 1$ and high if $s_{t+1} = 2$ (given that the transition probabilities $P_{11}, P_{22} \geq 0.5$). The variation is especially large when the one-period interest rate shocks are highly persistent (which implies that $exp(y_1^{T-1}) >> exp(y_2^{T-1})$).

The borrower could instead embed this price risk into payoffs by using a *time-homogeneous* debt rollover strategy based on *one-period risky debt*. Specifically, suppose the state in period t is 1. The borrower wants to raise B_1 in that date and state. Choose B_2 so that (B_1, B_2) is an eigenvector of the matrix Q (defined in (1)) associated with its maximal eigenvalue $exp(-r_{long})$. Then, (B_1, B_2) satisfies.

$$\left[\begin{array}{c} B_1 \\ B_2 \end{array} \right] = Q \left[\begin{array}{c} B_1 exp(r_{long}) \\ B_2 exp(r_{long}) \end{array} \right]$$

As long as (B_1, B_2) are both less than L_0, the borrower can use the following rollover strategy:

- issue a one-period bond with state-contingent payments $exp(r_{long})(B_1, B_2)$.
- repay the state-contingent period $(t+1)$ bond by again issuing a one-period risky bond with state-contingent payments $exp(r_{long})(B_1, B_2)$.
- continue ad infinitum.

Note that this strategy gives the borrower free resources to spend in every date and state, since $exp(r_{long}) < 1$.

The time homogeneity of this latter class of financing strategies based on one-period risky debt makes them much easier to manage technically. Hence, the paper's analysis focuses on this notion of (risky) debt rollover.

2.2.2 Zero r_{long}

This subsection assumes that $r_{long} = 0$ (that is, the growth rate of lending capacity). Under this assumption, for any finite T, Section 4 shows in Proposition 3 that there is a positive probability that the yield on a T-period bond is positive. Hence, the borrower cannot raise B by planning to roll over finite-horizon debt. But the borrower can use one-period risky debt as described in the prior subsection. Specifically, suppose the borrower wants to raise B_1 at some date in the high-interest rate state 1. Since $r_{long} = 0$, the maximal eigenvalue of Q is 1. Define the corresponding eigenvector (B_1, B_2), where $B_2 > 0$ and:

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = Q \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

As long as $B_i < L_0$, $i = 1, 2$, the borrower can raise B_1 in state 1 by following the one-period infinite debt rollover strategy described in the prior subsection.

The borrower cannot construct a sustainable rollover using finite-horizon riskfree bonds. However, she can use a zero-coupon infinite horizon bond - that is, money. The idea here is that the value of money is governed by the eigenvector (B_1, B_2). Thus, the borrower can raise B_1 in state 1 by issuing money which, in any future state j, has value B_j. However, this monetary implementation is more delicate than that based on one-period risky debt. The one-period risky debt implementation is anchored by the borrower's promises to refinance and repay. The monetary implementation relies instead on co-ordinated beliefs among the investors about the future value of the infinite-horizon asset.

2.2.3 Positive r_{long}

Now suppose instead that the long-term yield r_{long} is positive. Section 4 shows (in Proposition 3) that for any T, there is a positive probability that $y_1^T > r_{long} > 0$. Suppose the borrower attempts an infinite debt rollover using T-period bonds. If she encounters a long consecutive sequence of state 1 realizations, then her repayments will grow over time at rate y_1^T. They will eventually exceed the available loanable funds. Hence, when $r_{long} > 0$, no sustainable infinite debt rollover using riskfree bonds is possible.

The argument only treats safe (but longer maturity) debt, and so leaves open the possibility that an infinite debt rollover could be sustained using one-period *risky* debt, as was discussed in the two prior subsections. But that argument relied on $exp(r_{long}) \leq 1$. The paper shows that, once $exp(r_{long}) > 1$, it is also impossible to construct a sustainable infinite debt rollover using risky debt.

2.3 The Determinants of r_{long}

This subsection illustrates through a numerical example that r_{long} tends to be low when the one-period bond yield is risky and persistent.

Suppose that the Markov chain for the one-period bond prices takes the form:

$$q_1^1 = exp(-0.06); q_2^1 = exp(0.02)$$

$$P = [\begin{matrix} 0.5(1+\rho) & 0.5(1-\rho) \\ 0.5(1-\rho) & 0.5(1+\rho) \end{matrix}], \rho \in [0, 1)$$

For any value of ρ, state 2 is equally likely to occur as state 1 (under the stationary distribution implied by P). The autocorrelation of one-period bond prices/yields is given by ρ.

The following table uses the eigenvalue formula (2) for r_{long} to describe how it depends on the autocorrelation[7] ρ.

Table 1: Dependence of r_{long} on ρ

ρ	0	0.9	0.95	0.99	0.998
r_{long}	0.019	0.007	-0.002	-0.015	-0.019

The table reveals that the long-term yield is low in two distinct senses:

1. For any value of ρ, the average one-period yield (under the stationary distribution) is equal to 2.0%. The long-term yield r_{long} is less than that average for any value of ρ.

2. As ρ nears 1 - so that the one-period yield becomes increasingly persistent - r_{long} approaches the smallest possible realization of the one-period yield.

The paper shows that these properties are highly general.

We can gain intuition into these two properties by considering a related environment in which a fair coin is flipped every N periods to determine a new level for the one-period bond price (from the set $\{q_1^1, q_2^1\}$). In this setting, the price of a (τN)-period bond at the time of a coin flip satisfies:

$$q_j^{\tau N} = q_j^1 (q_j^1)^{(N-1)} (0.5(q_1^1)^N + 0.5(q_2^1)^N)^{(\tau-1)}, j = 1, 2,$$

Taking logs:

$$-\frac{1}{\tau N} ln(q_j^{\tau N}) = -\frac{1}{\tau} ln(q_j^1) - \frac{(\tau - 1)}{\tau N} ln(0.5(q_1^1)^N + 0.5(q_2^1)^N), j = 1, 2.$$

Taking limits with respect to τ, we obtain:

$$r_{long} = -\frac{1}{N} ln(0.5(q_1^1)^N + 0.5(q_2^1)^N). \quad (3)$$

[7] This table is meant only to be illustrative. Nonetheless, it may be worth noting that the estimated autocorrelation of interest rates, real or nominal, is typically found to be near one. The maximal autocorrelation in this table (0.998) corresponds to the estimated autocorrelation for one-year LIBOR yields in Bali, et al. (2009, Table 1).

Jensen's inequality then implies that for any N :

$$r_{long} < -0.5ln(q_1^1) - 0.5ln(q_2^1) = 0.02.$$

As well, when N is large (so that the one-period interest rates are highly persistent), the term in (3) inside the logarithm of is dominated by $(q_2^1)^N$ and so:

$$r_{long} \approx -ln(q_2^1) = -0.02.$$

Note that the fairness of the coin flip plays no role in this last conclusion.

Above, we have seen how an infinite rollover of one-period risky debt is sustainable if $r_{long} < 0$. The relevant debt promises large repayments in state 2, when one-period yields are low, and small repayments in state 1, when one-period yields are high. The following table 2 describes the ratio of the two repayments. (Its first row is the same as that of Table 1). It illustrates that, as is intuitive, the riskiness of the one-period debt increases with ρ.

Table 2: Dependence of Debt Risk and r_{long} on ρ

ρ	0	0.9	0.95	0.99	0.998
r_{long}	0.019	0.007	-0.002	-0.015	-0.019
B_2/B_1	no sustainable rollover	no sustainable rollover	3.55	16.6	83.3

When the short-term bond yields are highly persistent, the debt rollover is based on a very large repayment in state 2 relative to that in state 1.

As discussed in Section 2.2.1, when $r_{long} < 0$, an infinite rollover of riskfree *finite-maturity* debt is sustainable. For the particular parameterization of P and q^1 in this subsection, and if $\rho = 0.99$, the borrower can use 200-period bonds. In the high-yield state 1, those bonds have a yield of -0.2%, while their yield is -1.6% in state 2. The 200-period yield in state 1 is much lower than the one-period interest rate in that state, because of the (very!) small probability of switching to the highly persistent low-interest-rate state 2.

3 Model

This section presents the baseline general model in which asset prices are exogenous. The relevant growth rate (of loanable funds) is treated as deterministic until Section 5.

Time is discrete. Let (Ω, F, Pr) be a probability space.[8] The risk-neutral probability measure implied by asset prices is denoted by Pr^*. As is standard, Pr^* and Pr are assumed to be equivalent, meaning that the characterization "almost everywhere" has the same content for the two measures. Economically, this means that a claim to consumption in some event has a positive price if and only if that event has a positive true probability of occurring.

Let the stochastic process $\{x_t\}_{t=1}^{\infty}$ be a time-homogeneous Markov process[9], with respect to Pr^*, that has state space X. Consider a one-period bond that pays off one unit without risk. (I am deliberately agnostic about what "units" mean here, so that the bonds could be nominal or real.) I assume that the price of this one-period bond at any date is a time-invariant positive function q^1 of the Markov state, so that:

$$q_t^1 = q^1(x_t) \equiv exp(-y^1(x_t))$$

Here, $y^1(x_t) = -ln(q^1(x_t))$ is the yield on the one-period bond in state x_t.

Consider an N-period zero-coupon bond trading at date t that pays off one unit without

[8]This formalism allows for the possibility that bond yields are continuous random variables (so as to be consistent with much of the fixed income literature). But, as suggested by the prior section, some readers may find that the core intuitions are clearer in settings in which uncertainty is governed by a finite state Markov chain.

[9]Formally, let X be a Borel subset of a Euclidean space and $B(X)$ represent the Borel subsets of X. Suppose there is a Markov kernel p^* so that:

$$p^* : X \times B(X) \to [0,1]$$

For any B in $B(X)$,$p^*(., B)$ is Borel-measurable

For any x, $p^*(x, .)$ is a probability measure over $B(X)$.

Let μ_1 be a probability measure over $B(X)$. Then, the joint probability Pr^* of any event $(A_1 \times A_2 \times A_3 \times ... \times A_t)$, where A_t is in $B(X)$, can be computed as:

$$\int_{A_1}\int_{A_2}...\int_{A_{t-1}}\int_{A_t} p^*(x_{t-1}, dx_t)p^*(x_{t-2}, dx_{t-1})...p^*(x_1, dx_2)\mu_1(dx_1).$$

risk in period $(t+N)$. Denote its price by q_t^N and its continuously compounded yield by:

$$y_t^N = -ln(q_t^N)/N$$

A N-period bond is a one-period promise to receive an $(N-1)$-period bond. Hence, yields satisfy the following recursion:

$$exp(-Ny_t^N) = exp(-y^1(x_t))E^*(exp(-(N-1)y_{t+1}^{N-1})|x_t).$$

It follows that the yield y_t^N at date t is a time-and-state-invariant function y^N of the Markov state x_t, which takes the form:

$$\begin{aligned} y^N(x_t) &= -ln(E^*(\prod_{s=1}^{N} exp(-y^1(x_{t+s-1})|x_t))/N \\ &= -ln(E^*(exp(-\sum_{s=1}^{N} y^1(x_{t+s-1}))|x_t))/N \end{aligned}$$

With these representations in hand, I make the following assumption about long-term bond yields.

Assumption 1: There exists a constant r_{long} and a bounded function $\phi : X \to \mathbb{R}$ such that

$$lim_{N\to\infty} N(y^N(x) - r_{long}) = \phi(x) \tag{4}$$

almost everywhere.

Recall that Pr and Pr^* agree on what they imply about "almost everywhere". Hence, Assumption 1 can be viewed as being stated in terms of either the risk-neutral or true probabilities.

Assumption 1 immediately implies that the (very) long-term yield does not vary with the state x:

$$lim_{N\to\infty} y^N(x) = r_{long}.$$

However, it also has the stronger requirement that the rate of convergence of $y^N(x)$ to r_{long}, with respect to N, is sufficiently fast so that the sequence $N(y^N(x) - r_{long})$ does not explode to infinity (in absolute value) as N grows to infinity.

The following proposition applies the Perron-Frobenius Theorem to show that Assumption 1 is satisfied whenever x_t follows a finite-state Markov chain with a positive[10] transition matrix P^*. The statement of the proposition uses the notation $max(eig(M))$ to refer to the maximal eigenvalue of a matrix M.

Proposition 1. *Suppose that, under Pr^*, $\{x_t\}_{t=1}^{\infty}$ is governed by a Markov chain with a state space $\{1, 2, ..., J\}$ and a positive transition matrix P^*. Let q_i^1 be the price of a one-period bond in state i. Define a (positive) matrix Q^* via $Q^*_{ij} = P^*_{ij} q_i^1, i, j = 1, ..., J$. Then Assumption 1 is satisfied, with:*

$$r_{long} = -ln(max(eig(Q^*))).$$

Note that in the statement[11] of Proposition 1, the matrix element Q^*_{ij} is equal to the stochastic discount factor from state i to state j.

More generally, Hansen and Scheinkman (2009, p. 214) show that a version of Assumption 1 is a property of equilibrium in their (much broader) set of model environments.[12] It is also similar to Assumption 1 in Alvarez and Jermann (2005, p. 1982). It is implied too by the long-run risk asset pricing model of Bansal and Yaron (2004), which is itself a generalization

[10]The proposition also applies to any primitive transition matrix (so that there exists a natural number τ such that $P^{*\tau}$ is a positive matrix). In this way, it can be extended to cover second-order Markov chains.

[11]The connection made in Proposition 1 between r_{long} and the maximal eigenvalue of Q^* was earlier noted by Martin and Ross (2019, p. 692) and Bloise, et al. (2017, p. 1148).

[12]The variable r_{long} in this paper is equivalent to the negative of ρ in Hansen and Scheinkman (2009). While they write on page 214, "ρ is typically negative, " it does not seem that the negativity of ρ plays any role in their formal analysis. Of course, the current paper's focus is quite distinct, as it is all about exploring the consequences of negative $(r_{long} - g)$.

of the standard representative agent power utility model.[13]

Like Newell and Pizer (2003), Gollier (2015) argues that short-term interest rates should be viewed as following a random walk. This specification violates Assumption 1. However, Assumption 1 does accommodate processes that are very close to being random walks (as illustrated by the example in the prior section).

It will sometimes be useful to strengthen Assumption 1 by adding the following uniform boundedness restriction.

Assumption 1*: Assumption 1 is satisfied. In addition, there exists a constant $k > 0$ such that if $N \geq N^*$, then $|Ny^N(x) - Nr_{long} - \phi(x)| \leq k$ almost everywhere, where ϕ is defined as in (4).

Readers who are willing to proceed under the assumption that the state space X is finite can ignore Assumption 1*, as it is implied by Assumption 1 in that case.

4 Infinite Debt Rollover

This section contains the main results. It considers a setting in which growth is deterministic but bond yields are stochastic, and asks under what conditions an infinite debt rollover is sustainable. It shows that an infinite debt rollover is sustainable if and only if the growth rate is no smaller than r_{long}, the interest rate at the far right of the yield curve. It shows too that r_{long} is, in at least a couple of senses, low relative to one-period yields.

[13]See Bansal and Shaliastovich (2013) for technical details. Their model (equations (26)-(27) on page 17) implies that for both real and nominal yields,

$$lim_{N\to\infty}(Ny_t^N - Nr_{long})$$

is a time-invariant linear function (which is ϕ in Assumption 1) of the four state variables (which constitute x_t in Assumption 1). The long-run yield r_{long} can be computed as $lim_{N\to\infty}(B_{0,N} - B_{0,N-1})$ in their equation (A13) on page 31.

4.1 Definition

This subsection defines what is meant by a sustainable debt rollover. As noted in the introduction, the relevant growth is not that of the borrower's income: the borrower's position is self-financing, and so they need never use their income. What matters instead is that the growth of available loanable funds is sufficiently large to allow the borrower to keep rolling over their debt.

Suppose that the available loanable funds $\{L_t\}_{t=1}^{\infty}$ grow at a constant rate g, so that:

$$L_t = exp(tg)L_0$$

for some positive L_0. There is a *sustainable infinite debt rollover* if the borrower can construct a self-financing chain of debt issues while keeping the debt to available loanable funds ratio bounded. The relevant debt is not restricted in terms of either maturity or risk.

Mathematically, an infinite debt rollover is sustainable if there exists a real number $\lambda \geq 1$ and a bounded function $v : X \to \mathbb{R}$ so that:

$$exp(v(x)) = \frac{exp(-y^1(x))exp(g)}{\lambda} E^*(exp(v(x'))|x). \tag{5}$$

Here, $exp(v(x_t))L_t$ represents the amount of debt issued in period t in state x_t. Notice that v is only determined up to an arbitrary constant[14], so that the upper bound on $exp(v(x))$ can be made as small as is deemed plausible.[15] The use of exponentials insures that the value of the outstanding debt is always positive, which implies that some amount of debt is indeed being rolled over in every date and state.

Why is (5) the appropriate formulation to think about infinite debt rollover? Consider a borrower who owes $\frac{exp(v(x_t))L_t}{\lambda}$ units in state x_t. The borrower sells a bond that promises to

[14] Consider the linear functional operator $W(f)(x) = exp(-y^1(x))exp(g)E^*(f(x')|x)$. Then, $exp(v)$ is an eigenfunction of this operator W with eigenvalue λ.

[15] In particular, the upper bound can be chosen to be less than one.

pay:

$$(exp(v(x_{t+1}))L_{t+1}/\lambda)$$

in period $(t+1)$. That sale will raise:

$$L_t exp(g)\frac{exp(-y^1(x_t))E^*(exp(v(x_{t+1}))|x_t)}{\lambda} = exp(v(x_t))L_t$$

units in state x_t. The borrower can use that to pay off its obligations in state x_t, because:

$$exp(v(x_t))L_t \geq exp(v(x_t))L_t/\lambda.$$

In this sense, debt can be rolled over forever.

4.2 Main Result

The following (main) proposition demonstrates that, as illustrated in Section 2.2, an infinite debt rollover is sustainable if and only if g is no larger than r_{long}.

Proposition 2. *Suppose that Assumption 1* is satisfied. An infinite debt rollover is sustainable if and only if the growth rate g is no smaller than the long-term yield r_{long}. In that case, the rollover factor λ (in (5)) is equal to $exp(g - r_{long})$.*

Thus, it is possible to keep rolling over debt as long as the growth rate is no smaller than the long-term yield. It is important to emphasize that, under Assumption 1 (or, as needed, Assumption 1*), r_{long} is a time and state-invariant constant. Hence, the difference between risk-neutral and objective probabilities plays no role in the statement or proof of the proposition.

4.3 An Upper Bound For r_{long}

This subsection shows that r_{long} is weakly bounded from above by the risk-neutral expectation of shorter-term yields. The bound is strict if they are volatile (and if the Markov process satisfies a weak regularity condition).

The first result derives a (weak) upper bound in terms of S-period bond yields, for $S \geq 1$.

Proposition 3. *Let $\{x_t\}_{t=1}^{\infty}$ be strictly stationary under Pr^*, and suppose Assumption 1* is satisfied. Then:*

$$E^*(y^S(x_t)) \geq r_{long}$$

for any $S \geq 1$.

Using E^* rather than E eliminates the impact of risk on yields. The convexity effect then tilts the risk-neutral expectation of the yield curve downward.

Mathematically, Proposition 3 is a simple consequence of Jensen's inequality (as was illustrated in Section 2.3). With that motivation in mind, the following corollary shows that the upper bound for S-period yields becomes strict if they are volatile and the Markov process satisfies a regularity restriction, so that x_t does not always fully reveal x_{t+S}. (The regularity condition is satisfied by any Markov chain with a positive transition matrix.)

Corollary 1. *Suppose that the hypotheses of Proposition 3 are satisfied, and that, for some $S \geq 1$, the yield y^S satisfies $Var^*(y^S(x_t)) > 0$. Suppose in addition that the Markov process $\{x_t\}_{t=1}^{\infty}$ satisfies the restriction that for any $S \geq 1$ and for any $f : X \rightarrow \mathbb{R}$, $Var^*(f(x_t)) > 0 \Rightarrow E^*(Var^*(f(x_{t+S})|x_t)) > 0$. Then $E^*(y^S(x_t)) > r_{long}$.*

The upper bound in Proposition 3 is attained in the case of non-random one-period interest rates. Corollary 1 shows that, given the "variance -> conditional variance" restriction on the Markov process, the upper bound is attained only in that case.

Proposition 3 and Corollary 1 are formulated in terms of $E^*(y_t^S)$. In general, they do *not* apply to $E(y_t^S)$ (the unconditional expectation with respect to the true probability measure Pr). Under (substantive) stationary and ergodicity assumptions, the latter can be estimated consistently using only time-series averages of observed yields. Estimating the former requires an additional model of risk. If S-period bonds have positive risk premia, then $E(y_t^S) < E^*(y_t^S)$

and r_{long} may be larger than (the estimable) true expectation of S-period yields. Of course, under Assumption 1*, both expectations converge to r_{long} as S converges to infinity.

4.4 A Lower Bound for r_{long}

This subsection provides an extreme lower bound for r_{long}. More importantly, it shows in two important classes of models that this extreme lower bound is approximately attained when the process for one-period yields is highly persistent.

The following proposition shows that the long-term yield r_{long} can be no smaller than the lowest realization of one-period yields.

Proposition 4. *Suppose that* $r_{min} = inf_{x \in X} y^1(x)$. *Then* $r_{long} \geq r_{min}$.

The lower bound is intuitive. What is more interesting is that the lower bound is approximately attained when the process for short-term yields is highly persistent. The intuition is similar to that discussed in the introduction, in Section 2, and and in Weitzman (1998). The next two propositions make this point more formally and are, as far as I know, new to this paper. The first deals with Markov chains, while the second handles the case in which the one-period yield follows an autoregression (akin to Vasicek (1977)'s Gaussian model, but with bounded support).

4.4.1 Lower Bound: Markov Chain Case

To establish the Markov chain result, suppose (as in Proposition 1) that $\{x_t\}_{t=1}^{\infty}$ is governed by a Markov chain with state space $\{1, 2, ..., J\}$. Consider a sequence of economies indexed by the natural numbers. In any economy m, the (state-contingent) one-period bond prices are given by $\{q_i^1\}_{i=1}^{J}$ (and so are independent of m), and the positive[16] transition matrix (with respect to Pr^*) is P_m^*. Then, the following proposition describes what happens as the Markov chain becomes increasingly persistent.

[16] As with Proposition 1, this restriction can be relaxed to the requirement that P_m^* is primitive for all m.

Proposition 5. *Suppose that the sequence* $\{P_m^*\}_{m=1}^{\infty}$ *of Markov matrices converges (in the sup-norm) to the identity matrix. Let* $r_{long,m}$ *be the long-term yield in economy* m*. Then*

$$lim_{m\to\infty} r_{long,m} = r_{min} \equiv min_j - ln(q_j^1).$$

When m is large, both P and Q are near-diagonal. The largest eigenvalue of Q is then approximately its largest diagonal element, and (by Proposition 1) r_{long} is approximately r_{min}.[17]

4.4.2 Lower Bound: Autoregression Case

The next proposition covers the case in which one-period bond yields follow an autoregression. The analysis is necessarily more technical, as it requires the use of a continuous state space.

To be specific, suppose that the one-period bond yields are governed by the process:

$$y_{t+1}^1 = (1-\rho)\mu_y + \rho y_t^1 + \varepsilon_{t+1}(1-\rho^2)^{1/2}, t \geq 1, 0 < \rho < 1 \tag{6}$$

where, under Pr^*, $\{\varepsilon_t\}_{t=1}^{\infty}$ is an i.i.d sequence of random variables that have mean zero and have bounded support. (Here, the notation ρ is obviously distinct from that used in Section 2.3.) The initial bond yield y_1^1 is drawn from the stationary distribution for bond yields (which is given by the distribution of $(1-\rho^2)^{1/2}\sum_{n=0}^{\infty}\rho^n\varepsilon_n$, where $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables drawn from the distribution for ε_t).

Here, again, we will be interested in the properties of the long-run yield $r_{long}(\rho)$ when ρ is near 1. Note that the scaling factors $(1-\rho)$ and $(1-\rho^2)^{1/2}$ ensure that the first two moments of the stationary distribution of one-period bond yields are independent of ρ. This scaling helps clarify that it is the persistence of the process that is driving the result, not

[17] Proposition 5 is using a strong notion of persistence, as *all* eigenvalues of P_m^* are near 1 for m large. Intuitively, when all eigenvalues are close to 1, it takes a long time for the process to return to its stationary distribution, given any initial state. It would be more standard to measure persistence of a Markov chain through the size of the second-highest eigenvalue of the transition matrix. Proposition 5 is not valid with this notion of persistence.

its mean or variance. (However, the proposition remains valid even if the scaling factors are dropped.)

Proposition 6. *Consider a set of economies indexed by $\rho \in (0,1)$. Suppose that in economy ρ, the one-period bond yields follow the process (6). Suppose that (under Pr^*), ε has an atom at its lower bound $\varepsilon_{min} < 0$ OR it has a continuous density that is positive at ε_{min}. Then:*

$$lim_{\rho \to 1} \frac{r_{long}(\rho)}{r_{min}(\rho)} = 1,$$

where, in the economy indexed by ρ, $r_{min}(\rho)$ is the lowest possible realization of y_t^1 and $r_{long}(\rho)$ is the long-term yield.

Because of the scaling factors, the unconditional mean and variance of y_t^1 are both independent of ρ :

$$E^*(y_t^1) = \mu_y$$
$$Var^*(y_t^1) = Var^*(\varepsilon_t^1)$$

Nonetheless, the proof of the proposition shows that: -

$$lim_{\rho \to 1}(\mu_y - r_{min}(\rho)) = \infty$$
$$lim_{\rho \to 1}(\mu_y - r_{long}(\rho)) = \infty.$$

As ρ nears 1, the lowest possible realization of y_t^1 is becoming small (that is, highly negative) and so is the yield on a long-term bond. The point of the proposition is that their ratio nears 1 as ρ approaches 1.

The proposition (or at least its proof) requires that the (starred) distribution of ε_{t+1} has sufficient mass near ε_{min} (an atom or a positive continuous density). The needed assumption applies to a wide range of commonly used distributions with bounded support, including any truncations of unbounded distributions. Note that the restriction is equivalent whether stated in terms of Pr^* or Pr.

5 Stochastic Growth

Until now, the growth of available loanable funds has been assumed to be deterministic. This section extends the above analysis to the case in which that growth is stochastic.

5.1 A Basic Equivalence

Suppose that the growth rate of loanable funds L_{t+1} is a fixed bounded function g of the state x_{t+1} :

$$L_{t+1}/L_t = exp(g(x_{t+1})).$$

Given this definition, we can extend (5) to say that an infinite debt rollover is sustainable if there exists a bounded function v and a constant $\lambda \geq 1$ such that:

$$exp(v(x)) = \frac{exp(-y^1(x))}{\lambda} E^*(exp(v(x'))exp(g(x'))|x) \tag{7}$$

for almost all x. By multiplying through by $exp(g(x))$, we can say that an infinite debt rollover is sustainable if there is a bounded function $\hat{v}(= v + g)$ and a constant $\lambda \geq 1$ such that:

$$exp(\hat{v}(x)) = \frac{exp(-y^1(x))exp(g(x))}{\lambda} E^*(exp(\hat{v}(x'))|x). \tag{8}$$

In (8), the (logged) ratio of period $(t + 1)$ debt to period t (lagged) loanable funds is given by $\hat{v}(x)$ (which is only determined up to a constant).

The above expression suggests that adding stochastic growth is equivalent to subtracting growth rates from yields. Along those lines, consider a *detrended economy* in which growth $\hat{g} = 0$ (so that L_t does not have a stochastic or deterministic trend) and the one-period bond yield is given by:

$$\hat{y}^1(x) = y^1(x) - g(x)$$

in state x. The following proposition shows that the sustainability of infinite rollover is equivalent in the original economy and the detrended economy.

Proposition 7. *Consider a detrended economy in which the available loanable funds are constant over time ($g = 0$) and the one-period bond yield is given by:*

$$\hat{y}^1(x) = y^1(x) - g(x).$$

Then, an infinite debt rollover is sustainable in the detrended economy if and only if an infinite debt rollover is sustainable in the original economy (with stochastic growth).

The absence of arbitrage implies that in the detrended economy, the yield to an N-period bond in state x_t is given by:

$$\begin{aligned}\hat{y}^N(x_t) &= N^{-1}ln(E^*(exp(\sum_{s=1}^{N}\hat{y}^1(x_{t+s-1})|x_t) \\ &= N^{-1}ln(E^*(exp(\sum_{s=1}^{N}y^1(x_{t+s-1}) - g(x_{t+s-1}))|x_t)\end{aligned}$$

I assume that Assumption 1 (and, when necessary, Assumption 1*) apply to the detrended yields $\hat{y}^N$. Hence, there exists a constant $\hat{r}_{long}$ such that:

$$lim_{N\to\infty}\hat{y}^N(x) = \hat{r}_{long}.$$

Here, $\hat{r}_{long}$ is the (very) long-term (zero-coupon bond) yield in the detrended economy. As before, Assumption 1 ensures that it is independent of the state x.

Note that we can readily extend Proposition 1 to show that Assumption 1 is satisfied in the detrended economy if x_t is governed by a Markov chain with a strictly positive transition matrix P^* under the risk-neutral measure Pr^*. In that case:

$$\hat{r}_{long} = -ln(max(eig(\hat{Q}^*)),$$

where $\hat{Q}^*$ is defined via:

$$\hat{Q}^*_{ij} = P^*_{ij} exp(-\hat{y}^1_i) \text{ for all } i,j$$
$$= P^*_{ij} exp(-y^1_i) exp(g_i) \text{ for all } i,j$$

It is possible to use data (on a sufficiently rich set of long-dated assets) from the original economy to estimate[18] $\hat{r}_{long}$. Consider a T-period *growth asset* traded in period t which has a single payoff in period $(t+T)$ defined by:

$$exp(\sum_{s=0}^{T-1} g(x_{t+s})).$$

The payoff of a T-period growth asset is the cumulative growth in loanable funds from the current period t through period $(t+T-1)$. Hence, growth assets are wholly distinct from a Lucas tree based on the income of the borrower.

The price of a T-period growth asset is given by:

$$\hat{q}^T(x_t) = E^*(exp(\sum_{s=0}^{T-1} g(x_{t+s}) - y(x_{t+s}))|x_t).$$

But this implies (under Assumption 1*) that:

$$\hat{r}_{long} = lim_{T\to\infty} T^{-1} ln(\frac{1}{\hat{q}^T_t(x_t)}). \tag{9}$$

We can estimate the right hand side of (9) with the prices of very long-dated growth assets. This estimate does not hinge on any information about the underlying risk-neutral (or "objective") probabilities beyond knowing that Assumption 1* is satisfied.

[18] I thank a referee for this excellent suggestion.

5.2 Ensuring A Sustainable Infinite Debt Rollover

The following proposition is a simple application of Proposition 2.

Proposition 8. *Suppose Assumption 1* is satisfied in the detrended economy. An infinite debt rollover is sustainable in the economy with stochastic growth if and only if the long-term yield* $\hat{r}_{long}$ *in the detrended economy is non-positive. The factor* λ *(in (7)) is equal to* $exp(-\hat{r}_{long})$.

In thinking about Proposition 8, it is important to note that the right-hand side of (9) is the (normalized and logged) reciprocal of the *price* of a growth asset, not the *yield* or return on a growth asset. For example, suppose that the growth is deterministic (so that $g(x_t) = \bar{g}$ for all x_t). Then, the long-dated growth asset is free of risk and its *yield* is the same as r_{long} (the yield on a very long-term risk-free bond). But the right-hand side of (9) is $(r_{long} - \bar{g})$, and Proposition 8 then implies that there is an infinite debt rollover if $r_{long} < \bar{g}$. Indeed, it is readily seen using (9) that this same conclusion applies under the much weaker assumption that loanable capacity is stationary around a long-run deterministic trend[19] defined by the growth rate $\bar{g}$.

Perhaps more interestingly, suppose that (with respect to Pr^*) the one-period growth rate g_t is independent and identically distributed (i.i.d.) over time, and is stochastically independent from one-period bond yields. Then (9) implies that:

$$\begin{aligned}\hat{r}_{long} &= -lim_{T\to\infty}\frac{ln(E^*(exp(\sum_{s=0}^{T-1} g(x_{t+s})|x_t))}{T} - \frac{ln(E^*(exp(-\sum_{s=0}^{T-1} y(x_{t+s})|x_t))}{T} \\ &= -ln(E^*(exp(g(x_t))) + r_{long}.\end{aligned}$$

In conjunction with Proposition 8, Jensen's inequality immediately implies that there is a sustainable infinite debt rollover whenever the long riskfree bond yield r_{long} is less than the risk-neutral expectation $E^*(g(x_t))$ of the i.i.d. growth rate of loanable capacity. However,

[19] More precisely, the loanable capacity L_t evolves according to $L_t = L_0 exp(\bar{g}t + \varepsilon_t)$, where $|\varepsilon_t|$ is uniformly bounded.

the existence of priced risk[20] in the growth of loanable capacity would imply that $E(g(x_t)) > E^*(g(x_t))$. This differential creates the possibility that there may not be an infinite debt rollover even if $r_{long} < E(g(x_t)))$.

More generally, there are natural analogs of the upper and lower bounds on r_{long} derived in Sections 4.3-4.4 that apply to $\hat{r}_{long}$. In terms of the lower bounds, if $(y_t^1 - g_t)$ is governed by a highly persistent Markov chain, then $\hat{r}_{long} \approx min_x(y^1(x) - g(x))$. In terms of the upper bounds, an analog to Proposition 3 implies that (under appropriate regularity conditions):

$$E^*(y^1(x) - g^1(x)) > \hat{r}_{long}$$

and (from Proposition 8), there is an infinite debt rollover if $E^*(y^1(x)) < E^*(g^1(x))$.

6 Public Debt Bubbles in Dynamic Stochastic General Equilibrium

The above results are all for a partial equilibrium setting, in which the borrower treats asset prices as given. This section translates these prior findings about infinite debt rollover into similarly weak conditions for the existence of public debt bubbles in dynamic stochastic general equilibrium settings.[21] The first subsection deals with overlapping generations (OG) economies[22], while the second treats a (simple) class of Aiyagari (1994)-Bewley (1977)-Huggett (1993) (ABH) model economies.

[20]In considering this possibility, it is important to keep in mind that in models with bubbles, there is no connection between the growth rate of loanable funds and the growth rate of investors' marginal utilities that shapes E^* . This point is illustrated by the models in the next section.

[21]The discussion focuses on sufficient conditions for the existence of time homogeneous equilibria in which the bubble has positive value. As is typical in economies with bubbles, there is a large set of other equilibria.

[22]My analysis of OG economies in this paper abstracts from capital and considerations of dynamic efficiency. Abel and Panageas (2022) show how, in the presence of aggregate risk, an infinite debt rollover can be Pareto improving even though the level of capital is dynamically efficient.

6.1 An Overlapping Generations Model With Growth Shocks

This subsection analyzes overlapping generations models with shocks to lifetime income growth and to cohort-to-cohort income growth. The former growth rate determines the relevant one-period bond yields (and E^*), while the latter growth rate determines the evolution of loanable funds.

6.1.1 Model Description

Consider an overlapping generations economy in which a unit measure of agents are born at each date, and live for two periods ("young" and "old"). Their utility functions are given by:

$$ln(c_y) + E(ln(c_o))$$

where (c_y, c_o) are consumptions when young and old, respectively. In period t, young agents have an endowment e_{yt} and old agents have an endowment e_{ot}. There is an initial old cohort who prefers to consume more to less. The endowments evolve stochastically according to the laws of motion:

$$ln(e_{y,t}) = g_t + ln(e_{y,t-1}), t \geq 1$$
$$ln(e_{o,t+1}) = r_t + ln(e_{y,t}), t \geq 1$$

given $e_{y1} = 1$. I assume that (g_t, r_t) follow a time-homogeneous Markov chain with J states $(\bar{g}_i, \bar{r}_i)_{i=1}^J$ and a positive $J \times J$ transition matrix P.

6.1.2 Benchmark Autarkic Equilibrium

Consider an autarkic equilibrium in this overlapping generations economy. In this equilibrium, the (shadow) one-period bond yield in period t is given by:

$$y_t^1 = ln(e_{o,t+1}/e_{y,t}) = r_t$$

Hence, it evolves stochastically according to the Markov chain. Since the young agents face no future endowment risk, there is no difference between E^* and E. As well, the young agents are the ones that buy government debt, and so the growth rate g_t of their endowments from cohort to cohort determines the growth rate of available loanable funds.

With these considerations in mind, we can (as in Section 4) define:

$$\hat{Q}_{ij} = P_{ij} exp(-\bar{r}_i + \bar{g}_i)$$

As in Proposition 1, the autarkic long-term yield in the detrended version of this economy is then:

$$\hat{r}^{aut}_{long} = -ln(max(eig(\hat{Q}))). \quad (10)$$

As in Proposition 3 (and given the equivalence between E and E^* in this example economy), the autarkic long-term yield is less than the expectation of $(r - g)$, calculated using the stationary distribution implied by P.

6.1.3 Monetary (Public Debt Bubble) Equilibrium

Now suppose that each initial old agent is endowed with one unit of a divisible durable good called money. Money is intrinsically useless, as it provides no direct utility to any agent.

The following proposition shows that if $\hat{r}^{aut}_{long} < 0$, there exists an equilibrium in the *monetary OG economy* in which money has positive value. Since there are no tax collections in the economy, the positive value of money implies that there is a public debt bubble.[23]

Proposition 9. *Suppose $\hat{r}^{aut}_{long}$ (defined as in (10)) is negative. Then there is an equilibrium in the monetary OG economy in which money has a strictly positive price $\Gamma_i e_{y,t-1}$ in period t as a function of the Markov state i and the young agents' endowment in the prior period.*

In the equilibrium described in Proposition 9, if the difference $(\bar{r}_i - \bar{g}_i)$ is persistent, the

[23]The primary deficit is zero in the equilibrium described in Proposition 9. But the proposition can be generalized to incorporate a positive primary deficit in the form of lump-sum transfers to the old funded via money creation.

price of money moves inversely with $(\bar{r}_i - \bar{g}_i)$. Thus, inflation is high when the economy transits to a high $(r-g)$ state.

6.2 Tail Risk Economy

In this subsection, I add aggregate shocks to an ABH model similar to that analyzed in Kocherlakota (2022). There is no growth in this setup. I build on Proposition 3 to show that there is a monetary equilibrium in this economy if the *autarkic* long-term yield is negative (that is, less than the growth rate of 0). As above, positively valued money represents a public debt bubble.

6.2.1 Model Description

Suppose there is a unit measure of infinitely-lived agents. The agents' individual states evolve according to stochastically independent Markov chains[24] with state space $\{H, L\}$ and positive transition matrix Π, where $\Pi_{HL} = \Pi_{LH} = p$. There is always an equal fraction of agents in the two states.

In state H, an agent is endowed with e_H units of consumption and has momentary utility function over consumption:

$$ln(c_H).$$

An agent in state L in period t is endowed with $e_L > 0$ units of consumption and has momentary utility function:

$$\bar{\nu}_{t-1} c_L.$$

Here, ν_{t-1} is a *common* shock revealed in period $(t-1)$; assume $\nu_0 = 1$. It then follows a Markov chain with state space $\{\bar{\nu}_1, \bar{\nu}_2, ..., \bar{\nu}_J\}$ and transition matrix P. Agents in state L are hand-to-mouth, so that they consume all available resources at each date.[25] The common

[24] The label L in this example is entirely distinct from its use to represent lending capacity earlier in the paper.

[25] In contrast, in Kocherlakota (2022), agents in state L are allowed to choose their asset positions freely subject to a borrowing limit. The hand-to-mouth restriction simplifies the construction of a monetary equi-

shocks affect the precautionary demand for savings at the different dates.

6.2.2 Autarkic Benchmark Equilibrium

Kocherlakota (2022) shows that if $J = 1$, and there exists $\Delta > 0$ such that:

$$\frac{1}{e_H - \Delta} = \frac{\beta(1-p)}{e_H - \Delta} + \beta\bar{\nu}_1 p$$

then there is a bubbly stationary equilibrium in which the real interest rate is constant at zero, or equivalently where the price of money is constant at some $\Gamma > 0$. In this equilibrium, any agent in state H chooses to buy Δ/Γ additional units of money. They then divest their moneyholdings immediately upon transiting to state L.

Under what conditions does this kind of equilibrium exist if $J > 1$ (so that there are aggregate shocks to the precautionary demand for savings)? To address this question, consider an autarkic equilibrium in this economy, under the presumption that the agents in state H are the marginal asset buyers. In this equilibrium, the risk-neutral and true expectations are aligned (because the agents in state H know their marginal utility next period). The one-period yield evolves according to:

$$exp(-r_i^{ABH}) = \beta(1-p) + \beta p\bar{\nu}_i e_H, i = 1..., J.$$

The growth of available loanable funds (from agents in state H) is deterministically zero.

Given these elements of the equilibrium, define (as in Section 2) a $(J \times J)$ matrix Q^{ABH} where:

$$Q_{ij}^{ABH} = P_{ij} exp(-r_i^{ABH}), i, j = 1, ..., J$$

The long-term yield in the autarkic equilibrium is then:

$$r_{long}^{ABH} = -ln(max(eig(Q^{ABH})). \tag{11}$$

librium in the presence of aggregate shocks.

Again, as in Proposition 3, r_{long}^{ABH} is smaller than $\sum_{i=1}^{J} \mu_j r_j^{ABH}$, where $(\mu_j)_{j=1}^{J}$ is the stationary density implied by P.

6.2.3 Existence of a Monetary (Public Debt Bubble) Equilibrium

As suggested by Proposition 2, the following proposition shows that there is a stationary monetary equilibrium of the kind discussed above if $r_{long}^{ABH} < 0$.

Proposition 10. *Suppose $r_{long}^{ABH} < 0$, where r_{long}^{ABH} is defined as in (11). Then there is a monetary equilibrium in which the price of money across aggregate states is a positive vector $(\Gamma_j)_{j=1}^{J}$ that satisfies:*

$$\frac{\Gamma_i}{e_H - \Gamma_i} = \beta \sum_{j=1}^{J} P_{ij}((1-p)(\frac{\Gamma_j}{e_H - \Gamma_j}) + p\bar{\nu}_i \Gamma_j), i = 1, ..., J$$

In the equilibrium characterized in Proposition 10, each agent in state H uses their resources to buy one unit of money and then divests their moneyholdings upon transit to (the hand-to-mouth) state L. The resulting distribution of money is geometric: a measure $p(1-p)^{n-1}/2$ agents have n units of money. (Note that this distribution is unaffected by the aggregate shocks.)

The per-capita quantity of money in this equilibrium is $0.5/p$. But money is neutral. Hence, if the quantity of money is fixed at $M > 0$, there is an equilibrium with a state-contingent vector of money prices given by:

$$\frac{0.5/p}{M}\Gamma$$

In this case, each agent in state H in date t buys $\frac{M}{0.5/p}$ additional units of money (regardless of the price of money).

7 Conclusions

This paper provides a one-parameter check for whether asset prices and growth rates are such that an infinite debt rollover is sustainable. A natural next step for future research is to use information in asset pricing data to measure this single parameter.

Some work along these lines has been done.[26] The results are mixed. In the context of a nonparametric recursive utility model of risk, Christensen (2017) estimates r_{long} - the real yield on a zero-coupon bond with infinite maturity - to be between 1.5% and 2% per year in the US (the parameter y in Table III on p. 1525 of his paper). His estimate makes no use of yield data for bonds that have maturities longer than 90 days.[27]

In contrast, Balter, Pelsser, and Schotman (2021) use only data on euro swap rates with five-year and twenty-year maturities to estimate r_{long} (nominal) for Europe. They find that the implied bond yields are highly persistent. As a result, the point estimate of r_{long} (what they call θ in the tables on p. 208 of their paper) is notably negative, but the associated standard errors are enormous.

Both of these papers represent attempts to estimate r_{long} in the context of arbitrage-free asset pricing models. As discussed in Section 5, if loanable capacity is known to be stationary around a deterministic trend defined by $\bar{g}$, then an infinite debt rollover is possible if and only if $r_{long} < \bar{g}$. More generally, though, Section 5 emphasizes the need to estimate, and test the non-positivity of, $\hat{r}_{long}$ - the long-term bond yield in a detrended economy in which realized growth rates (of available loanable funds) are netted out of short-term yields. A first step in estimating $\hat{r}_{long}$ (or at least an upper bound for $\hat{r}_{long}$) is the specification and estimation of a model of the *prices* of the long-horizon growth assets discussed in Section 5.1.

[26] There is of course an enormous literature on yield curve estimation. I have chosen two recent papers that treat r_{long} as a fixed parameter and estimate that parameter in the context of arbitrage-free models.

[27] Perhaps not unrelatedly, Christensen's estimates seem high to me. Corollary 1 uses an arbitrage relationship between short-term and long-term bonds to show that $E^*(y_t^1) > r_{long}$. Christensen's estimation cannot make use of this relationship. In US data, a typical sample average of real yields on 90 day Treasury bills is less than 1%. It seems unlikely that these short-lived assets have sufficient risk to nearly double the starred expectations of their yields over their unstarred expectations.

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Appendix

This appendix collects the proofs.

Proof of Proposition 1

The Perron-Frobenius Theorem implies that the maximal eigenvalue of Q^* is positive, and so $ln(max(eig(Q^*))$ is well-defined. The rest of the proof shows that $r_{long} = -ln(max(eig(Q^*)))$ satisfies Assumption 1.

The vector q^N of state-contingent prices of N-period zero-coupon bonds can be recursively calculated as:

$$q^N = Q^* q^{N-1}.$$
$$= (Q^*)^{N-1} q_1$$

Define the matrix $Q^{**} = Q^* exp(r_{long})$. Since $exp(-r_{long})$ is the maximal eigenvalue of Q^*, repeated exponentiation of this matrix results in a well-defined limit:

$$lim_{N\to\infty}(Q^{**})^N = Q^\infty$$

where Q^∞ is a positive matrix. Then:

$$lim_{N\to\infty} q^N exp(N r_{long})$$
$$= lim_{N\to\infty} exp(r_{long})(Q^{**})^{N-1} q^1$$
$$= exp(r_{long}) Q^\infty q^1.$$

Taking logs (on a component by component basis) proves the proposition:

$$\begin{aligned} & lim_{N\to\infty} N(-y^N + r_{long}) \\ = & lim_{N\to\infty}(ln(q^N) + Nr_{long}) \\ = & ln(Q^\infty q^1). \end{aligned}$$

Proof of Proposition 2

Proof. Suppose an infinite debt rollover is sustainable. Then there exists $\lambda \geq 1$ and a bounded real-valued function v such that:

$$\begin{aligned} exp(v(x_t)) &= \lambda^{-1} exp(-y^1(x_t)) exp(g) E^*(exp(v(x_{t+1}))|x_t) \\ &= \lambda^{-N} exp(Ng) E^*(\prod_{n=1}^{N} exp(-y^1(x_{t+n-1})) exp(v(x_{t+n}))|x_t) \end{aligned}$$

where the second equality is a consequence of recursing forwards. The function v is bounded from above by some v_{max} and from below by some v_{min}. Hence, taking logs:

$$v(x_t) \leq -Nln(\lambda) + Ng - Ny^N(x_t) + v_{max}$$
$$v(x_t) \geq -Nln(\lambda) + Ng - Ny^N(x_t) + v_{min}$$

Divide by N on both sides and take limits. We get:

$$0 = -ln(\lambda) + g - lim_{N\to\infty} y^N(x_t)$$
$$0 = -ln(\lambda) + g - r_{long}.$$

Hence, $g - r_{long} = ln(\lambda) \geq 0$.

Now suppose that $g \geq r_{long}$ and define $\hat{\lambda} = exp(g - r_{long}) \geq 1$. Then:

$$
\begin{aligned}
lim_{N\to\infty}\hat{\lambda}^{-N}exp(Ng)E^*(\prod_{n=1}^{N} exp(-y^1(x_{t+n-1}))|x_t) \\
= lim_{N\to\infty}\hat{\lambda}^{-N}exp(Ng)exp(-Ny^N(x_t)) \\
= lim_{N\to\infty}exp(Nr_{long})exp(-Ny^N(x_t)) \\
= lim_{N\to\infty}exp(-N(y^N(x_t) - r_{long})) \\
= exp(-\phi(x_t))
\end{aligned}
$$

where ϕ is defined as in (4). Hence, ϕ satisfies:

$$
\begin{aligned}
exp(-\phi(x_t)) &= \hat{\lambda}^{-1}exp(g)lim_{N\to\infty}\hat{\lambda}^{-N+1}exp((N-1)g)E^*(\prod_{n=1}^{N} exp(-y^1(x_{t+n-1}))|x_t) \\
&= \hat{\lambda}^{-1}exp(g)exp(-y^1(x_t))lim_{N\to\infty}\hat{\lambda}^{-N+1}exp((N-1)g)E^*(((\prod_{n=2}^{N} exp(-y^1(x_{t+n-1}))|x_{t+1})|x_t) \\
&= \hat{\lambda}^{-1}exp(g)exp(-y^1(x_t))lim_{N\to\infty}E^*(exp((N-1)r_{long} - (N-1)y^{N-1}(x_{t+1}))|x_t) \\
&= \hat{\lambda}^{-1}exp(g)exp(-y^1(x_t))E^*(exp(-\phi(x_{t+1})|x_t)
\end{aligned}
$$

The last step is an application of the bounded convergence theorem (justified by Assumption 1*) to the sequence:

$$\{exp((N-1)r_{long} - (N-1)y^{N-1}(x_{t+1}))\}_{N=1}^{\infty}$$

It follows that $v = -\phi$ and $\lambda = \hat{\lambda}$ jointly satisfy the restriction (5). □

Proof of Proposition 3

A N-period zero-coupon bond is a promise to receive a $(N-S)$-period zero coupon bond in S periods. Hence, the yields satisfy the recursion:

$$-Ny^N(x_t)+Nr_{long} = -Sy^S(x_t)+Sr_{long}+ln(E^*(exp(-(N-S)y^{N-S}(x_{t+S})+(N-S)r_{long})|x_t))$$

Let N converge to infinity, and (using assumption 1*) substitute in terms of the bounded function ϕ defined in (4):

$$-\phi(x_t) = -Sy^S(x_t) + Sr_{long} + ln(E^*(exp(-\phi(x_{t+S}))|x_t))$$

Jensen's inequality implies that:

$$-\phi(x_t) \geq -Sy^S(x_t) + Sr_{long} - E^*(\phi(x_{t+S})|x_t).$$

Take unconditional expectations on both sides:

$$-E^*(\phi(x_t)) \geq -E^*(Sy^S(x_t)) + Sr_{long} - E^*(\phi(x_{t+s})).$$

The stationarity of $\{x_t\}_{t=1}^{\infty}$ implies that the two unconditional expectations are equal. Thus, we arrive at:

$$E^*(y^S(x_t)) \geq r_{long}.$$

Proof of Corollary 1

As in the proof of Proposition 3, we can use Assumption 1* to derive the relationship:

$$-\phi(x_t) = (-Sy^S(x_t) + Sr_{long}) + ln(E^*(exp(-\phi(x_{t+S}))|x_t)). \tag{12}$$

Suppose $\phi(x)$ is constant for almost all x. Then, (12) implies that $y^S(x)$ is equal to r_{long} almost everywhere, which contradicts the hypothesis in the proposition. So, we can conclude that $Var^*(exp(\phi(x))) > 0$. Under the restriction on the Markov process, there is some positive (starred) probability such that the conditional (starred) variance:

$$Var^*(exp(-\phi(x_{t+S}))|x_t) > 0.$$

Applying Jensen's inequality to (12), we obtain:

$$-\phi(x_t) \geq -Sy^S(x_t) + Sr_{long} - E^*(\phi(x_{t+S})|x_t) \tag{13}$$

where the inequality is strict with some positive starred probability.

Taking unconditional expectations, we get:

$$-E^*(\phi(x_t)) > -E^*(Sy^S(x_t)) + Sr_{long} - E^*(\phi(x_{t+1}))$$

Given the stationarity of $\{x_t\}_{t=1}^{\infty}$, we can cancel to arrive at the desired conclusion:

$$E^*(y^S(x_t)) > r_{long}.$$

Proof of Proposition 4

The supposition implies that for all x :

$$exp(-y^1(x)) \leq exp(-r_{min}).$$

Recall that:

$$
\begin{aligned}
-r_{long} &= lim_{N\to\infty} y^N(x) \\
&= lim_{N\to\infty} N^{-1} ln(E_t^* exp(\sum_{n=0}^{N-1} -y^1(x_{t+n}))) \\
&\leq -r_{min}
\end{aligned}
$$

which proves the proposition.

Proof of Proposition 5

In each economy m, define the matrix Q_m^* by setting $Q_{m,ij}^* = P_{m,ij}^* q_i^1$. We know from Proposition 1 that:

$$r_{long,m} = -ln(max(eig(Q_m^*))$$

The limit $Q_\infty^* = lim_{m\to\infty} Q_m^*$ is a diagonal matrix, with $Q_{\infty,ii}^* = q_i^1$. Hence:

$$
\begin{aligned}
lim_{m\to\infty} r_{long,m} &= -ln(max(eig(Q_\infty^*)) \\
&= -ln(max_i q_i^1) \\
&= min_i - ln(q_i^1) \\
&= r_{min},
\end{aligned}
$$

which proves the proposition.

Proof of Proposition 6

In the economy indexed by ρ, the minimal one-period bond yield is given by:

$$r_{min}(\rho) = \mu_y + \frac{\varepsilon_{min}(1+\rho)^{1/2}}{(1-\rho)^{1/2}}.$$

To find $r_{long}(\rho)$, let y_t^N be the yield on a N-period bond. It satisfies the recursive relationship:

$$Ny_t^N = y_t^1 - ln(E_t^* exp(-(N-1)y_{t+1}^{N-1})).$$

Given the Markovian structure, Ny_t^N is a time-homogeneous function of y_t^1. We can guess and verify that this function is affine, so that:

$$Ny_t^N = A_0^N + A_1^N y_t^1.$$

where $A_0^1 = 0$ and $A_1^N = 1$. Plugging this representation into the recursive relationship, we get:

$$\begin{aligned} & A_0^N + A_1^N y_t^1 \\ &= y_t^1 - ln(E_t^* exp(-A_0^{N-1} - A_1^{N-1}(1-\rho)\mu_y - A_1^{N-1}\rho y_t^1 - A_1^{N-1}\varepsilon_{t+1}(1-\rho^2)^{1/2}) \\ &= y_t^1 + A_1^{N-1}\rho y_t^1 + A_0^{N-1} + A_1^{N-1}(1-\rho)\mu_y - ln(E_t^* exp(-A_1^{N-1}\varepsilon_{t+1}(1-\rho^2)^{1/2})) \end{aligned}$$

Hence, the constants $\{A_0^N, A_1^N\}_{N=1}^{\infty}$ satisfy the recursive restrictions:

$$\begin{aligned} A_1^N &= 1 + \rho A_1^{N-1} \\ A_0^N &= A_0^{N-1} + A_1^{N-1}(1-\rho)\mu_y - ln(E_t^* exp(-A_1^{N-1}\varepsilon_{t+1}(1-\rho^2)^{1/2})). \end{aligned}$$

where $A_1^N = 1$ and $A_0^N = 0$. It follows that:

$$\begin{aligned} lim_{N\to\infty} A_1^N &= \frac{1}{(1-\rho)} \\ lim_{N\to\infty}(A_0^N - A_0^{N-1}) &= \mu_y - ln(E_t^* exp(-\varepsilon_{t+1}\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})). \end{aligned}$$

Hence, we can find the long-term yield as:

$$\begin{aligned} r_{long}(\rho) &= lim_{N\to\infty} y_t^N \\ &= lim_{N\to\infty} N^{-1}(A_0^N + A_1^N y_t^1) \\ &= lim_{N\to\infty} N^{-1} A_0^N \\ &= \mu_y - ln(E_t^* exp(-\varepsilon_{t+1}\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})). \end{aligned}$$

Now, rewrite by adding and subtracting $\frac{\varepsilon_{min}(1+\rho)^{1/2}}{(1-\rho)^{1/2}} = (r_{min}(\rho) - \mu_y)$:

$$r_{long}(\rho) - \mu_y = (r_{min}(\rho) - \mu_y) - ln(E_t^* exp(-\varepsilon_{t+1}\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}} + \varepsilon_{min}\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})). \tag{14}$$

We next derive a lower bound on the last term of (14). The nature of the lower bound depends on the two distinct assumptions being made about the distribution of ε_{t+1}.

Case 1 (atom): Since ε_{t+1} has an atom at ε_{min}, $Pr^*(\varepsilon_{t+1} = \varepsilon_{min}) = \pi^* > 0$. This allows us to rewrite the last term of (14) as:

$$-ln(\pi^* + (1-\pi^*)E_t^*(exp((\varepsilon_{min} - \varepsilon_{t+1})\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})|\varepsilon_{t+1} > \varepsilon_{min})))$$

This term is bounded from above by:

$$-ln(\pi^*).$$

It follows that:

$$0 \leq (r_{long}(\rho) - \mu_y) - (r_{min}(\rho) - \mu_y)) \leq -ln(\pi^*).$$

Dividing through by $(r_{min}(\rho) - \mu_y) < 0$ and taking limits, we get:

$$0 \geq lim_{\rho\to 1}(\frac{r_{long}(\rho) - \mu_y}{r_{min}(\rho) - \mu_y} - 1) \geq lim_{\rho\to 1}\frac{-ln(\pi^*)}{\frac{\varepsilon_{min}(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} = 0$$

so that:

$$
\begin{aligned}
1 &= lim_{\rho\to 1}\frac{r_{long}(\rho)/r_{min}(\rho)-\mu_y/r_{min}(\rho)}{1-\mu_y/r_{min}(\rho)} \\
&= lim_{\rho\to 1}\frac{r_{long}(\rho)/r_{min}(\rho)}{1}
\end{aligned}
$$

which proves the proposition in the atom case.

Case 2 (positive continuous density): Suppose ε_{t+1} has a positive continuous density at ε_{min}. There then exists $h>0$ and $\delta^*>0$ such that the density of ε_{t+1} is above h over the interval $[\varepsilon_{min}, \varepsilon_{min}+\delta^*]$. It follows that:

$$
\begin{aligned}
&E_t^* exp(-(\varepsilon_{t+1}-\varepsilon_{min})\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}) \\
&\geq \int_0^{\delta^*} exp(-x\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})hdx \\
&= -\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}exp(-x\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})h|_0^{\delta^*} \\
&= \frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}h[1-exp(-\delta^*\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})].
\end{aligned}
$$

We can use this inequality to bound the last term of (14) from above:

$$
\begin{aligned}
&-ln(E_t^* exp(-\varepsilon_{t+1}\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}+\varepsilon_{min}\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}) \\
&\leq -ln(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}) - ln(h) - ln(1-exp(-\delta^*\frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))
\end{aligned}
$$

Rearranging (14), we get:

$$
\begin{aligned}
r_{long}(\rho) - \mu_y &\leq r_{min}(\rho) - \mu_y - ln(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}) - ln(h) - ln(1 - exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}})) \\
\frac{r_{long}(\rho) - \mu_y}{r_{min}(\rho) - \mu_y} &\geq 1 + \frac{-ln(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}) - ln(h) - ln(1 - exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))}{\varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} \\
1 - \frac{r_{long}(\rho) - \mu_y}{r_{min}(\rho) - \mu_y} &\leq \frac{-ln(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}) - ln(h) - ln(1 - exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))}{-\varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} \\
1 - \frac{r_{long}(\rho)/r_{min}(\rho) - \mu_y/r_{min}(\rho)}{1 - \mu_y/r_{min}(\rho)} &\leq \frac{-ln(\frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}}) - ln(h) - ln(1 - exp(-\delta^* \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}))}{-\varepsilon_{min} \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}}} \qquad (15)
\end{aligned}
$$

Note that $r_{long}(\rho)/r_{min}(\rho) \leq 1$ for ρ near 1 (because $r_{min}(\rho)$ is more negative than $r_{long}(\rho)$). Hence, taking limits of both sides of (15), we find that:

$$0 \geq lim_{\rho \to 1} 1 - \frac{r_{long}(\rho)/r_{min}(\rho)}{1} \geq 0$$

which proves the proposition for the positive continuous density case.

Proof of Proposition 7

It follows from a straightforward comparison of (5) and (8).

Proof of Proposition 8

Proposition 7 showed that an infinite debt rollover is sustainable in the economy with stochastic growth if and only if an infinite debt rollover is sustainable in the detrended economy without growth. But Proposition 2 implies that an infinite debt rollover is sustainable in the detrended economy if and only if the long-term yield $\hat{r}_{long}$ is non-positive.

Proof of Proposition 9

A monetary equilibrium Γ satisfies the Euler equations of the young agents in the various states:

$$\frac{\Gamma_i exp(-\bar{g}_i)}{1-\Gamma_i exp(-\bar{g}_i)} = \sum_{j=1}^{J} P_{ij} \frac{\Gamma_j}{exp(\bar{r}_i)+\Gamma_j}$$

or equivalently:

$$\frac{\Gamma_i}{exp(\bar{g}_i)-\Gamma_i} = \sum_{j=1}^{J} P_{ij} \frac{\Gamma_j}{exp(\bar{r}_i)+\Gamma_j}.$$

Define a nonlinear operator:

$$T : \mathbb{R}_+ \to \mathbb{R}_+$$

$$T_i((\Gamma_j)_{j=1}^J) = exp(\bar{g}_i)((\sum_{j=1}^{J} P_{ij} \frac{\Gamma_j}{exp(\bar{r}_i)+\Gamma_j})^{-1}+1)^{-1}.$$

A monetary equilibrium Γ is a fixed point of T.

The Jacobian of T at $\Gamma = 0$ is $\hat{Q}$. Hence, for Γ sufficiently close to zero:

$$T(\Gamma) \approx \hat{Q}\Gamma.$$

Because $\hat{r}_{long}^{aut} < 0$, $\hat{Q}$ has a maximal eigenvalue larger than 1. By the Perron-Frobenius Theorem, there is a strictly positive eigenvector w associated with this maximal eigenvalue. Since T is well-approximated by $\hat{Q}$ for Γ near zero, there exists a small but positive scalar ξ such that

$$T_i(\xi w) > \xi w_i$$

for all i.

T is a strictly monotone operator and its range is bounded above by $(exp(\bar{g}_i))_{i=1}^J$. Hence the sequence $\{T^N(\xi w)\}_{N=1}^{\infty}$ converges:

$$lim_{N\to\infty} T^N(\xi w) = w^*.$$

The limit w^* is a positive fixed point of T and so is the desired monetary equilibrium price vector.

Proof of Proposition 10

The restrictions in the proposition are the Euler equations for agents in state H that ensure that those agents' optimum involves buying one additional unit of money (at price Γ_i in state i).

Define a nonlinear operator $T : \mathbb{R}^J_+ \to \mathbb{R}^J_+$ by:

$$T_i((\Delta_j)_{j=1}^J) = e_H((\beta \sum_{j=1}^J P_{ij}((1-p)(\frac{\Delta_j}{e_H - \Delta_j}) + p\bar{\nu}_i\Delta_j))^{-1} + 1)^{-1}.$$

The Jacobian of T at $\Gamma = 0$ is given by Q^{ABH}, and so the behavior of T near $\Gamma = 0$ is well-approximated by Q^{ABH}.

Let $\overrightarrow{\delta} = (\delta_1, \delta_2, ..., \delta_J)$ be the positive eigenvector of Q^{ABH}, with unit Euclidean norm, associated with the eigenvalue $exp(-r^{ABH}_{long}) > 1$. (The Perron-Frobenius Theorem implies that such a positive eigenvector exists.) Since T is well-approximated by Q^{ABH} for Γ small, there is a sufficiently small but positive scalar ξ such that.

$$T_i(\xi\overrightarrow{\delta}) > \xi\delta_i$$

for all i. The operator T is strictly monotone and is bounded from above by y_H. Hence the sequence $\{T^N(\xi\overrightarrow{\delta})\}_{N=1}^\infty$ converges:

$$lim_{N\to\infty}T^N(\xi\overrightarrow{\delta}) = \Gamma.$$

The limit Γ is a positive fixed point of T and so satisfies the restriction in the proposition.