Online Appendix

A Proofs

A.1 Proof of Lemma 1

Lemma 1. *Under a given network α, the vector of log prices is given by*

$$p(\alpha) = -\mathcal{L}(\alpha)(\varepsilon + a(\alpha)), \tag{12}$$

and log GDP is given by

$$y(\alpha) = \omega(\alpha)^\top (\varepsilon + a(\alpha)), \tag{13}$$

where $a(\alpha) = (\log A_i(\alpha_i), \ldots, \log A_n(\alpha_n))$.

Proof. Combining the unit cost equation (8) with the equilibrium condition (10) and taking the log we find that, for all i,

$$p_i = -\varepsilon_i - a_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} p_j, \tag{43}$$

where $a_i(\alpha_i) = \log(A_i(\alpha_i))$. This is a system of linear equations whose solution is (12). The log price vector is also normally distributed since it is a linear transformation of normal random variable. Combining with (6) yields (13). □

A.2 Proof of Corollary 1

The proof of Corollary 1 is in Supplemental Appendix B in Kopytov et al. (2024).

A.3 Proof of Lemma 2

Lemma 2. *In equilibrium, the technique choice problem of the representative firm in sector i is*

$$\alpha_i^* \in \arg\max_{\alpha_i \in \mathcal{A}_i} a_i(\alpha_i) - \sum_{j=1}^{n} \alpha_{ij} \mathcal{R}_j(\alpha^*), \tag{17}$$

where

$$\mathcal{R}(\alpha^*) = \mathrm{E}[p(\alpha^*)] + \mathrm{Cov}[p(\alpha^*), \lambda(\alpha^*)] \tag{18}$$

is the vector of equilibrium risk-adjusted prices, and where

$$\mathrm{E}[p(\alpha^*)] = -\mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) \;\; \textit{and} \;\; \mathrm{Cov}[p(\alpha^*), \lambda(\alpha^*)] = (\rho - 1)\mathcal{L}(\alpha^*)\Sigma[\mathcal{L}(\alpha^*)]^\top \beta.$$

Proof. We first consider the stochastic discount factor. Equation (A.5) in Supplemental Appendix

A in Kopytov et al. (2024) shows that aggregate consumption can be written as a function of prices. Combining that equation with (5) we can write $\lambda = \log(\Lambda)$ as

$$\lambda(\alpha^*) = -(1-\rho)\sum_{i=1}^{n}\beta_i p_i(\alpha^*). \tag{44}$$

Taking the log of (8) yields

$$k_i(\alpha_i, \alpha^*) = -(\varepsilon_i + a(\alpha_i)) + \sum_{j=1}^{n}\alpha_{ij} p_j(\alpha^*). \tag{45}$$

Both $\lambda(\alpha^*)$ and $k_i(\alpha_i, \alpha^*)$ are normally distributed since they are linear combinations of ε and the log price vector, which is normally distributed by Lemma 1.

Turning to the firm problem 9, we can write

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathrm{E}\left[\Lambda \frac{\beta^\top \mathcal{L}(\alpha^*)\mathbf{1}_i}{P_i} K_i(\alpha_i, P)\right],$$

where we have used (A.7) from Supplemental Appendix A in Kopytov et al. (2024). We can drop $\beta^\top \mathcal{L}(\alpha^*)\mathbf{1}_i > 0$ since it is a deterministic scalar that does not depend on α_i. Rewriting this equation in terms of log quantities yields

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}\exp[\lambda(\alpha^*) - p_i(\alpha^*) + k_i(\alpha_i, \alpha^*)],$$

where we emphasize that λ and p_i depend only on the equilibrium technique choice α^*. The terms $\lambda(\alpha^*)$, $p_i(\alpha^*)$, and $k_i(\alpha_i, \alpha^*)$ are normally distributed. We can therefore use the expression for the expected value of a lognormal distribution and write

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \exp\left\{\mathrm{E}[\lambda(\alpha^*) - p_i(\alpha^*) + k_i(\alpha_i, \alpha^*)] + \frac{1}{2}\mathrm{V}[\lambda(\alpha^*) - p_i(\alpha^*) + k_i(\alpha_i, \alpha^*)]\right\}.$$

Taking away the exponentiation, as it is a monotone transformation, and $\mathbb{E}[\lambda(\alpha^*) - q_i(\alpha^*)]$ since it does not affect the minimization yields

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}[k_i(\alpha_i, \alpha^*)] + \frac{1}{2}\mathrm{V}[\lambda(\alpha^*) - p_i(\alpha^*) + k_i(\alpha_i, \alpha^*)]. \tag{46}$$

This expression can be written as

$$\begin{aligned}\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}[k_i(\alpha_i, \alpha^*)] + \frac{1}{2}\mathrm{V}[\lambda(\alpha^*)] + \frac{1}{2}\mathrm{V}[k_i(\alpha_i, \alpha^*) - p_i(\alpha^*)] \\ + \mathrm{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)] + \mathrm{Cov}[\lambda(\alpha^*), -p_i(\alpha^*)],\end{aligned}$$

where we can drop $\mathrm{V}[\lambda(\alpha^*)]$ and $\mathrm{Cov}[\lambda(\alpha^*), -p_i(\alpha^*)]$ as they do not affect the minimization. Finally, we can expand $\mathrm{V}[k_i(\alpha_i, \alpha^*) - p_i(\alpha^*)]$ to get

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}[k_i(\alpha_i, \alpha^*)] + \frac{1}{2}\mathrm{E}\left[(k_i(\alpha_i, \alpha^*) - p_i(\alpha^*) - \mathrm{E}[k_i(\alpha_i, \alpha^*) - p_i(\alpha^*)])^2\right] + \mathrm{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)].$$

Taking the first-order condition with respect to α_{ik}, we find

$$\frac{1}{2}\mathrm{E}\left[2(k_i(\alpha_i, \alpha^*) - p_i(\alpha^*) - \mathrm{E}[k_i(\alpha_i, \alpha^*) - p_i(\alpha^*)])\left(\frac{dk_i(\alpha_i, \alpha^*)}{d\alpha_{ik}} - \mathrm{E}\left[\frac{dk_i(\alpha_i, \alpha^*)}{d\alpha_{ik}}\right]\right)\right] + \mathrm{E}\left[\frac{dk_i(\alpha_i, \alpha^*)}{d\alpha_{ik}}\right] + \mathrm{Cov}\left[\lambda(\alpha^*), \frac{dk_i(\alpha_i, \alpha^*)}{d\alpha_{ik}}\right] + \gamma_i - \chi_{ik} = 0,$$

where $\gamma_i \geq 0$ is the Lagrange multiplier on $\sum_{j=1}^n \alpha_{ij} \leq \overline{\alpha}_i$ and $\chi_{ik} \geq 0$ is the multiplier on $\alpha_{ik} \geq 0$. At an equilibrium, $\alpha = \alpha^*$ and $k_i(\alpha_i^*, \alpha^*) = p_i(\alpha^*)$, and so

$$\mathrm{E}\left[\frac{dk_i(\alpha_i^*, \alpha^*)}{d\alpha_{ik}}\right] + \mathrm{Cov}\left[\lambda(\alpha^*), \frac{dk_i(\alpha_i^*, \alpha^*)}{d\alpha_{ik}}\right] + \gamma_i^* - \chi_{ik}^* = 0$$

describes the equilibrium choice of firm i. Notice that this equilibrium first-order condition can also come from the problem

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}[k_i(\alpha_i, \alpha^*)] + \mathrm{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)].$$

Finally, note that

$$\begin{aligned}\arg\min_{\alpha_i \in \mathcal{A}} \mathrm{E}[k_i(\alpha_i, \alpha^*)] + \mathrm{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)] &= \arg\min_{\alpha_i \in \mathcal{A}} -\mu_i - a_i(\alpha_i) + \sum_{j=1}^n \alpha_{ij}\mathrm{E}[p_j] \\ &\quad + \mathrm{Cov}\left[\lambda(\alpha^*), -\varepsilon_i - a_i(\alpha_i) + \sum_{j=1}^n \alpha_{ij} p_j\right] \\ &= \arg\min_{\alpha_i \in \mathcal{A}} -a_i(\alpha_i) + \sum_{j=1}^n \alpha_{ij}\mathrm{E}[p_j] \\ &\quad + \mathrm{Cov}[\lambda(\alpha^*), -\varepsilon_i] + \sum_{j=1}^n \alpha_{ij}\mathrm{Cov}[\lambda(\alpha^*), p_j] \\ &= \arg\min_{\alpha_i \in \mathcal{A}} -a_i(\alpha_i) + \sum_{j=1}^n \alpha_{ij}\mathcal{R}_j(\alpha^*),\end{aligned}$$

which completes the proof. □

A.4 Proof of Lemma 3

Lemma 3. *An efficient production network α^* solves*

$$\mathcal{W} \equiv \max_{\alpha \in \mathcal{A}} W(\alpha, \mu, \Sigma),$$

where $\mathcal{W}$ is a measure of the welfare of the household, and where

$$W(\alpha, \mu, \Sigma) \equiv \mathrm{E}[y(\alpha)] - \frac{1}{2}(\rho - 1)\mathrm{V}[y(\alpha)], \tag{21}$$

is welfare under a given network α.

Proof. Since we only have one agent in the economy, any Pareto efficient allocation must maximize the utility of the representative household. Under a given network and a given productivity shock ε the first welfare theorem applies, and the equilibrium is efficient. The consumption chosen by the planner is therefore given by (13). It follows that the efficient production network must solve

$$\begin{aligned} \max_{\alpha \in \mathcal{A}} \mathrm{E}[u(Y)] &= \max_{\alpha \in \mathcal{A}} \frac{1}{1-\rho} \mathrm{E}[\exp((1-\rho)\log Y)] \\ &= \max_{\alpha \in \mathcal{A}} \frac{1}{1-\rho} \exp\left((1-\rho)\mathrm{E}[\log Y] + \frac{1}{2}(1-\rho)^2 \mathrm{V}[\log Y]\right) \\ &= \max_{\alpha \in \mathcal{A}} \mathrm{E}[\log Y] - \frac{1}{2}(\rho - 1)\mathrm{V}[\log Y] \end{aligned} \tag{47}$$

where we have used the fact that $\log Y$ is normally distributed. □

A.5 Proof of Corollary 2

Corollary 2. *The efficient Domar weight vector ω^* solves*

$$\mathcal{W} = \max_{\omega \in \mathcal{O}} \underbrace{\omega^\top \mu + \bar{a}(\omega)}_{\mathrm{E}[y]} - \frac{1}{2}(\rho - 1)\underbrace{\omega^\top \Sigma \omega}_{\mathrm{V}[y]}, \tag{26}$$

where $\mathcal{O} = \left\{\omega \in \mathbb{R}^n_+ : \omega \geq \beta \text{ and } 1 \geq \omega^\top (\mathbf{1} - \bar{\alpha})\right\}$ and $\bar{a}(\omega)$ is given by (23).

Proof. Using (14) and the definition of Domar weights, the original planning problem (21) can be written

$$\mathcal{W} = \max_{\alpha \in \mathcal{A}, \omega} \omega^\top \mu + \omega^\top a(\alpha) - \frac{1}{2}(\rho - 1)\omega^\top \Sigma \omega,$$

subject to $\beta^\top (I - \alpha)^{-1} = \omega^\top$, which is equivalent to

$$\max_{\omega \in \mathcal{O}} \omega^\top \mu + \left[\max_{\alpha \in \mathcal{A}} \omega^\top a(\alpha)\right] - \frac{1}{2}(\rho - 1)\omega^\top \Sigma \omega,$$

where the inner problem is subject to $\beta^\top (I-\alpha)^{-1} = \omega^\top$ and where we can limit the feasible set of the outside problem to $\mathcal{O}$ since for $\omega \notin \mathcal{O}$ the inner constraints could never be satisfied. This last problem is the same as (26) because, by (23), $\bar{a}(\omega) = \max_{\alpha \in \mathcal{A}} \omega^\top a(\alpha)$ subject to $\beta^\top (I-\alpha)^{-1} = \omega^\top$. □

A.6 Proof of Lemma 4

Lemma 4. *The objective function of the planner's problem (26) is strictly concave. Furthermore, there is a unique vector of Domar weights ω^* that solves that problem, and there is a unique production network $\alpha(\omega^*)$ associated with that solution.*

Proof. We first show that the value function $\bar{a}(\omega)$ defined by (23) is strictly concave. Consider the maximization problem

$$\bar{a}(\omega) = \max_{\alpha \in \mathcal{A}} \omega^\top a(\alpha),$$

subject to $\beta^\top (I-\alpha)^{-1} = \omega^\top$. Since $I-\alpha$ is always invertible for $\alpha \in \mathcal{A}$ we can rewrite this constraint as the affine relationship

$$\alpha^\top \omega = \omega - \beta. \tag{48}$$

Take two feasible points ω^0 and ω^1, and let $\alpha^0 \in \mathcal{A}$ and $\alpha^1 \in \mathcal{A}$ be their respective maximizers. Consider the convex combination α^t defined component-by-component as

$$\alpha_i^t = \frac{\omega_i^0}{\omega_i^0 + \omega_i^1}\alpha_i^0 + \frac{\omega_i^1}{\omega_i^0 + \omega_i^1}\alpha_i^1.$$

We will show that α^t is a feasible point for $\omega^t = \frac{\omega^0 + \omega^1}{2}$. First notice that $\alpha^t \geq 0$ and that

$$\sum_j \alpha_{ij}^t = \frac{\omega_i^0}{\omega_i^0 + \omega_i^1}\sum_j \alpha_{ij}^0 + \frac{\omega_i^1}{\omega_i^0 + \omega_i^1}\sum_j \alpha_{ij}^1 \leq \bar{\alpha}_i,$$

so that $\alpha^t \in \mathcal{A}$. Next, since (48) holds for (α^0, ω^0) and (α^1, ω^1), we can write

$$\sum_j \omega_j^0 \alpha_{ji}^0 = \omega_i^0 - \beta_i, \text{ and } \sum_j \omega_j^1 \alpha_{ji}^1 = \omega_i^1 - \beta_i.$$

Summing these two equations up, we get

$$\sum_j \left(\omega_j^0 \alpha_{ji}^0 + \omega_i^1 \alpha_{ji}^1\right) = \omega_i^0 + \omega_i^1 - 2\beta_i,$$

$$\sum_j \frac{\omega_j^0 + \omega_j^1}{2} \left(\frac{\omega_j^0}{\omega_j^0 + \omega_j^1}\alpha_{ji}^0 + \frac{\omega_i^1}{\omega_j^0 + \omega_j^1}\alpha_{ji}^1\right) = \frac{\omega_i^0 + \omega_i^1}{2} - \beta_i,$$

$$\sum_j \omega_j^t \alpha_{ji}^t = \omega_i^t - \beta_i,$$

which implies that (48) holds for $\left(\alpha^t, \omega^t\right)$. Therefore, α^t is a feasible point for ω^t.

Consider the value function at ω^t:

$$\bar{a}\left(\omega^t\right) = \bar{a}\left(\frac{\omega^0 + \omega^1}{2}\right) \geq \sum_i \omega_i^t a\left(\alpha_i^t\right) = \sum_i \frac{\omega_i^0 + \omega_i^1}{2} a_i \left(\frac{\omega_i^0}{\omega_i^0 + \omega_i^1}\alpha_i^0 + \frac{\omega_i^1}{\omega_i^0 + \omega_i^1}\alpha_i^1\right),$$

where the inequality follows since α^t might not be a maximizer for ω^t. From the strict concavity of a_i, we find

$$\bar{a}\left(\omega^t\right) > \sum_i \frac{\omega_i^0 + \omega_i^1}{2} \left(\frac{\omega_i^0}{\omega_i^0 + \omega_i^1} a_i\left(\alpha_i^0\right) + \frac{\omega_i^1}{\omega_i^0 + \omega_i^1} a_i\left(\alpha_i^1\right)\right) = \frac{1}{2}\bar{a}\left(\omega^0\right) + \frac{1}{2}\bar{a}\left(\omega^1\right).$$

This holds for any feasible ω^0 and ω^1, and so $\bar{a}$ is midpoint strictly concave. By the Theorem of Maximum $\bar{a}$ is also continuous, and so $\bar{a}$ is therefore strictly concave. It follows that the objective function (26) is also strictly concave, which proves the first part of the statement.

Since the objective (26) is strictly concave, the feasible set is convex, there is a unique maximizer so there is a unique solution ω^* to the planner's problem. Now, notice that the objective function (23) is strictly concave since a_i is strictly concave for all i. The feasible set (the intersection of (48) and $\mathcal{A}$) is convex so there is once again a unique maximizer. It follows that for each ω there is a unique α that solves (23), and there is therefore a unique α^* associated with ω^*. □

A.7 Proof of Proposition 1

Proposition 1. *There exists a unique equilibrium, and it is efficient.*

Proof. For a given production network α and a given draw of the random TFP vector ε, the economy is standard, and the equilibrium is unique. The first welfare theorem also applies and so the allocation is efficient. We therefore only need to focus on the choice of network under uncertainty. An equilibrium network $\alpha^* \in \mathcal{A}$ is fully characterized by a solution to (17) and where

$\mathcal{R}(\alpha^*)$ is given by (18) which can be written in terms of primitives as

$$\mathcal{R}(\alpha^*) = \underbrace{-\mathcal{L}(\alpha^*)(\mu + a(\alpha^*))}_{\mathrm{E}[p]} + \underbrace{(\rho - 1)\mathcal{L}(\alpha^*)\Sigma[\mathcal{L}(\alpha^*)]^\top \beta}_{\mathrm{Cov}(p,\lambda)}.$$

Since the objective function is strictly concave and the constraint set is defined by affine functions, it follows that $\alpha^* \in \mathcal{A}$ is an equilibrium network if there exists Lagrange multipliers $\chi_{ij}^e \geq 0$ and $\gamma_i^e \geq 0$ such that 1) the first-order conditions of the firms

$$\frac{\partial a_i}{\partial \alpha_i}(\alpha^*) + \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) - (\rho - 1)\mathcal{L}(\alpha^*)\Sigma\mathcal{L}(\alpha^*)^\top \beta + \chi_i^e - \gamma_i^e \mathbf{1} = 0, \tag{49}$$

evaluated at α^* are satisfied, and 2) the complementary slackness conditions

$$-\chi_{ij}^e \alpha_{ij}^* = 0, \tag{50}$$

$$\gamma_i^e \left(\sum_{j=1}^n \alpha_{ij}^* - \overline{\alpha}_i \right) = 0, \tag{51}$$

are satisfied for all i, j.

Next, consider the social planner's problem given by (26) and subject to the constraints in Corollary 2. By Lemma 4, the objective function is strictly concave and the constraint set is defined by affine function. It follows that an allocation $\alpha \in \mathcal{A}$ is efficient if there exist nonnegative Lagrange multipliers $\hat{\chi}$ and $\hat{\gamma}$ such that 1) the first-order conditions

$$\mu + \nabla \bar{a} - (\rho - 1)\Sigma\omega + \hat{\chi} - \hat{\gamma}(\mathbf{1} - \bar{\alpha}) = 0, \tag{52}$$

where $\omega^\top = \beta^\top \mathcal{L}(\alpha)$ and where $\nabla \bar{a}$ is the derivative of the aggregate TFP shifter (23), are satisfied and the 2) complementary slackness conditions

$$-\hat{\chi}_i(\omega_i - \beta_i) = 0, \tag{53}$$

$$\hat{\gamma}\left(\omega^\top(\mathbf{1} - \bar{\alpha}) - 1\right) = 0, \tag{54}$$

are satisfied for all i. To derive $\nabla \bar{a}$, we can use the problem (23). The objective function of this problem is strictly concave (see proof of Lemma 4) and the constraint set is convex.[34] It follows that the unique maximizer is characterized by the first-order condition

$$\omega_i \frac{\partial a_i}{\partial \alpha_{ij}} - \zeta_j \omega_i + \check{\chi}_{ij} - \check{\gamma}_i = 0 \Leftrightarrow \zeta_j = \frac{\partial a_i}{\partial \alpha_{ij}} + \tilde{\chi}_{ij} - \tilde{\gamma}_i, \tag{55}$$

[34] Recall that the constraint set is given by $\alpha \in \mathcal{A}$ and an affine function (48).

and the complementary slackness conditions

$$\alpha^\top \omega - \omega + \beta = 0, \tag{56}$$

$$-\check{\chi}_{ij}\alpha_{ij} = 0, \tag{57}$$

$$\check{\gamma}_i \left(\sum_{j=1}^{n} \alpha_{ij} - \overline{\alpha}_i \right) = 0, \tag{58}$$

for all i, j and where $\tilde{\chi}_{ij} = \frac{\check{\chi}_{ij}}{\omega_i}$ and $\tilde{\gamma}_i = \frac{\check{\gamma}_i}{\omega_i}$. Applying the envelope theorem to (23), we obtain

$$\nabla \bar{a} = a(\alpha) + (I - \alpha)\zeta = a(\alpha) + (I - \alpha)\left(\frac{\partial a_i}{\partial \alpha_i} + \tilde{\chi}_i - \tilde{\gamma} \right),$$

where we use (55) to express ζ. Plugging this expression in (52), we get

$$\mu + a(\alpha) + (I - \alpha)\left(\frac{\partial a_i}{\partial \alpha_i} + \tilde{\chi}_i - \tilde{\gamma} \right) - (\rho - 1)\Sigma\omega + \hat{\chi} - \hat{\gamma}(\mathbf{1} - \bar{\alpha}) = 0 \Leftrightarrow$$

$$\frac{\partial a_i}{\partial \alpha_i} + \mathcal{L}(\alpha)(\mu + a(\alpha)) - (\rho - 1)\mathcal{L}(\alpha)\Sigma\mathcal{L}(\alpha)^\top \beta + \tilde{\chi}_i + \mathcal{L}(\alpha)\hat{\chi} - (\hat{\gamma}\mathcal{L}(\alpha)(\mathbf{1} - \bar{\alpha}) + \tilde{\gamma}) = 0, \tag{59}$$

where the second line follows from the first by left-multiplying by $\mathcal{L}(\alpha) = (I - \alpha)^{-1}$.

Now, we will show that the equilibrium and efficiency conditions coincide. Suppose that we have a solution to the planner's problem $(\alpha^p, \omega^p, \tilde{\chi}, \hat{\chi}, \tilde{\gamma}, \hat{\gamma}, \zeta)$. Consider the candidate equilibrium $(\alpha^e, \omega^e, \chi^e, \gamma^e)$ where $\alpha^e = \alpha^p$, $\omega^e = \omega^p$, $\chi_i^e = \tilde{\chi}_i + \mathcal{L}(\alpha^p)\hat{\chi}$ for all i, and $\gamma^e = \hat{\gamma}\mathcal{L}(\alpha^p)(\mathbf{1} - \bar{\alpha}) + \tilde{\gamma}$. First, note that since $\tilde{\chi}, \hat{\gamma}, \tilde{\gamma}, \hat{\gamma}$ are nonnegative, so are χ^e, γ^e. Next, the candidate equilibrium satisfies the first-order condition (49). The first complementary slackness condition (50) is also satisfied. Indeed, suppose that $\alpha_{ij}^p > 0$, which implies that $\omega_i^p > \beta_i$, then $\tilde{\chi}_{ij} = 0$ and $\hat{\chi}_j = 0$, such that $\chi_{ij}^e = 0$, and the condition is satisfied. If instead the constraint $\alpha_{ij}^p \geq 0$ binds such that $\tilde{\chi}_{ij} > 0$, we have $\chi_{ij}^e > 0$. Furthermore, from the first-order condition (49) $\alpha_{ij}^e = 0$, so the condition is satisfied. For the second complementary slackness condition (51), if $\sum_{j=1}^n \alpha_{ij}^p < \overline{\alpha}_i$ for some i then $\tilde{\gamma}_i = 0$ and $\hat{\gamma} = 0$. It follows that $\gamma^e = 0$ and the condition is satisfied. If instead the constraint binds such that $\sum_{j=1}^n \alpha_{ij}^p = \overline{\alpha}_i$ for some i, then $\sum_{j=1}^n \alpha_{ij}^e = \overline{\alpha}_i$, and the second complementary slackness condition (51) is satisfied. It follows that any efficient allocation can be decentralized as an equilibrium allocation. Since we know that an efficient allocation exists (it is the outcome of an optimization problem on a compact set), this proves that an efficient equilibrium exists.

Suppose instead that we have an equilibrium $(\alpha^e, \omega^e, \chi^e, \gamma^e)$ where $\omega^e = \mathcal{L}(\alpha^e)^\top \beta$, and consider the candidate efficient allocation $(\alpha^p, \omega^p, \tilde{\chi}, \hat{\chi}, \tilde{\gamma}, \hat{\gamma}, \zeta)$, where $\alpha^p = \alpha^e$, $\omega^p = \omega^e$, $\tilde{\chi} = \chi^e$, $\hat{\chi} = 0$, $\tilde{\gamma} = \gamma^e$, $\hat{\gamma} = 0$, $\zeta = -\mathcal{L}(\alpha^e)(\mu + a(\alpha^e)) + (\rho - 1)\mathcal{L}(\alpha^e)\Sigma\mathcal{L}(\alpha^e)^\top \beta$. Note that the first-order conditions (59) of the planner are satisfied. Next, notice that the complementary slackness conditions (53)-

(54) are always satisfied. Finally, the first-order conditions (55) and the complementary slackness conditions (56)-(58) are also satisfied. For the condition (56), note that since $\omega^e = \mathcal{L}(\alpha^e)^\top \beta$ we have $(\alpha^p)^\top \omega^p - \omega^p + \beta = 0$ and so the condition is satisfied. For the condition (57), if $\alpha^e_{ij} > 0$ then $\chi^e_{ij} = \tilde{\chi}_{ij} = 0$ and the condition is satisfied. If instead the constraint $\alpha^e_{ij} \geq 0$ binds such that $\alpha^e_{ij} = 0$, then $\alpha^p_{ij} = 0$ and the condition is also satisfied. Finally, for the condition (58), if $\sum_{j=1}^n \alpha^e_{ij} < \bar{\alpha}_i$ then $\gamma^e_i = 0$ and so $\tilde{\gamma} = 0$, so that the condition is satisfied. If instead $\sum_{j=1}^n \alpha^p_{ij} = \bar{\alpha}_i$, it must be that $\sum_{j=1}^n \alpha^e_{ij} = \bar{\alpha}_i$, and so (58) is satisfied. We have therefore shown that any equilibrium corresponds to an efficient allocation. By Lemma 4, the objective function of the planner is strictly concave and its constraint set is convex. It follows that there is a unique efficient allocation and therefore a unique equilibrium. □

A.8 Proof of Proposition 2

Proposition 2. *The Domar weight ω_i of sector i is (weakly) increasing in μ_i and (weakly) decreasing in Σ_{ii}.*

Proof. Note that by the maximum theorem applied to (26), ω_i is a continuous function of μ and Σ. We consider the comparative statics with respect to μ_i first. We proceed by contradiction. Suppose that ω_i is not an increasing function of μ_i. Then, by continuity of ω_i as a function of μ_i, there exists a point (μ^0, Σ^0) and an interval (μ^0_i, μ^1_i) such that $\omega_i(\mu_i, \mu^0_{-i}, \Sigma^0) < \omega_i(\mu^0_i, \mu^0_{-i}, \Sigma^0)$ for any $\mu_i \in (\mu^0_i, \mu^1_i)$. Denote the optimal network at (μ, Σ) by $\alpha^*(\mu, \Sigma)$. Now, consider an increase in μ_i from μ^0_i to μ^1_i (holding other elements of μ^0 and Σ^0 fixed). From Corollary 4, we can write the change in welfare as

$$\mathcal{W}(\mu^1_i, \mu^0_{-i}, \Sigma^0) = \mathcal{W}(\mu^0_i, \mu^0_{-i}, \Sigma^0) + \int_{\mu^0_i}^{\mu^1_i} \omega_i(\mu_i, \mu^0_{-i}, \Sigma^0)\, d\mu_i.$$

Suppose instead that the network is fixed at its original value $\alpha^*(\mu^0_i, \mu^0_{-i}, \Sigma^0)$. Equations (14) imply that under a fixed network the change in μ_i affects welfare only through its impact on expected log GDP. By Corollary 1, the change in welfare can thus be written as

$$W(\alpha^*(\mu^0_i, \mu^0_{-i}, \Sigma^0); \mu^1_i, \mu^0_{-i}, \Sigma^0) = W(\alpha^*(\mu^0_i, \mu^0_{-i}, \Sigma^0); \mu^0_i, \mu^0_{-i}, \Sigma^0) + \omega_i(\mu^0_i, \mu^0_{-i}, \Sigma^0)(\mu^1_i - \mu^0_i).$$

But since the initial network $\alpha^*(\mu^0_i, \mu^0_{-i}, \Sigma^0)$ is feasible at $(\mu^1_i, \mu^0_{-i}, \Sigma^0)$, welfare maximization implies that $\mathcal{W}(\mu^1_i, \mu^0_{-i}, \Sigma^0) = W(\alpha^*(\mu^1_i, \mu^0_{-i}, \Sigma^0); \mu^1_i, \mu^0_{-i}, \Sigma^0) \geq \mathcal{W}(\alpha^*(\mu^0_i, \mu^0_{-i}, \Sigma^0); \mu^1_i, \mu^0_{-i}, \Sigma^0)$, and so

$$\int_{\mu^0_i}^{\mu^1_i} \omega_i(\mu_i, \mu^0_{-i}, \Sigma^0)\, d\mu_i \geq \omega_i(\mu^0_i, \mu^0_{-i}, \Sigma^0)(\mu^1_i - \mu^0_i). \tag{60}$$

Since we have assumed by contradiction that $\omega_i(\mu_i, \mu^0_{-i}, \Sigma^0) < \omega_i(\mu^0_i, \mu^0_{-i}, \Sigma^0)$ for all $\mu_i \in (\mu^0_i, \mu^1_i)$, it follows that

$$\int_{\mu_i^0}^{\mu_i^1} \omega_i\left(\mu_i, \mu_{-i}^0, \Sigma^0\right) d\mu_i < \int_{\mu_i^0}^{\mu_i^1} \omega_i\left(\mu_i^0, \mu_{-i}^0, \Sigma^0\right) d\mu_i = \omega_i\left(\mu_i^0, \mu_{-i}^0, \Sigma^0\right)\left(\mu_i^1 - \mu_i^0\right),$$

which contradicts (60). Therefore, ω_i is an increasing function of μ_i.

For the second part of the proposition, recall that $\frac{d\mathcal{W}}{d\Sigma_{ii}} = -\frac{1}{2}(\rho-1)\omega_i^2$ by Corollary 4. Using analogous steps, we can then establish the second part of this proposition. □

A.9 Proof of Proposition 3

Proposition 3. *Let γ denote either the mean μ_i or an element of the covariance matrix Σ_{ij}. If $\omega \in \operatorname{int}\mathcal{O}$, then the response of the equilibrium Domar weights to a change in γ is given by*

$$\frac{d\omega}{d\gamma} = \underbrace{-\mathcal{H}^{-1}}_{\textit{propagation}} \times \underbrace{\frac{\partial \mathcal{E}}{\partial \gamma}}_{\textit{impulse}}, \tag{31}$$

where the $n \times n$ negative definite matrix $\mathcal{H}$ is given by

$$\mathcal{H} = \nabla^2 \bar{a} + \frac{d\mathcal{E}}{d\omega}, \tag{32}$$

and where the matrix $\nabla^2\bar{a}$ is the Hessian of the aggregate TFP shifter function $\bar{a}$, and $\frac{d\mathcal{E}}{d\omega} = \frac{d\operatorname{Cov}[\varepsilon,\lambda]}{d\omega} = -(\rho-1)\Sigma$ is the Jacobian matrix of the risk-adjusted TFP vector $\mathcal{E}$.

Proof. At an interior solution, the first-order conditions of (26) are

$$F(\omega, \mu, \Sigma) := \mu + \nabla\bar{a} - (\rho-1)\Sigma\omega = 0,$$

where $\nabla\bar{a}$ is the gradient of $\bar{a}$. Differentiating with respect to ω, we find that

$$\frac{dF}{d\omega} = \nabla^2\bar{a} - (\rho-1)\Sigma,$$

where $\nabla^2\bar{a}$ is the Hessian matrix of $\bar{a}$. From the implicit function theorem, it follows that

$$\frac{d\omega}{d\gamma} = -\left[\nabla^2\bar{a} - (\rho-1)\Sigma\right]^{-1}\frac{\partial F}{\partial \gamma}.$$

If $\gamma = \mu_i$, we have

$$\frac{\partial F}{\partial \gamma} = \frac{\partial F}{\partial \mu_i} = \mathbf{1}_i = \frac{\partial \mathcal{E}}{\partial \mu_i},$$

where $\mathbf{1}_i$ is a column vector of zeros except for 1 at element i. If $\gamma = \Sigma_{ij}$, we have

$$\frac{\partial F}{\partial \gamma} = \frac{\partial F}{\partial \Sigma_{ij}} = -\frac{1}{2}(\rho-1)(\omega_j\mathbf{1}_i + \omega_i\mathbf{1}_j) = \frac{\partial \mathcal{E}}{\partial \Sigma_{ij}},$$

where, if $i \neq j$, we differentiate with respect to Σ_{ij} and Σ_{ji} simultaneously to preserve the symmetry of the covariance matrix and divide by two to preserve the scale. Finally, in the proof of Lemma 4, we show that $\nabla^2 \bar{a}$ is negative definite. It follows from (32) that $\mathcal{H}$ and its inverse are also negative definite. □

Proposition 3 can be extended to handle the case in which some of the constraints $\omega_i \geq \beta_i$ bind with strictly positive Lagrange multipliers. We show how this can be done in Supplemental Appendix D in Kopytov et al. (2024).

A.10 Proofs of Corollary 3, Lemmas 5, 6 and 7, and of Proposition 4

These proofs are in Supplemental Appendix B in Kopytov et al. (2024).

A.11 Proof of Lemma 8

Lemma 8. *If $\omega \operatorname{int} \mathcal{O}$, the equilibrium Domar weights are approximately given by*

$$\omega = \omega^\circ - [\mathcal{H}^\circ]^{-1} \mathcal{E}^\circ + O\left(\|\omega - \omega^\circ\|^2\right), \tag{35}$$

where the superscript $\circ$ indicates that $\mathcal{H}$ and $\mathcal{E}$ are evaluated at ω°.

Proof. At an interior solution, the first-order conditions of (26) are

$$\mu + \nabla \bar{a}(\omega) - (\rho - 1)\Sigma\omega = 0. \tag{61}$$

The first-order Taylor expansion of $\nabla \bar{a}(\omega)$ around ω° is

$$\nabla \bar{a}(\omega) = \nabla \bar{a}(\omega^\circ) + \nabla^2 \bar{a}(\omega^\circ)(\omega - \omega^\circ) + O\left(\|\omega - \omega^\circ\|^2\right).$$

Plugging it into (61), we get

$$\omega - \omega^\circ = -\left[\nabla^2 \bar{a}(\omega^\circ) - (\rho - 1)\Sigma\right]^{-1} \left[\mu - (\rho - 1)\Sigma\omega^\circ + \nabla \bar{a}(\omega^\circ)\right] + O\left(\|\omega - \omega^\circ\|^2\right).$$

From (32), we can write $\mathcal{H}^\circ = \nabla^2 \bar{a}(\omega^\circ) - (\rho - 1)\Sigma$. From (27), $\mathcal{E}^\circ = \mu - (\rho - 1)\Sigma\omega^\circ$. Therefore,

$$\omega - \omega^\circ = -[\mathcal{H}^\circ]^{-1}\left[\mathcal{E}^\circ + \nabla \bar{a}(\omega^\circ)\right] + O\left(\|\omega - \omega^\circ\|^2\right).$$

Next, by the envelope theorem applied to (23) we find

$$\nabla \bar{a}(\omega^\circ) = a(\alpha^\circ) + (I - \alpha^\circ)\zeta = 0, \tag{62}$$

where ζ is the vector of Lagrange multipliers associated with the constraint $\alpha^{\top}\omega^{\circ} = \omega^{\circ} - \beta$. To find these multipliers, recall from (55) that the first-order conditions of the problem (23) are

$$\zeta_i = \frac{\partial a_i}{\partial \alpha_{ij}} + \tilde{\chi}_{ij} - \tilde{\gamma}_i. \tag{55}$$

Now recall that $\frac{\partial a_i}{\partial \alpha_{ij}} (\alpha_i^{\circ}) = 0$ for all i, j by the definition of α_i°. It follows that if set $\alpha_i = \alpha_i^{\circ}$ for all i and all the Lagrange multipliers $\tilde{\chi}_{ij}$, $\tilde{\gamma}_i$ equal to zero, then the first-order conditions and complementary slackness conditions are satisfied, and by definition of ω°,the constraint $(\alpha^{\circ})^{\top} \omega^{\circ} = \omega^{\circ} - \beta$ is also satisfied. This is therefore the (unique) solution to that optimization problem. It follows from (62) that $\nabla \bar{a} (\omega^{\circ}) = a (\alpha^{\circ}) = 0$ where the last equality comes from our normalization.

□

A.12 Proof of Lemma 9

The proof of Lemma 9 is a special case of Proposition 9 in Supplemental Appendix G in Kopytov et al. (2024).

A.13 Proof of Proposition 5

Proposition 5. *Let γ denote either the mean μ_i or an element of the covariance matrix Σ_{ij}. Under an endogenous network, welfare responds to a marginal change in γ as if the network were fixed at its equilibrium value α^*, that is*

$$\frac{d\mathcal{W}(\mu, \Sigma)}{d\gamma} = \frac{\partial W(\alpha^*, \mu, \Sigma)}{\partial \gamma}.$$

Proof. Recall from Lemma 3 that the equilibrium α^* solves the welfare-maximization problem

$$\mathcal{W}(\mu, \Sigma) = \max_{\alpha \in \mathcal{A}} W(\alpha, \mu, \Sigma), \tag{63}$$

where

$$W(\alpha, \mu, \Sigma) = \mathrm{E}[y(\alpha)] - \frac{1}{2}(\rho - 1)\mathrm{V}[y(\alpha)], \tag{64}$$

is welfare under a given network α and beliefs (μ, Σ). Both $\mathrm{E}[y]$ and $\mathrm{V}[y]$ depend on beliefs through (14). Since the objective function (63) and its associated constraints are continuously differentiable functions of α, and since the constraint $\alpha \in \mathcal{A}$ does not depend on beliefs, the envelope theorem immediately implies that

$$\frac{d\mathcal{W}(\mu, \Sigma)}{d\gamma} = \frac{\partial W(\alpha^*, \mu, \Sigma)}{\partial \gamma},$$

where the right-hand side is the change in welfare keeping the network constant at α^*. □

A.14 Proof of Corollary 4

Corollary 4. *The impact of an increase in μ_i on welfare is given by*

$$\frac{d\mathcal{W}}{d\mu_i} = \omega_i, \tag{39}$$

and the impact of an increase in Σ_{ij} on welfare is given by

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho - 1)\,\omega_i\omega_j. \tag{40}$$

Proof. Combining (64) with Corollary 14, it is immediate to show that

$$\frac{\partial W(\alpha^*, \mu, \Sigma)}{\partial \mu_i} = \omega_i,$$

and

$$\frac{dW(\alpha^*, \mu, \Sigma)}{d\Sigma_{ij}} = -\frac{1}{2}(\rho - 1)\,\omega_i\omega_j.$$

Putting these expressions together with Proposition 5 yields the result. □

A.15 Proof of Proposition 6

Proposition 6. *The presence of uncertainty lowers expected log GDP, in the sense that* $\mathrm{E}[y]$ *is largest when* $\Sigma = 0$.

Proof. The proof follows from Corollary 3. From (14), define

$$\mathcal{Y}(\alpha, \mu, \Sigma) = \mathrm{E}[y(\alpha)] = \omega(\alpha)^\top (\mu + a(\alpha)), \tag{65}$$

and

$$\mathcal{V}(\alpha, \mu, \Sigma) = \mathrm{V}[y(\alpha)] = \omega(\alpha)^\top \Sigma\omega(\alpha), \tag{66}$$

as the expected value and the variance of log GDP under the network α and the beliefs (μ, Σ). Let $\alpha^*(\mu, \Sigma)$ denote an optimal network (a solution to (21)) under the beliefs (μ, Σ).

Fix μ. We first establish that $\alpha^*(\mu, 0)$ maximizes $\mathcal{Y}(\alpha, \mu, 0)$. To see this, note that (66) implies that $\mathcal{V}(\alpha, \mu, 0) = 0$ for all pairs (α, μ). The problem (21) of the social planner with $\Sigma = 0$ can therefore be written as

$$\max_{\alpha\in\mathcal{A}} \mathcal{Y}(\alpha, \mu, 0) - \frac{1}{2}(\rho - 1)\,\mathcal{V}(\alpha, \mu, 0) = \max_{\alpha\in\mathcal{A}} \mathcal{Y}(\alpha, \mu, 0) = \mathcal{Y}(\alpha^*(\mu, 0), \mu, 0),$$

where the second equality comes from the definition of $\alpha^*(\mu, 0)$.

Next, notice that

$$\mathcal{Y}\left(\alpha^*\left(\mu,0\right),\mu,0\right) \geq \mathcal{Y}\left(\alpha^*\left(\mu,\Sigma\right),\mu,0\right) = \mathcal{Y}\left(\alpha^*\left(\mu,\Sigma\right),\mu,\Sigma\right), \tag{67}$$

where the inequality comes from the fact that $\alpha^*(\mu,0)$ maximizes $\mathcal{Y}(\alpha,\mu,0)$, and the equality comes from the fact that $\mathcal{Y}(\alpha,\mu,\Sigma)$, given by (65), does not explicitly depend on Σ. Since (67) holds for any Σ, it follows that expected log GDP $\mathcal{Y}(\alpha^*(\mu,\Sigma),\mu,\Sigma)$ is maximized at $\Sigma = 0$, which is the desired result. □

A.16 Proof of Corollary 5

The proof of Corollary 5 is in Supplemental Appendix B in Kopytov et al. (2024).

A.17 Proof of Proposition 7

Proposition 7. *If $\omega \in \operatorname{int}\mathcal{O}$, the following holds.*

1. *The impact of an increase in μ_i on log GDP is given by*

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} = \underbrace{\omega_i}_{\text{Fixed network}} - (\rho-1)\,\omega^\top\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\mu_i}, \quad \text{and} \quad \frac{d\,\mathrm{V}[y]}{d\mu_i} = \underbrace{0}_{\text{Fixed network}} -2\omega^\top\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\mu_i}.$$

2. *The impact of an increase in Σ_{ij} on log GDP is given by*

$$\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} = \underbrace{0}_{\text{Fixed network}} - (\rho-1)\,\omega^\top\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\Sigma_{ij}}, \quad \text{and} \quad \frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} = \underbrace{\omega_i\omega_j}_{\text{Fixed network}} -2\omega^\top\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\Sigma_{ij}}.$$

Proof. Differentiating (14) with respect to μ_i yields

$$\frac{d\,\mathrm{V}[y]}{d\mu_i} = 2\omega^\top\Sigma\frac{d\omega}{d\mu_i},$$

which, together with (31), yields

$$\frac{d\,\mathrm{V}[y]}{d\mu_i} = -2\omega^\top\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\mu_i}. \tag{68}$$

Next, from (39) we find

$$\frac{d\mathcal{W}}{d\mu_i} = \omega_i = \frac{d\,\mathrm{E}[y]}{d\mu_i} - \frac{1}{2}(\rho-1)\frac{d\,\mathrm{V}[y]}{d\mu_i},$$

which we can combine with the previous equation to get

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} = \omega_i - (\rho-1)\,\omega^\top\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\mu_i}.$$

Similarly, differentiating (14) with respect to Σ_{ij} yields

$$\frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} = \omega_i\omega_j + 2\omega^\top \Sigma \frac{d\omega}{d\Sigma_{ij}} = \omega_i\omega_j - 2\omega^\top \Sigma \mathcal{H}^{-1} \frac{\partial \mathcal{E}}{\partial \Sigma_{ij}}.$$

From (40), we can write

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho-1)\,\omega_i\omega_j = \frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} - \frac{1}{2}(\rho-1)\frac{d\,\mathrm{V}[y(\alpha)]}{d\Sigma_{ij}},$$

which we can combine with the previous equation to find

$$\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} = -(\rho-1)\,\omega^\top \Sigma \mathcal{H}^{-1} \frac{\partial \mathcal{E}}{\partial \Sigma_{ij}}.$$

□

A.18 Proof of Corollary 6

Corollary 6. *Without uncertainty ($\Sigma = 0$) the moments of GDP respond to changes in beliefs as if the network were fixed, such that*

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} = \frac{\partial\,\mathrm{E}[y]}{\partial \mu_i} = \omega_i, \qquad \textit{and} \qquad \frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} = \frac{\partial\,\mathrm{V}[y]}{\partial \Sigma_{ij}} = \omega_i\omega_j.$$

Proof. When $\Sigma = 0$, the problem of the planner (3) becomes

$$\mathcal{W} = \mathrm{E}[y(\alpha^*)] = \max_{\alpha\in\mathcal{A}} \mathrm{E}[y(\alpha)] = \max_{\alpha\in\mathcal{A}} \omega(\alpha)^\top (\mu + a(\alpha)).$$

The envelope theorem then implies that $\frac{d\,\mathrm{E}[y]}{d\mu_i} = \omega_i$ which proves the first part of the corollary. The envelope theorem also implies $\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} = 0$ which leads to the second part of the corollary when combined with Proposition 5 and Corollary 1. □

A.19 Proofs of Corollaries 7 and 8

These proofs are in Supplemental Appendix B in Kopytov et al. (2024).

B Additional results related to the calibrated economy

In this appendix, we provide additional information about 1) the data used in the calibration of Section 8, 2) our calibration strategy, 3) how well the model fits the data, 4) the quantitative importance of the mechanism, and 5) robustness exercises.

B.1 Data

The Bureau of Economic Analysis (BEA) provides sectoral input-output tables that allow us to compute the intermediate input shares as well as the shares of final consumption expenditure accounted for by different sectors. We rely on the harmonized tables constructed by Vom Lehn and Winberry (2022) that provide consistent annual data for $n = 37$ sectors over the period 1948-2020. Table III provides the list of the sectors included in this data set.

TABLE III. The 37 sectors used in our analysis

Mining	Utilities
Construction	Wood products
Nonmetallic minerals	Primary metals
Fabricated metals	Machinery
Computer and electronic manufacturing	Electrical equipment manufacturing
Motor vehicles manufacturing	Other transportation equipment
Furniture and related manufacturing	Misc. manufacturing
Food and beverage manufacturing	Textile manufacturing
Apparel manufacturing	Paper manufacturing
Printing products manufacturing	Petroleum and coal manufacturing
Chemical manufacturing	Plastics manufacturing
Wholesale trade	Retail trade
Transportation and warehousing	Information
Finance and insurance	Real estate and rental services
Professional and technical services	Management of companies and enterprises
Administrative and waste management services	Educational services
Health care and social assistance	Arts and entertainment services
Accommodation	Food services
Other services	

Notes: Sectors are classified according to the NAICS-based BEA codes. See Vom Lehn and Winberry (2022) for details of the data construction.

From these data, we can compute the input shares α_{ijt} of each sector in each year t. The typical share α_{ij} in the data has an average of 0.0128 and a standard deviation over time of 0.0048, for a coefficient of variation of 0.37. We also use the input-output tables to compute sectoral total factor productivity, following the procedure in Vom Lehn and Winberry (2022) closely. Specifically, sectoral TFP is measured as the Solow residual, i.e. the residual that remains after removing the contribution of input factors from a sector's gross output. We make three departures from Vom Lehn and Winberry (2022) in constructing the TFP series. First, to be consistent with our model, we let the input shares α_{ijt} vary over time. Second, we do not smooth the resulting Solow residuals. Finally, we update the time series to include the years up to 2020.

B.2 Calibration procedure

The three groups of parameters that we need to calibrate are 1) the household's preferences, i.e. the consumption shares β and the risk-aversion ρ, 2) the parameters of the TFP shifter function (2), and 3) the processes for the exogenous sectoral productivity shocks, i.e. μ_t and Σ_t. Some of these parameters can be computed directly from the data. The other ones are estimated using a combination of indirect inference and standard time-series methods. Below, we describe the exact procedure used for each set of parameters.

Household preferences

Since the preference parameter β_i corresponds to the household's expenditure share of good i, we pin down its value directly from the data by averaging the consumption share of good i over time. The sectors with the largest consumption shares are "Real estate" (14%), "Retail trade" (12%) and "Health care" (11%). See Supplemental Appendix H in Kopytov et al. (2024) for a version of the calibrated economy with time-varying β's.

The relative risk aversion parameter ρ determines to what extent firms are willing to trade off higher input prices for access to more stable suppliers. The literature uses a broad range of values for ρ and it is unclear a priori which one is best for our application. We therefore estimate ρ using a method of simulated moments (MSM) described below.

Endogenous productivity shifter

We specialize the TFP shifter function (2) to

$$\log A_i\left(\alpha_i\right) = a_i^\circ - \sum_{j=1}^{n} \kappa_{ij}\left(\alpha_{ij} - \alpha_{ij}^\circ\right)^2 - \kappa_{i0}\left(\sum_{j=1}^{n}\alpha_{ij} - \sum_{j=1}^{n}\alpha_{ij}^\circ\right)^2, \tag{69}$$

where the last term can provide a penalty from deviating from an ideal labor share. We denote by κ the matrix with typical element κ_{ij}. This functional form takes as inputs the ideal shares α_{ij}°, the actual shares α_{ijt}, the coefficients κ_{ij} and the constant a_i°. The ideal shares α_{ij}° are set to the time average of the input shares observed in the data.[35] We set the constant a_i° equal to the average TFP of sector i. The coefficients κ_{ij}, which determine how costly it is to deviate from the ideal shares in terms of productivity, are estimated using the MSM procedure described below. Without any restrictions the matrix κ would have $n \times (n+1) = 1406$ elements. To reduce the number of free parameters to estimate, we restrict κ to be of the form $\kappa = \kappa^i \kappa^j$ where κ^i is an $n \times 1$ column vector and κ^j is an $1 \times (n+1)$ row vector. The kth element of κ^i then scales the cost for producer k of changing the share of any of its inputs, and the lth element in κ^j scales the cost of changing the share of input l for any producer. We normalize the first element in κ^i to pin down the scale of κ^i and κ^j. The matrix κ then contains only $2n = 74$ free parameters to estimate.

[35] We experimented with an alternative calibration in which we include and estimate a j-specific shifter to α_{ij}°. The results are similar to our baseline calibration.

Exogenous productivity process

The source of uncertainty in the model is the vector of productivity shocks $\varepsilon_t \sim \mathcal{N}(\mu_t, \Sigma_t)$. In the calibrated model, we allow μ_t and Σ_t to vary over time to account for changes in the stochastic process for ε_t over the sample period. To parameterize the evolution of μ_t and Σ_t, we first filter out the endogenous productivity shifter $A_i(\alpha_{it})$ and the normalization term $\zeta(\alpha_{it})$ from the measured sectoral TFP, $e^{\varepsilon_{it}} A_i(\alpha_{it}) \zeta(\alpha_{it})$, implied by the production function (1). We then estimate the evolution of μ_t and Σ_t from the remaining component. To do so, we assume that ε_t follows a random walk with drift,

$$\varepsilon_t = \gamma + \varepsilon_{t-1} + u_t, \tag{42}$$

where γ is an $n \times 1$ vector of deterministic drifts and $u_t \sim \text{iid } \mathcal{N}(0, \Sigma_t)$ is a vector of shocks. We estimate γ by computing the average of the productivity growth rates $\Delta\varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$ over time.

When making decisions in period t, firms know the past realizations of ε_t so that the conditional mean of ε_t is given by $\mu_t = \gamma + \varepsilon_{t-1}$. The covariance Σ_t of the innovation u_t is estimated using a rolling window that puts more weight on more recent observations to allow for time-varying uncertainty about sectoral productivity. Specifically, we estimate the covariance between sector i and j at time t by computing $\Sigma_{ijt} = \sum_{s=1}^{t-1} \phi^{t-s-1} u_{is} u_{js}$, where $0 < \phi < 1$ is a parameter that determines the relative weight of more recent observations. Its value is set to the sectoral average of the corresponding parameters of a GARCH(1,1) model estimated on each sector's productivity innovation u_{it}. In the calibrated economy, its value is $\phi = 0.47$. Note that this procedure implies that the time series for ε_t depends on the parameters of the TFP shifters. Therefore, the estimation of the stochastic process for sectoral productivity has to be done jointly with the estimation of κ.

Matching model and data moments

We use an indirect inference approach and estimate the parameters $\Theta \equiv \{\rho, \kappa\}$ by minimizing

$$\hat{\Theta} = \arg\min_{\Theta} \left(m(z) - m(\Theta)\right)^\top W \left(m(z) - m(\Theta)\right),$$

where $m(z)$ is a vector of moments computed from the data, and $m(\Theta)$ is the vector of corresponding model-implied moments conditional on the parameters Θ. The moments that we target are the time series of the production shares α_{ijt}, normalized by their average in the data, and the demeaned time series of aggregate consumption growth, normalized by the average of its absolute value in the data. We target consumption since the stochastic discount factor of the household is central to the trade-off that firms face when choosing production techniques.[36]

[36] To strike a balance between matching both the shares and consumption growth reasonably well, the weighting matrix W assigns a weight of $\left(n^2 \times T\right)^{-1}$ to the shares moments (recall that there are n^2 shares time series, each of length T) and a weight of $(T-1)^{-1}$ to the consumption growth moment (the length of the consumption growth time series is $T-1$).

We match $n^2 \times T + T - 1$ moments with only $2n+1$ free parameters. The model is thus strongly over-identified. We use particle swarm optimization to find the global minimizer $\hat{\Theta}$ (Kennedy and Eberhart, 1995). The estimated coefficient of relative risk aversion $\hat{\rho}$ is 4.27, which is similar to values used or estimated in the macroeconomics literature.

B.3 The calibrated economy

We want our model to fit key features of the data that relate to 1) the structure of the production network, 2) how the network responds to changes in beliefs, and 3) how this response affects macroeconomic aggregates. As we have seen earlier, the Domar weights, and how they react to changes in μ_t and Σ_t, play a central role for these mechanisms. In this section, we first describe the evolution of μ_t and Σ_t in the calibrated economy. We then report unconditional moments of the model-implied Domar weights and how they compare to the data. Finally, we look at the relationship between the Domar weights and the beliefs μ_t and Σ_t and verify that the correlations predicted by the mechanisms of the model are present in the data.

Evolution of beliefs in the data

Our estimation procedure provides a time-series for μ_t and Σ_t. To illustrate the overall evolution of beliefs over our sample period, we compute two measures that capture the aggregate impact of changes in μ_t and Σ_t. The first measure is the Domar-weighted average growth in the conditional mean of productivity, defined as

$$\Delta\bar{\mu}_t = \sum_{j=1}^{n} \omega_{jt} \Delta\mu_{jt}. \tag{70}$$

We use the Domar weights ω_{jt} in this equation to properly reflect the importance of a sector for GDP, as implied by (13). The solid blue line in Figure 5 shows the evolution of $\Delta\bar{\mu}_t$ over the sample period. As expected, $\Delta\bar{\mu}_t$ tends to go below zero during NBER recessions and is positive during expansions.

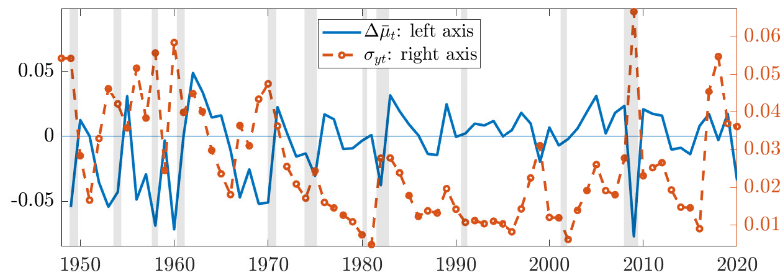
To describe how aggregate uncertainty evolves in the calibrated economy, we also compute the within-period perceived standard deviation of log GDP. From (14), this can be written as

$$\sigma_{yt} = \sqrt{\mathrm{V}[y]} = \sqrt{\omega_t^\top \Sigma_t \omega_t}. \tag{71}$$

The red dashed line in Figure 5 represents the evolution of σ_{yt} over the sample period. While uncertainty is on average relatively low, especially during the Great Moderation era, spikes are clearly visible in the earlier years and, in particular, during the Great Recession of 2007-2009.[37]

[37] σ_{yt} pertains only to uncertainty about the stochastic part of TFP ε. As such, it does not capture overall economic uncertainty, which might also be affected by changes in employment, investment, monetary and fiscal policy, etc.

Figure 5. Domar-weighted TFP and uncertainty changes



Notes: Solid blue line: Domar-weighted average growth in the conditional mean of productivity, $\Delta\bar{\mu}_t = \sum_{j=1}^{n} \omega_{jt}\Delta\mu_{jt}$. Red dashed line: Domar-weighted conditional variance of productivity, $\sigma_{yt} = \sqrt{\omega_t^\top \Sigma_t \omega_t}$. Shaded areas represent NBER recessions.

Unconditional Domar weights

Figure 6 shows the average Domar weight of each sector in the data (blue bars) and in the model (black line). The sectors with the highest Domar weights in the data are "Real estate", "Food and beverage", "Retail trade", "Finance and insurance" and "Health care". According to our theory (Corollary 4), changes in the expected level and variance of productivity in those sectors will have the largest effects on welfare.

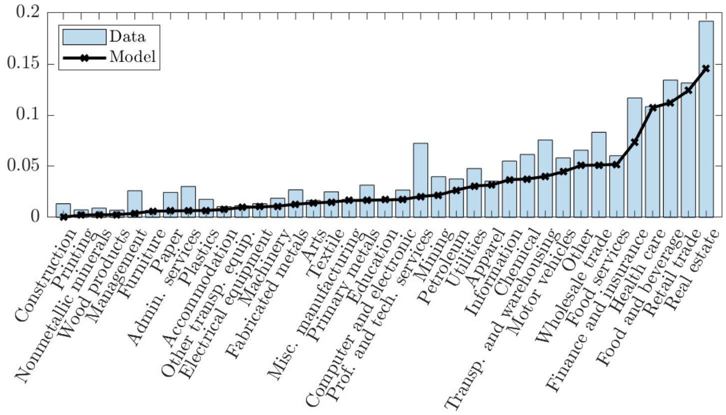
The cross-sectional correlation between the average Domar weights in the model and in the data is 0.96, so that the calibrated model fits this important feature of the production network well. However, the average Domar weight in the model (0.032) is lower than its counterpart in the data (0.047). This is because the estimation also targets aggregate consumption growth. Given the observed variation in TFP, if the model were to match the Domar weights perfectly, consumption would be too volatile compared to the data. Under our calibration, the volatility of consumption growth in the model is 2.73%, close to its data target of 2.65% (row (6) of Table IV).[38]

The model can account for about 40% of the observed average standard deviation of the Domar weights over time, as shown in row (2) of Table IV. Row (3) also reports that the coefficient of variation of the Domar weights in the model is 0.07 compared to 0.11 in the data. Once we take into account their relative scale, the model can thus account for a sizable portion of the variation in a key moment that characterizes the production network.[39]

[38]Since there is no investment and that the only primary factor of production (labor) is in fixed supply, consumption and aggregate TFP are equal in the model. It follows that we cannot match the volatility of both quantities and the model somewhat overpredicts TFP volatility (see Table IV). Including an investment margin in the model, so that GDP no longer equals consumption, might improve the fit of the Domar weights while keeping consumption growth in the model as volatile as in the data.

[39]One reason why the Domar weights are less volatile in the model than in the data is that we assume that the $\{A_i\}_{i=1}^{n}$ functions are time invariant. In reality, technological changes might affect the shape of these functions which would translate into additional variation in the Domar weights.

Figure 6. Sectoral Domar weights in the data and the model



Notes: The Domar weights are computed for each sector in each year and then averaged over all time periods.

TABLE IV. Domar weights, consumption and TFP in the model and in the data

	Statistic	Data	Model
(1)	Average Domar weight $\bar{\omega}_j$	0.047	0.032
(2)	Standard deviation $\sigma(\omega_j)$	0.0050	0.0021
(3)	Coefficient of variation $\sigma(\omega_j)/\bar{\omega}_j$	0.107	0.066
(4)	Corr (ω_{jt}, μ_{jt})	0.08	0.08
(5)	Corr $(\omega_{jt}, \Sigma_{jjt})$	−0.37	−0.31
(6)	Consumption growth volatility	2.65%	2.73%
(7)	TFP growth volatility	1.83%	2.73%

Notes: For each sector, we compute the time series of its Domar weight ω_{jt}, as well as its standard deviation $\sigma(\omega_j)$ and its mean $\bar{\omega}_j$. Rows (1) and (2) report cross-sectional averages of these statistics. Row (3) is the ratio of rows (2) and (1). Each period, we compute cross-sectional correlations of the Domar weights ω_{jt} with μ_{jt} and Σ_{jjt} (mean and variance of exogenous TFP ε_{jt}). Rows (4) and (5) report time-series averages of these correlations. Rows (6) and (7) compare consumption growth and TFP growth volatilities across the model and the data. The TFP data comes from Fernald (2014) and is not adjusted for capacity utilization.

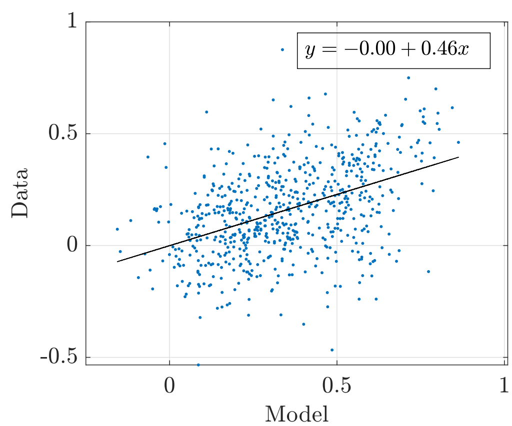
Domar weights and beliefs

One of the key mechanisms of the model predicts that a decline in the expected productivity of a sector, or an increase in its variance, should lead firms to reduce the importance of that sector as an input provider, leading to a decline in its Domar weight. Proposition 2 makes this point formally for a single change in μ_i or Σ_{ii}. Of course, in the data multiple changes in μ_t and Σ_t occur at the same time, and it would be difficult to isolate the impact of a single change on the Domar weights. Instead, we look at simple cross-sector correlations between the Domar weights ω_{it} and the first (μ_{it}) and the second moments (Σ_{iit}) of sectoral TFPs, both in the data and in the model. These correlations provide a straightforward, albeit noisy, measure of the interrelations between ω_t, μ_t and Σ_t. As can be seen in rows (4) and (5) of Table IV, the predictions of the model are borne out in the data. The model is thus able to capture well the impact of beliefs on the structure of the production network.

Sectoral correlations

The model is also able to replicate features of the correlation between sectoral outputs. We focus on growth rates to accommodate different trends in the data and in the model. For each pair of sectors, we compute the correlation in their output growth in the model and in the data, and plot them in Figure 7. The model reasonably captures cross-sectoral comovements: We find that the correlation between the data- and model-implied values is 0.44. On average, sectoral outputs are positively correlated in the model and in the data, although the model correlation is somewhat weaker on average (see the first column of Table V).

Figure 7. Cross-sector correlations in the model and in the data



Notes: For each pair of sectors, we compute correlations in the growth rates of sectoral output in the model and in the data. Each dot in the graph shows the value of this correlation in the model (X-axis) and in the data (Y-axis). The solid black line results from the ordinary least square analysis.

Table V also reports averages of these correlations during periods of low and high TFP growth and uncertainty growth, as measured by (70) and (71). We see that in the data these correlations are lower in good times, when TFP growth is high and uncertainty growth is low. The model is able to replicate this ranking. Intuitively, in bad times consumption is low and so the household is particularly worried about bad shocks. To avoid them, firms rely more on the most stable producers. As firms are mostly purchasing from the same sectors, sectoral outputs become more correlated.

B.4 Counterfactual exercises

In this appendix we provide more information about the counterfactual exercises of Section 8.2.

TABLE V. Correlations in sectoral sales growth

	All years	TFP growth		Uncertainty growth	
		Low	High	Low	High
Model	0.18	0.22	0.13	0.16	0.20
Data	0.36	0.37	0.34	0.32	0.38

Notes: For each sector pair (i,j), we compute correlations in the growth rates of sectoral output in the model and in the data. We then take averages across all sectors. TFP growth and Uncertainty growth are measured as in Figure 5. We use high/low to refer to years with TFP growth or uncertainty growth above/below corresponding median levels.

Long-run moments

Table VI provides differences in long-run moments between our baseline model and the three alternative economies described in the main text. In the "known ε_t", $\mathrm{E}[y]$ and $\mathcal{W}$ collapse to realized GDP and $\mathrm{V}[y] = 0$. In Table VI, we compute instead these moments *before* ε_t is known but still assuming that the production network is chosen optimally for the realized draw of ε_t

TABLE VI. Uncertainty, GDP and welfare in the post-war sample

	Baseline model compared to...		
	Fixed network	as if $\Sigma_t = 0$	Known ε_t
Expected log GDP, $\mathrm{E}[y]$	+2.12%	−0.01%	+0.68%
Expected st. dev. of log GDP, $\sqrt{\mathrm{V}[y]}$	+0.13%	−0.10%	−0.22%
Expected welfare, $\mathcal{W}$	+2.11%	+0.01%	+0.71%
Realized log GDP, y	+1.61%	+0.07%	−0.54%

Notes: Baseline variables minus their counterparts in the "fixed network", the "as if $\Sigma_t = 0$", the "known ε_t" alternatives.

Time series under the "known ε_t" alternative economy

In the "known ε_t", beliefs (μ_t, Σ_t), and in particular uncertainty, play no role in shaping the network and, from the planner's problem, the optimal network is simply the one that maximizes (realized) consumption. It follows that consumption (or GDP) is always larger than in the baseline model (bottom right panel in Figure 8).[40] Unsurprisingly, the difference is particularly pronounced during episodes of high uncertainty, when knowing ε_t provides a larger advantage, and reaches a high of 3% during the Great Recession. On average, GDP is 0.54% larger than in the baseline economy suggesting a sizable impact of uncertainty on the economy (bottom row in Table VI).

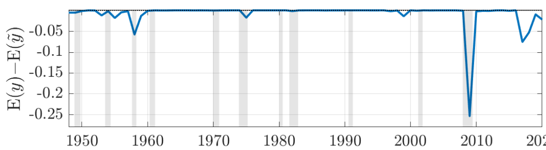
The top three panels in the right column of Figure 8 show how the baseline and alternative economies differ in terms of expected log GDP, the standard deviation of log GDP, and (expected)

[40] Again, here we report the moments *before* ε_t is known but still assuming that the production network is chosen optimally for the realized draw of ε_t.

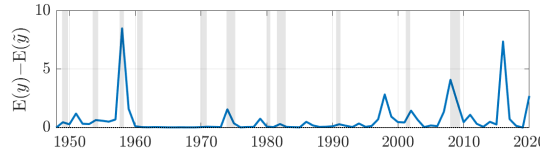
welfare. Crucially, these measures are evaluated *before* ε is realized.[41] Welfare $\mathcal{W}$ is always lower in the alternative economy because, by construction, $\mathcal{W}$ is what the network in the baseline model maximizes. Furthermore, the optimal network in this economy does not seek to increase $\mathrm{E}[y]$ and reduce $\mathrm{V}[y]$. As a result, $\mathrm{E}[y]$ is on average lower and $\mathrm{V}[y]$ is on average higher (right column in Table VI).

Figure 8. The role of uncertainty in the postwar period

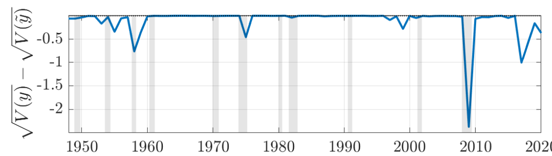
Left column: the "as if $\Sigma_t = 0$" alternative | Right column: the "known ε_t" alternative



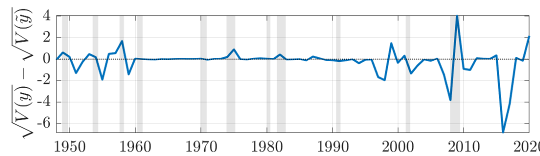
(a) Difference in expected log GDP [%]



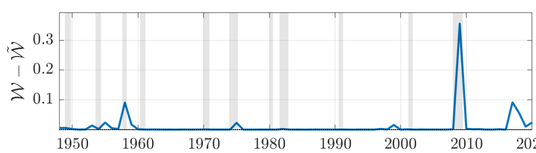
(b) Difference in expected log GDP [%]



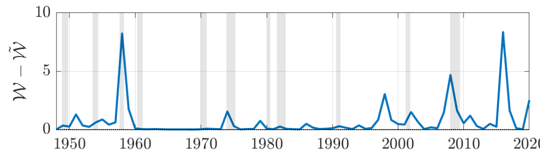
(c) Difference in expected st. dev. of log GDP [%]



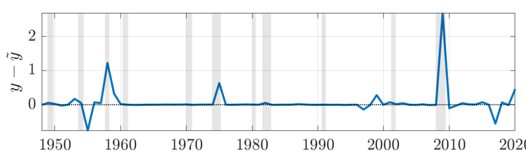
(d) Difference in expected st. dev. of log GDP [%]



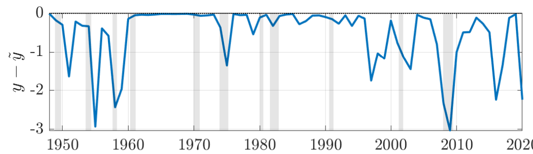
(e) Difference in expected welfare [%]



(f) Difference in expected welfare [%]



(g) Difference in realized log GDP [%]



(h) Difference in realized log GDP [%]

Notes: The differences between the series implied by the baseline model (without tildes) and the two alternatives (marked by tildes): the "as if $\Sigma_t = 0$" alternative (left column) and the "known ε_t" alternative (right column). All economies are hit by the same shocks that are filtered out from the TFP data under our baseline model. All differences are expressed in percentage terms. Expected log GDP $\mathrm{E}[y]$ and expected standard deviation of log GDP $\sqrt{\mathrm{V}[y]}$ are evaluated before ε_t is realized.

[41] Note that *realized* welfare in this economy is simply equal to realized log GDP.