Supplement to: "Comparative statics with linear objectives: normality, complementarity, and ranking multi-prior beliefs"

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Abstract

This supplement contains additional results related to Dziewulski and Quah (2023). These notes should be read in conjunction with the article.

In this supplement, we present proofs of some of the claims made in the main article. In addition, in Section S.11 we use the set-theoretic notion of first order stochastic dominance defined in Section 5 of the main article to study comparative statics in problems of dynamic choice under ambiguity. Throughout this supplement, we employ the notation introduced in the main article.

S.1 Anti-symmetry of the parallelogram order

As we argued in Remark 2.3 of the main paper, the parallelogram order is transitive and reflexive in the class of compact and convex subsets of $\mathbb{R}^\ell$. In this section we show that it is also anti-symmetric, i.e., for any compact and convex sets A, $A' \subseteq \mathbb{R}^\ell$, if A' dominates A, and A dominates A' by the parallelogram order for $K = \{1, \ldots, \ell\}$, then the two sets are equal.

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Our proof uses the following auxiliary result on extreme points. We say that $x \in A$ is an *extreme point* of A if it is not a convex combination of any other points in A.

Lemma S.1. *Take any convex sets $A, A' \subseteq \mathbb{R}^\ell$ such that $A \nsubseteq A'$. Then, there is an extreme point of* $\text{co}(A \cup A')$ *that belongs to $x \in A \setminus A'$.*

Proof. Recall that a convex hull of a set consists of all convex combinations of its extreme points. Specifically, it must be that $\text{co}(A \cup A')$ consists of convex combinations of extreme points in A and A'. Towards contradiction, suppose that all such extreme points are in A'. Since A' is convex, we have $A \subseteq \text{co}(A \cup A') = \text{co}\, A' = A'$, yielding a contradiction. Therefore, there must be at least one extreme point of $\text{co}(A \cup A')$ in $A \setminus A'$. $\square$

We continue with our main argument. Towards contradiction, suppose that A' dominates A, and A dominates A' by parallelogram order, but $A \nsubseteq A'$. By the lemma above, there is an extreme point $x \in \text{co}(A \cup A')$ such that $x \in A \setminus A'$. By Theorem 12.7 in Soltan (2015), there are vectors $p^1, \dots, p^N$, such that $\Phi_A^n = \text{argmax}\left\{p^n \cdot y : y \in \Phi_A^{n-1}\right\}$, for all $n = 1, \dots, N$, and $\{x\} = \Phi_A^N$, where $\Phi_A^0 = A$. Let $\Phi_{A'}^N$ be the set induced as above for A', for the same vectors $p^1, \dots, p^N$. By successive application of Theorem 2 of the main paper, it must be that $\Phi_{A'}^N$ dominates Φ_A^N by the parallelogram order. In particular, there must be some $x' \in \Phi_{A'}^N$ such that $x' \geq x$. Similarly, the set $\Phi_A^N = \{x\}$ dominates $\Phi_{A'}^N$ by the parallelogram order. Thus, it must be that $x \geq x'$. However, the two inequalities imply that $x = x'$, which contradicts that $x \notin A$.

S.2 Asymptotic cones, convex hulls, and closed sets

In this section we discuss the relationship between asymptotic cones, convex hulls and closed sets. We prove Proposition S.1 which we use to prove that statement (iv) in Theorem 3 implies statement (i) (see the Appendix to the main article). We also prove Proposition S.2, which provides sufficient conditions under which the optimisation problems discussed in Sections 2–4 have a solution.

We prove Proposition S.1 through three lemmas that follow. Recall that an *asymptotic cone* $\mathbf{A}Y$ of the set $Y \subseteq \mathbb{R}^\ell$ is the set of limits of all sequences of the form $\{\lambda_n x_n\}$, for positive numbers $\lambda_n \to 0$ and $x_n \in Y$, for all n.

Lemma S.1. *Suppose the set $Y \subset \mathbb{R}^\ell$ satisfies $\mathbf{A}Y \subseteq \mathbb{R}^\ell_+$ and the sequence $\{x_n\}$ is given by $x_n = \sum_{i=1}^k \alpha_n^i y_n^i$, where $\alpha_n^i \geq 0$ and $y_n^i \in Y$, for all i and n. If $\{x_n\}$ and (for every i) $\{\alpha_n^i\}$ are bounded sequences, then the sequence $\{\alpha_n^i y_n^i\}$ is also bounded.*

Proof. Towards contradiction, suppose there is a set $I = \{1, \ldots, m\}$ such that the sequence $\{\alpha_n^i y_n^i\}$ is unbounded, for all $i \in I$. Note that, the set must have at least two elements; otherwise $\{x_n\}$ would be unbounded. Similarly, the sum $\sum_{i \in I} \alpha_n^i y_n^i$ must be bounded. After taking subsequences if necessary, suppose that the sequence $\alpha_n^1 y_n^1 / L_n^1$ converges to $y^1 \neq 0$, where L_n^1 denotes the norm of $\alpha_n^1 y_n^1$. The limit y^1 must belong to $\mathbf{A}Y \subseteq \mathbb{R}^\ell_+$ since $\alpha_n^1 / L_n^1 \to 0$. Thus, the sequence

$$\frac{\sum_{i=2}^m \alpha_n^i y_n^i}{L_n^1}$$

converges to $-y^1 < 0$, since $\sum_{i \in I} \alpha_n^i y_n^i / L_n^1 \to 0$. If each term $\alpha_n^i y_n^i / L_n^1$ for $i \neq 1$ is bounded, then one of them will have a limit outside of $\mathbb{R}^\ell_+ \supseteq \mathbf{A}Y$, yielding a contradiction. Alternatively, suppose that $\alpha_n^2 y_n^2 / L_n^1$ is unbounded, without loss of generality. As previously, the sequence $\alpha_n^2 x_n^2 / (L_n^1 \, L_n^2)$, where L_n^2 denotes the norm of $\alpha_n^2 y_n^2 / L_n^1$, has a limit in $\mathbf{A}Y = \mathbb{R}^\ell_+$, which implies that the sequence

$$\frac{\sum_{i=3}^m \alpha_n^i y_n^i}{L_n^1 \, L_n^2}$$

has a limit in $\mathbb{R}^\ell_- \setminus \{0\}$. If each sequence $\alpha_n^i x_n^i / (L_n^1 \, L_n^2)$ is convergent, then one of them has a limit that is not in $\mathbb{R}^\ell_+ \supseteq \mathbf{A}Y$, yielding a contradiction. Otherwise, we can continue the argument which will eventually lead to a contradiction. □

The next lemma introduces a general class of sets that admit a *closed* convex hull.

Lemma S.2. *Whenever the set $Y \subseteq \mathbb{R}^\ell$ is closed, upward comprehensive, and $\mathbf{A}Y = \mathbb{R}^\ell_+$, then its convex hull $\operatorname{co} Y$ is closed.*[1]

Proof. Let $\{x_n\}$ be a sequence in $\operatorname{co} Y$ converging to x. By Carathéodory's theorem, we may assume that $x_n = \sum_{i=1}^{\ell+1} \alpha_n^i y_n^i$, for $y_n^i \in Y$, $\alpha_n^i \geq 0$, and $\sum_{i=1}^{\ell+1} \alpha_n^i = 1$, for all $i = 1, \ldots, \ell + 1$ and n, without loss of generality. Moreover α_n^i converges to $\alpha^i \geq 0$, for

[1] Whenever $Y \subseteq \mathbb{R}^\ell$ is upward comprehensive and $\mathbf{A}Y \subseteq \mathbb{R}^\ell_+$, then $\mathbf{A}Y = \mathbb{R}^\ell_+$. Take any $y \in Y$ and $x \in \mathbb{R}^\ell_+$. Since Y is upward comprehensive, we have $(y + 1/\lambda x) \in Y$, for any $\lambda > 0$. Moreover, $\lambda(y + 1/\lambda x) \to x$ as $\lambda \to 0$. Since x was arbitrary, this proves that $\mathbb{R}^\ell_+ \subseteq \mathbf{A}Y$.

all $i = 1, \dots, \ell + 1$. By shifting Y by a constant if necessary, we can also assume that $x = 0$. It suffices to show that $x \in \operatorname{co} Y$.

We partition the sequences of indices $i = 1, \dots, \ell + 1$ into two groups: (i) those i for which the sequence $\{y_n^i\}$ is bounded, and (ii) those i for which the sequence $\{\alpha_n^i y_n^i\}$ is bounded, but $\{y_n^i\}$ is not. Denote the two sets by I, I', respectively. By Lemma S.1, these are the only two cases that we need to consider.

For each $i \in I$, we may assume that the sequence $\{y_n^i\}$ has the limit y^i which belongs to Y (since Y closed). For each $i \in I'$, denote the limit of $\{\alpha_n^i x_n^i\}$ by z^i, which exists by assumption. In particular, it must be that $\alpha_n^i \to 0$, and so $z^i \in \mathbf{A}Y \subseteq \mathbb{R}_+^\ell$. As a result, we have $z = \sum_{i \in I'} z^i \geq 0$ and $z + \sum_{i \in I} \alpha^i y^i = x = 0$. Thus, we have $\sum_{i \in I} \alpha^i y^i = -z \leq 0$. Since we can always re-normalise the weights so that $\sum_{i \in I} \alpha^i y^i \in \operatorname{co} Y$, and since $\operatorname{co} Y$ is upward comprehensive, this suffices to show that $x = 0 \in \operatorname{co} Y$. □

Next, we establish a relationship between asymptotic cones and convex hulls.

Lemma S.3. *If* $\mathbf{A}Y \subseteq \mathbb{R}_+^\ell$ *then* $\mathbf{A}(\operatorname{co} Y) \subseteq \mathbb{R}_+^\ell$, *for any* $Y \subseteq \mathbb{R}^\ell$.

Proof. Suppose that $\lambda_n x_n \to z$, where $x_n \in \operatorname{co} Y$, for all n, and $\lambda_n \to 0$. We claim that $z \geq 0$. By Carathéodory's theorem, we may assume (without loss of generality) that $x_n = \sum_{i=1}^{\ell+1} \alpha_n^i y_n^i$, where $\alpha_n^i \geq 0$, $y_n^i \in Y$, and $\sum_{i=1}^{\ell+1} \alpha_n^i = 1$, for all $i = 1, \dots, \ell + 1$ and n. Thus, $\lambda_n \alpha_n^i \to 0$, for all $i = 1, \dots, \ell + 1$. Moreover, by Lemma S.1 and our assumption, each sequence $\{(\lambda_n \alpha_n^i) y_n^i\}$ is convergent to some $z^i \in \mathbf{A}Y \subseteq \mathbb{R}_+^\ell$, and so $\lambda_n x_n = \lambda_n \sum_{i=1}^{\ell+1} \alpha_n^i y_n^i$ converges to $z = \sum_{i=1}^{\ell+1} z^i \geq 0$. □

The next proposition follows from the previous two lemmas.

Proposition S.1. *If* $Y \subseteq \mathbb{R}^\ell$ *is closed, upward comprehensive, and* $\mathbf{A}Y = \mathbb{R}_+^\ell$, *then* $\mathbf{A}(\operatorname{co} Y) = \mathbb{R}_+^\ell$.

Proof. By Lemmas S.2, S.3, the set $\operatorname{co} Y$ is closed and $\mathbf{A}(\operatorname{co} Y) \subseteq \mathbb{R}_+^\ell$. Since $\operatorname{co} Y$ is upward comprehensive, we have $\mathbb{R}_+^\ell \subseteq \mathbf{A}(\operatorname{co} Y)$, proving the claim. □

The following proposition establishes sufficient conditions under which the minimum of any strictly positive linear functional over a set Y is well-defined. This is used extensively in Sections 2–4, where we focus on minimisation problems with linear objectives.

Proposition S.2. *Let $Y \subseteq \mathbb{R}^\ell_+$ be closed and $\mathbf{A}Y \subseteq \mathbb{R}^\ell_+$. Then, for all $p \in \mathbb{R}^\ell_{++}$, the set* $\operatorname{argmin}\left\{p \cdot y : y \in Y\right\}$ *is nonempty and closed, and (thus) the function $f : \mathbb{R}^\ell_{++} \to \mathbb{R}$, where $f(p) := \min\left\{p \cdot y : y \in Y\right\}$ is well-defined.*

Proof. Suppose there is some $\bar{p} \gg 0$ for which the minimization problem has no solution. Choose any $\bar{y} \in Y$ and consider the set $Y' = \left\{y \in Y : \bar{p} \cdot y \leq \bar{p} \cdot \bar{y}\right\}$. If the minimization problem has no solution then Y' is unbounded; indeed, if it is bounded then it is both closed and bounded and there will be y^* that minimizes $\bar{p} \cdot y$ in Y', and also in Y.

If Y' is unbounded, then it contains an unbounded sequence $\{y_n\}$. Let $\hat{y}_n = y_n/\|y_n\|$ have a limit given by $\hat{y} \neq 0$, which is in $\mathbf{A}Y$. Since $\bar{p} \cdot \hat{y}_n \leq \bar{p} \cdot \bar{y}/\|y_n\|$, by taking limits we obtain $\bar{p} \cdot \hat{y} \leq 0$, which is impossible since $\bar{p} \gg 0$ and $\hat{y} > 0$. □

At the beginning of Section 4 of the main paper, we claim that the profit function $\pi(p) := \max\left\{F(x) - p \cdot x : x \in X\right\}$ of the firm is well-defined for any strictly positive price p, whenever the asymptotic cone of the production possibility set

$$P = \left\{(z, y) \in \mathbb{R}^\ell \times \mathbb{R} : (z, y) \leq \big(-x, F(x)\big),\ \text{for } x \in X\right\}$$

is contained in $\mathbb{R}^{\ell+1}_-$. Indeed, since

$$\pi(p) \ := \ \max\Big\{F(x) - p \cdot x : x \in X\Big\} \ = \ \max\Big\{(p, 1) \cdot (z, y) : (z, y) \in P\Big\},$$

Proposition S.2 guarantees that the function is well-defined for any $p \in \mathbb{R}^\ell_{++}$.

S.3 Continuation of Remark 2.4

Theorem 1 in the main article tell us that, if two constraint sets are ordered by the parallelogram order, then so are the minimizers of a linear objective over those sets. This feature makes our results applicable to decision procedures where linear objectives are sequentially applied.

For example, in the context of factor demand, the firm may choose, among those bundles that minimize cost, the ones that minimize the usage of one or a combination of factors belonging to $C \subseteq \{1, 2, \ldots, \ell\}$ (perhaps for environmental reasons, or to minimize the firm's vulnerability to supply disruptions). In that case, the firm's factor demand at factor prices $p \in \mathbb{R}^\ell_{++}$ and output q is $H^*(p,q) := \operatorname{argmin}\big\{\sum_{i\in C} x_i : x \in H(p,q)\big\}$. By Theorem 1, if F is $\mathcal{P}$-increasing, then $H(p,\cdot)$ is $\mathcal{P}$-increasing, and so is $H^*(p,\cdot)$.

Another application is to guarantee normality for efficient bundles. For a given production function F, a bundle $x \in \mathbb{R}^\ell_+$ is *efficient at producing* q if $x \in U(q)$ and $x' < x$ implies $x' \notin U(q)$, for any alternative x'. Let $E(q)$ denote the set of bundles that produce q efficiently. Given $x \in E(q)$ and $q' > q$, we ask whether there is $x' \in E(q')$ such that $x' \geq x$. For example, suppose x represents the effort levels of ℓ team members in a joint project, and gives an efficient way of allocating the effort among the team members to produce q. If the output target is now higher at q', is there a way of producing this efficiently, while ensuring that no team member contributes strictly less? This is not always possible. For example, suppose $U(q) = \big\{(x_1,x_2) \in \mathbb{R}^2_+ : x_1 x_2 = 1\big\}$ and $U(q') = \big\{(x_1,x_2) \in \mathbb{R}^2_+ : \min\{x_1,x_2\} \geq 2\big\}$. Then $\{(2,2)\} = E(q')$ and $(3,1/3) \in E(q)$, but clearly $(2,2) \not\geq (3,1/3)$.

The situation in this example cannot occur when the upper contour sets are convex sets ranked by the parallelogram order. Indeed, according to Che et al. (2020), if $x \in E(q)$ and $U(q)$ is convex, then there is a sequence of non-zero weights $p^1, p^2, \ldots, p^{m-1} \in \mathbb{R}^\ell_+$ and $p^m \in \mathbb{R}^\ell_{++}$ such that $x \in \Phi^m(q)$, where $\Phi^k(q) := \operatorname{argmin}\big\{p^k \cdot x : x \in \Phi^{k-1}(q)\big\}$ and $\Phi^0(q) = U(q)$. If $U(q')$ dominates $U(q)$ in the parallelogram order then, by consecutive application of Theorem 1, we know that $\Phi^m(q')$ dominates $\Phi^m(q)$ in the parallelogram order. Thus (subject to standard conditions guaranteeing that $\Phi^m(q')$ is nonempty), there is $x' \in \Phi^m(q')$ such that $x' \geq x$. Since $p^m \gg 0$, the bundle $x' \in E(q')$.

There are other manifestations of this result. For example, suppose V' and V are two convex sets representing the utility-possibilities of ℓ agents. If V' dominates V in the parallelogram order, then (using essentially the same argument) we may conclude that for every Pareto optimal utility allocation $v' \in V'$ there is a Pareto optimal allocation $v \in V$ such that $v' \geq v$. In other words, suppose the initial allocation is v' and the economy shrinks from V' to V; then there is a new allocation v which is Pareto optimal and which involves all agents sharing in the loss – no one is strictly better off.

S.4 Continuation of Example 5

Let $X \subseteq \mathbb{R}^\ell$ be a convex lattice. In the main article we claim that a function $F : X \to \mathbb{R}$ is increasing in the $\mathcal{C}$-flexible order for $K \subseteq \{1, \ldots, \ell\}$ if it is continuous, increasing, supermodular, and concave in x_{-i}, for all $i \in K$. This result can be found in Quah (2007); we prove it here for easy reference.

Take any $q' \geq q$ and $x, x' \in X$ such that $x'_K \not\geq x_K$ and $F(x) \geq q$, $F(x') \geq q'$. We show that there is a $\lambda \in [0, 1]$ satisfying $F\big(\lambda x' + (1-\lambda)(x \wedge x')\big) \geq q$, $F\big(\lambda x + (1-\lambda)(x \vee x')\big) \geq q'$. This suffices for F to be increasing in the $\mathcal{C}$-flexible order for K.

Consider two cases. (i) If $F(x \wedge x') \geq q$, set $\lambda = 0$. By monotonicity of F, we have $F(x \wedge x') \geq F(x') \geq q'$. Alternatively, let (ii) $F(x \wedge x') < q$. Since $q \leq q' \leq F(x')$, by continuity of F there is some $\lambda \in [0, 1]$ such that $F\big(\lambda x' + (1 - \lambda)(x \wedge x')\big) = q$. Denote $v = x' - (x \wedge x') = (x \vee x') - x$, which is a positive vector. Since $x'_K \not\geq x_K$, there is some $i \in K$ such that $v_i = 0$. In particular, we obtain

$$
\begin{aligned}
q' - q \;&\leq\; F(x') - F\big(\lambda x' + (1 - \lambda)(x \wedge x')\big) \;=\; F(x') - F\big(x \wedge x' + \lambda v\big) \\
&\leq\; F(x \vee x') - F\big(x + \lambda v\big) \;\leq\; F\big((x \vee x') - \lambda v\big) - F(x),
\end{aligned}
$$

where the second inequality follows from supermodularity of F and the third is implied by the fact that F is concave in x_{-i} and $v_i = 0$.[2] Therefore, since $F(x) \geq q$, it must be that $q' \leq F\big((x \vee x') - \lambda v\big) = F\big(\lambda x + (1 - \lambda)(x \vee x')\big)$. This suffices to show that $\mathcal{C}$-flexible order is stronger that parallelogram order. $\square$

S.5 Substitutes in production

In Section 4 of the main article we mentioned in a footnote that our results can be applied to study technologies that exhibit substitutability of inputs. Here, we substantiate this claim. As in Section 4, let X be a non-empty and closed subset of $\mathbb{R}^\ell_+$, and $F : X \to \mathbb{R}$ be a regular production function (as in Definition 4 in the main paper). At factor prices $p \in \mathbb{R}^\ell_{++}$, the firm's (unconditional) *input/factor demand* is given by $\mathcal{H}(p) := \operatorname{argmax}\big\{F(x) - p \cdot x : x \in X\big\}$.

[2] We are making use of the fact that when f is concave in direction v, we have $f(x - v) - f(x) \geq f(x - v - tv) - f(x - tv)$, for any $x \in X$ and scalar $t > 0$.

For any two distinct factors $i, j = 1, \ldots, \ell$, we say that i is a *substitute* of j, if, for any $p, p' \in \mathbb{R}^{\ell}_{++}$ satisfying $p_{-j} = p'_{-j}$, and any $x \in \mathcal{H}(p)$, there is $x' \in \mathcal{H}(p')$ such that $x'_i \geq x_i$ if $p'_j \geq p_j$, and $x'_i \leq x_i$ if $p'_j \leq p_j$. The next result states that the substitutes property between any two factors is symmetric and equivalent to the submodularity of the profit function with resepct to the prices of those two factors.

Proposition S.1. *Let $X \subseteq \mathbb{R}^{\ell}_{+}$ be a closed set and $F : X \to \mathbb{R}$ be a regular function. For any distinct $i, j = 1, \ldots, \ell$, these statements are equivalent:* (i) *factor i is a substitute of j;* (ii) *factor j is a substitute of i;* (iii) *the profit function $\pi : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$, where $\pi(p) := \max\big\{F(x) - p \cdot x : x \in X\big\}$, is submodular in (p_i, p_j) (keeping other prices fixed).*

Proof. Analogously to our argument in Section 4 of the main paper, we answer this question by defining the correspondence Γ^j with the domain $T = \mathbb{R}_+$, by

$$\Gamma^j(t_j) \;:=\; \Big\{(y, v) \in \mathbb{R}^{\ell+1} : y \geq x \text{ and } v \leq F(x) - t_j x_j, \text{ for some } x \in X\Big\}. \tag{S.1}$$

It is straightforward to check that, for any $p \in \mathbb{R}^{\ell}_{++}$,

$$x \in \mathcal{H}(p_j + t_j, p_{-j}) \iff \big(x, -F(x) + t_j x_j\big) \in \operatorname{argmin}\Big\{(p, 1) \cdot y : y \in \Gamma^j(t_j)\Big\}$$

and thus $\pi(p_j + t_j, p_{-j}) = -\min\big\{(p, 1) \cdot y : y \in \Gamma^j(t_j)\big\}$. Theorem 3 (with $K = \{i\}$) guarantees that the following are equivalent: (i) $\operatorname{co}\Gamma^j$ is $\mathcal{P}$-increasing in $\{i\}$; (ii) i is a substitute of j; and (iii) $-\pi(p_j + t_j, p_{-j})$ has increasing differences in (t_j, p_i). Notice that condition (iii) is equivalent to π being submodular in (p_j, p_i), other prices being fixed. Since submodularity is a symmetric property, we conclude that i is a substitute of j if, and only if, j is a substitute of i, with both equivalent to submodularity of π in (p_j, p_i). $\square$

S.6 FSD and increasing value under ambiguity

We mentioned in a footnote in Section 5 of the main article that the set-generalization of first order stochastic dominance studied there is designed for monotone comparative statics and is distinct from the generalization needed for comparing utility levels. In the latter case, we are interested in conditions on the belief correspondence $\Lambda : T \to \triangle_S$ such that, for any increasing function $u : S \to \mathbb{R}$, the value function $v : T \to \mathbb{R}$, given by

$$v(t) \;:=\; \min\left\{\int u(s) d\lambda(s) : \lambda \in \Lambda(t)\right\}, \tag{S.2}$$

is increasing in t. Below, we characterize this property.

Proposition S.1. *Suppose the correspondence $\Lambda : T \to \triangle_S$ has compact and convex values. Then, the following statements are equivalent.*

(i) *Correspondence Λ satisfies the following property:*

(F) *if $t' \geq_T t$, then for any $\lambda' \in \Lambda(t')$ there is some $\lambda \in \Lambda(t)$ such that $\lambda' \succeq \lambda$.*

(ii) *For any increasing function $u : S \to \mathbb{R}$, the function v in (S.2) increases in t.*

Proof. To show that (i) $\Rightarrow$ (ii), take any $t' \geq_T t$ and $\lambda' \in \Lambda(t')$. By (F), there is some $\lambda \in \Lambda(t)$ such that $\lambda' \succeq \lambda$. Thus, for any increasing u,

$$\int_S u(s)d\lambda'(s) \;\geq\; \int_S u(s)d\lambda(s) \;\geq\; \min\left\{\int_S u(s)d\nu(s) : \nu \in \Lambda(t)\right\}.$$

Taking the minimum over the left term gives us the result.

To show (ii) $\Rightarrow$ (i), suppose (F) fails. Then there is $t' \geq t$ and $\lambda' \in \Lambda(t')$ such that $\lambda' \not\succeq \lambda$, for all $\lambda \in \Lambda(t)$. Let $V = \{y \in \mathbb{R}^\ell : y_i \geq \lambda'(s_i),\ \text{for } i = 1, \ldots, \ell\}$. Since $V \cap \Lambda(t') = \emptyset$ and $\big(V - \Lambda(t')\big)$ is closed and convex, by the strong separating hyperplane theorem, $\min\left\{\sum_{i=1}^\ell \hat{p}_i y_i : y \in V\right\} > \max\left\{\sum_{i=1}^\ell \hat{p}_i \lambda(s_i) : \lambda \in \Lambda(t')\right\}$, for some $\hat{p} \in \mathbb{R}^\ell$. Given that V is upward comprehensive, $\hat{p} > 0$ and $\sum_{i=1}^\ell \hat{p}_i \lambda'(s_i) = \min\{\hat{p} \cdot y : y \in V\}$. Define $u : S \to \mathbb{R}$ by $u(s_1) = \hat{p}_1$ and $u(s_{i+1}) = u(s_i) + \hat{p}_{i+1}$, for $i = 1, \ldots, \ell$, which is an increasing function. Since $\int_S u(s)d\mu(s) = u(s_{\ell+1}) - \sum_{i=1}^\ell \hat{p}_i \mu(s_i)$, for any $\mu \in \triangle_S$,

$$\begin{aligned}
\min\left\{\int_S u(s)d\lambda(s) : \lambda \in \Lambda(t)\right\} &= u(s_{\ell+1}) - \max\left\{\sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda(t)\right\} \\
> u(s_{\ell+1}) - \sum_{i=1}^{\ell} \hat{p}_i \lambda'(s_i) &\geq u(s_{\ell+1}) - \max\left\{\sum_{i=1}^{\ell} \hat{p}_i \lambda(s_i) : \lambda \in \Lambda(t')\right\} \\
&= \min\left\{\int_S u(s)d\lambda(s) : \lambda \in \Lambda(t')\right\}.
\end{aligned}$$

Thus (F) is indeed necessary for monotone maxmin utility. $\square$

Notice that, property (F) is strictly weaker than $\mathcal{P}_{\mathrm{FSD}}$-increasing property. Clearly, any correspondence that increases in the latter sense satisfies (F), but the converse does not hold. In fact, as we show below, (F) is not sufficient for monotone comparative statics, i.e., this property alone does not guarantee that the set of maximisers of the function $f(x,t) := \min\left\{\int g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ with respect to x is increasing in the parameter t, for all supermodular functions g. Therefore, it is not sufficient for $f(x,t)$ to be supermodular, for all supermodular functions g.

Example S.1. Suppose that $X = \{0,1\}$ and $S = \{s_1, s_2, s_3\}$. The distribution λ is given by $\lambda(s_1) = 1/2$ and $\lambda(s_2) = 3/4$, while λ' satisfies $\lambda'(s_1) = \lambda'(s_2) = 1/2$ and μ is given by $\mu(s_1) = 1/4$, $\mu(s_2) = 7/8$. Suppose that $T = \{t, t'\}$, where $t' >_T t$, and $\Lambda(t') = \{\lambda'\}$ and $\Lambda(t) = \text{co}\{\lambda, \mu\}$. Since $\lambda' \succeq \lambda$, correspondence Λ obeys stochastic dominance in the sense given by (F). Let $g : X \times S \to \mathbb{R}$ be such that $g(0, s_1) = g(0, s_2) = 5$, $g(0, s_3) = 21$, $g(1, s_1) = 0$, $g(1, s_2) = 8$, and $g(1, s_3) = 24$; note that $g(x, s)$ is increasing in s and supermodular in (x, s). Since $\int_S g(0, s) d\lambda'(s) > \int_S g(1, s) d\lambda'(s)$, we have $\{0\} = \text{argmax} \left\{ f(x, t') : x \in X \right\}$. However, given that

$$\int_S g(0,s)d\lambda(s) \; > \; \int_S g(1,s)d\mu(s) \; = \; \int_S g(1,s)d\lambda(s) \; > \; \int_S g(0,s)d\mu(s),$$

it must be that $\{1\} = \text{argmax} \left\{ f(x, t) : x \in X \right\}$.

Even though property (F) is not sufficient for monotone comparative statics within a general class of supermodular functions g, it may suffice in certain special cases of g. For example, suppose X consists of only two actions – 0 and 1 – with $g(1, s)$ increasing in s and $g(0, s)$ decreasing in s, then obviously $f(1, t) - f(0, t)$ is increasing in t if Λ satisfies (F), since $f(1, t)$ and $f(0, t)$ are separately increasing and decreasing in t. In the study of global games with ambiguity by Ui (2015), this is precisely the assumption imposed on (what we call) g, which then allows the author to conclude that the higher action is chosen by players in the game when they receive a higher signal.

S.7 FSD and single crossing differences

In Proposition 8 of the main article we concluded that the belief correspondence Λ is $\mathcal{P}_{\text{FSD}}$-increasing if, and only if, the resulting value function $f(x, t) := \min \left\{ \int g(x, s) d\lambda(s) : \lambda \in \Lambda(t) \right\}$ is supermodular in (x, t), for any supermodular function g. As we pointed out in a footnote in Section 5, the $\mathcal{P}_{\text{FSD}}$-increasing property is also necessary for the function f to satisfy a weaker condition — *single crossing differences* — for any supermodular function g.[3] As shown in Milgrom and Shannon (1994), single crossing differences alone

[3] The function $g : X \times S \to \mathbb{R}$ has single crossing differences if $g(x', s') \geq (>) \, g(x, s')$ implies $g(x', s) \geq (>) g(x, s)$, for any $x' \geq x$ and $s' \geq s$, where we assume that $X, S \subseteq \mathbb{R}$.

is sufficient for the set of maximisers of f with respect to x to be increasing in t. Below, we provide the formal proof of our claim.

Suppose that $\tilde{f}(\tilde{x},t) := \min\left\{\int_S \tilde{g}(\tilde{x},s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ violates increasing differences, for some function $\tilde{g}$. In particular, for some $x' \geq x$ and $t' \geq t$,

$$v := \tilde{f}(x',t) - \tilde{f}(x,t) > \tilde{f}(x',t') - \tilde{f}(x,t').$$

Define the function g by $g(y,s) = \tilde{g}(y,s)$, for $y \leq x$, and $g(y,s) = \tilde{g}(y,s) - v$ otherwise. Clearly, g is supermodular, but f given by $f(\tilde{x},t) := \min\left\{\int_S g(\tilde{x},s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ violates single crossing differences since $0 = f(x',t) - f(x,t) > f(x',t') - f(x,t')$. Therefore, the maxmin value function violates increasing differences for some supermodular function $\tilde{g}$ if, and only if, it violates single-crossing differences for another function. Clearly, this suffices to show that the beliefs Λ are $\mathcal{P}_{\text{FSD}}$-increasing if, and only if, the value function obeys single-crossing differences, for any supermodular function g.

S.8 Continuation of Remark 5.2

Next, we turn to the claim in Remark 5.2. Recall that, whenever the function $g(x,s)$ is increasing in s, one can assume that the belief correspondence Λ has upward comprehensive values, without affecting the maxmin representation of preferences. In such a case, the $\mathcal{P}_{\text{FSD}}$ monotonicity remains necessary for $f(x,t) := \min\left\{\int g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ to have increasing differences (in (x,t)), for all $g(x,s)$ that are supermodular in (x,s) and increasing in s.

Proposition S.1. *Suppose that correspondence $\Lambda : T \to \triangle_S$ has compact, convex, and upward comprehensive values. Then the following statements are equivalent.*

(i) Λ *is $\mathcal{P}_{\text{FSD}}$-increasing.*

(ii) *The function f in (8) is supermodular in (x,t), for all supermodular functions g that are increasing in s.*

Proof. Implication (i) $\Rightarrow$ (ii) follows from Proposition 8. We prove the converse in two steps. First, using Theorem 3 and an argument analogous to the one in the proof of Proposition 8, we can show that the function f satisfies increasing differences only if the correspondence $\Gamma : T \to \mathbb{R}^\ell$, defined as

$$\Gamma(t) \ := \ \Big\{ y \in \mathbb{R}^\ell : y_i \geq -\lambda(s_i), \text{ for all } i = 1, \ldots, \ell \text{ and some } \lambda \in \Lambda(t) \Big\},$$

increases in the parallelogram order. This means that for any $t' \geq_T t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$, θ and $\theta' \in \mathbb{R}^\ell$ such that $\theta_i \leq \mu(s_i)$, $\theta'_i \leq \mu'(s_i)$, $\lambda(s_i) + \lambda'(s_i) = \theta_i + \theta'_i$, and $\theta_i \geq \lambda'(s_i)$ for all i. Therefore, Λ has the following property, which we shall refer to as ($\star$): for any $t' \geq_T t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that $(1/2)\lambda + (1/2)\lambda' \succeq (1/2)\mu + (1/2)\mu'$ and $\lambda' \succeq \mu$.

To complete the proof we show that ($\star$) implies $\mathcal{P}_{\text{FSD}}$ monotonicity when Λ is upper comprehensive. ($\star$) states that for any $t' \geq t$, $\lambda \in \Lambda(t)$, and $\lambda' \in \Lambda(t')$, there is $\mu \in \Lambda(t)$ and $\mu' \in \Lambda(t')$ such that $\mu(s_i) \geq \lambda'(s_i)$ and $\mu(s_i) + \mu'(s_i) \geq \lambda(s_i) + \lambda'(s_i)$ for all i. We modify μ and μ' state-by-state such that the condition holds with equality. Suppose $\mu(s_1) + \mu'(s_1) > \lambda(s_1) + \lambda'(s_1)$. If it is possible, choose $\nu^1(s_1)$ in the interval $\big[\lambda'(s_1), \mu(s_1)\big]$ such that $\nu^1(s_1) + \mu'(s_1) = \lambda(s_1) + \lambda'(s_1)$ and then set $\nu'^1(s_1) = \mu'(s_1)$. If, after setting $\nu^1(s_1) = \lambda'(s_1)$, we have $\nu^1(s_1) + \mu'(s_1) > \lambda(s_1) + \lambda'(s_1)$, then set $\nu'^1(s_1) = \lambda(s_1)$. Let $\nu^1(s_i) = \mu(s_i)$ and $\nu'^1(s_i) = \mu(s_i)$ for $i \geq 2$. Note that ν^1 and ν'^1 are bona fide distributions (i.e., both functions are increasing with the state) and, since Λ is upper comprehensive, $\nu^1 \in \Lambda(t)$, $\nu'^1 \in \Lambda(t')$. Furthermore, ν^1 and ν'^1 satisfy the conditions required by ($\star$) and $\nu^1(s_1) + \nu'^1(s_1) = \lambda(s_1) + \lambda'(s_1)$. Now define ν^2 and ν'^2 by $\nu^2(s_i) = \nu^1(s_i)$ and $\nu'^2(s_i) = \nu'^1(s_i)$, for all $i \neq 2$. If possible, set $\nu^2(s_2) \in \big[\max\{\lambda'(s_2), \nu^1(s_1)\}, \mu(s_2)\big]$ so that $\nu^2(s_1) + \nu'^1(s_2) = \lambda(s_2) + \lambda'(s_2)$ and then set $\nu'^2(s_2) = \nu'^1(s_2)$. Otherwise, set $\nu^2(s_2) = \max\{\lambda'(s_2), \nu^1(s_1)\}$ and set $\nu'^2(s_2)$ so that $\nu^2(s_2) + \nu'^2(s_2) = \lambda(s_2) + \lambda'(s_2)$. Note that both ν^2 and ν'^2 are distributions, with $\nu^2 \in \Lambda(t)$, $\nu'^2 \in \Lambda(t')$, and $\nu(s_i) \geq \lambda'(s_i)$ for all i; furthermore, $\nu^2(s_i) + \nu'^2(s_i) \geq \lambda(s_i) + \lambda'(s_i)$ for all i, with equality in the case of $i = 1, 2$. By repeating this process we eventually obtain $\nu \in \Lambda(t)$ and $\nu' \in \Lambda(t')$ with the required property. Thus, $\mathcal{P}_{\text{FSD}}$ monotonicity holds. $\square$

S.9 Continuation of Example 13

In this section we revisit the class of multi-prior beliefs presented in Example 13 of the main article. As we have shown, such correspondences are $\mathcal{P}_{\text{FSD}}$-increasing; however, in general, they do not increase in the strong set order or in the $\mathcal{C}$-flexible sense.

For example, let $\Omega = \{\omega_1, \omega_2\}$, $\pi(\omega_1) = \pi(\omega_2) = 1/2$, and $T = \{t, t'\}$, with $t' >_T t$. Let the correspondence A be given by $A(\omega_1, t) = \{0\}$, $A(\omega_1, t') = \{0, 3\}$, and $A(\omega_2, t) = A(\omega_2, t') = \{1, 4\}$. Therefore, the set $A(\omega, t')$ dominates $A(\omega, t)$ in the strong sense, for all $\omega \in \Omega$. Let $\Lambda^\omega(\tilde{t})$ denote the set of degenerate probability distributions over $A(\omega, \tilde{t})$, and $\Lambda(\tilde{t}) = \sum_{i=1,2} \pi(\omega_i)\Lambda^\omega(\tilde{t})$, for all $t \in T$. We claim that the correspondence Λ does not increase in the strong set order. Take distributions

$$\lambda(z) \;=\; \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{2} & \text{if } 0 \le z < 4 \\ 1 & \text{otherwise;} \end{cases} \quad \text{and} \quad \lambda'(z) \;=\; \begin{cases} 0 & \text{if } z < 1 \\ \frac{1}{2} & \text{if } 1 \le z < 3 \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, we have $\lambda \in \Lambda(t)$ since the measure is obtained by mixing degenerate measures concentrated at 0 and 4 with weights equal to $\pi(\omega_1)$ and $\pi(\omega_2)$, respectively. Similarly, we have $\lambda' \in \Lambda(t')$. However, $\lambda \wedge \lambda'$ and $\lambda \vee \lambda'$ are given by

$$(\lambda \wedge \lambda')(z) \;=\; \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{2} & \text{if } 0 \le z < 3 \\ 1 & \text{otherwise;} \end{cases} \quad \text{and} \quad (\lambda \vee \lambda')(z) \;=\; \begin{cases} 0 & \text{if } z < 1 \\ \frac{1}{2} & \text{if } 1 \le z < 4 \\ 1 & \text{otherwise.} \end{cases}$$

Since the support of $\lambda \wedge \lambda'$ is $\{0, 3\}$, it could not belong to $\Lambda(t)$ consisting of distributions with the support in $\{0, 1, 4\}$. For the same reason, there is no convex combination of $\lambda \wedge \lambda'$ and λ' that belongs to $\Lambda(t)$, since the supports of $\lambda \wedge \lambda'$ and λ' contain 3. Hence, the correspondence increases neither in the strong set order, nor in the $\mathcal{C}$-flexible sense.

S.10 Optimizing over beliefs

In the examples involving optimal saving and portfolio choice presented in Section 5 of the main paper, x is the choice variable and t is the parameter. We present below an example with a different flavor: it has both x and t as choice variables and exploits the fact that supermodularity is preserved by the sum.

Consider a firm operating in uncertain market conditions must decide how much to produce and how much to spend on advertising. In period 1, the marginal cost of production is $c > 0$ and the marginal cost of advertising is $a > 0$. If the firm chooses t units of advertising, its belief on the demand for its output s is given by a set of distributions $\Lambda(t)$. We assume that higher advertising leads to greater demand in the sense that Λ is $\mathcal{P}_{\text{FSD}}$-increasing. For an example of how this could arise, see Example 13 in the main paper.

In period 2, s is realized and the firm has to meet this demand even if it exceeds its period 1 output; the profit in period 2 is $\pi(x,s) := s - \kappa(\max\{s-x,0\})$. Function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ should be interpreted as the cost of producing the additional units to meet demand in period 2. At the same time, goods for which there is no demand can be freely disposed. Also, notice that $\pi(x,s)$ need not be increasing in s.

The firm chooses $x \geq 0$ and $t \geq 0$ in period 1 to maximize

$$\Pi(x,t,c,a) \;:=\; \min\left\{\int_S \pi(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\} - cx - at.$$

It is straightforward to check that π is supermodular if κ is increasing, convex, and $\kappa(0) = 0$.[4] Proposition 8 guarantees that $f(x,t) = \min\left\{\int_S \pi(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ is a supermodular function of (x,t) and therefore Π is supermodular in (x,t). Furthermore, Π has increasing differences in $\big((x,t),(-c,-a)\big)$. Thus $\operatorname{argmax}_{(x,t)\in\mathbb{R}^2_+} \Pi(x,t,c,a)$ decreases with (c,a) in the strong set order, i.e., a fall in advertising cost or a fall in the period 1 cost of production leads to more advertising and greater output.

S.11 Dynamic programming under ambiguity

Hopenhayn and Prescott (1992) use the tools of monotone comparative statics to analyze stationary dynamic optimization problems. In this section, we show how their results can be extended to the case where the agent has a multi-prior belief, by applying the concepts and results from Section 5 of the main article.

Consider an agent who faces a stochastic control problem where X and S are the sets of endogenous and exogenous state variables, respectively. To keep the exposition simple, we shall assume that X is a sublattice of a Euclidean space and S is a subset of another Euclidean space. The evolution of s over time follows a Markov process with the transition function λ. The agent's problem can be formulated in the following way (see Stokey et al., 1989). At each period τ, given the current state $(x_\tau, s_\tau) \in X \times S$,

[4] Take any $x' \geq x$ and consider three cases. If (i) $s \leq x$, then $\delta(s) := \big[\pi(x',s) - \pi(x,s)\big] = 0$; whenever (ii) $x < s \leq x'$, then $\delta(s) = \kappa(s-x)$; and finally (iii) $s > x'$ implies $\delta(s) = \kappa(s-x) - \kappa(s-x')$. In either case, under the assumptions imposed on κ, the function δ is increasing in s.

the agent chooses the endogenous variable $x_{\tau+1}$ for the following period; $x_{\tau+1}$ is chosen from a non-empty feasible set $B(x_\tau, s_\tau) \subseteq X$ which may depend on the current state. The single-period return is given by the function $F : X \times S \times X \to \mathbb{R}$; $F(x, s, y)$ is the payoff when (x, s) is the state variable in period τ and y is the endogenous state variable in period $\tau + 1$ chosen in period τ. Finally, we assume that the payoffs are discounted by a constant factor $\beta \in (0, 1)$.

The agent's objective is to maximize her expected discounted payoffs over an infinite horizon, given the initial condition (x, s). We denote the value of this optimization problem by $v^*(x, s)$. Under standard assumptions — in particular, the continuity and boundedness of F and the continuity of B — this problem admits a recursive representation, where $v = v^*$ is the unique solution to the Bellman equation

$$v(x,s) \;=\; \max\left\{F(x,s,y) + \beta \int_S v(y,\tilde{s})d\lambda(\tilde{s},s) : y \in B(x,s)\right\},$$

where $\lambda(\cdot, s)$ is a cumulative probability distribution over states of the world in the following period, conditional on the current state s.[5] The function v^* is bounded and continuous. Moreover, whenever we define operator $\mathscr{T} : \mathcal{B} \to \mathcal{B}$ by

$$\big(\mathscr{T}v\big)(x,s) \;=\; \max\left\{u(x,s,y) + \beta \int_S v(y,\tilde{s})d\lambda(\tilde{s},s) : y \in B(x,s)\right\},$$

that maps the space $\mathcal{B}$ of bounded and continuous real-valued functions over $X \times S$ into itself, then beginning at *any* bounded and continuous function $v \in \mathcal{B}$, function $(\mathscr{T}^n v)$ converges uniformly to v^* as n tends to infinity.[6] Furthermore, the set

$$\Phi(x,s) \;:=\; \arg\max\left\{F(x,s,y) + \beta \int_S v^*(y,\tilde{s})d\lambda(\tilde{s},s) : y \in B(x,s)\right\}$$

is non-empty and compact, for all $(x, s) \in X \times S$, and the correspondence $\Phi : X \times S \to X$ is upper hemi-continuous. We refer to any optimal control problem in which v^* and Φ have the properties listed in this paragraph as a *well-behaved* problem.

Given a well-behaved problem, Hopenhayn and Prescott (1992) (henceforth HP) apply Theorem 4.3 in Topkis (1978) to show that the value $v^*(x, s)$ *is supermodular in x and has increasing differences in (x, s)* under the following assumptions: (i) $F(x, s, y)$ is supermodular in (x, y) and has increasing differences in $\big((x, y), s\big)$; (ii) the graph of B is

[5] See Theorem 9.6 in Stokey et al. (1989) for details.

[6] By $\mathscr{T}^n$ we denote the n'th orbit of the operator $\mathscr{T}$, i.e., we have $(\mathscr{T}^{n+1}v) = \mathscr{T}\big(\mathscr{T}^n v\big)$.

a sublattice of $X \times S \times X$; (iii) $\lambda(\cdot, s)$ is increasing in s with respect to the first order stochastic dominance. The properties of v^* in turn guarantee that the function

$$f(x,s,y) \ := \ F(x,s,y) + \beta \int_S v^*(y,\tilde{s}) d\lambda(\tilde{s},s)$$

is supermodular in y and has increasing differences in $\big(y,(x,s)\big)$. By Theorem 6.1 in Topkis (1978), $\Phi(x,s)$ is a compact sublattice of X and is increasing in (x,s).[7] This in turn guarantees the existence of the greatest optimal selection

$$\phi(x,s) \ := \ \Big\{ y \in \Phi(x,s) : y \geq_X z, \text{ for all } z \in \Phi(x,s) \Big\},$$ [8]

that is increasing and Borel measurable. Lastly, the policy function ϕ induces a Markov process on $X \times S$, where, for measurable sets $Y \subseteq X$ and $T \subseteq S$, the probability of $Y \times T$ conditional on (x,s) is the probability of T conditional on s if $\phi(x,s) \in Y$, and it is zero otherwise. HP make use of the monotonicity of ϕ to guarantee that this Markov process has a stationary distribution.[9] We now consider a stochastic control problem identical to the one we just described, except that we allow the agent to be ambiguity averse. Since at each period τ the exogenous variable is drawn from the set S, the set of all possible realizations of the exogenous variable over time is given by S^∞. An expected utility maximizer behaves as though she is guided by a distribution over S^∞; to obtain the utility of a given plan of action, the agent evaluates the discounted utility on every possible path, i.e., over every element in S^∞ and takes the average across paths, weighing each path with its probability. When the agent has a maxmin preference, her behavior can be modeled by a *set* of distributions $\mathcal{M}$ over S^∞. The utility of a plan is then given by the minimum of the expected discounted utility for every distribution in $\mathcal{M}$.

In contrast to expected discounted utility, it is known that the agent's utility in the maxmin model will not generally have a recursive representation. However, there is a condition on $\mathcal{M}$ called *rectangularity* which is sufficient (and effectively necessary) for

[7] Condition (ii) on B guarantees that $B(x,s)$ is sublattice of X and that it increases with (x,s) in the strong set order. Given with the properties on f, we know that $\Phi(x,s)$ is a sublattice and that it increases with (x,s); this follows from Theorem 6.1 in Topkis (1978).

[8] Function is well-defined because Φ is compact-valued and a sublattice.

[9] The focus in this section is on primitive conditions guaranteeing the monotonicity of the policy function. Readers who are interested in how the distribution over (x,s) evolves over time (under monotonicity or weaker assumptions) should consult Huggett (2003). HP and Stachurski and Kamihigashi (2014) also discuss uniqueness and other issues relating to the stationary distribution.

this to hold (see Epstein and Schneider, 2003). Furthermore, it is known that a time-invariant version of rectangularity is also sufficient to guarantee that the agent's problem can be solved through the Bellman equation, in a way analogous to that for expected discounted utility (see Iyengar, 2005). This condition says that the agent's belief over the possible value of the exogenous variable in the following period, after observing s in the current period, is given by a *set* of distribution functions $\Lambda(s)$; this set depends on the current realization of the exogenous variable and is time-invariant. The set $\mathcal{M}$, given an initial value s_0, is obtained by concatenating the transition probabilities. Therefore, the probability of a path $(s_1, s_2, s_3, \ldots)$ is $\prod_{i=1}^{\infty} p_i$, where p_1 is the probability of s_1 for some distribution in $\Lambda(s_0)$, p_2 is the probability of s_2 for some distribution in $\Lambda(s_2)$, etc.

With this assumption on $\mathcal{M}$ in place, and some other standard conditions, one could guarantee that the value $v^*(x,s)$ of the control problem with the initial state (x,s), is the unique solution to the Bellman equation

$$v(x,s) \;=\; \max\Big\{F(x,s,y) + \beta(Av)(y,s) : y \in B(x,s)\Big\},$$

where $(Av)(y,s) = \min\big\{\int_S v(y,s)d\lambda(s) : \lambda \in \Lambda(s)\big\}$ (see Iyengar, 2005). Furthermore, the problem is *well-behaved* in the sense defined at the beginning of this section.

With this basic set-up, we are almost in a position to recover a monotone result of the HP type; all that is needed is a condition guaranteeing that $(Av)(y,s)$ is a supermodular function of (y,s), whenever v is supermodular. When X and S are one-dimensional, Proposition 8 tells us that this holds if belief $\Lambda(t)$ is $\mathcal{P}_{\text{FSD}}$-increasing.

Proposition S.1. *Consider a well-behaved optimal control problem where X, $S \subseteq \mathbb{R}$, with X compact and S finite. Let $F(x,s,y)$ be supermodular in (x,s,y), $\Lambda : S \to \triangle_S$ be $\mathcal{P}_{\text{FSD}}$-increasing, and the graph of $B : X \times S \to X$ be a sublattice; then the value function $v^*(x,s)$ is supermodular, and the correspondence $\Phi : X \times S \to \mathbb{R}$, where*

$$\Phi(x,s) \;:=\; \arg\max\Big\{F(x,s,y) + \beta(Av^*)(y,s) : y \in B(x,s)\Big\}$$

is sublattice-valued and increasing in the strong set order. Finally, the greatest selection $\phi : X \times S \to \mathbb{R}$ of Φ is well-defined, increasing, and Borel measurable.

Proof. Let $v : X \times S \to \mathbb{R}$ be a continuous and bounded function. Since the problem is well-behaved we know that the function $(\mathscr{T}v)$, given by

$$\big(\mathscr{T}v\big)(x,s) \;=\; \max\Big\{F(x,s,y) + \beta(Av)(y,s) : y \in B(x,s)\Big\},$$

is a continuous function on $X \times S$ and $\mathscr{T}^n v$ converges uniformly to v^* as $n \to \infty$. By Proposition 8 in the main paper, whenever function v is supermodular, then so is Av. This implies that $F(x, s, y) + \beta(Av)(y, s)$ is supermodular over $X \times S \times X$. Given that the graph of correspondence B is a sublattice, by Theorem 4.3 in Topkis (1978), the function $\mathscr{T} v$ is supermodular in (x, s). Since supermodularity is preserved under uniform convergence, we conclude that $v^* = \mathscr{T} v^*$ is a supermodular function of (x, s). The set $\Phi(x, s)$ consists of elements y that maximize $F(x, s, y) + \beta(Av^*)(x, s)$ over $B(x, s)$. Since the objective function is supermodular, while values of correspondence B are complete sub-lattices of X, by Theorem 6.1 in Topkis (1978), set $\Phi(x, s)$ is a complete sub-lattice of X. Furthermore, since B increases over $X \times S$ in the strong set order, so does Φ. As the problem is well-behaved, $\Phi(x, s)$ admits the greatest selection $\phi(x, s)$ and this selection is increasing. That ϕ is Borel measurable follows from standard arguments (see HP). $\square$

Below we discuss an application of this result.

Example S.2. Consider the following dynamic optimization problem of a firm. In each period, the firm collects revenue $\pi(x, s)$, where $s \in S$ denotes the realized exogenous state of the world and $x \in \mathbb{R}_+$ is the level of capital stock currently available to the firm. Once s is revealed to the firm and the revenue collected, the firm may invest $a \in [0, K]$ at a cost $c(a)$, K being a finite positive number. With this investment, capital stock in the next period is $y = \delta x + a$, where $\delta \in [0, 1]$ denotes the fraction of non-depreciated capital. Therefore, the dividend in each period is

$$F(x, s, y) \ := \ \pi(x, s) - c(y - \delta x),$$

where the firm chooses y from the interval $B(x, s) = [\delta x, \delta x + K]$. We know from HP that if the firm is an expected utility maximizer and the optimal control problem is well-behaved, the firm has a policy function that is increasing in (x, s) under these additional conditions: the transition *function* $\Lambda : S \to \triangle_S$ is increasing with respect to first order stochastic dominance and F is supermodular; the latter is guaranteed if π is supermodular and c is concave. Proposition S.1 goes further by saying that this remains true if the firm has a maxmin preference, so long as the belief Λ is $\mathcal{P}_{\text{FSD}}$-increasing.

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