SUPPLEMENT TO “DYNAMIC CONCERN FOR MISSPECIFICATION”

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1. MISSING PROOFS

Proof of Lemma 2. Fix $a \in A$ and define $\bar{u} = \max_{y \in Y} u(a,y) - \min_{y \in Y} u(a,y)$. For every $n \in \mathbb{N}$, $\min_{p_a \in \Delta(Y)} \left(\mathbb{E}_{p_a}[u(a,y)] + \frac{R(p_a \| q_{a,n})}{\lambda_n} \right) \subseteq [\min_{y \in Y} u(a,y), \max_{y \in Y} u(a,y)]$, so possibly restricting to a subsequence, I can assume that the limit in the LHS of the statement is well defined. The statement is then proved by showing that any such subsequence converges to the RHS. In particular, I show that it is impossible to have

$$\lim_{n \to \infty} \min_{p_a \in \Delta(Y)} \left(\mathbb{E}_{p_a}[u(a,y)] + \frac{R(p_a \| q_{a,n})}{\lambda_n} \right) < \mathbb{E}_{q_a}[u(a,y)]. \tag{1}$$

This is sufficient as $\lim_{n \to \infty} \mathbb{E}_{q_{a,n}}[u(a,y)] = \mathbb{E}_{q_a}[u(a,y)]$ and $\left(\mathbb{E}_{q_{a,n}}[u(a,y)]\right)_{n \in \mathbb{N}}$ is a sequence pointwise larger than the sequence whose limit is taken on the LHS. If equation (1) held, there would be an $\varepsilon \in \mathbb{R}_{++}$ with

$$\lim_{n \to \infty} \min_{p_a \in \Delta(Y)} \left(\mathbb{E}_{p_a}[u(a,y)] + \frac{R(p_a \| q_{a,n})}{\lambda_n} \right) = \mathbb{E}_{q_a}[u(a,y)] - \varepsilon. \tag{2}$$

For every $n \in \mathbb{N}$, let $p_a^n \in \Delta(Y)$ be an arbitrary element of

$$\operatorname{argmin}_{p_a \in \Delta(Y)} \left(\mathbb{E}_{p_a}[u(a,y)] + \frac{R(p_a \| q_{a,n})}{\lambda_n} \right).$$

Since Y is a compact metric space so is $\Delta(Y)$, and therefore, I can assume (by restricting to a subsequence) that p_a^n converges to some $\hat{p}_a \in \Delta(Y)$. By equation (2) and the fact that $\lim_{n \to \infty} p_a^n = \hat{p}_a$, $\mathbb{E}_{\hat{p}_a}[u(a,y)] \le \mathbb{E}_{q_a}[u(a,y)] - \varepsilon$. Therefore,

$$\begin{aligned}
&\int_0^{\bar{u}} 1 - \hat{p}_a \left(\left\{ y \in Y : u(a,y) - \min_{\bar{y} \in Y} u(a,\bar{y}) \le x \right\} \right) \mathrm{d}x + \frac{3}{4}\varepsilon = \mathbb{E}_{\hat{p}_a}[u(a,y)] + \frac{3}{4}\varepsilon - \min_{\bar{y} \in Y} u(a,\bar{y}) \\
&\le \mathbb{E}_{q_a}[u(a,y)] - \min_{\bar{y} \in Y} u(a,\bar{y}) = \int_0^{\bar{u}} 1 - q_a \left(\left\{ y \in Y : u(a,y) - \min_{\bar{y} \in Y} u(a,\bar{y}) \le x \right\} \right) \mathrm{d}x.
\end{aligned} \tag{3}$$

CLAIM 7: *There exist $M \in \mathbb{R}$ and $L \in \mathbb{R}_{++}$ such that*

$$\hat{p}_a(\{y \in Y : u(a,y) \le M - L\}) - q_a(\{y \in Y : u(a,y) \le M\}) \ge \frac{\varepsilon}{2\bar{u}}. \tag{4}$$

Proof of the Claim. Suppose that for every $M \in \mathbb{R}$ and $L \in \mathbb{R}_{++}$ equation (4) does not hold. Then for every $L \in \mathbb{R}_{++}$

$$\int_0^{\bar{u}} 1 - \hat{p}_a \left(\left\{ y \in Y : u(a,y) - \min_{\bar{y} \in Y} u(a,\bar{y}) \le x \right\} \right) \mathrm{d}x$$

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$$\geq \int_0^{\bar{u}+L} 1 - q_a\left(\left\{y \in Y : u(a,y) - \min_{\bar{y}\in Y} u(a,\bar{y}) \leq x\right\}\right) - \frac{\varepsilon}{2\bar{u}}\mathrm{d}x - L$$
$$= \int_0^{\bar{u}} 1 - q_a\left(\left\{y \in Y : u(a,y) - \min_{\bar{y}\in Y} u(a,\bar{y}) \leq x\right\}\right)\mathrm{d}x - \varepsilon/2 - L\frac{\varepsilon}{2\bar{u}} - L.$$

Since L is arbitrarily small,

$$\int_0^{\bar{u}} 1 - \hat{p}_a\left(\left\{y \in Y : u(a,y) - \min_{\bar{y}\in Y} u(a,\bar{y}) \leq x\right\}\right)\mathrm{d}x$$
$$\geq \int_0^{\bar{u}} 1 - q_a\left(\left\{y \in Y : u(a,y) - \min_{\bar{y}\in Y} u(a,\bar{y}) \leq x\right\}\right)\mathrm{d}x - \frac{\varepsilon}{2},$$

a contradiction with equation (3). □

The claim, in turn, implies that there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$p_a^n\left(\left\{y \in Y : u(a,y) \leq M - \frac{L}{2}\right\}\right) - q_{a,n}\left(\left\{y \in Y : u(a,y) \leq M - \frac{L}{2}\right\}\right) \geq \frac{\varepsilon}{4\bar{u}}.$$

Then by Theorem 1.24 in Liese and Vajda (1987)

$$\min_{p_a\in\Delta(Y)} \mathbb{E}_{p_a}[u(a,y)] + \frac{1}{\lambda_n}R(p_a||q_{a,n})$$
$$\geq \min_{y\in Y} u(a,y) + \left(p_a^n\left(\left\{y \in Y : u(a,y) \leq M - \frac{L}{2}\right\}\right)\log\frac{p_a^n\left(\left\{y \in Y : u(a,y) \leq M - \frac{L}{2}\right\}\right)}{q_{a,n}\left(\left\{y \in Y : u(a,y) \leq M - \frac{L}{2}\right\}\right)}\right)/\lambda_n$$
$$+\left(p_a^n\left(\left\{y \in Y : u(a,y) > M - \frac{L}{2}\right\}\right)\log\frac{p_a^n\left(\left\{y \in Y : u(a,y) > M - \frac{L}{2}\right\}\right)}{q_{a,n}\left(\left\{y \in Y : u(a,y) > M - \frac{L}{2}\right\}\right)}\right)/\lambda_n.$$

But, the last term diverges to $+\infty$, a contradiction with $\min_{p_a\in\Delta(Y)} \mathbb{E}_{p_a}[u(a,y)] + \frac{R(p_a||q_{a,n})}{\lambda_n} \leq \max_{y\in Y} u(a,y) < \infty$. ■

Proof of Lemma 5. Recall that weak and vague convergence for probability measures are equivalent when the state space is compact, and that Y is indeed compact. Therefore, by Assumption 1 the assumptions of Theorem 15.7.3 in Kallenberg (1973) are satisfied for the sequence of integrand functions and probability measures $(\log(\tilde{q}_{a,n}), p_a^n)_{n\in\mathbb{N}}$. ■

Proof of Lemma 7. Observe that by Assumption 1

$$LLR(h_t, Q) = -\log\left(\frac{\max_{q\in Q}\prod_{\tau=1}^{t}\tilde{q}_{a_\tau}(y_\tau)}{\max_{r\in N(Q)}\prod_{\tau=1}^{t}\tilde{r}_{a_\tau}(y_\tau)}\right) = \log\left(\frac{\max_{r\in N(Q)}\prod_{\tau=1}^{t}\tilde{r}_{a_\tau}(y_\tau)}{\max_{q\in Q}\prod_{\tau=1}^{t}\tilde{q}_{a_\tau}(y_\tau)}\right)$$

$$= \log\left(\frac{\prod_{\tau=1}^{t}\tilde{r}_{a_\tau}(y_\tau)}{\prod_{\tau=1}^{t}\tilde{q}'_{a_\tau}(y_\tau)}\right) = \log\left(\prod_{\tau=1}^{t}\tilde{r}_{a_\tau}(y_\tau)\right) - \log\left(\prod_{\tau=1}^{t}\tilde{q}'_{a_\tau}(y_\tau)\right)$$

$$= \log\left(\prod_{a\in A}\prod_{y\in\operatorname{supp} p_a^{h_t}}\tilde{r}_a(y)^{\sum_{\tau=1}^{t}\mathbb{I}_{\{a\}}(a_\tau)p_a^{h_t}(\{y\})}\right) - \log\left(\prod_{a\in A}\prod_{y\in\operatorname{supp} p_a^{h_t}}\tilde{q}'_a(y)^{\sum_{\tau=1}^{t}\mathbb{I}_{\{a\}}(a_\tau)p_a^{h_t}(\{y\})}\right)$$

$$= \sum_{a\in A}\sum_{\tau=1}^{t}\mathbb{I}_{\{a\}}(a_\tau)\sum_{y\in\operatorname{supp} p_a^{h_t}} p_a^{h_t}(\{y\})\log(\tilde{r}_a(y)) - \sum_{a\in A}\sum_{\tau=1}^{t}\mathbb{I}_{\{a\}}(a_\tau)\sum_{y\in\operatorname{supp} p_a^{h_t}} p_a^{h_t}(\{y\})\log(\tilde{q}'_a(y))$$

$$= \sum_{a\in A}\sum_{\tau=1}^{t}\mathbb{I}_{\{a\}}(a_\tau)\sum_{y\in\operatorname{supp} p_a^{h_t}} p_a^{h_t}(\{y\})\left(\log(\tilde{r}_a(y)) - \log(\tilde{q}'_a(y))\right)$$

$$= \sum_{a\in A}\sum_{\tau=1}^{t}\mathbb{I}_{\{a\}}(a_\tau)\int_Y \log\left(\frac{\tilde{r}_a(y)}{\tilde{q}'_a(y)}\right)\mathrm{d}p_a^{h_t}(y).$$

■

Proof of Lemma 8. I first show that the function

$$\begin{array}{ll}\Delta(A)\times Q & \to \mathbb{R} \\ (\alpha, q) & \mapsto \sum_{a\in A}\alpha(a)\,R(p_a^*\|q_a)/c\end{array} \tag{5}$$

is continuous. Fix an $a \in A$ and let $(q_n)_{n\in\mathbb{N}} \in Q^{\mathbb{N}}$ be a sequence that converges to $q \in Q$. By Assumption 1 (ii), $\tilde{q}_{a,n}(y)$ is converging to $\tilde{q}_a(y)$ for p_a^* almost every y. Then

$$|R(p_a^*\|q_{a,n}) - R(p_a^*\|q_a)| = \left|\int_Y \log\left(\frac{\tilde{q}_a(y)}{\tilde{q}_{a,n}(y)}\right)\mathrm{d}p_a^*(y)\right|$$

and observe that the integrand on the right-hand side is dominated by a constant by Assumption 1 (i). Therefore, by the dominated convergence theorem $|R(p_a^*\|q_{a,n}) - R(p_a^*\|q_a)|$ converges to 0. Since A is finite and the function in equation (5) is linear in α, I have obtained the desired continuity. With this, the statement follows from the Maximum Theorem.

2. AXIOMATIZATION

The agent evaluates simple acts, i.e., measurable and finite ranged maps from a state space S into the set of simple probability measures $X = \Delta(Z)$ over a set of prizes Z, where S is endowed with a σ-algebra of events Σ. The set of those acts is denoted as $\mathcal{F}$. Given any $x \in X$, $x \in \mathcal{F}$ is the act that delivers x in every state, and in this way, I identify X as the subset of constant acts in $\mathcal{F}$. If $f, g \in \mathcal{F}$, and $E \in \Sigma$, I denote as gEf the simple act that yields $g(s)$ if $s \in E$ and $f(s)$ if $s \notin E$. Since X is convex, for every $f, g \in \mathcal{F}$, and $\gamma \in (0,1)$, I denote as $\gamma f + (1-\gamma)g \in \mathcal{F}$ the simple act that pays $\gamma f(s) + (1-\gamma)g(s)$ for all $s \in S$. Denote as $\Delta(S)$ the space of probability distributions endowed with the topology of setwise convergence (with respect to the measurable sets in Σ).

I model the agent's preference with a binary relation $\succsim$ on $\mathcal{F}$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$. An event E is *null* if $fEh \sim gEh$ for every $f, g, h \in \mathcal{F}$.

An event is *nonnull* if it is not null. For every $E\in\Sigma$, the conditional preference relation $\succsim_E$ is defined by $f\succsim_E g$ if $fEh\succsim gEh$ for some $h\in\mathcal{F}$. An event is *strongly nonnull* if for every $x,x'\in X$ with $x\succ x'$, $x\succ x'Ex$. Let $B_0(\Sigma)$ denote the set of all real-valued Σ-measurable simple functions endowed with the supnorm. The subset of functions in $B_0(\Sigma)$ that take values in $C\subseteq\mathbb{R}$ is denoted as $B_0(\Sigma,C)$. A functional $I:\Phi\to\mathbb{R}$ defined on a nonempty subset Φ of $B_0(\Sigma)$ is a *niveloid* if for every $\varphi,\psi\in\Phi$, $I(\varphi)-I(\psi)\leq\sup(\varphi-\psi)$. It is *translation invariant* if $I(\alpha\varphi+(1-\alpha)k\mathbb{I}_S)=I(\alpha\varphi)+(1-\alpha)k$ for all $\alpha\in[0,1]$, $\varphi\in\Phi$, and $k\in\mathbb{R}$ such that $\alpha\varphi+(1-\alpha)k\mathbb{I}_S$ and $\alpha\varphi$ are in Φ. A niveloid is *normalized* if $I(k\mathbb{I}_S)=k$ for all $k\in\mathbb{R}$ such that $k\mathbb{I}_S\in\Phi$. A function $c:\Delta(S)\to\mathbb{R}_+$ is *grounded* if $c^{-1}(0)\neq\emptyset$.

When formalized in terms of a binary relation, the average robust control decision criterion reads as follows.

DEFINITION 8: *A tuple (u,Q,μ,λ) is an* average robust control representation *of the preference relation $\succsim$ if $u:X\to\mathbb{R}$ is a nonconstant affine function, $Q\subseteq\Delta(S)$ is a nonempty set, $\mu\in\Delta(Q)$, $\lambda\geq 0$, and for all $f,g\in\mathcal{F}$*

$$f\succsim g\Longleftrightarrow\mathbb{E}_\mu\left[\min_{p\in\Delta(S)}\left(\int_S u(f)\,dp+\frac{R(p||q)}{\lambda}\right)\right]\geq\mathbb{E}_\mu\left[\min_{p\in\Delta(S)}\left(\int_S u(g)\,dp+\frac{R(p||q)}{\lambda}\right)\right].\tag{6}$$

The average robust control representation is the counterpart of (1) when expressed over acts. An apparent difference is that u here takes outcomes as input only instead of pairs of actions and consequences. However, this discrepancy is inconsequential, as in Section 2, I can define a larger space of consequences $\hat{Y}=A\times Y$ that includes both actions and outcomes and transforming each model $p\in\Delta(Y)^A$ into an element of $\hat{p}\in\Delta\left(\hat{Y}\right)^A$ such that $\hat{p}_a(a',y)=0$ if $a'\neq a$ and $\hat{p}_a(a,y)=p_a(y)$ for all $y\in Y$. Still, this embedding of actions into outcomes muddles the interpretation of the learning results significantly. Therefore I opted to maintain the distinction explicit.

The first axiomatic step is a static one. I characterize in terms of behavioral axioms an agent that evaluates according to equation (6) the acts whose consequences are obtained in the same period and before any new information is received.

AXIOM 1—Variational Axiom: ***Weak Order.***
Weak Certainty Independence. *If $f,g\in\mathcal{F}$, $x,x'\in X$, $\gamma\in(0,1)$, and $\gamma f+(1-\gamma)x\succsim\gamma g+(1-\gamma)x$, then $\gamma f+(1-\gamma)x'\succsim\gamma g+(1-\gamma)x'$.*
Continuity. *If $f,g,h\in\mathcal{F}$ the sets $\{\gamma\in[0,1]:\gamma f+(1-\gamma)g\succsim h\}$ and $\{\gamma\in[0,1]:h\succsim\gamma f+(1-\gamma)g\}$ are closed.*
Monotonicity. *If $f,g\in\mathcal{F}$, and $f(s)\succsim g(s)$ for all $s\in S$, then $f\succsim g$.*
Uncertainty Aversion. *If $f,g\in\mathcal{F}$, $\gamma\in(0,1)$, and $f\sim g$, then $g+\gamma(f-g)\succsim f$.*
Nondegeneracy. *$f\succ g$ for some $f,g\in\mathcal{F}$.*
Weak Monotone Continuity. *If $f,g\in\mathcal{F}$, $x\in X$, $(E_n)_{n\in\mathbb{N}}\in\Sigma^{\mathbb{N}}$ with $f\succ g$, $E_1\supseteq E_2\supseteq\ldots$ and $\cap_{n\in\mathbb{N}}E_n=\emptyset$, then there exists $n_0\in\mathbb{N}$ such that $xE_{n_0}f\succ g$.*

Maccheroni, Marinacci, and Rustichini (2006) show that Axiom 1 characterizes the class of variational preferences and discuss the axiom content.

2.0.0.1. *Structured Preferences* I consider agents who face two levels of uncertainty: the uncertainty on the best structured description of the DGP and whether each description is exact.

A representation is structured if it allows separating these two layers. In particular, to achieve this separation, I consider a state space S that admits the decomposition $S = \Omega \times \Delta(\Omega)$ for some finite Ω endowed with its Borel sigma-algebra.

DEFINITION 9: *An average robust control representation* (u, Q, μ, λ) *is* structured *if* Q *is finite and there exists* $\left(\rho_q\right)_{q \in Q}$ *such that for every* $q \in Q$*, and* $\omega \in \Omega$*,* $\rho_q \in \Delta(\Omega)$ *and* $q\left(\left\{\omega, \rho_q\right\}\right) = \rho_q(\omega)$.

The interpretation of a structured representation is that the state space can be factored in two components, the realization of the single period consequence $\omega \in \Omega$ and a component $\rho \in \Delta(\Omega)$ that pins down the distribution over states each period. An event E is *structured* if $E = \Omega \times B$ for some $B \in \mathcal{B}(\Delta(\Omega))$. The sigma-algebra generated by the structured events is denoted as Σ_s.[1]

I say that an event $E \subseteq S$ satisfies the sure-thing principle if, for all $f, g, h, h' \in \mathcal{F}$, $fEh \succsim gEh$ implies $fEh' \succsim gEh'$. I denote by Σ_{st} the set of events that satisfy the sure-thing principle.

AXIOM 2—Structured Savage: *(i) There is a finite set* $E \subseteq S$ *such that* $S \setminus E$ *is null. (ii)* ***P2.*** $\Sigma_s \subseteq \Sigma_{st}$. *(iii)* ***P4.*** *If* $E, E' \in \Sigma_s$ *and* $x, y, w, z \in X$ *are such that* $x \succ y$ *and* $w \succ z$*, then*

$$xEy \succ xE'y \Rightarrow wEz \succ wE'z.$$

Structured Savage requires that (i) the agent posits a finite number of models and (ii) guarantees that when evaluating acts that only depend on the identity of the structured model, the agent satisfies the Sure-Thing Principle. It also (iii) guarantees that when an agent faces alternatives whose outcomes depend only on whether the DGP belongs to two sets of models, their choices consistently reveal the one deemed more likely.

AXIOM 3—Intramodel Sure-Thing Principle: *For every* $f, g, h, h' \in \mathcal{F}$,

$$fWh \succsim_\rho gWh \implies fWh' \succsim_\rho gWh' \qquad \forall W \subseteq \Omega, \forall \rho \in \Delta(\Omega).$$

Structured Savage's P2 and the Intramodel STP imply that bets *between* models and preference over acts *within* a model satisfy the STP. However, they admit violations of the STP for acts whose payoff depends on both the model's identity and the outcome realization within the model, as the ones of the original Ellsberg's paradox.

The case I study is when the relative likelihood of the structured models is only captured by the belief μ. In particular, the agent is equally concerned about how much each model departs from the actual DGP.

AXIOM 4—Uniform Misspecification Concern: *For every* $\rho, \rho' \in \Delta(\Omega)$ *and* $f, g \in \mathcal{F}$ *such that*

$$\rho(\{\omega : f(\omega, \rho) = y\}) = \rho'(\{\omega : g(\omega, \rho') = y\}) \quad \forall y \in X$$

and $\Omega \times \{\rho\}$, $\Omega \times \{\rho'\}$ *are nonnull,*

$$f \succsim_\rho x \Longleftrightarrow g \succsim_{\rho'} x \qquad \forall x \in X.$$

[1]With a slight abuse of notation for every $B \in \mathcal{B}(\Delta(\Omega))$ and $W \subseteq \Omega$ I denote as $\succsim_B$ and $\succsim_W$ the binary relations $\succsim_{\Omega \times B}$ and $\succsim_{W \times \Delta(\Omega)}$ and I write fBg and fWg for $f(\Omega \times B)g$ and $f(W \times \Delta(\Omega))g$.

This axiom requires that if acts f and g induce identical outcome distributions under ρ and ρ', they are compared with a safe alternative in the same way conditional on the best-fitting model being revealed to be ρ or ρ'.

DEFINITION 10: *The state space is adequate if: (i) there exist $k \in (0,1)$ and $(W_\rho)_{\rho\in\Delta(\Omega)} \in (2^\Omega)^{\Delta(\Omega)}$ such that for all $\rho \in \Delta(\Omega)$ with $\Omega \times \{\rho\}$ nonnull, $\rho(W_\rho) = k$, (ii) for every $\omega, \omega' \in \Omega$, and $\rho \in \Delta(\Omega)$ such that $\{\omega\} \times \{\rho\}$ and $\{\omega'\} \times \{\rho\}$ are nonnull, $\rho(\omega) = \rho(\omega')$.*

All the agent's structured models have an event with the same probability and are uniform over a model-specific set of outcomes. It is well-known that equal probability requirements are essential for probabilistic sophistication with respect to a finite measure over states to have a bite.

AXIOM 5—Uncertainty Neutrality Over Models: *Let $x, y, w, z \in X$, $\rho \in \Delta(\Omega)$, and $\gamma \in (0,1)$. Then $[\gamma x + (1-\gamma) y]_\rho w \sim y_\rho z$ if and only if $x_\rho w \sim [(1-\gamma) x + \gamma y]_\rho z$.*

Uncertainty Neutrality over Models guarantees that at the level of bets over models, the agent is "risk-neutral", as changing the performance under ρ by $(x-y)\gamma$ has an impact that does not depend on the level of utility under that model. It is immediate from the proof of Theorem 4 that if dropped, it leads to a more general representation with a nonlinear utility index U over the performance of each robust control model.

THEOREM 4: *Suppose that S is adequate, there at least three disjoint nonnull events in Σ_s, and every nonnull $E \in \Sigma_s$ contains at least three disjoint nonnull events. The following are equivalent:*
1. $\succsim$ admits a structured average robust control representation (u, Q, μ, λ);
2. $\succsim$ satisfies Axioms 1-5.
Moreover, in this case, every two structured average robust control representations share the same μ.

Proof of Theorem 4. (Only if) That $\succsim$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, Nondegeneracy, and Weak Monotone Continuity follows from Lemma 9.

Let $\rho \in \Delta(\Omega)$, $W \subseteq \Omega$, $f, g, h, h' \in \mathcal{F}$, and $fWh \succsim_\rho gWh$. If $\Omega \times \{\rho\}$ is null then trivially $fWh' \succsim_\rho gWh'$. Therefore, suppose $\Omega \times \{\rho\}$ is nonnull. By Lemma 12, and since $q \mapsto \rho_q$ is injective, there exists $\bar{q} \in \Delta(S)$ with $\rho_{\bar{q}} = \rho$, $\mu(\{\bar{q}\}) > 0$, and $\min_{p\in\Delta(S)} \int_S u(fWh)\mathrm{d}p + \frac{1}{\lambda} R(p\|\bar{q}) \geq \min_{p\in\Delta(S)} \int_S u(gWh)\mathrm{d}p + \frac{1}{\lambda} R(p\|\bar{q})$. By Proposition 1.4.2 in Dupuis and Ellis (2011) this is equivalent to $\int_S \phi(u(fWh))\mathrm{d}\bar{q} \geq \int_S \phi(u(gWh))\mathrm{d}\bar{q}$ with $\phi(\cdot) = -\exp(-\lambda(\cdot))$. This is also equivalent to $\int_{W\times\Delta(\Omega)} \phi(u(f))\mathrm{d}\bar{q} \geq \int_{W\times\Delta(\Omega)} \phi(u(g))\mathrm{d}\bar{q}$. But then, by reversing all the steps with h' in place of h I get $fWh' \succsim_\rho gWh'$ and therefore $\succsim$ satisfies Intramodel Sure-Thing Principle.

Moreover, $\succsim$ satisfies Uniform Misspecification Concern by Lemma 13. That there is a finite set $B \subseteq \Delta(\Omega)$ such that $\Omega \times (\Delta(\Omega) \setminus B)$ is null immediately follows from the representation and part 2 of Lemma 12. Let $\Omega \times B \in \Sigma_s$ and $f, g, h, h' \in \mathcal{F}$. If $\Omega \times B$ is null, clearly $\Omega \times B \in \Sigma_{st}$. Suppose $\Omega \times B$ is nonnull, then

$$f(\Omega \times B)h \succsim g(\Omega \times B)h$$
$$\iff$$

$$\int_{\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(f\right)\right]+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right) \quad \geq \quad \int_{\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(g\right)\right]+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right)$$

$$\Longleftrightarrow$$

$$f\left(\Omega\times B\right)h' \quad \succsim \quad g\left(\Omega\times B\right)h'$$

where the two equivalences follow by Lemma 12. This proves that $\Omega\times B\in\Sigma_{st}$. Since B was chosen to be an arbitrary measurable subset of $\Delta\left(\Omega\right)$, $\Sigma_s\subseteq\Sigma_{st}$, and Structured Savage P2 holds.

That $\succsim$ satisfies Structured Savage P4 and Uncertainty Neutrality over Models immediately follows from Lemma 12 and the representation.

(If) By Structured Savage's P2, $\Sigma_s\subseteq\Sigma_{st}$. Suppose $E\in\Sigma_s$ is nonnull, and let $x,x'\in X$ with $x\succ x'$. Then there exist $f,g,h\in\mathcal{F}$ such that $fEh\succ gEh$. Since f and g are simple acts, they assume finitely many values, and by Weak Order, there exist $\bar{x},\underline{x}\in X$ with $\bar{x}\succsim f\left(s\right),g\left(s\right)\succsim\underline{x}$ for all $s\in E$. Since $E\in\Sigma_s\subseteq\Sigma_{st}$, $fE\bar{x}\succ gE\bar{x}$. By the Monotonicity and Weak Order parts of the Variational Axiom, $\underline{x}\emptyset\bar{x}=\bar{x}E\bar{x}\succsim fE\bar{x}\succ gE\bar{x}\succsim\underline{x}E\bar{x}$. Therefore, by Structured Savage P4, $x=x'\emptyset x\succ x'Ex$. Since $E\in\Sigma_s$ and $x,x'\in X$ were arbitrarily chosen, each nonnull $E\in\Sigma_s$ is also strongly nonnull.

Next, fix a finite $B\subseteq\Delta\left(\Omega\right)$, such that for each $\rho\in B$, $\Omega\times\{\rho\}$ is nonnull, and such that $S\setminus\{\Omega\times B\}$ is null. Such a set exists by the Structured Savage axiom, and the cardinality of B is at least 3 by assumption of the theorem. For every $\rho\in B$, by the previous part of the proof $\Omega\times\{\rho\}$ is strongly nonnull and so by Lemma 11,

$$f\succsim_\rho g\Longleftrightarrow\min_{p\in\Delta(S)}\int_S\hat{u}\left(f\right)\mathrm{d}p+\frac{1}{\lambda_\rho}R\left(p||q_\rho\right) \tag{7}$$

for some $q_\rho\in\Delta\left(S\right)$ with support contained in $\Omega\times\{\rho\}$ and a nonconstant affine $\hat{u}$.

CLAIM 8: $q_\rho=\rho\times\delta_\rho$.

Proof. Since the space is adequate, there exists $v_\rho\in(0,1)$ such that $\rho\left(\omega\right)\in\{0,v_\rho\}$. In particular, by applying Uniform Misspecification Concern with $\rho=\rho'$, I obtain that $q_\rho\left(\omega,\rho\right)=v_\rho\iff\rho\left(\omega\right)=v_\rho$, and the desired conclusion follows. □

Let $Q=\{q_\rho\in\Delta(S):\rho\in B\}$. Identify each act $f\in\mathcal{F}$ with the real-valued function $\hat{f}:Q\to\hat{u}\left(X\right)$ with $\hat{f}\left(q_\rho\right)=\min_{p\in\Delta(S)}\int_S\hat{u}\left(f\right)\mathrm{d}p+\frac{1}{\lambda_\rho}R\left(p||q_\rho\right)$ for all $\rho\in B$ where λ_ρ is given by equation (7).

I now show that

$$\hat{f}=\hat{g}\implies f\sim g \qquad \forall f,g\in\mathcal{F}.$$

I partition S in $\left\{\{\Omega\times\rho\}_{\rho\in B},S\setminus\{\Omega\times B\}\right\}$ and establish the claim by induction on the number of cells of the partition on which f and g are not identical. Let f and g be such that $\hat{f}=\hat{g}$ and they differ on one element of the partition, say E. Then $f=fEg\sim g$ by definition of $\sim_E$ and Structured Savage P2, so $f\sim g$. For the inductive step, suppose that whenever f and g are such that $\hat{f}=\hat{g}$ and they differ at most on $n\in\mathbb{N}$ elements of the partition, $f\sim g$. Let f and g be such that $\hat{f}=\hat{g}$ and they differ on $n+1\in\mathbb{N}$ elements of the partition. Let E be an element of the partition on which they differ. Then, fEg and g differ on one element of the partition, and fEg and f differ on n elements of the partition. Therefore, by the inductive hypothesis, $g\sim fEg\sim f$.

Moreover, it is immediate to see that $\hat{u}(X)^Q \subseteq \left\{\hat{f} : f \in \mathcal{F}\right\}$. Therefore, with a slight abuse of notation I let $\succsim$ denote also the binary relation on $\hat{u}(X)^Q$ defined by $\hat{f} \succsim \hat{g}$ if and only if $f \succsim g$.

CLAIM 9: *For every* $v, v', w, z \in \hat{u}(X)$*,* $\rho \in B$*, and* $\gamma \in (0,1)$

$$v_\rho w \succsim (\gamma v + (1-\gamma) v')_\rho z \iff ((1-\gamma) v + \gamma v')_\rho w \succsim v'_\rho z.$$

Proof. If $v = v'$ the equivalence is obvious. Suppose without loss of generality that $v' > v$.

1. Let $v_\rho w \succsim (\gamma v + (1-\gamma) v')_\rho z$. By Monotonicity, Structured Savage, and since $\Omega \times \{\rho\}$ is strongly nonnull, this implies that $w > z$. By Continuity, Structured Savage, and since $\Omega \times \{\rho\}$ is strongly nonnull there is $\alpha \in [0,1]$ with $v_\rho (\alpha w + (1-\alpha) z) \sim (\gamma v + (1-\gamma) v')_\rho z$. By Uncertainty Neutrality over Models, $((1-\gamma) v + \gamma v')_\rho (\alpha w + (1-\alpha) z) \sim v'_\rho z$. By Monotonicity, this implies that $((1-\gamma) v + \gamma v')_\rho w \succsim v'_\rho z$.

2. Let $((1-\gamma) v + \gamma v')_\rho w \succsim v'_\rho z$. By Monotonicity, Structured Savage, and since $\Omega \times \{\rho\}$ is strongly nonnull, this implies that $w > z$. Then, by Continuity, Structured Savage, and the fact that $\Omega \times \{\rho\}$ is strongly nonnull there exists $\alpha \in [0,1]$ with $((1-\gamma) v + \gamma v')_\rho (\alpha w + (1-\alpha) z) \sim v'_\rho z$. By Uncertainty Neutrality over Models, this implies that $v_\rho (\alpha w + (1-\alpha) z) \sim (\gamma v + (1-\gamma) v')_\rho z$. By Monotonicity, this implies that $v_\rho w \succsim (\gamma v + (1-\gamma) v')_\rho z$. □

By the previous claim, Continuity, Structured Savage, and Theorem VII.3.5 in Wakker (2013) there exists $\mu \in \Delta(Q)$ such that for all $\psi, \psi' \in \hat{u}(X)^Q$

$$\psi \succsim \psi' \iff \sum_{q \in Q} \psi(q) \mu(q) \geq \sum_{q \in Q} \psi'(q) \mu(q).$$

Moreover, by Observation VII.3.5 in Wakker (2013), μ is uniquely identified. But then, by definition of $\succsim$, I obtain that for all $f, g \in \mathcal{F}$

$$f \succsim g \iff \mathbb{E}_\mu \left[\min_{p \in \Delta(S)} \mathbb{E}_p [u(f)] + \frac{R(p||q_\rho)}{\lambda_\rho} \right] \geq \mathbb{E}_\mu \left[\min_{p \in \Delta(S)} \mathbb{E}_p [u(g)] + \frac{R(p||q_\rho)}{\lambda_\rho} \right].$$

Moreover, by Lemma 13 Uniform Misspecification Concern implies that $\lambda = \lambda_\rho$ for all $\rho \in B$, proving the result. ■

The theorem characterizes the representation (u, Q, μ, λ) with probabilistic uncertainty *about* the model (Structured Savage), probabilistic sophistication *given* a model (Intramodel Sure-Thing Principle), and *incomplete trust* in any model (Uncertainty Aversion).

I next provide axioms that characterize the dynamic adjustment of preferences in the face of information. In particular, I look at joint axioms on a collection of history-dependent binary relations $(\succsim^h)_{h \in \mathcal{H}}$ indexed by the realized history. Recall that the relevant set of length $t \in \mathbb{N}$ histories for structured preferences is Ω^t.

AXIOM 6—Constant Preference Invariance: *For every* $x, x' \in X$ *and* $h \in \mathcal{H}$*,*

$$x \succsim^h x' \Leftrightarrow x \succsim^\emptyset x'.$$

This axiom captures the fact that I am not considering the problem of an agent discovering their taste. The preferences over state-independent alternatives are fixed and do not react to new information.

AXIOM 7—Dynamic Consistency over Models: *Let $f,g \in \mathcal{F}$ be Σ_s-measurable, $t \in \mathbb{N}$, $\omega^t \in \Omega^t$ and $\bar{z}, \underline{z} \in X$ be such that $\bar{z} \succsim f(s) \succsim \underline{z}$ and $\bar{z} \succsim g(s) \succsim \underline{z}$ for all $s \in S$. Let*

$$h^{\omega^t}(\omega,\rho) = \gamma_{h(\omega,\rho)} \prod_{i=1}^{t} \rho(\omega_i)\, \bar{z} + \left(1 - \gamma_{h(\omega,\rho)} \prod_{i=1}^{t} \rho(\omega_i)\right) \underline{z} \qquad \forall (\omega,\rho) \in S, \forall h \in \{f,g\}$$

where $\gamma_{h(\omega,\rho)}$ satisfies $h(\omega,\rho) \sim \bar{z}\gamma_{h(\omega,\rho)} + \left(1 - \gamma_{h(\omega,\rho)}\right) \underline{z}$. Then,

$$f \succsim^{\omega^t} g \Longleftrightarrow f^{\omega^t} \succsim g^{\omega^t}.$$

The second dynamic axiom requires Bayesian rationality when considering acts whose consequences only depend on the structured model. Formally, it requires that when comparing acts that only bet on the identity of the model, at a given history, I can reduce the comparison to acts evaluated ex-ante. To do so, the payoff conditional to each model must be scaled proportionally to the amount of evidence that has been generated in favor of that model.[2]

To single out the *quantitative* speed at which the concern for misspecification is adjusted, I need a quantitative measure of similarity. For every $x, y \in X$ with $x \succ y$ and $E \in \Sigma$ let $\boldsymbol{\gamma}_{\succsim}^{xEy}$ be defined by

$$\boldsymbol{\gamma}_{\succsim}^{xEy} x + \left(1 - \boldsymbol{\gamma}_{\succsim}^{xEy}\right) y \sim xEy.$$

That is, $\boldsymbol{\gamma}_{\succsim}^{xEy}$ is the weight to alternative x in the certain equivalent to act xEy. It captures both the confidence in event E and the attitudes towards uncertainty. It is easy to see that under the Variational Axiom $\boldsymbol{\gamma}_{\succsim}^{xEy}$ always exists and is unique.

For every $x, y \in X$, $E \in \Sigma$, $\varepsilon \in (0,1)$, and $\succsim$ and $\succsim'$ that satisfy the Variational Axiom, I say that $\succsim$ is (x,y,E,ε)-*similar* to $\succsim'$ if $\left|\boldsymbol{\gamma}_{\succsim}^{xEy} - \boldsymbol{\gamma}_{\succsim'}^{xEy}\right| \leq \varepsilon$. That is, the certain equivalent of the binary act xEy is ε close under preferences $\succsim$ and $\succsim'$.

AXIOM 8—Asymptotic Frequentism: *For every $\rho \in \Delta(S)$, $x, y \in X$ with $x \succ^{\emptyset} y$, $\varepsilon > 0$, and $E \in \Sigma$ there is $\tau \in \mathbb{N}$ and $\varepsilon' > 0$ such that if $\min\{t, t'\} \geq \tau$ and $h_t, h_{t'}$ have outcome frequencies that are ε' close to ρ then $\succsim^{h_t}$ is (x,y,E,ε)-similar to $\succsim^{h_{t'}}$.*

The axiom requires that for every binary act xEy, a sufficiently long sequence of outcomes with similar empirical frequency stabilizes the certain equivalent.

PROPOSITION 3: *Suppose that: (i) For every $h \in \mathcal{H}$, $\succsim^h$ satisfies the axioms of Theorem 4 and (ii) $(\succsim^h)_{h \in \mathcal{H}}$ satisfies Constant Preference Invariance, Dynamic Consistency over Models, and Asymptotic Frequentism. Then for every $h \in \mathcal{H}$, $\succsim^h$ admits an average robust control representation $(u, Q, \mu(\cdot|h), \lambda_h)$ and for every sequence $(h_t)_{t \in \mathbb{N}}$ with outcome frequency converging to some $\rho \notin \{\rho_q : q \in Q\}$,*

$$\lim_{t \to \infty} \lambda_{h_t} / \left(\frac{LLR(h_t, Q)}{t}\right) \tag{8}$$

[2]This axiom can lead to implications beyond our average robust control decision criterion, as it implies Bayesian updating for each decision criterion that performs an average of model-specific evaluations. In this way, it would complement the elegant theory of subjective learning developed in Dillenberger, Lleras, Sadowski, and Takeoka (2014), which does not require that the analyst observes the same information as the agent.

exists. Moreover, if for some $q \in Q$, $x \succ^{\emptyset} y$, *and* $E \subseteq \Omega$ *with* $\rho_q(E) > 0$

$$\liminf_{t\to\infty} \gamma_{\succsim^{h_t}}^{x(E\times\{\rho_q\})y} > 0,$$

the limit is finite, and if

$$\limsup_{t\to\infty} \gamma_{\succsim^{h_t}}^{x(E\times\{\rho_q\})y} < \rho_q(E)\,\mu(q),$$

it is strictly positive.

Proof of Proposition 3. By Proposition 5, $\succsim^h$ admits an average robust control representation $(u, Q, \mu(\cdot|h), \lambda_h)$ for every $h \in \mathcal{H}$. Observe that since the outcome frequency is converging along the sequence $(h_t)_{t\in\mathbb{N}}$, by Lemma 7, $\lim_{t\to\infty} \frac{LLR(h_t,Q)}{t} = 1/c$ for some $c \in \mathbb{R}_{++}$. Suppose by way of contradiction that

$$l:\ = \liminf_{n\to\infty} c\lambda^{h_t} = \liminf_{t\to\infty} \frac{\lambda_{h_t}}{\frac{LLR(h_t,Q)}{t}} < \limsup_{t\to\infty} \frac{\lambda_{h_t}}{\frac{LLR(h_t,Q)}{t}} = \limsup_{t\to\infty} c\lambda_{h_t} =:\ L.$$

Let $\bar{q} \in Q$ be such that $\Omega \times \{\rho_{\bar{q}}\}$ is nonnull and so that $\bar{q} \in \min_{q\in Q} R(\rho||q)$. Since $\Omega \times \{\rho_{\bar{q}}\}$ contains at least three nonnull events, by Lemma 12, there is $E \subseteq W$ and $r \in (0,1)$ with $\rho_{\bar{q}}(E) = r$. Let $x, z \in X$, $\gamma^*, \gamma_* \in (0,1)$, and $\lambda^*, \lambda_* \in \left(\frac{l}{c}, \frac{L}{c}\right)$ be such that $x \succ^{\emptyset} z$, $\lambda^* > \lambda_*$,

$$\frac{-\mu(\bar{q})\log(r\exp(-\lambda^*(u(z))) + (1-r)\exp(-\lambda^*(u(x))))}{\mu\left(\min_{q\in Q} R(\rho||q)\right)\lambda^*} + \left(1 - \frac{\mu(\bar{q})}{\mu\left(\min_{q\in Q} R(\rho||q)\right)}\right) u(z)$$
$$= u(\gamma^* x + (1-\gamma^*) z),$$

and

$$\frac{-\mu(\bar{q})\log(r\exp(-\lambda_*(u(z))) + (1-r)\exp(-\lambda_*(u(x))))}{\mu\left(\min_{q\in Q} R(\rho||q)\right)\lambda_*} + \left(1 - \frac{\mu(\bar{q})}{\mu\left(\min_{q\in Q} R(\rho||q)\right)}\right) u(z)$$
$$= u(\gamma_* x + (1-\gamma_*) z),$$

where the existence of such γ_*, γ^* is guaranteed by u being affine. Moreover, it is easy to see that $\gamma_* > \gamma^*$. Consider a subsequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} c\lambda^{h_{t_n}} = l$. Moreover, let $M \in \mathbb{N}$ be such that for all $n \geq M$

$$\lambda^{h_{t_n}} < \frac{\lambda_* + \frac{l}{c}}{2}.$$

Similarly, let $(t_{\tilde{n}})_{\tilde{n}\in\mathbb{N}}$ such that $\lim_{\tilde{n}\to\infty} c\lambda^{h_{t_{\tilde{n}}}} = L$. Moreover, let $\tilde{M} \in \mathbb{N}$ be such that for all $\tilde{n} \geq \tilde{M}$

$$\lambda^{h_{t_{\tilde{n}}}} > \frac{\lambda^* + \frac{L}{c}}{2}.$$

With this, by Proposition 5 and Proposition 1.4.2 in Dupuis and Ellis (2011), for all $n \geq M$ and $\tilde{n} \geq \tilde{M}$

$$\gamma_{\succsim^{h_{t_n}}}^{x(E\times\{\rho_{\bar{q}}\})z} > \gamma_* \text{ and } \gamma_{\succsim^{h_{t_{\tilde{n}}}}}^{x(E\times\{\rho_{\bar{q}}\})z} < \gamma^*.$$

But this in turn implies that $\succsim^{h_{t_n}}$ is never $\left(x, y, \left(E \times \{\rho_{\bar{q}}\}\right), (\gamma_* - \gamma^*)\right)$-similar to $\succsim_{h_{t_{\tilde{m}}}}$ for

$$\min\{n, \tilde{n}\} \geq \max\left\{M, \tilde{M}\right\},$$

a contradiction. This shows that either λ_{h_t} converges or it diverges to plus infinity. The last part of the statement immediately by taking E in the first part of the proof to be equal to the one whose existence is asserted in the statement, and by the construction of γ_* and γ^* above. ■

2.1. *Ancillary Lemmas for the Representation*

LEMMA 9: *Suppose that there exist a nonconstant affine function $u : X \to \mathbb{R}$, a nonempty and finite $Q \subseteq \Delta(S)$, $\mu \in \Delta(Q)$, and $(\lambda_q)_{q\in Q} \in \mathbb{R}_+^Q$ such that for all $f, g \in \mathcal{F}$*

$$f \succsim g \Longleftrightarrow \mathbb{E}_\mu\left[\min_{p\in\Delta(S)} \mathbb{E}_p[u(f)] + \frac{R(p||q)}{\lambda_q}\right] \geq \mathbb{E}_\mu\left[\min_{p\in\Delta(S)} \mathbb{E}_p[u(g)] + \frac{R(p||q)}{\lambda_q}\right]. \quad (9)$$

Then $\succsim$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, Nondegeneracy, Weak Monotone Continuity, and admits the representation

$$f \succsim g \Longleftrightarrow \min_{p\in\Delta(S)} \int_S \hat{u}(f)\, dp + \hat{c}(p) \geq \min_{p\in\Delta(S)} \int_S \hat{u}(g)\, dp + \hat{c}(p) \quad (10)$$

for some nonconstant affine $\hat{u} : X \to \mathbb{R}$ and a grounded, convex, and lower semicontinuous function $\hat{c} : \Delta(S) \to [0, \infty]$. Moreover, I can choose $\hat{u} = u$ and $\hat{c}$ is such that $\hat{c}^{-1}(0) = \mathbb{E}_\mu[q]$.

Proof. I first observe that without loss of generality I can take u to be such that $0 \in \operatorname{int} u(X)$ in the representation of equation (9). Indeed, since u is nonconstant and affine, there exists $x \in X$ with $u(x) \in \operatorname{int} u(X)$. Define $u'(y) = u(y) - u(x)$ for all $y \in X$. Then,

$$\begin{aligned} f \succsim g &\Longleftrightarrow \mathbb{E}_\mu\left[\min_{p\in\Delta(S)} \mathbb{E}_p[u(f)] + \frac{R(p||q)}{\lambda_q}\right] \geq \mathbb{E}_\mu\left[\min_{p\in\Delta(S)} \mathbb{E}_p[u(g)] + \frac{R(p||q)}{\lambda_q}\right] \\ &\Longleftrightarrow \mathbb{E}_\mu\left[\min_{p\in\Delta(S)} \mathbb{E}_p[u'(f)] + \frac{R(p||q)}{\lambda_q}\right] \geq \mathbb{E}_\mu\left[\min_{p\in\Delta(S)} \mathbb{E}_p[u'(g)] + \frac{R(p||q)}{\lambda_q}\right] \end{aligned}$$

and $0 \in \operatorname{int} u'(X)$.

Fix $q \in Q$. The functional $I_q : B_0(\Sigma, u(X)) \to \mathbb{R}$ defined as

$$I_q(\varphi) := \min_{p\in\Delta(S)} \int_S \varphi(s)\, \mathrm{d}p + \frac{1}{\lambda_q} R(p||q) \qquad \forall \varphi \in B_0(\Sigma, u(X))$$

is easily seen to be monotone, translation invariant, and concave by the concavity of the minimum. Since Q is finite,

$$\hat{I}(\varphi) := \int_Q I_q(\varphi)\, \mathrm{d}\mu(q) \qquad \forall \varphi \in B_0(\Sigma, u(X))$$

is well-defined and $\hat{I}$ is monotone, concave, and represents $\succsim$.[3] Let $\varphi \in B_0(\Sigma, u(X))$, $k \in u(X)$, and $\gamma \in (0,1)$. Since u is affine, X is convex, and $0 \in \operatorname{int} u(X)$, $\gamma\varphi + (1-\gamma)k \in B_0(\Sigma, u(X))$, $\gamma\varphi \in B_0(\Sigma, u(X))$, and

$$\hat{I}(\gamma\varphi + (1-\gamma)k) = \int_Q I_q(\gamma\varphi + (1-\gamma)k)\,\mathrm{d}\mu(q) = \int_Q I_q(\gamma\varphi) + (1-\gamma)k\mathrm{d}\mu(q) = \hat{I}(\gamma\varphi) + (1-\gamma)k.$$

But then, notice that

$$\int_Q \left(\min_{p\in\Delta(S)} \int_S u(f)\,\mathrm{d}p + \frac{1}{\lambda_q} R(p\|q) \right) \mathrm{d}\mu(q) = \int_Q I_q(u(f))\,\mathrm{d}\mu(q) = \hat{I}(u(f))$$

where $\hat{I}$ is monotone and translation invariant. Therefore, by Lemma 25 in Maccheroni, Marinacci, and Rustichini (2006), $\hat{I}$ is a concave niveloid, and it is clearly normalized. With this, by Lemma 28 and Footnote 15 in Maccheroni, Marinacci, and Rustichini (2006) $\succsim$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, and Nondegeneracy.

Fix $f, g \in \mathcal{F}$, $x \in X$, and $(E_i)_{i\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $E_1 \supseteq E_2 \supseteq \ldots$, $\cap_{i\geq 1} E_i = \emptyset$, and $f \succ g$. Then, for all $q \in Q$, $\lim_{i\to\infty} q(E_i) = 0$ for all $i \in \mathbb{N}$ and by Proposition 1.4.2 in Dupuis and Ellis (2011)

$$-e^{-\lambda_q\left(I_q\left(u(x)\mathbb{I}_{E_i} + u(f)\mathbb{I}_{S\setminus E_i}\right)\right)} = -\int_{S\setminus E_i} \exp(-\lambda_q u(f(s)))\,\mathrm{d}q(s) - \int_{E_i} \exp(-\lambda_q u(x))\,\mathrm{d}q(s).$$

But then

$$\lim_{i\to\infty} -\exp\left(-\lambda_q\left(I_q\left(u(x)\mathbb{I}_{E_i} + u(f)\mathbb{I}_{S\setminus E_i}\right)\right)\right) = \int_S -e^{-\lambda_q u(f(s))}\mathrm{d}q(s)$$

that is $\lim_{i\to\infty} I_q\left(u(x)\mathbb{I}_{E_i} + u(f)\mathbb{I}_{S\setminus E_i}\right) = \frac{-\log\left(\int_S \exp\left(-\lambda_q u(f(s))\right)\mathrm{d}q(s)\right)}{\lambda_q} = I_q(u(f))$. Since the statement holds for every q in the finite Q and $\hat{I}(u(g)) < \hat{I}(u(f)) = \int_Q I_q(u(f))\mathrm{d}\mu(q)$ there exists $i \in \mathbb{N}$ such that $\hat{I}\left(u(x)\mathbb{I}_{E_i} + u(f)\mathbb{I}_{S\setminus E_i}\right) > \hat{I}(u(g))$ proving that $\succsim$ satisfies Weak Monotone Continuity. Thus, by Theorem 3 and Lemma 30 in Maccheroni, Marinacci, and Rustichini (2006) it admits the representation in equation (10).

By the first part of the lemma, $u(x) \geq u(x') \iff x \succsim x' \iff \hat{u}(x) \geq \hat{u}(x')$ and therefore, by the uniqueness up to a positive affine transformation of $\hat{u}$ guaranteed by Corollary 5 in Maccheroni, Marinacci, and Rustichini (2006) and the fact that every two affine functions that represent $\succsim$ on X are positive affine transformations of each other, I can choose $u = \hat{u}$. Finally, by (ii) $\Longrightarrow$ (iii) of Lemma 32 in Maccheroni, Marinacci, and Rustichini (2006) for every $q \in Q$, and $k \in u(X)$, $\partial I_q(k) = \{q\}$. Let $\bar{k} \in \operatorname{int} u(X) \neq \emptyset$ and observe that since Q is finite, $\lim_{\alpha\downarrow 0} \frac{\hat{I}(\bar{k}+\alpha\varphi) - \hat{I}(\bar{k})}{\alpha} = \lim_{\alpha\downarrow 0} \mathbb{E}_\mu\left[\frac{I_q(\bar{k}+\alpha\varphi) - I_q(\bar{k})}{\alpha}\right] = \mathbb{E}_\mu\left[\int_S \varphi\mathrm{d}q\right]$. Now, applying (iii) $\Longrightarrow$ (ii) of Lemma 32 in Maccheroni, Marinacci, and Rustichini (2006), I obtain that the unique $\hat{c}$ identified by the choice of $\hat{u}$ has $\hat{c}^{-1}(0) = \{\mathbb{E}_\mu[q]\}$. ■

LEMMA 10: *If $E \in \Sigma_{st}$ is strongly nonnull and $\succsim$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, and Weak Monotone Continuity,*

[3] Here, represents is slightly abused to mean that $f \succsim g$ if and only if $\hat{I}(u(f)) \geq \hat{I}(u(g))$.

then $\succsim_E$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, Nondegeneracy, and Weak Monotone Continuity.

Proof. Let $f,g,h \in \mathcal{F}$. By Completeness of $\succsim$ at least one between $fEh \succsim gEh$ and $gEh \succsim fEh$ holds. Therefore, by definition of $\succsim_E$ at least one between $f \succsim_E g$ and $g \succsim_E f$ holds.

Let $f,f',f'' \in \mathcal{F}$, with $f \succsim_E f'$ and $f' \succsim_E f''$. By definition of $\succsim_E$, there exist $h',h'' \in \mathcal{F}$ such that $fEh' \succsim f'Eh'$ and $f'Eh'' \succsim f''Eh''$. Since $E \in \Sigma_{st}$, $fEh'' \succsim f'Eh''$. By Transitivity of $\succsim$, $fEh'' \succsim f''Eh''$, and so by definition of $\succsim_E$, $f \succsim_E f''$.

Let $f,g \in \mathcal{F}$, $x,x' \in X$, and $\gamma \in (0,1)$, be such that $\gamma f + (1-\gamma)\,x \succsim_E \gamma g + (1-\gamma)\,x$. Since $E \in \Sigma_{st}$, $(\gamma f + (1-\gamma)\,x)\,Ex \succsim (\gamma g + (1-\gamma)\,x)\,Ex$. By Weak Certainty Independence of $\succsim$ I get $(\gamma f + (1-\gamma)\,x')\,E\,(\gamma x + (1-\gamma)\,x') \succsim (\gamma g + (1-\gamma)\,x')\,E\,(\gamma x + (1-\gamma)\,x')$. But then by definition of $\succsim_E$, $\gamma f + (1-\gamma)\,x' \succsim_E \gamma g + (1-\gamma)\,x'$, proving that $\succsim_E$ satisfies Weak Certainty Independence.

Let $f,g,h,h' \in \mathcal{F}$. Since $E \in \Sigma_{st}$,

$$\{\gamma \in [0,1] : \gamma f + (1-\gamma)\,g \succsim_E h\} = \{\gamma \in [0,1] : \gamma fEh' + (1-\gamma)\,gEh' \succsim hEh'\}$$

and

$$\{\gamma \in [0,1] : h \succsim_E \gamma f + (1-\gamma)\,g\} = \{\gamma \in [0,1] : hEh' \succsim \gamma fEh' + (1-\gamma)\,gEh'\}$$

where the sets on the RHSs are closed by Continuity of $\succsim$, proving that $\succsim_E$ satisfies Continuity.

Let $f,g,h \in \mathcal{F}$ and $f(s) \succsim_E g(s)$ for all $s \in S$. For every $s \in S$, since E is strongly nonnull, it is impossible to have $g(s) \succ f(s)$, as otherwise it would hold that $g(s) \succ f(s)\,Eg(s)$, a contradiction with $f(s) \succsim_E g(s)$. Then, $fEh \succsim gEh$ by Monotonicity of $\succsim$. Therefore, by definition of $\succsim_E$, $f \succsim_E g$ and so $\succsim_E$ satisfies Monotonicity.

Let $f,g,h \in \mathcal{F}$, $\gamma \in (0,1)$ and $f \sim_E g$. Since $E \in \Sigma_{st}$, $fEh \sim gEh$ and by Uncertainty Aversion, $(\gamma f + (1-\gamma)\,g)\,Eh = \gamma fEh + (1-\gamma)\,gEh \succsim fEh$. By definition of $\succsim_E$, this implies that $\gamma f + (1-\gamma)\,g \succsim_E f$ and so $\succsim_E$ satisfies Uncertainty Aversion.

Since E is nonnull, there exist $f,g,h \in \mathcal{F}$ such that $fEh \succ gEh$. But then, since $E \in \Sigma_{st}$, there is no $h' \in \mathcal{F}$ with $gEh' \succsim fEh'$. Therefore, by definition of $\succsim_E$, $f \succ_E g$ and $\succsim_E$ satisfies Nondegeneracy.

Let $f,g,h \in \mathcal{F}$, $x \in X$, $(E_i)_{i\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $E_1 \supseteq E_2 \supseteq \ldots$ and $\cap_{n\geq 1} E_n = \emptyset$, and $f \succ_E g$. Since $E \in \Sigma_{st}$, $fEh \succ gEh$. Moreover, $(E_i')_{i\in\mathbb{N}}$ where $E_i' = E_i \cap E$ is such that $E_1' \supseteq E_2' \supseteq \ldots$ and $\cap_{n\geq 1} E_n' \subseteq \cap_{n\geq 1} E_n = \emptyset$. Then $(xE_i'f)\,Eh = xE_i'\,(fEh)$ for all $i \in \mathbb{N}$ and by Weak Monotone Continuity and the fact that $fEh \succ gEh$ there exists $n_0 \in \mathbb{N}$ such that $\left(xE_{n_0}'f\right)Eh \succ gEh$. But notice that $(xE_{n_0}f)\,Eh = \left(xE_{n_0}'f\right)Eh \succ gEh$ and therefore $xE_{n_0}f \succ_E g$, as $E \in \Sigma_{st}$. ■

LEMMA 11: *Let $\Omega \times \{\rho\} \in \Sigma_{st}$ be strongly nonnull and contain at least three disjoint nonnull events, and suppose $\succsim$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, Nondegeneracy, Weak Monotone Continuity, the Intramodel Sure-Thing Principle, and Structured Savage. For every $f,g \in \mathcal{F}$,*

$$f \succsim_\rho g \Longleftrightarrow \min_{q\in\Delta(S)} \mathbb{E}_q[u_\rho(f)] + \frac{R(q||p_\rho)}{\lambda_\rho} \geq \min_{q\in\Delta(S)} \mathbb{E}_q[u_\rho(g)] + \frac{R(q||p_\rho)}{\lambda_\rho} \tag{11}$$

where u_ρ is a nonconstant affine function, $\lambda_\rho \in \mathbb{R}_+$, and $p_\rho \in \Delta(S)$. Moreover, u_ρ can be chosen to be the same for all such ρ and $\operatorname{supp} p_\rho \subseteq \Omega \times \{\rho\}$.

Proof. By Lemma 10 $\succsim_\rho$ satisfies Weak Order, Weak Certainty Independence, Continuity, Monotonicity, Uncertainty Aversion, Nondegeneracy, and Weak Monotone Continuity. I now show that for every $f,g,h,\bar{h} \in \mathcal{F}$ and $E \in \Sigma$, $fEh \succsim_\rho gEh \implies fE\bar{h} \succsim_\rho gE\bar{h}$. Observe that by definition of $\succsim_\rho$, $fEh \succsim_\rho gEh$ implies that there exists $\hat{h} \in \mathcal{F}$ such that $(fEh)\,\rho\hat{h} \succsim (gEh)\,\rho\hat{h}$. But then, there exists $h' \in \mathcal{F}$ such that

$$
\begin{aligned}
& (fEh)\,\rho\hat{h} \succsim (gEh)\,\rho\hat{h} \\
\implies & (f\{(\omega,\rho):(\omega,\rho)\in E\}\,h)\,\rho\hat{h} \succsim (g\{(\omega,\rho):(\omega,\rho)\in E\}\,h)\,\rho\hat{h} \\
\implies & (f\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,h)\,\rho\hat{h} \succsim (g\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,h)\,\rho\hat{h} \\
\implies & (f\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,h) \succsim_\rho (g\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,h) \\
\implies & \left(f\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,\bar{h}\right) \succsim_\rho \left(g\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,\bar{h}\right) \\
\implies & \left(f\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,\bar{h}\right)\rho h' \succsim \left(g\{(\omega,\rho'):\rho'\in\Delta(\Omega),(\omega,\rho)\in E\}\,\bar{h}\right)\rho h' \\
\implies & \left(f\{(\omega,\rho):(\omega,\rho)\in E\}\,\bar{h}\right)\rho h' \succsim \left(g\{(\omega,\rho):(\omega,\rho)\in E\}\,\bar{h}\right)\rho h' \implies fE\bar{h} \succsim_\rho gE\bar{h}
\end{aligned}
$$

where the third, fifth, and seventh implications follow from the definition of $\succsim_\rho$, the fourth implication follows from the Intramodel Sure-Thing Principle, and the other implications only rewrite the acts involved.

Next, observe that if $E \subseteq \Omega \times \{\rho\}$ is nonnull, then there exist $f,g,h \in \mathcal{F}$ with $(fEh)\,\rho h = fEh \succ gEh = (gEh)\,\rho h$. By Structured Savage P2, this implies that $fEh \succ_\rho gEh$, so that E is nonnull for the preference $\succsim_\rho$. With this, the first part follows from Theorem 1 in Strzalecki (2011). For the second part, notice that by Theorem 3 and Lemma 30 in Maccheroni, Marinacci, and Rustichini (2006), $\succsim$ admits a variational representation:

$$
f \succsim g \iff \min_{p\in\Delta(S)}\left(\int u(f)\,\mathrm{d}p + c(p)\right) \geq \min_{p\in\Delta(S)}\left(\int u(g)\,\mathrm{d}p + c(p)\right) \tag{12}
$$

for some nonconstant affine $u : X \to \mathbb{R}$ and a lower semicontinuous and grounded function $c : \Delta(S) \to [0,\infty]$.

Next, notice that $\succsim$ and $\succsim_\rho$ coincide on X. Indeed, let $x \succ x'$. Since $\Omega \times \{\rho\}$ is strongly nonnull $x \succ x'\rho x$ and given that $\Omega \times \{\rho\} \in \Sigma_{st}$ it follows that $x \succ_\rho x'$. Conversely, let $x \succsim x'$, then by equation (12) $u(x) \geq u(x')$. Since c is grounded, there exists $q^* \in \Delta(S)$ with $c(q^*) = 0$. But then

$$
u(x) \geq \min_{q\in\Delta(S)} \left(u(x')\,q(\Omega\times\{\rho\}) + (1 - q(\Omega\times\{\rho\}))\,u(x) + c(q)\right)
$$

that is, $x(\Omega\times\{\rho\})\,x \succsim x'(\Omega\times\{\rho\})\,x$, and $x \succsim_\rho x'$. Therefore, by the uniqueness up to a positive affine transformation of u guaranteed by Corollary 5 in Maccheroni, Marinacci, and Rustichini (2006) and the fact that every two affine functions that represent $\succsim$ on X are positive affine transformations of each other, I can choose $u = u_\rho$. Suppose by way of contradiction that there exists $E \in \Sigma$ such that $E \cap (\Omega\times\{\rho\}) = \emptyset$ and $p_\rho(E) > 0$. Let $x, y \in X$ with $x \succ y$. Then, $u(x) > u(y)\,p_\rho(E) + u(x)\,(1 - p_\rho(E)) \geq \min_{q\in\Delta(S)} \int u(yEx)\,\mathrm{d}q + \frac{1}{\lambda_\rho} R(q\|p_\rho)$ and so by equation (11), $x \succ_\rho yEx$. But since $x = x(\Omega\times\{\rho\})\,x$, $x = (yEx)(\Omega\times\{\rho\})\,x$ and $\Omega\times\{\rho\} \in \Sigma_{st}$ this would imply $x \succ x$, a contradiction to the Weak Order of $\succsim$. ■

LEMMA 12: *Suppose that the assumptions of Theorem 4 hold. Let* $\succsim$ *be such that for all* $f,g\in\mathcal{F}$, $f\succsim g\Longleftrightarrow\mathbb{E}_{\mu}\left[\min_{p\in\Delta(S)}\mathbb{E}_{p}\left[u\left(f\right)\right]+\frac{R(p||q)}{\lambda_q}\right]\geq\mathbb{E}_{\mu}\left[\min_{p\in\Delta(S)}\mathbb{E}_{p}\left[u\left(g\right)\right]+\frac{R(p||q)}{\lambda_q}\right]$ *where* $u:X\rightarrow\mathbb{R}$ *is a nonconstant affine function,* $Q\subseteq\Delta\left(S\right)$ *is a finite and nonempty set such that*

$$q\left(\left\{\omega,\rho_q\right\}\right)=\rho_q\left(\omega\right)\qquad\forall q\in Q,\forall\omega\in\Omega,\tag{13}$$

for some $\rho_q\in\Delta\left(\Omega\right)$, $\mu\in\Delta\left(Q\right)$, *and* $\left(\lambda_q\right)_{q\in Q}\in\mathbb{R}_{+}^{Q}$. *Then:*
1. *For every* $\Omega\times B\in\Sigma_s$ *and* $f,h\in\mathcal{F}$,

$$\begin{aligned}&\int_Q\min_{p\in\Delta(S)}\int_S u\left(f_{\Omega\times B}h\right)dp+\frac{R\left(p||q\right)}{\lambda_q}d\mu\left(q\right)\\=&\int_{\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(f\right)\right]+\frac{R\left(p||q\right)}{\lambda_q}d\mu\left(q\right)\\&+\int_{Q\setminus\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(h\right)\right]+\frac{R\left(p||q\right)}{\lambda_q}d\mu\left(q\right).\end{aligned}$$

2. *For every* $\Omega\times B\in\Sigma_s$, *if* $\mu\left(\left\{q\in Q:\rho_q\in B\right\}\right)=0$, *then* $\Omega\times B$ *is null.*

Proof. 1) Let $\Omega\times B\in\Sigma_s$ and $f,h\in\mathcal{F}$. Observe that

$$\begin{aligned}&\int_Q\min_{p\in\Delta(S)}\int_S u\left(f_{\Omega\times B}h\right)\mathrm{d}p+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right)\\=&\int_{\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S):q\gg p}\int_S u\left(f_{\Omega\times B}h\right)\mathrm{d}p+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right)\\&+\int_{Q\setminus\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S):q\gg p}\int_S u\left(f_{\Omega\times B}h\right)\mathrm{d}p+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right)\\=&\int_{\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(f\right)\right]+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right)\\&+\int_{Q\setminus\left\{q\in Q:\rho_q\in B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(h\right)\right]+\frac{R\left(p||q\right)}{\lambda_q}\mathrm{d}\mu\left(q\right)\end{aligned}$$

where the second equality follows from the fact that by equation (13) $q\gg p$ and $\rho_q\in B$ imply $\mathrm{supp}p\subseteq\mathrm{supp}q\subseteq\Omega\times B$ (and conversely $q\gg p$ and $\rho_q\notin B$ imply $\mathrm{supp}p\subseteq\mathrm{supp}q\subseteq S\setminus(\Omega\times B)$).

2) It follows from 1), since in this case for every $f,g,h\in\mathcal{F}$, $f_{\Omega\times B}h\succsim g_{\Omega\times B}h\Longleftrightarrow\int_{\left\{q\in Q:\rho_q\notin B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(h\right)\right]+\frac{R(p||q)}{\lambda_q}\mathrm{d}\mu\left(q\right)\geq\int_{\left\{q\in Q:\rho_q\notin B\right\}}\min_{p\in\Delta(S)}\mathbb{E}_p\left[u\left(h\right)\right]+\frac{R(p||q)}{\lambda_q}\mathrm{d}\mu\left(q\right)$ and the RHS is always trivially satisfied as an equality. ■

LEMMA 13: *Suppose that the assumptions of Theorem 4 hold and for all* $f,g\in\mathcal{F}$, $f\succsim g\Longleftrightarrow\mathbb{E}_{\mu}\left[\min_{p\in\Delta(S)}\mathbb{E}_{p}\left[u\left(f\right)\right]+\frac{R(p||q)}{\lambda_q}\right]\geq\mathbb{E}_{\mu}\left[\min_{p\in\Delta(S)}\mathbb{E}_{p}\left[u\left(g\right)\right]+\frac{R(p||q)}{\lambda_q}\right]$ *where* $u:X\rightarrow\mathbb{R}$ *is a nonconstant affine function,* $Q\subseteq\Delta\left(S\right)$ *is finite, nonempty, with* $q\left(\left\{\omega,\rho_q\right\}\right)=\rho_q\left(\omega\right)$, *for all* $q\in Q$, $\omega\in\Omega$ *and some* $\rho_q\in\Delta\left(\Omega\right)$, $\mu\in\Delta\left(Q\right)$, *and* $\left(\lambda_q\right)_{q\in Q}\in\mathbb{R}_{+}^{Q}$. *Then* $\succsim$

satisfies Uniform Misspecification Concern if and only if there exists λ^* *with* $\lambda_q = \lambda^*$ *for all* $q \in \operatorname{supp}\mu$.

Proof. (If) Let $\rho, \rho' \in \Delta(\Omega)$, $f, g \in \mathcal{F}$, and $x \in X$ be such that $\Omega \times \{\rho\}$ and $\Omega \times \{\rho'\}$ are nonnull,

$$\rho(\{\omega : f(\omega, \rho) = y\}) = \rho'(\{\omega : g(\omega, \rho') = y\}) \quad \forall y \in X, \tag{14}$$

and $f \succsim_{\Omega \times \{\rho\}} x$. Since $\Omega \times \{\rho\}$ and $\Omega \times \{\rho'\}$ are nonnull, by part 2 of Lemma 12 there exist $q, q' \in Q$ with $\mu(\{q\}) > 0$, $\mu(\{q'\}) > 0$, $\rho_q = \rho$, and $\rho_{q'} = \rho'$. Let $\phi(c) = -\exp(-\lambda^* c)$ for all $c \in u(X)$ and $\xi \in \Delta(X)$ be the probability measure such that for all $y \in X$, $\xi(y) = q\left(\left\{\left(\omega, \rho_q\right) : f\left(\omega, \rho_q\right) = y\right\}\right)$, then $\int_\Omega \phi(u(f))\mathrm{d}q = \int_X \phi(u(y))\mathrm{d}\xi(y)$. Moreover, equation (14) implies $\int_\Omega \phi(u(g))\mathrm{d}q' = \int_X \phi(u(y))\mathrm{d}\xi(y)$. Therefore, by Lemma 12 both $f \succsim_{\Omega \times \{\rho\}} x$ and $g \succsim_{\Omega \times \{\rho'\}} x$ mean that $\int_X \phi(u(y))\mathrm{d}\xi(y) \geq \phi(u(x))$ proving that $\succsim$ satisfies Uniform Misspecification Concern.

(Only if) Suppose by way of contradiction that there exist $q, q' \subseteq Q$ and $k \in \mathbb{R}_{++}$ with $\mu(\{q\}) > 0$, $\mu(\{q'\}) > 0$, and

$$\lambda_q > k > \lambda_{q'}. \tag{15}$$

Since the state space is adequate there exist $W_q \subseteq \Delta(\Omega), W_{q'} \subseteq \Delta(\Omega)$ and $c \in (0,1)$ with $\rho_q(W_q) = \rho_{q'}(W_{q'}) = c$. Moreover, $q\left(W_q \times \left\{\rho_q\right\}\right) = c = q'\left(W_{q'} \times \left\{\rho_{q'}\right\}\right)$ and $q\left(W_{q'} \times \left\{\rho_{q'}\right\}\right) = 0 = q'\left(W_q \times \left\{\rho_q\right\}\right)$. Pick $z, y \in X$ with $z \succ y$. For all $x \in X$,

$$\rho_q\left(\left\{\omega : z\left(\left(W_q \times \left\{\rho_q\right\}\right) \cup \left(W_{q'} \times \left\{\rho_{q'}\right\}\right)\right) y\left(\omega, \rho_q\right) = x\right\}\right)$$

is equal to $\rho_{q'}\left(\left\{\omega : z\left(\left(W_q \times \left\{\rho_q\right\}\right) \cup \left(W_{q'} \times \left\{\rho_{q'}\right\}\right)\right) y\left(\omega, \rho_{q'}\right) = x\right\}\right)$. By the convexity of X and Lemma 12 there is $\hat{x} \in X$ with $z \succ \hat{x} \succ y$ and $z\left(\left(W_q \times \left\{\rho_q\right\}\right) \cup \left(W_{q'} \times \left\{\rho_{q'}\right\}\right)\right) y \sim_{\rho_{q'}} \hat{x}$. But by equation (15) and Lemma 12, $\hat{x} \succ_{\rho_q} z\left(\left(W_q \times \left\{\rho_q\right\}\right) \cup \left(W_{q'} \times \left\{\rho_{q'}\right\}\right)\right) y$, a violation of Uniform Misspecification Concern. ■

PROPOSITION 5: *Suppose: (i) For every* $h \in \mathcal{H}$, $\succsim^h$ *satisfies the axioms of Theorem 4 and (ii)* $(\succsim^h)_{h \in \mathcal{H}}$ *satisfies Constant Preference Invariance and Dynamic Consistency over Models. Then for every* $h \in \mathcal{H}$, $\succsim^h$ *admits an average robust control representation* $(u, Q, \mu(\cdot|h), \lambda_h)$ *where* $q\left(\left\{\omega, \rho_q\right\}\right) = \rho_q(\omega)$ *for all* $q \in Q, \omega \in \Omega$.

Proof. That each $\succsim^h$ admits an average robust control representation $(u_h, Q_h, \mu_h, \lambda_h)$ where $q\left(\left\{\omega, \rho_q\right\}\right) = \rho_q(\omega)$ for all $q \in Q_h$ and $\omega \in \Omega$ for some $\rho_q \in \Delta(\Omega)$ follows from (the proof of) Theorem 4. That $u_h = u$ for some constant affine u follows from Constant Preference Invariance.

I now prove that Dynamic Consistency over Models implies $\mu(\cdot|h_t) = \mu_{h_t}$ for all $h_t = (\omega_i)_{i=1}^t \in \mathcal{H}_t$ such that $\prod_{i=1}^t \rho_q(\omega_i) > 0$ for some $q \in Q$. By definition, $\mu_{h_t} = \mu$ for the empty history. Let f and g be measurable with respect to Σ_s. Then I can suppress the dependence on ω in $f(\omega, \rho)$ and $g(\omega, \rho)$ and $f \succsim^{h_t} g$ if and only if $f^{h_t} \succsim^{\emptyset} g^{h_t}$.

But by construction, the latter is equivalent to

$$\mathbb{E}_\mu\left[\gamma_{f(\rho_q)} \prod_{i=1}^t \rho_q(\omega_i)\left(u(\bar{z}) - u(\underline{z})\right)\right] \geq \mathbb{E}_\mu\left[\gamma_{g(\rho_q)} \prod_{i=1}^t \rho_q(\omega_i)\left(u(\bar{z}) - u(\underline{z})\right)\right]$$

. Dividing both sides by the strictly positive ex-ante probability of history h_t, I obtain

$$\frac{\mathbb{E}_\mu\left[\gamma_{f(\rho_q)}\prod_{i=1}^{t}\rho_q(\omega_i)\left(u(\bar{z})-u(\underline{z})\right)\right]}{\int_{\Delta(\Delta(S))}\prod_{i=1}^{t}\rho_q(\omega_i)\,\mathrm{d}\mu(q)} \geq \frac{\mathbb{E}_\mu\left[\gamma_{g(\rho_q)}\prod_{i=1}^{t}\rho_q(\omega_i)\left(u(\bar{z})-u(\underline{z})\right)\right]}{\int_{\Delta(\Delta(S))}\prod_{i=1}^{t}\rho_q(\omega_i)\,\mathrm{d}\mu(q)}.$$

But then, by the formula for Bayesian updating, this is equivalent to

$$\int_{\Delta(\Delta(S))}\gamma_{f(\rho_q)}\left(u(\bar{z})-u(\underline{z})\right)\mathrm{d}\mu(q|h_t) \geq \int_{\Delta(\Delta(S))}\gamma_{g(\rho_q)}\left(u(\bar{z})-u(\underline{z})\right)\mathrm{d}\mu(q|h_t)$$

that is $\int_{\Delta(\Delta(S))} u\left(f\left(\rho_q\right)\right)\mathrm{d}\mu(q|h_t) \geq \int_{\Delta(\Delta(S))} u\left(g\left(\rho_q\right)\right)\mathrm{d}\mu(q|h_t)$. Stated differently, $\succsim^{h_t}$ admits an SEU representation of the acts measurable with respect to Σ_s with Bernoulli utility u and probability measure $\mu(\cdot|h_t)$. Since for the histories $h_t = (\omega_i)_{i=1}^{t} \in \mathcal{H}_t$ where $\prod_{i=1}^{t}\rho_q(\omega_i) = 0$ for all $q \in Q$ Bayesian updating does not impose any restriction, the result follows. ■

3. EXAMPLES AND REMARKS

EXAMPLE 4: *To provide a simple illustration of dynamic inconsistency, I consider the two-period truncated problem. An urn contains black* (b) *or green* (g) *balls. At period, the DM is asked to bet* 1 *dollar on the color of the ball drawn from the urn or to opt out* (o) *and observe the drawn ball with a certain payoff of* 0.6*. That is,* $u(a,y) = \mathbb{I}_{\{y\}}(a)$ *if* $a \in \{b,g\}$ *and* $u(o,y) = 0.55$*. Suppose that at period* 0*, the level of concern for misspecification is* $\Lambda(h_0) = 0$ *and that the agent considers two models,* q, q'*, that assign respectively probability* 0.7 *and* 0.3 *to drawing a black ball, independently of the agent action. The prior* μ *assigns equal probability to these two models.*

To illustrate the possibility of dynamic inconsistencies of a forward-looking agent, I introduce a discount factor equal to $\delta = 0.9$ *and suppose that* $\Lambda((0,b)) = 2$*. In this case, at time* 0*, the decision maker would like to commit to the following plan: opt out in the first period and then, in the second period, bet on the color of the ball drawn in the first period. However, the increase in concern for misspecification created by the observation of the black drawn makes this plan not feasible: at history* $(0,b)$*, the agent will opt out.*

EXAMPLE 5—Unsafe SEU: *Suppose* $A = \{$*Bet Heads, Bet Tails, Out*$\}$ *and* $Y = \{$*Heads, Tails*$\}$*. The utility is* 0 *if Out,* 1 *if the action matches the outcome, and* -1 *if there is a mismatch. Each agent's model is an action-independent probability of Heads. So identify* $Q = \{0.9, 0.4\}$*, and let* $p_a^*(\text{Heads}) = 0.6$ *for all* $a \in A$*, and* $\mu(0.9) = \frac{1}{2} = \mu(0.4)$*. The actions of a Bayesian SEU maximizer converge to Bet Tails with an average performance of* -0.2 *versus a safe payoff of* 0 *under action Out.*

REMARK 3: *A few subtleties about our use of Theorem 11.4.1 in Dudley (2018) in the proof of Theorem 1 are needed. That's Varadarajan Theorem about the convergence of the empirical measure to the theoretical measure in a separable metric space, and it is used here to guarantee that,* $\mathbb{P}_\Pi$*-a.s.,* $\lim_{n\to\infty} p_a^{\mathbf{h}_{t_n}} = p_a^*$ *for all* $a \in \operatorname{supp}\bar{\alpha}$*. As explained in Dudley (2018), the proof of that theorem has two components. The first part shows that it is enough to check the* $\mathbb{P}_\Pi$*-a.s. convergence of* $\mathbb{E}_{p_a^{\mathbf{h}_{t_n}}}[f(y)]$ *to* $\mathbb{E}_{p_a^*}[f(y)]$ *for* $f : Y \to \mathbb{R}$ *coming from a countable set of* *bounded and Lipschitz functions. That is a property of the weak convergence of measures and of the separability of the space* Y*, which has nothing to do with the nature of the measure* $p_a^{\mathbf{h}_{t_n}}$ *and* p_a^* *involved, and so it applies without changes here. The second step then invokes the SLLN to guarantee that fixed any bounded and Lipschitz* f*,* $\mathbb{E}_{p_a^{\mathbf{h}_{t_n}}}[f(y)]$ *converges to* $\mathbb{E}_{p_a^*}[f(y)]$ *if* $p_a^{\mathbf{h}_{t_n}}$ *is an empirical measure and* p_a^* *is the true law. That step needs instead a minor adaptation because, for us, the empirical measure is the one conditional on the periods in which action* a *is played, and therefore the SLLN for i.i.d. random variables invoked by Theorem 11.4.1 in Dudley (2018) cannot be directly applied. Still, the desired convergence is obtained by considering the martingale process*

$$\mathbf{X}_{t+1}^a = \begin{cases} \mathbf{X}_t^a + f(\mathbf{y}_{t+1}) - \mathbb{E}_{p_a^*}[f(\mathbf{y}_{t+1})] & \mathbf{a}_{t+1} = a \\ \mathbf{X}_t^a & \textit{otherwise} \end{cases}$$

that $\mathbb{P}_\Pi$*-a.s., converge to a finite limit (as* f *is continuous so each* $\mathbf{X}_t^a$ *is guaranteed to be integrable). With this, define*

$$\mathbf{Y}_t^a :=\begin{cases} \frac{\mathbf{X}_t^a}{\sum_{\tau=1}^t \mathbb{I}_{\{a\}}(\mathbf{a}_\tau)} & \sum_{\tau=1}^t \mathbb{I}_{\{a\}}(\mathbf{a}_\tau) > 1 \\ 0 & \textit{otherwise} \end{cases}$$

Then $\mathbb{P}_\Pi$*-a.s.,* $\mathbf{Y}_{t_n}^a$ *converge to* 0 *on the histories in which* $\lim_{n\to\infty}\sum_{\tau=1}^t \mathbb{I}_{\{a\}}(\mathbf{a}_\tau) = \infty$ *proving that on those histories* $\lim_{n\to\infty}\mathbf{Y}_{t_n}^a = \mathbb{E}_{p_a^{\mathbf{h}_{t_n}}}[f(y)] - \mathbb{E}_{p_a^*}[f(y)] = 0$*. Since* $a \in \operatorname{supp}\lim_{n\to\infty}\alpha_{t_n}(\mathbf{h}_{t_n})$ *implies* $\lim_{t\to\infty}\sum_{\tau=1}^t \mathbb{I}_{\{a\}}(\mathbf{a}_\tau) = \infty$*, the desired convergence is obtained.*

3.1. *Computations supporting Example 3*

The condition for not switching from action 0 to an action a with $a \geq 0$ in a Berk-Nash equilibrium is $p_a^*(s \leq a)\left(\mathbb{E}_{p_a^*}(v) - a\right) \leq 0$. By Proposition 1.4.2 in Dupuis and Ellis (2011), the condition for not switching from action 0 to an action a with $a \geq 0$ in a c-robust equilibrium is $-\frac{c}{R\left(p_a^* \| q_a\right)} \log \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left(-\frac{R\left(p_a^* \| q_a\right)}{c}[v-a]\,\mathbb{I}_{[0,a]}(s)\right) \mathrm{d}p_a^*(s)\mathrm{d}p_a^*(v) \leq 0$. By Jensen inequality, the LHS is lower in the second case, and I obtain the desired conclusion.

(Chapter head:)*

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