CAP-AND-TRADE AND CARBON TAX MEET ARROW-DEBREU

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Abstract. We propose two general equilibrium models, quota equilibrium and emission tax equilibrium. Government specifies quotas or taxes on emissions, then refrains from further action. All results remain valid regardless of how government chooses its emissions target. Quota equilibrium exists; the allocation of emission property rights impacts the distribution of welfare. If the only externality arises from total net emissions, quota equilibrium is Pareto optimal among all feasible outcomes with the same total net emissions. For certain tax rates, emission tax equilibrium may not exist. Every quota equilibrium can be realized as an emission tax equilibrium and vice versa. However, different quota prices may arise in equilibrium from a single quota, and different emission levels may arise in equilibrium from a single tax rate. This leads to inequivalence between quota and emission tax equilibria.

Key Words: Cap-and-Trade, Carbon Tax, Arrow-Debreu Model, Equilibrium Existence, Pareto Optimality, Equivalence and Inequivalence of Cap and Tax.

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1. Introduction

The mitigation of climate change requires the reduction of greenhouse gas emissions. In the absence of regulation, market mechanisms have proven insufficient to achieve the necessary reduction. Regulatory schemes—notably "carbon taxes" and cap-and-trade—have been implemented. However, these regulatory schemes have not yet been incorporated into a general equilibrium (GE) model of the rigor and generality of Arrow and Debreu (1954). In this paper, we do so by defining quota equilibrium and emission tax equilibrium.[1]

The government first selects certain commodities (for example, carbon dioxide CO_2, methane CH_4, and chlorofluorocarbons CFCs) to be regulated. In "Cap-and-trade", each private firm is assigned an emissions quota, a property right to emit regulated commodities up to the quota. A firm may exceed its emissions quota, provided that it buys the right to do so from another firm whose emissions fall short of its quota.[2] Our notion of quota equilibrium generalizes cap-and-trade by allowing the government to assign itself a quota. Quota equilibrium thus includes two important polar cases: cap-and-trade, in which all the quota is assigned to private firms, and global quota equilibrium, in which all of the quota is assigned to the government. An *emission tax* (commonly called a "carbon tax") is a tax per unit of emissions of a regulated commodity. Note that the Walrasian auctioneer has no flexibility to alter that price in order to equilibrate supply and demand.[3]

We embed quota and emission taxes into a generalization of the Arrow-Debreu model. We study how to implement *any* emissions target, not necessarily a socially optimal target, via a quota or a tax. In our quota model, the government sells the quota it has assigned to itself; in the emission tax model, the government generates tax revenue from emissions. In both models, the government specifies how the revenue will be rebated to consumers. The government specifies a quota (along with its allocation between the private and the government firm), a tax rate, and a rebate scheme. Then, it abstains from further action,

[1]Although our discussion and examples are heavily focused on CO_2 emissions, our model allows for very general externalities in agents' preferences. In particular, our main results apply to a wide variety of externalities, including the regulation of other forms of pollution.

[2]An alternative to cap-and-trade is a *fixed quota* model, in which the government prohibits each private firm from exceeding its emissions quota. Fixed quotas are used in the regulation of local forms of pollution, which primarily affect a small geographic region. For example, fixed quotas limit the discharge of toxic chemicals from a given plant into ground water immediately adjacent to that plant.

[3]An alternative to emission tax is a *fuel tax*, which is a tax t per unit of a fuel, such as gasoline. A fuel tax is an add-on tax; if a consumer ω buys a gallon of gasoline from a firm j, ω pays a price $p+t$; the firm remits t to the government, and retains revenue p, so there is a wedge of t between the price the consumer pays and the price the firm receives. Although t is fixed, market forces can cause p to vary.

allowing market forces to determine a price that equilibrates supply and demand. Our key results are as follows.

(1) **Existence of Quota Equilibrium**: Take any quota whose total emissions are consistent with a feasible consumption-production pair. Under mild assumptions, we show in Theorem 1 that there is at least one associated quota equilibrium, and every quota equilibrium yields total net emissions equal to the chosen quota. Example 1 compares quota equilibrium with equilibrium in the associated unregulated economy. Example 2 demonstrates that the quota allocation has a major impact on the optimal quota level, the consumptions, and the distribution of welfare among agents;

(2) **Welfare Properties of Quota Equilibrium**: Because carbon dioxide emissions disperse uniformly through the atmosphere and are highly persistent, the resulting climate change is determined by the history of aggregate past emissions, regardless of where they were emitted. The effects of global warming are different in different regions, but the effects in every region depend only on the level of atmospheric carbon dioxide. For this reason, we consider the case in which the externality depends only on the total net emissions.

In the presence of externalities, the First Welfare Theorem generally does not hold, and it is too much to hope that quota equilibria are Pareto Optimal. Indeed, imposing a quota of zero net emissions today would result in an immediate return to a preindustrial society. Thus, for badly chosen quotas, quota equilibrium may lead to very bad outcomes. However, Theorem 2 is a version of the First Welfare Theorem, asserting that quota equilibria are Constrained Pareto optimal, i.e. Pareto optimal among all outcomes with the same total net emissions, so there is no way to achieve a Pareto Improvement without moving to a different target.[4]

Countries *have* adopted emission reduction targets spanning decades.[5] Once a target has been set, how can it best be implemented? Theorem 2 says that market forces alone are sufficient to ensure efficient implementation of the targets, without additional regulation. Theorem 2 formalizes an important intuition that motivates cap-and-trade. Given an emissions target, one would like to achieve it at the minimum

[4]In practice, the planner cannot hope to get the social optimum exactly right. Indeed, there is no reason to think the social planner will identify a Pareto Optimal allocation, and the literature does not provide any version of the First Welfare Theorem in this case.

[5]Under the Paris Agreement of 2015, the US pledged to reduce emissions by 50% by 2030, while the the EU pledged to reduce emissions by 55% by 2030 and achieve carbon neutrality by 2050. China has set a goal that net emissions will peak in 2030, and achieving zero net emissions by 2060.

cost. Cap-and-trade allows firms that can readily achieve emission reductions to do so, and sell their unused emission rights to other firms that find emission reductions more costly. Theorem 2 formalizes this important intuition for allowing the trading of emission rights, rather than assigning binding quotas to each firm.

(3) **Emission tax Equilibrium Model**: We formulate the emission tax equilibrium model in which the government allows for arbitrary net emissions of regulated commodities, but charges an emission tax, which specifies the emission price of each unit of regulated commodities. Example 3 and Example 4 show that emission tax equilibrium may not exist for certain tax rates. We show under mild assumptions that every quota equilibrium is an emission tax equilibrium for a carefully chosen government rebate scheme (see Theorem 3). Moreover, every emission tax equilibrium is a global quota equilibrium, i.e., the equilibrium of an economy in which all of the quota is assigned to the government (see Theorem 4). This makes the intuition in the literature precise that carbon taxes and emission quota are "almost" equivalent;[6] However, as we shall see, such equivalencce only holds in a narrow sense;[7]

(4) **Multiplicity of Equilibria**: Multiplicity of equilibria is a robust phenomenon in general equilibrium models. It occurs in static GE models, even with perfectly well-behaved prederences; see Example 15.B.2 of Mas-Colell, Whinston, and Green (1995), Example 3 and Example 5. It can also arise from linear production technology, which is commonly assumed in the Pigouvian tax literature: see Example 4 and Example 6.

Example 3 and Example 4 show that there may be multiple emissions tax equilibria with different total net emissions associated with a single tax rate. If the tax rate is set in the hope of achieving the lowest emissions, the actual emissions could be the highest. Similarly, Example 5 and Example 6 show that there may be multiple quota equilibria with different quota prices associated with the same quota. Quota equilibria achieve the target emissions, but leave uncertainty on quota prices. Emission tax equilibria fix the emissions price, but leave uncertainty on quantity of emissions;

[6]For example, Chapter 8 of Nordhaus (1979) is a pioneering study of strategies for controlling CO_2 emissions. Nordhaus writes, "In the real world, the policy can take the form either of taxing carbon emissions, or of physical controls (such as rationing). In an efficient solution, the two are interchangeable in principle"[page 136]. It is not clear whether Nordhaus, in this statement, contemplated trading of the emission rations.

[7]Previous literature (See, e.g., Weitzman (1974), Nordhaus (1979), Goulder and Schein (2013), and Stavins (2019)) suggests that quota and tax regulatory schemes are equivalent in some aspects, but inequivalent in others. There is no consensus on the exact nature of the inequivalence. For example, Goulder and Schein (2013) write, "We show that the various options are equivalent along more dimensions than often are recognized. In addition, we bring out important dimensions along which the approaches have very different impacts. Several of these dimensions have received little attention in prior literature."

(5) **Inequivalence between Quota and Emission Tax**: Multiplicity of equilibria actually leads to an important inequivalence between quota and emission tax regulatory schemes. Theorem 3 does not rule out the possibility that multiple quota equilibria with different quotas are mapped to emission tax equilibria with the same tax rate, while Theorem 4 does not rule out the possibility that multiple emission tax equilibria with different tax rates are mapped to quota equilibria with the same quota. The government does not require information on agent preferences and production technology to achieve the emissions target via quota. On the other hand, the government needs full information of agents' demand and supply correspondences in order to compute a tax rate that is consistent with the emissions target. As Example 5 and Example 6 illustrate, even with full information, due to multiplicity of equilibria, it may still be impossible for the government to achieve the emissions target through an emission tax. Previous work (Weitzman (1974)) has focused on price/quantity uncertainty arising from the regulator's incomplete information about the production functions. Here, we obtain uncertainty from a different source, multiplicity of equilibria, even if the regulator has complete information.

1.1. **Structure of the Paper.** Section 2 compares our framework and results with the existing literature. Section 3 furnishes the rigorous framework of the quota equilibrium model. The existence and constrained Pareto optimality of quota equilibrium are presented in Theorem 1 and Theorem 2, respectively. Furthermore, Example 1 compares quota equilibrium to equilibrium in the associated unregulated economy. Example 2 illustrates that the allocation of quota can have a major impact on the distribution of welfare among agents. Section 4 presents the emission tax equilibrium model. We study the connection between quota and emission tax equilibrium in Section 5. In Section 5.1, we show that every quota equilibrium can be realized as an emission tax equilibrium and vice versa (see Theorem 3 and Theorem 4). In Section 5.2, we study multiplicity of emission tax equilibrium. We show, in Example 3 and Example 4, that emission tax equilibrium specifies the price for emissions, but leaves uncertainty on emissions levels. In Section 5.3, we illustrate, in Example 5 and Example 6, the inequivalence between quota and emission tax regulatory schemes. Section 6 concludes the paper and sets out some promising future directions. Appendix A contains rigorous proofs of all theorems. The Supplementary Material, Anderson and Duanmu (2025), contains additional results and detailed analysis of examples.

2. Literature Review

Cap-and-trade and emission taxes have been extensively studied in *partial equilibrium*.[8] We discuss the limitations of partial equilibrium in Section 1.1 of the Supplementary Material. Here, we compare our formulation to previous general equilibrium approaches in the literature.

2.1. **Optimal Taxation.**

In the presence of externalities, the social cost of a production or consumption activity does not equal the private cost faced by a firm or agent, and hence the equilibrium need not satisfy the first-order conditions for Pareto Optimality. Sandmo (1975) initiates a line of research on Pigouvian taxation with externalities through general equilibrium analysis, in a representative agent setting. Subsequent literature applies Sandmo (1975)'s framework to environmental policy.[9]

(1) In this literature, the social planner has full information about the representative agent's utility function and the production technology, then computes a socially optimal outcome. The social planner equates first-order conditions to identify the *optimal Pigouvian tax rate*, under which the competitive equilibrium allocation coincides with the socially optimal outcome. In practice, the social planner can at best estimate a social optimum, and this estimated social optimum is never exactly correct. Indeed, the social planner might use her own utility function, resulting in a planner's optimum. The estimated social optimum, or planner's optimum, need not satisfy the agents' first-order conditions, and hence need not be an equilibrium. There may be no equilibrium associated with the estimated optimal Pigouvian tax; if equilibrium exists, it need not be Pareto optimal;

(2) Due to multiplicity of equilibrium, even if the optimal Pigouvian tax rate supports a given social optimum as an equilibrium, it may also support other equilibrium outcomes, which need not be Pareto Optimal.

Consideration of the Arrow-Debreu model allows us to address these issues. Sandmo (1975)'s representative agent model is a special case of our model.

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[8] See, e.g. Barnett (1980), Fischer (2003), Ciaian and Swinnen (2006) and Flachsland, Marschinski, and Edenhofer (2009). In addition, Sandmo (1975) points out that "most of the literature on optimal taxation is partial equilibrium in nature and does not contain satisfactory treatment of the benefits resulting from control of externalities."

[9] See, e.g. Bovenberg and van der Ploeg (1994), Bovenberg and Mooij (1994), Bovenberg and Goulder (1996), Golosov et al. (2014), Goulder, Hafstead, and Williams (2016), and Goulder et al. (2019). Goulder et al. (2019) departs from Sandmo's framework by combining two models. The first is a representative agent model that is used to determine the prices of commodities, both final consumption goods and inputs to production. The second model has heterogeneous agents (5 in the numerical simulations). Meshing these two models allows them to estimate the effects of a carbon tax policy on different income quintiles.

2.2. **Lindahl Equilibrium.** In an economy with a public good, a Lindahl equilibrium (see Foley (1970), del Mercato and Nguyen (2023) and Bonnisseau, del Mercato, and Siconolfi (2023)) is a system of individualized prices charged per unit of public good, such that all agents agree on a common level of the public good. Lindahl equilibrium is Pareto efficient. However, since computing the individualized prices requires knowing the demand functions of all agents, there are serious difficulties in implementing Lindahl equilibrium in practice.

In contrast to Lindahl equilibrium, cap-and-trade and emission taxes have been implemented in practice to reduce emissions.[10] Our equilibrium concepts formalize the cap-and-trade and emission tax regulatory schemes. In both of our equilibrium concepts, the prices of emissions are uniform across payers, rather than individualized as in Lindahl equilibrium. Thus, Lindahl equilibrium has very different properties from emission taxes and from cap-and-trade.

2.3. **Fixed Price Equilibrium.** Drèze (1975) considers exchange economies in which the prices of certain commodities are subject to inequality constraints, so it may not be possible to equilibrate supply and demand through prices alone. He defines a Fixed Price Equilibrium[11] in which rationing is used to equilibrate supply and demand, at fixed prices. Thus, a Fixed Price Equilibrium sets constraints on both prices *and* quantities in a single equilibrium. Both cap-and-trade and emissions taxes set quantities or prices, but not both.[12] Drèze does not allow for externalities, which are the essential motivation for regulating pollution.

2.4. **General Equilibrium with Commodity Taxation.** Shafer and Sonnenschein (1976) use an alternative tax formulation, which does not include our emission tax equilibrium model.[13] Shafer and Sonnenschein (1976)'s formulation, in particular, includes imposing fuel taxes on commodities that generate pollution, rather than on pollution *per se*. For example, a tax on gasoline is a fuel tax: the consumer pays the equilibrium price and the tax to the seller, who keeps the equilibrium price and remits the tax to the government. However, fuel taxes may be inefficient because they tax all uses of a commodity, regardless of the emissions generated. For example, natural gas may be burned, releasing a large quantity of CO_2, or may be used in the production of chemicals, releasing a much smaller quantity of CO_2. Indeed,

[10]See `https://www.c2es.org/content/cap-and-trade-basics/`.

[11]This terminology arose in the subsequent literature; see Laroque (1978).

[12]However, some jurisdictions have implemented hybrid cap-and-trade schemes incorporating limits on emission prices. Karp and Traeger (2021) propose a smart cap scheme under which the cap adjusts based on the emission price, and a smart tax scheme under which the tax adjusts based on the emission quantity. We do not consider such hybrid schemes in this paper.

[13]Our emission tax equilibrium model specifies the emission price of each unit of regulated commodities, and our formulation does not include theirs.

carbon sequestration relies on using some of the electricity generated by burning coal in order to reduce CO_2 emissions. A fuel tax on coal discourages sequestration by taxing the portion of the coal devoted to electricity generation needed for sequestration. We provide an example (see Example 2.3 in the Supplementary Material) in which fuel tax equilibrium is Pareto dominated by emission tax equilibrium. Shafer and Sonnenschein (1976) also provide an existence theorem for their general equilibrium model with commodity taxation. However, we provide a natural example (see Example 2.6 in the Supplementary Material) in which fuel tax equilibrium does not exist for certain tax rates. Thus, the assumptions in Shafer and Sonnenschein (1976) are not entirely innocuous.

Geanakoplos and Polemarchakis (2008) considered an exchange economy with externalities. Their paper is different in spirit from our work. The main theorem of Geanakoplos and Polemarchakis (2008) states that generically, in the presence of separable externalities, every Walrasian equilibrium is Pareto dominated by some tax equilibrium; we do not have a comparable result. On the other hand, their exchange setting does not permit an analysis of the effects of regulation on the production process, but CO_2 emissions are generated largely as byproducts of production. They consider only separable externalities, i.e. given agent i, the consumption of other agents does not change the marginal rates of substitution in i's preference over i's own consumption. Note that an increase in global temperature *does* affect the marginal rate of substitution between housing in Minneapolis and housing in Miami, so the externality arising from CO_2 emissions does not fit their model.

3. Quota Production Economy

In this section, we present a general equilibrium model that incorporates quota regulatory schemes on net pollution emissions, and define quota equilibrium. Our model covers two important polar cases: cap-and-trade equilibrium (the emissions quota are allocated entirely to private firms) and global quota equilibrium (the emission quota is vested in the government).

All our results apply regardless of how the emission quota is chosen.[14] Take any quota whose total emissions are consistent with a feasible consumption-production pair. Theorem 1 asserts there exists at least one quota equilibrium. Theorem 2 asserts that every quota equilibrium consumption-production pair is *constrained Pareto optimal*, i.e., it is Pareto optimal among all feasible consumption-production pairs with the same total net emissions. Example 1 compares quota equilibrium with equilibrium in the associated unregulated economy. Example 2 shows

[14] In particular, the government need not know anything about agents' preferences and demands.

that the allocation of quotas affects the set of Pareto optimal quota levels, the distribution of consumption, and the welfare of individual agents. We first formulate agents' preferences on their own consumption, following Hildenbrand (1974):

Definition 3.1. The set $\mathcal{P}$ of *own-consumption preferences* on the Euclidean space $\mathbb{R}^\ell$ consists of elements of the form $(X, \succ)$, where

- The *consumption set* $X \subset \mathbb{R}^\ell_{\geq 0}$ is closed and convex;
- The *own-consumption preference relation* $\succ$ is a continuous, irreflexive and acyclic[15] binary relation defined on X.

For every $(X, \succ) \in \mathcal{P}$ and $a, b \in X$, $a \succ b$ means a is strictly preferred to b. Note that we do not require the completeness or transitivity of $\succ$. An own-consumption preference relation $\succ$ on X is *continuous* if, for every $x, y \in X$ with $x \succ y$, there exist relatively open sets $U \ni x$ and $V \ni y$ such that $a \succ b$ for all $a \in U$ and $b \in V$. The set $\mathcal{P}$ is a compact metric space in the topology of closed convergence, as in Hildenbrand (1974). For two elements $x_1, x_2 \in \mathbb{R}^\ell$, we abuse the notation and write $(x_1, x_2) \in (X, \succ)$ if $x_1, x_2 \in X$ and $x_1 \succ x_2$. An own-consumption preference $P = (X, \succ)$ is *convex* if $\{y \in X : y \succ x\}$ is convex for every $x \in X$, and we use $\mathcal{P}_H$ to denote the set of convex own-consumption preferences from $\mathcal{P}$. Let $\Delta = \{p \in \mathbb{R}^\ell : \|p\| = \sum_{i=1}^{l} |p_i| = 1\}$ be the set of all prices.[16] Note that we allow for negative prices, which can be interpreted as prices for emissions of bads.

Definition 3.2. A quota production economy

$$\mathcal{E} \equiv \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (m^{(j)})_{j \in J}, \mathcal{Z}(m)\} :$$

is a list such that:

(1) Ω is a finite set of agents. For every agent $\omega \in \Omega$, its consumption set $X(\omega)$ is a nonempty, closed, and convex subset of $\mathbb{R}^\ell_{\geq 0}$. We write X_ω for $X(\omega)$;
(2) J is a finite set of producers. There is a single government firm, denoted as firm 0, with production set $Y_0 = \{0\}$. Each private firm $j \in J$, $j \neq 0$, has a production set $Y_j \subset \mathbb{R}^\ell$ is a non-empty subset. We write $Y = \prod_{j \in J} Y_j$;
(3) The set of allocations is $\mathcal{A} = \prod_{\omega \in \Omega} X_\omega$, equipped with the product topology;
(4) Let $M = \mathcal{A} \times Y \times \Delta$. The *global preference relation* of agent ω is $\succ_\omega \subset M \times M$. For $m, m' \in M$, $m \succ_\omega m'$ means that the agent ω strictly prefers m over m'. $\succ_\omega$ represents

[15] Irreflexive means that $a \not\succ a$. Acyclic means that if $a \succ b$, then $b \not\succ a$.
[16] Hart and Kuhn (1975) established the existence of equilibrium with the price set Δ through a fixed and antipodal point theorem (see Section 2 of Hart and Kuhn (1975)).

the agent's preference on the allocation (including the agent's own consumption), production, prices. $\succ_\omega$ is essential for studying welfare properties and Pareto rankings among consumption-production pairs;

(5) Although the global preference relation is essential for Pareto comparisons, we only need the induced *preference map* to establish the existence of equilibrium. The preference map $P_\omega : \mathcal{A}_{-\omega} \times Y \times \Delta \to \mathcal{P}$[17] of agent ω is defined by[18]

$$P_\omega(x, y, p) = (X_\omega, \{(a, b) \in X_\omega \times X_\omega | (x_{+\omega}(a), y, p) \succ_\omega (x_{+\omega}(b), y, p)\}).$$

We can write $P_\omega(x, y, p)$ as $(X_\omega, \succ_{x,y,\omega,p})$. $P_\omega(x, y, p)$ is the own-consumption preference of ω, given x, y, p. As P_ω depends on other agents' consumption, production, and prices, we allow for very general externalities.[19] However, since the agent has no control over other agents' consumption, production, and prices, she chooses her consumption bundle according to her own-consumption preference $P_\omega(x, y, p)$. For every $\omega \in \Omega$, we assume P_ω is continuous in the norm topology on $\mathcal{A}_{-\omega} \times Y \times \Delta$;

(6) $\theta(\omega)(j) \geq 0$ is agent ω's share of firm j such that $\sum_{\omega\in\Omega} \theta(\omega)(j) = 1$. We sometimes write $\theta_{\omega j}$ for $\theta(\omega)(j)$. The agents' shares of private firms are exogenous to the model. The government chooses the agents' shares $\theta(\omega)(0)$ of the government firm; this choice allows the government to determine how government revenues are rebated to agents;

(7) $e \in (\mathbb{R}^\ell_{\geq 0})^\Omega$ is the vector of initial endowments for the agents;

(8) The government chooses to regulate the first $k \leq \ell$ commodities and assigns quotas to the firms.[20] Define $m^{(j)} \in \mathbb{R}^k_{\leq 0}$ to be the negative of the quota for firm j. Let $m = \sum_{j\in J} m^{(j)}$. The *quota-compliance region* $\mathcal{Z}(m) = \{m\} \times \prod_{k<n\leq\ell} \mathcal{Z}(m)_n$ is a convex subset of $\mathbb{R}^\ell_{\leq 0}$, where $\mathcal{Z}(m)_n$ is $\{0\}$ or $\mathbb{R}_{\leq 0}$ for $k < n \leq \ell$.

Remark 3.3. The main features of our model are:

(1) The quota-compliance region $\mathcal{Z}(m)$ reflects the society's choice on which commodities to dispose, and at what quantity.[21] $-m = -\sum_{j\in J} m^{(j)}$ is the quota on the total net

[17]We adopt the standard microeconomic theory notation $\mathcal{A}_{-\omega} = \prod_{i\neq\omega} X_i$.

[18]In the following displayed formula, note that x is a $|\Omega| - 1$ dimensional vector. $x_{+\omega}(a)$ is the $|\Omega|$ dimensional vector such that its ith coordinate equals x_i for $i \neq \omega$ and its ωth coordinate is a.

[19]Following Florenzano (2003), our externalities enter via agents' preferences. We do not allow externalities to alter the production technology. See, for example, Starrett (1972), Golosov et al. (2014), Nordhaus (2017), and Kotlikoff et al. (2021) for treatment of externalities in the production process.

[20]The government may assign zero quota for a commodity, thereby prohibiting emissions. For example, CFCs and HCFCs are bads that deplete ozone, and are now prohibited under the Montréal Protocol.

[21]The quota-compliance region is closely related to the *disposal cone* introduced in Florenzano (2003). In Section 2.1 of the Supplementary Material, we compare these two concepts.

emissions of regulated commodities. $-m^{(j)}$ represents firm j's property right to emit regulated commodities. The profits attributable to this property right flow through to the shareholders of firm j. The government determines the shareholdings of the government firm and thereby determines the distribution of its profits; for example, it could choose to distribute those profits equally to all agents. The allocation of quotas between the government firm and the private firms thus plays a key role in the effect of the quota scheme on income distribution. In equilibrium, the emissions of a firm are determined endogenously and may be above or below that firm's quota. When the government firm's quota is 0, our model captures *cap-and-trade*, in which the profits from reducing emissions accrue to the shareholders of the firms that reduce those emissions. When private firms' quotas are 0, we refer to our model as a *global quota economy*; profits from reducing emissions accrue to agents, according to the shareholdings of the government firm chosen by the government. The allocation of quotas between the government firm and the private firms may be different for different regulated commodities; for example, the government might impose cap-and-trade on one regulated commodity, while assigning the quota for another regulated commodity entirely to the government firm. The government may also split the quota for a given regulated commodity between the government firm and the private firms;

(2) As $\mathcal{Z}(m)_n$ is either $\mathbb{R}_{\leq 0}$ or $\{0\}$ for $k < n \leq \ell$, our model allows for free disposal in some, and exact market clearing in other, unregulated commodities.[22] The government may choose to tolerate some bads, and allow some goods (such as atmospheric oxygen) to be left unused at price zero. Some goods are fully consumed at a nonnegative price;

(3) Our model has a single government, say the European Union. Note that in our quota equilibrium model, the regulated commodity is CO_2 emissions, not electricity. We can treat electricity consumed in France as a different commodity from electricity consumed in Italy. This would be consistent with country-varying prices for electricity, and a uniform price for CO_2 emissions throughout the European Union. Indeed, since France relies on nuclear generation of electricity, a low CO_2 emissions technology, it is important that emissions, not electricity, be the regulated commodity; see Example B.3 on Pareto inferiority of fuel tax equilibrium.

Definition 3.4. Given a quota production economy $\mathcal{E}$ with k regulated commodities, the *associated unregulated economy* $\mathcal{E}'$ is defined as:

[22] In particular, Theorem 1 remains valid if we require non-free-disposal for all unregulated commodities.

(1) $\mathcal{E}'$ has the same set of commodities and agents as $\mathcal{E}$. Agents' endowments, consumption sets and preferences are the same for $\mathcal{E}$ and $\mathcal{E}'$;
(2) $\mathcal{E}'$ and $\mathcal{E}$ have the same government firm, private firms and agents' shareholdings;
(3) In $\mathcal{E}'$, the first k commodities are no longer regulated commodities; since these commodities are no longer regulated, they can be freely disposed at equilibrium. That is, the quota-compliance region $\mathcal{Z}$ is $\mathbb{R}^k_{\leq 0} \times \prod_{k<n\leq \ell} \mathcal{Z}(m)_n$.

The associated unregulated economy $\mathcal{E}'$ can be viewed as a quota production economy without regulated commodities. The associated unregulated economy is a benchmark against which the effects of quota regulation can be measured.

We now turn to the equilibrium concept for quota production economies. Let proj_k be the projection onto the first k coordinates. We define the *quota price* as the negative of the prices for the regulated commodities. For every $\omega \in \Omega$, $p \in \Delta$ and $y \in Y$, the *quota budget set* $B^m_\omega(y,p)$ is:

$$\{z \in X_\omega : p \cdot z \leq p \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j}\big(p \cdot y(j) + \text{proj}_k(p) \cdot m^{(j)}\big)\}.$$

Each private firm j's profit at p is $p \cdot y(j) + \text{proj}_k(p) \cdot m^{(j)}$.[23] The government firm's profit at p is $p \cdot y(0) + \text{proj}_k(p) \cdot m^{(0)} = \text{proj}_k(p) \cdot m^{(0)}$; this comes solely from selling its quota. For each $(x,y,p) \in \mathcal{A} \times Y \times \Delta$ and each $\omega \in \Omega$, the *quota demand set* $D^m_\omega(x,y,p)$ is[24]:

$$D^m_\omega(x,y,p) = \{z \in B^m_\omega(y,p) : v \succ_{x_{-\omega},y,\omega,p} z \implies v \notin B^m_\omega(y,p)\}.$$

The quota demand set consists of all elements in the quota budget set $B^m_\omega(y,p)$ that maximize the agent's own-consumption preference, given (x,y,p).

Given a price p, firm j's supply set $S^m_j(p)$ is $\underset{z\in Y_j}{\text{argmax}}\big(p \cdot z + \text{proj}_k(p) \cdot m^{(j)}\big) = \underset{z\in Y_j}{\text{argmax}}\, p \cdot z$; the value of the firm's assigned quota does not depend on the firm's production plan.[25] All firms' profits depend only on prices and their own production.[26] However, the total quota and the allocation of quotas affect firms' production decisions indirectly through prices, as quotas affect equilibrium prices.

[23] If a private firm emits less than its quota, the firm generates revenue by selling its remaining quota to other firms. If a private firm emits more than its quota, the firm must purchase quota from other firms.
[24] In the following displayed formula, $x_{-\omega}$ is the vector x without the ωth coordinate, hence $x_{-\omega} \in \mathcal{A}_{-\omega}$.
[25] The assignment of quotas to firms bears an analogy to the use of firms' endowments in Geanakoplos et al. (1990) and del Mercato and Platino (2017).
[26] We assume producers are profit maximizers. Makarov (1981) established a general equilibrium existence theorem that allows firm objectives other than profit maximization.

Definition 3.5. Let $\mathcal{E}$ be a quota production economy. A $\mathcal{Z}(m)$-compliant quota equilibrium is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that

(1) $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$ (every agent's consumption is in her demand set);
(2) $\bar{y}(j) \in S^m_j(\bar{p})$ for all $j \in J$ (every firm is profit maximizing given the price $\bar{p}$);
(3) $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m)$.

As we have emphasized in the Introduction, the equilibrium price $\bar{p}$ emerges endogenously[27] after the government has established the quota (along with its allocation) and shareholdings of the government firm. Our feasibility constraint $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m)$ implies, at equilibrium, that the total net emissions of the regulated commodities equals the prespecified total quota, which is the sum of the (government and private) firms' quotas.

The associated unregulated economy in Definition 3.4 is a special case of quota production economy, and the equilibrium concept for the associated unregulated economy is a special case of quota equilibrium in Definition 3.5. For the convenience of the reader, we provide an explicit definition of equilibrium for the associated unregulated economy. For every $\omega \in \Omega$, $p \in \Delta$ and $y \in Y$, since there is no regulated commodity, the *budget set* $B_\omega(y, p)$ is:

$$\{z \in X_\omega : p \cdot z \leq p \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j} p \cdot y(j)\}.$$

For each $(x, y, p) \in \mathcal{A} \times Y \times \Delta$ and each $\omega \in \Omega$, the *demand set* $D_\omega(x, y, p)$ is:

$$D^m_\omega(x, y, p) = \{z \in B_\omega(y, p) : v \succ_{x_{-\omega}, y, \omega, p} z \implies v \notin B_\omega(y, p)\}.$$

The demand set consists of all elements in the budget set $B_\omega(y, p)$ that maximize the agent's own-consumption preference, given (x, y, p).

Since there is no regulated commodity, each private firm j's profit at price p is $p \cdot y(j)$, and the government firm's profit at the price p is $p \cdot y(0) = 0$. Thus, given a price p, firm j's supply set $S_j(p)$ is $\operatorname*{argmax}_{z\in Y_j} p \cdot z$.

Definition 3.6. Let $\mathcal{E}$ be a quota production economy and $\mathcal{E}'$ be the associated unregulated economy. A $\mathcal{Z}$-compliant equilibrium of $\mathcal{E}'$ is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that

(1) $\bar{x}(\omega) \in D_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$ (every agent's consumption is in her demand set);
(2) $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$ (every firm is profit maximizing given the price $\bar{p}$);
(3) $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}$.

[27]In particular, the equilibrium price of the quota is determined through market forces.

We now present examples that illustrate the important features of Definition 3.2. Our first example compares quota equilibrium to equilibrium in the associated unregulated economy.

Example 1. Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ be a quota production economy such that[28]

(1) $\mathcal{E}$ has three commodities CO_2, coal and electricity. CO_2 is the regulated commodity;

(2) The agent space is $\Omega = \{\omega_1, \omega_2\}$. The consumption set for both agents is $\{0\} \times \mathbb{R}^2_{\geq 0}$. Agent ω_1's endowment is $(0, 1, 0)$, while agent ω_2's endowment is $(0, 0, 0)$. Given the total net CO_2 emissions v, agent ω_1's utility function is $f_v(x_{11}, x_{21}, x_{31}) = x_{31} - v^2$, where x_{11}, x_{21}, x_{31} are agent ω_1's consumption of CO_2, coal and electricity. Agent ω_2's utility function is $g_v(x_{12}, x_{22}, x_{32}) = x_{22} - v^2$, where x_{12}, x_{22}, x_{32} are agent ω_2's consumption of CO_2, coal and electricity. The agent ω_1 derives utility from electricity, is indifferent to coal consumption, and suffers a negative externality from the total net CO_2 emissions. The agent ω_2 derives utility from coal, is indifferent to electricity consumption, and suffers a negative externality from the total net CO_2 emissions;

(3) The government firm's production set Y_0 is the singleton $\{0\}$. There is a single private firm with production set $Y_1 = \{(r, -r, r) : r \in \mathbb{R}_{\geq 0}\}$. So the private firm has the production technology to burn r units of coal to generate r units of electricity and r units of CO_2 as a byproduct. The government firm distributes all profit to agent ω_2[29] and each consumer owns an equal share of the private firm. That is, $\theta_{\omega_1}(0) = 0$, $\theta_{\omega_2}(0) = 1$, and $\theta_{\omega_1}(1) = \theta_{\omega_2}(1) = \frac{1}{2}$;

(4) The government firm is assigned quota $-m$ while the private firm is assigned quota 0. The quota-compliance region $\mathcal{Z}(m)$ is $\{m\} \times \{0\}^2$. We let $-m$ range over $(0, 1]$ to study how quota equilibrium changes in response to changes in quota levels.

Suppose the government chooses a quota $-m \in (0, 1]$. Then, there is a unique $\mathcal{Z}(m)$-compliant quota equilibrium such that:

- agents ω_1 and ω_2's equilibrium consumption are $(0, 0, -m)$ and $(0, 1 + m, 0)$;
- the private firm's equilibrium production is $(-m, m, -m)$;
- the equilibrium price is $(\frac{-m-1}{2}, \frac{-m}{2}, \frac{1}{2})$.

Once the government sets the quota $-m$, the total net CO_2 emissions v must equal $-m$. Hence, the government can achieve any desired positive emissions level by setting the quota

[28]The detailed analysis of this example is presented in Section 3.1 of the Supplementary Material.
[29]This is not essential but makes the computation easier.

to equal that emissions level.[30] The equilibrium utility of agent ω_1 is $-m - m^2$. Hence, agent ω_1's equilibrium utility is a increasing function for $-m \in (0, \frac{1}{2}]$, and is a decreasing function for $-m \in (\frac{1}{2}, 1]$. The equilibrium utility of agent ω_2 is $(1+m) - m^2$, which is a decreasing function of $-m \in (0, 1]$. For $-m = 1$, agent ω_1's utility is 0 while agent ω_2's utility is -1. Both agent ω_1 and ω_2 have lowest utility when the quota $-m = 1$.

We now consider the associated unregulated economy $\mathcal{E}'$ in which free disposal of CO_2 is allowed at equilibrium. $\mathcal{E}'$ has a unique equilibrium:

- agents ω_1 and ω_2's equilibrium consumption are $(0, 0, 1)$ and $(0, 0, 0)$, respectively;
- the private firm's equilibrium production is $(1, -1, 1)$;
- the equilibrium price is $(0, \frac{1}{2}, \frac{1}{2})$

The equilibrium price of CO_2 is 0. Since the government firm's revenue is 0, the agent ω_2's income is 0 and agent ω_1 sells all coal to the private firm. The total net emissions of CO_2 is at the highest possible level. At this unique equilibrium of $\mathcal{E}'$, agent ω_1's utility is 0 while agent ω_2's utility is -1. For every quota level in $(0, 1)$, the associated quota equilibrium has *lower* CO_2 *emissions* and *Pareto dominates* the unique equilibrium in the unregulated economy $\mathcal{E}'$.

Our next example shows that the allocation of quotas between the government and private firms affects agents' equilibrium consumption.

Example 2. Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, (m^{(j)})_{j \in J}, \mathcal{Z}(m)\}$ be a quota production economy such that

(1) $\mathcal{E}$ has three commodities CO_2, coal and electricity. CO_2 is the regulated commodity;

(2) The agent space is $\Omega = \{1, 2, 3\}$. For each $\omega \in \Omega$, agent ω's consumption set is $\{0\} \times \mathbb{R}^2_{\geq 0}$, and the endowment is $(0, \omega, 0)$. Every agent derives utility from electricity, is indifferent to coal consumption, and the only externality on agents' preferences arises from the total net CO_2 emissions;

(3) The government firm's production set Y_0 is $\{0\}$. There are two private firms with production sets $Y_1 = \{(r, -r, r) : r \in \mathbb{R}_{\geq 0}\}$ and $Y_2 = \{(-2r, 0, -r) : r \in \mathbb{R}_{\geq 0}\}$. Firm 1 has the production technology to burn r units of coal to generate r units of electricity and r units of CO_2 as a byproduct. Firm 2 has the production technology to use r units of electricity to sequester[31] $2r$ units of CO_2. In other words, r units of electricity

[30]There does not exist any quota equilibrium associated with the 0 quota level. When the quota is set to 0, $\mathcal{E}$ does not satisfy Assumption 1. Hence, there is no violation of our main existence result Theorem 1.

[31]Sequestration is an emerging technology in which CO_2 is extracted from the exhaust gases of electricity generation, and placed in long-term storage underground. We could equally well consider carbon offsets, in

and $2r$ units of CO_2 are the inputs to firm 2's production process, and the output is 0.[32] For appropriate prices, in particular a negative price for CO_2 emissions, firm 2 is motivated to purchase electricity and sequester CO_2, with zero output;

(4) Let m be the negative of the total quota on CO_2 emissions. The quota-compliance region is $\mathcal{Z}(m) = \{m\} \times \{0\}^2$. We let the quota levels and allocations vary to compare the associated quota equlibria. The second private firm is assigned a 0 quota;

(5) The government firm distributes profit equally among agents and the second private firm is owned equally by all agents. That is, $\theta_\omega(0) = \theta_\omega(2) = \frac{1}{3}$ for all $\omega \in \Omega$. For the first private firm, we have $\theta_\omega(1) = \frac{\omega}{6}$ for all $\omega \in \Omega$.

Our goal is to reduce CO_2 emissions, so we consider the case where the quota $-m < 6$. Once the government sets the quota $-m$, the total net CO_2 emissions v must equal $-m$. There is a unique equilibrium price $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, which is independent of the allocation of quotas.[33] The allocation of quotas affects the equilibrium consumption and welfare of agents.

Global Quota Economy: We first consider the case where all quotas are allocated to the government firm. For total net CO_2 emissions $v = -m \in [0, 6)$ and each agent $\omega \in \Omega$, her equilibrium consumption is $(0, 0, \frac{\omega}{2} + \frac{v}{6})$.

Cap-and-Trade: We now consider the case where all quotas are allocated to the first private firm. For total net CO_2 emissions $v = -m \in [0, 6)$ and each agent $\omega \in \Omega$, her equilibrium consumption is $(0, 0, \frac{\omega}{2} + \frac{v\omega}{12})$. For all $v = -m \in [0, 6)$, agent $\omega = 3$ is better off in the cap-and-trade case while agent $\omega = 1$ is better off in the global quota case.

3.1. **Existence of Quota Equilibrium.** In this section, for any given quota, we establish the existence of a quota equilibrium for quota production economies as in Definition 3.2. Our result generalizes Proposition 3.2.3 in Florenzano (2003), in which the quota-compliance region is a convex cone. While a quota-compliance cone allows the society to prohibit the emissions of a given pollutant, it does not allow for setting a positive cap on net emissions.

Definition 3.7. The set of quota-compliant consumption-production pair of $\mathcal{E}$ is

$$\mathcal{O} = \left\{ (x, y) \in \mathcal{A} \times Y : \sum_{\omega \in \Omega} x(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} y(j) \in \mathcal{Z}(m) \right\}.$$

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which a firm is paid to plant trees; the input is CO_2, which has a negative price, and the outputs are carbon (in the form of wood) and oxygen. The analysis of sequestration is simpler than that of carbon offsets.

[32] This may seem a little strange at first, but it is the natural way of describing the sequestration technology, which removes CO_2 from the combustion exhaust of power plants.

[33] We give computation of the equilibrium price and allocations in Section 3.2 of the Supplementary Material.

The set $\hat{Y}_j$ of quota-compliant production plans for the j-th producer is

$$\left\{y_j \in Y_j : \exists (x, y') \in \mathcal{A} \times \prod_{i \neq j} Y_i, \sum_{\omega \in \Omega} x(\omega) - \sum_{\omega \in \Omega} e(\omega) - y_j - \sum_{i \neq j} y'(i) \in \mathcal{Z}(m)\right\}.$$

The set $\hat{X}_i$ of quota-compliant consumption for the i-th agent is

$$\left\{x_i \in X_i : \exists (x', y) \in \prod_{\omega \neq i} X_\omega \times \prod_{j \in J} Y_j, x_i + \sum_{\omega \neq i} x'(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} y(j) \in \mathcal{Z}(m)\right\}.$$

We make the following survival assumption.

Assumption 1. Let $\mathcal{E}$ be a quota production economy as in Definition 3.2:

(1) there exists an agent $\omega_0 \in \Omega$ such that the set $X_{\omega_0} - \sum_{j \in J} \theta_{\omega_0 j}(Y_j + \{E(m^{(j)})\})$[34] has non-empty interior U_{ω_0} and $e(\omega_0) \in U_{\omega_0}$;

(2) there exists a commodity $s \in \{1, 2, \ldots, \ell\}$[35] such that:
- let π_s be the projection onto the s-th coordinate. The projection $\pi_s(X_{\omega_0})$ is unbounded, and the agent ω_0 has a strongly monotone own-consumption preference on the commodity s,[36] where ω_0 is the agent specified in Item 1;
- for every $\omega \in \Omega$, there is an open $V_\omega \subset \mathbb{R}$ containing the s-th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j \in J} \theta_{\omega j}(Y_j + \{E(m^{(j)})\})$[37] for all $v \in V_\omega$;

As usual in GE production models, we need a complex assumption to ensure that every quasi-equilibrium is an equilibrium. Following the previous literature, we could have assumed: for every $\omega \in \Omega$, $e_\omega \in \operatorname{int}\big(X_\omega - \sum_{j \in J} \theta_{\omega j}(Y_j + \{E(m^{(j)})\})\big)$.[38] However, that assumption would rule out the case of an agent who has no shareholding of any private firm and who is unable to consume a given commodity, and hence is restrictive.[39] Our Assumption 1 is less restrictive, in that it allows for agents who are not endowed with certain commodities, have no shareholdings of private firms and are unable to consume regulated commodities. Item 1 in Assumption 1 requires there be a single agent ω_0 whose endowment is in the interior

[34]For all $j \in J$, $E(m^{(j)}) \in \mathbb{R}^\ell_{\leq 0}$ is the vector such that its projection to the first k coordinates is $m^{(j)}$ and its other coordinates are 0.

[35]The commodity s will typically be an unregulated commodity.

[36]Given any $(x, y, p) \in \mathcal{A}_{-\omega} \times Y \times \Delta$, we have $(a, a') \in P_{\omega_0}(x, y, p)$ if $a_s > a'_s$ and $a_t = a'_t$ for all $t \neq s$.

[37]$(e(\omega)_{-s}, v)$ is the vector such that its s-th coordinate is v, and its t-th coordinate is the same as the the t-th coordinate of $e(\omega)$ for all $t \neq s$.

[38]As in the previous literature, the interior is taken with respect to the topology of $\mathbb{R}^\ell$.

[39]Poor people generally do not have any shareholdings of private firms. Moreover, as poor individuals are incapable of sequestering emissions of regulated commodities, it is reasonable to assume that the projections of their consumption sets to regulated commodities are singletons; see Footnote 49 for further explanation.

of $X_{\omega_0} - \sum_{j\in J}\theta_{\omega_0 j}(Y_j + \{E(m^{(j)})\})$,which implies that the agent ω_0's budget at any quota quasi-equilibrium is strictly positive. Item 1 in Assumption 1 and the first bullet of Item 2 in Assumption 1 imply that the quota quasi-equilibrium price of the commodity s is strictly positive. Hence, by the second bullet of Item 2 in Assumption 1, every agent has a positive budget at any quota quasi-equilibrium, which shows that every quota quasi-equilibrium is a quota equilibrium. We present a complete argument in Appendix A.1. As in the existing literature, Assumption 1 is not needed for the existence of quota quasi-equilibrium and is sufficient, but not necessary, for the existence of quota equilibrium.

We now present our main existence theorem.[40]

Theorem 1. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ be a quota production economy as in Definition 3.2. Suppose $\mathcal{E}$ satisfies Assumption 1, and the following conditions:*

(1) for $\omega \in \Omega$, we have $0 \in X_\omega$, $\hat{X}_\omega$ being compact and the preference map P_ω taking value in $\mathcal{P}_H$, the set of convex own-consumption preferences from $\mathcal{P}$;[41]

(2) for $\omega \in \Omega$ and $(x, y) \in \mathcal{O}$, there is $u \in X_\omega$ with $(u, x_\omega) \in \bigcap_{p\in\Delta\cap(\mathcal{Z}')^0} P_\omega(x_{-\omega}, y, p)$, where $\mathcal{Z}' = \{v \in \mathbb{R}^\ell : (\forall n \leq k)(v_n = 0) \wedge (\forall k < n \leq \ell)(v_n \in \mathcal{Z}(m)_n)\}$ and $(\mathcal{Z}')^0 = \{p \in \Delta : (\forall z \in \mathcal{Z}')(p \cdot z \leq 0)\}$ is the polar cone of $\mathcal{Z}'$;

(3) the aggregate production set $\bar{Y} = \left\{\sum_{j\in J} y(j) : y \in Y\right\}$ is closed and convex, and for each $j \in J$, the set $\hat{Y}_j$ is relatively compact;

(4) the set $\bar{Y} + \mathcal{Z}(m)$ is closed.

Then $\mathcal{O}$ is nonempty, i.e., it is feasible to achieve the quota, and there exists a quota equilibrium. Moreover, the equilibrium price is in $(\mathcal{Z}')^0$.

Remark 3.8. Theorem 1 generalizes Proposition 3.2.3 in Florenzano (2003) by not requiring the quota-compliance region $\mathcal{Z}(m)$ to be a cone.[42] Otherwise, the assumptions of Theorem 1 are similar to those of the existing GE literature. Assumption 1 weakens the classical survival assumption, while Item 2 in Theorem 1 weakens the classical non-satiation assumption:

(1) Closedness and convexity of consumption and production sets are standard assumptions. Compactness of quota-compliant consumption sets and relative compactness of

[40]Kubler (2024) defines an *emission-cap equilibrium*, which builds on our quota equilibrium notion. He establishes the existence of emission-cap equilibrium in incomplete markets with two periods.

[41]As noted in Florenzano (2003), this condition can be weakened to the following condition: for each $(x, y, p) \in \mathcal{O} \times (\Delta \cap \mathcal{Z}^0)$ and all $\omega \in \Omega$, $(x(\omega), x(\omega)) \notin \mathrm{conv}(P_\omega(x_{-\omega}, y, p))$, where $\mathrm{conv}(P_\omega(x_{-\omega}, y, p))$ denotes the convex hull of $P_\omega(x_{-\omega}, y, p)$.

[42]Bonnisseau, del Mercato, and Siconolfi (2023) also rely on transforming their economy into a production economy that satisfies Proposition 2.2.2 in Florenzano (2003).

quota-compliant production sets play essential roles in establishing the existence of equilibrium, as in Debreu (1959);[43]

(2) Florenzano (2003) does not require that firms' production sets contain zero. As a result, a firm may incur a loss at equilibrium, and this loss is passed on to the shareholders. We follow Florenzano (2003)'s formulation, to avoid specifying a bankruptcy mechanism for firms. In our setting, the relevant production set for firm j is $Y_j + E(m^{(j)})$, and we cannot rule out the possibility that some regulated commodities have positive equilibrium prices, so that the value of a firm's quota is negative. Thus, even if 0 is in Y_j, firm j's profit might be negative. It is essential for the existence of equilibrium that firm profits, whether positive or negative, be passed on to the agents. If one wished to make the firms limited liability, one would need to introduce a bankruptcy mechanism that specifies who bears firm losses. In all our examples, the equilibrium prices for regulated commodities are negative and 0 is an element of all production sets, which jointly imply that firms' profit at equilibrium are non-negative, so the bankruptcy issue does not arise;

(3) If the preference maps do not depend on prices, Item 2 in the assumptions of Theorem 1 is equivalent to assuming non-satiation for the set of quota-compliant consumption-production pairs;

(4) The assumption that $\bar{Y} + \mathcal{Z}(m)$ is closed follows from $\bar{Y} \cap (-\mathcal{Z}(m)) = \{0\}$, as indicated in Proposition 2.2.4 in Florenzano (2003). Note that $(-\mathcal{Z}(m)) \subset \mathbb{R}^{\ell}_{\geq 0}$. If there is no free production and each firm can choose not to produce, then $\bar{Y} \cap (-\mathcal{Z}(m)) = \{0\}$.

Theorem 1 shows that for any feasible quota, the quota production economy has a quota equilibrium, regardless of how the emissions target was chosen. At any equilibrium, the total net emissions of regulated commodities equal the prespecified quota. The proof of Theorem 1 is given in Appendix A.3. Moreover, every quota equilibrium can be realized as a global quota equilibrium by carefully setting the shareholdings of the government firm, in order to match the equilibrium distribution of the revenue from selling the quota.

Proposition 1. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ be a quota production economy as in Definition 3.2. Let $(\bar{x}, \bar{y}, \bar{p})$ be a quota equilibrium. Consider a quota production economy $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \tilde{\theta})_{\omega\in\Omega}, (Y_j)_{j\in J}, (\tilde{m}^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ where:*

[43] See Proposition 2.2.4 in Florenzano (2003) for primitive conditions implying these assumptions.

(1) all private firm's quota are 0*, and the government firm's quota is* $-m$*,*[44] *i.e.,* $\mathcal{F}$ *is a global quota economy;*

(2) $\tilde{\theta}(\omega)(j) = \theta(\omega)(j)$ *for every private firm* $j \in J$*, and* $\tilde{\theta}(\omega)(0) = \frac{\sum_{j\in J}\theta_{\omega j}\mathrm{proj}_k(\bar{p})\cdot m^{(j)}}{\mathrm{proj}_k(\bar{p})\cdot m}$.

Then $(\bar{x}, \bar{y}, \bar{p})$ *is a* $\mathcal{Z}(m)$*-compliant global quota equilibrium for* $\mathcal{F}$*.*

Note that in Proposition 1, the government firm's shareholdings in $\mathcal{F}$ depend on the equilibrium price in $\mathcal{E}$, and the excess demand is a different function of price in the two economies; they agree only at the equilibrium price, which is the same in $\mathcal{E}$ and $\mathcal{F}$. Thus, different quota equilibria in $\mathcal{E}$ are global quota equilibria in possibly different global quota economies. However, when there is only one regulated commodity, the excess demand is the same function of price in $\mathcal{E}$ and $\mathcal{F}$, therefore, every equilibrium of a quota production economy can be realized as a global quota equilibrium of a single global quota economy.

Proposition 2. *Let* $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ *be a quota production economy as in Definition 3.2 such that* $k = 1$*.*[45] *Consider another quota production economy* $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \tilde{\theta})_{\omega\in\Omega}, (Y_j)_{j\in J}, (\tilde{m}^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ *where:*

(1) all private firm's quota are 0*, and the government firm's quota is* $-m$*;*

(2) $\tilde{\theta}(\omega)(j) = \theta(\omega)(j)$ *for every private firm* $j \in J$*, and* $\tilde{\theta}(\omega)(0) = \frac{\sum_{j\in J}\theta_{\omega j}m^{(j)}}{m}$.

Then every quota equilibrium in $\mathcal{E}$ *is a global quota equilibrium in* $\mathcal{F}$*.*

We give proofs of Propositions 1 and 2 in Section 2.3 of the Supplementary Material.

3.2. **Welfare Theorem for Production Economies with Quota.** Since net CO_2 emissions affect the temperature, emissions are an important factor to consider in comparing a house in Minneapolis to one in Miami. In this section, we focus on this specific type of externality and investigate the welfare properties of quota equilibrium. In particular, we assume that the only externality arises from the total net emissions of the regulated commodities and establish that every quota equilibrium consumption-production pair is constrained Pareto optimal, i.e., it is Pareto optimal among all consumption-production pairs with the same total net emissions of the regulated commodities. As it is possible for a quota equilibrium consumption-production pair to Pareto dominate another quota equilibrium consumption-production pair with a different quota, a quota equilibrium consumption-production pair is, in general, not full Pareto optimal among all consumption-production pairs.

[44]That is, we have $\tilde{m}^0 = m$ and $\tilde{m}^j = 0$ for all $j \in J \setminus \{0\}$. Note that we have $\sum_{j\in J}\tilde{m}^{(j)} = \sum_{j\in J} m^{(j)} = m$.

[45]There is a single regulated commodity. Hence, we have $m^{(j)} \in \mathbb{R}_{\leq 0}$ for all $j \in J$.

Total net pollution emissions depend on agents' consumption and the aggregate production. In particular, for any $(x,y) \in \mathcal{A} \times Y$, $C(x,y) = \text{proj}_k\big(\sum_{\omega\in\Omega} e(\omega) + \sum_{j\in J} y(j) - \sum_{\omega\in\Omega} x(\omega)\big)$ is the total net emissions of the regulated commodities. The externality in agents' preferences needs to be taken into account in defining Pareto domination.

Definition 3.9. We say that *the only externality arises from the total net emissions of the regulated commodities* if, for all $(x,y,p) \in \mathcal{A} \times Y \times \Delta$ and $(x',y',p') \in \mathcal{A} \times Y \times \Delta$ such that $C(x,y) = C(x',y')$, $P_\omega(x_{-\omega},y,p) = P_\omega(x'_{-\omega},y',p')$ for all $\omega \in \Omega$.

When the only externality arises from the total net pollution emissions, an agent ω's global preference relation $\succ_\omega$ can be viewed as a binary relation defined on $X_\omega \times G$, where $G = \text{proj}_k\big(\sum_{\omega\in\Omega} e(\omega) + \bar{Y} - \bar{X}\big)$[46] denote the set of all possible total net emissions of the regulated commodities. We define *Pareto domination* and *full Pareto optimality* as:

Definition 3.10. For two consumption-production pairs $(x,y),(x',y') \in \mathcal{A} \times Y$, we say (x,y) *Pareto dominates* (x',y') if:

- for all $\omega \in \Omega$, $(x'(\omega), C(x',y')) \not\succ_\omega (x(\omega), C(x,y))$;
- there exists some $\omega_0 \in \Omega$ such that $(x(\omega_0), C(x,y)) \succ_{\omega_0} (x'(\omega_0), C(x',y'))$.

A consumption-production pair (g,h) *strongly Pareto dominates* another consumption-production pair (g',h') if $(g(\omega), C(g,h)) \succ_\omega (g'(\omega), C(g',h'))$ for all $\omega \in \Omega$. A consumption-production pair (x,y) is *(weakly) Pareto optimal* among $F \subset \mathcal{A} \times Y$ if no consumption-production pair in F (strongly) Pareto dominates (x,y). A consumption-production (x,y) is *(weakly) full Pareto optimal* if it is (weakly) Pareto optimal among $\mathcal{A} \times Y$.

It is too much to hope that quota equilibria are full Pareto optimal. After all, setting the quota to zero would likely result in an immediate return to a pre-industrial society. Thus, some quotas are associated with very bad equilibrium outcomes. Therefore, we need to introduce the following weakened notion of Pareto optimality to better characterize the social welfare properties of quota equilibrium.

Definition 3.11. A consumption-production pair $(x,y) \in \mathcal{A} \times Y$ is constrained *(weakly) Pareto optimal* if (x,y) is (weakly) Pareto optimal among all quota-compliant consumption-production pairs with the same total net emissions of the regulated commodities.

In the general case, constrained Pareto optimality is the most one can hope for. Most industrial countries have set emissions targets. The next theorem shows that, regardless

[46] $\bar{X} = \sum_{\omega\in\Omega} X_\omega$ is the aggregate consumption set.

of how the emissions target is set, constrained Pareto optimality of the quota equilibrium consumption-production pair is achieved through market forces.[47] Hence, there is no way to Pareto improve on a quota equilibrium without moving to a different emissions level.

Theorem 2. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ be a quota production economy as in Definition 3.2, and suppose that the only externality arises from the total net emissions of the first k regulated commodities. Suppose $\mathcal{Z}(m)_n = \{0\}$ for all $k < n \leq \ell$, i.e., we require non-free-disposal for unregulated commodities at equilibrium.*[48] *Let $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m)$-compliant quota equilibrium. Then:*

(1) $(\bar{x}, \bar{y})$ is constrained weakly Pareto optimal;

(2) Suppose $P_\omega(\bar{x}_{-\omega}, \bar{y}, \bar{p})$ is negatively transitive and locally non-satiated for all $\omega \in \Omega$. Then $(\bar{x}, \bar{y})$ is constrained Pareto optimal.

The proof of Theorem 2 is given in Appendix A.4. In Section 2.4 of the Supplementary Material, we present a special case in which the externality depends only on the *production* of regulated commodities.[49]

Note that the total net emissions of the regulated commodities is likely to affect agents' preferences. Thus, for a fixed rebate scheme, changing the quota alters the consumers' welfare, and the equilibrium consumption-production pair for one quota may Pareto dominate the equilibrium consumption-production pair for a different quota. As the next example illustrates, it may even be possible to identify a quota level such that every associated quota equilibrium is full Pareto optimal, and the allocation of quota may affect agents' utility, hence altering these quota levels. However, changing the rebate scheme may benefit some consumers and disadvantage others, but it cannot result in a Pareto improvement.

Example 2 (Continued)**.** For a given total net CO_2 emissions v, suppose that every agent's preference is characterized by the same utility function $u_v(x_1, x_2, x_3) = x_3 - \frac{v^2}{2}$, where

[47] In particular, more prescriptive regulations such as mandating the emissions level of each firm are not required, and indeed can be counter-productive.

[48] Even in the context of Arrow and Debreu (1954) and McKenzie (1959), the first welfare theorem may fail if some coordinates of the equilibrium prices are negative or the equilibrium allows free disposal. Since we allow for possible negative equilibrium prices for unregulated commodities, non-free-disposal for unregulated commodities is necessary for the validity of this theorem. In fact, if we allow for free disposal of electricity in Example 2, then there is a quota equilibrium that is not constrained Pareto optimal.

[49] In the classical general equilibrium model developed Arrow and Debreu (1954), equilibrium assigns ownership which conveys the right to consume the commodity, but does not entail the obligation to eliminate it. In the regulation of pollution, a firm that accepts payment to take ownership of emissions must be obligated to eliminate these emissions, for example, by sequestering CO_2. Moreover, individual consumers are technologically incapable of eliminating emissions, an industrial process available only to firms. Therefore, we assume that agents cannot consume the regulated commodities.

x_1, x_2, x_3 is the agent's consumption of CO_2, coal and electricity, respectively. As in Sandmo (1975), we assume that the government has full information about agents' endowments and utility functions as well as the production technology, and its objective is to find the quota which leads to full Pareto optimal quota equilibrium.

Global Quota Economy: Recall that agent ω's equilibrium consumption is $(0, 0, \frac{\omega}{2} + \frac{v}{6})$ in the global quota economy. Thus, the agent ω's utility is $(\frac{\omega}{2} + \frac{v}{6}) - \frac{v^2}{2}$. Taking the derivative with respect to v, we see that all agents prefer the same quota level, $-m = v = \frac{1}{6}$. Thus, the quota equilibrium associated with the quota level $\frac{1}{6}$ is the unique full Pareto optimal quota equilibrium. This quota equilibrium Pareto dominates all other (global) quota equilibria.

Cap-and-Trade: Recall that agent ω's equilibrium consumption is $(0, 0, \frac{\omega}{2} + \frac{v\omega}{12})$ in the cap-and-trade economy. For agent $\omega \in \Omega = \{1, 2, 3\}$, her utility is $(\frac{\omega}{2} + \frac{v\omega}{12}) - \frac{v^2}{2}$. Taking the derivative with respect to v, we see that agent ω's utility is a strictly increasing function of v on $[0, \frac{\omega}{12}]$, and a strictly decreasing function of v on $(\frac{\omega}{12}, 1]$. Agents with lower endowment prefer a lower quota, while agents with higher endowment prefer a higher quota. All quota equilibria associated with quota level $-m = v \in [\frac{1}{12}, \frac{1}{4}]$ are full Pareto optimal among all (cap-and-trade) quota equilibria. The society faces a choice among different quota levels.

4. Emission Tax Production Economy

An alternative to setting quotas is to set tax rates on pollution emissions. In this section, we present a general equilibrium model that incorporates a tax regulatory scheme on net pollution emissions. The government rebates emission tax revenue to agents according to a rebate scheme. A general equilibrium model that incorporates emission tax of pollution is:

Definition 4.1. An emission tax production economy

$$\mathcal{F} \equiv \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \lambda\}$$

is a list such that

(1) All firms are private firms. Consumption sets, preferences, endowments, production sets, and agents' shares of private firms are defined the same as in Definition 3.2;

(2) $\lambda \in \mathbb{R}^{\Omega}_{\geq 0}$ is the government's rebate scheme of agents with $\sum_{\omega\in\Omega} \lambda(\omega) = 1$.[50] The government rebates the emission tax revenue to agents according to λ;

[50] Both Shafer and Sonnenschein (1976) and Geanakoplos and Polemarchakis (2008) rebate tax revenue to agents. The rebate scheme in Geanakoplos and Polemarchakis (2008) provides an equal rebate to each agent.

(3) The compliance region $\mathcal{V}$ takes the form of $\prod_{n\leq \ell}\mathcal{V}_n$ where $\mathcal{V}_n=\mathbb{R}_{\leq 0}$ for all $n\leq k$ and $\mathcal{V}_n$ is either $\{0\}$ or $\mathbb{R}_{\leq 0}$ for all $n>k$;

(4) $t\in \mathrm{proj}_k(\Delta)$ is an emission tax rate on the emissions of the first k commodities.

Remark 4.2. The features of our general equilibrium model with emission tax are:

- The government rebate scheme λ plays the same role as agents' shareholdings of the government firm in production economies with quota, both of which are determined by the government. Moreover, the emission property rights of regulated commodities in an emission tax production economy coincide with the emission property rights in the corresponding global quota economy, since all rights are vested in the government, and the profits are distributed to agents according to the rebate scheme;
- As the first k coordinates of the compliance region $\mathcal{V}$ are $\mathbb{R}_{\leq 0}$, we allow for arbitrary net emissions for the regulated commodities, but charge a tax on the emissions of these commodities. As in production economies with quota, we allow for either free disposal or non-free-disposal of the unregulated commodities;

Recall that $C(x,y)=\mathrm{proj}_k\big(\sum_{\omega\in\Omega}e(\omega)+\sum_{j\in J}y(j)-\sum_{\omega\in\Omega}x(\omega)\big)$ is the total net emissions of the regulated commodities. For every $\omega\in\Omega$, $p\in\Delta$ and $(x,y)\in\mathcal{A}\times Y$, the *emission tax budget set* $B^t_\omega(x,y,p)$ is defined to be:

$$\{z\in X_\omega : p\cdot z\leq p\cdot e(\omega)+\sum_{j\in J}\theta_{\omega j}p\cdot y(j)+\lambda(\omega)t\cdot C(x,y)\}.$$

So an agent's budget consists of the value of her endowment, her dividends from firms and her rebate from the government's emission tax revenue. For each $\omega\in\Omega$ and $(x,y,p)\in\mathcal{A}\times Y\times\Delta$, the *emission tax demand set* $D^t_\omega(x,y,p)$ is defined to be:

$$\{z\in B^t_\omega(x,y,p) : v\succ_{x_{-\omega},y,\omega,p} z \implies v\notin B^t_\omega(x,y,p)\}.$$

The emission tax demand set consists of all elements in the emission tax budget set $B^t_\omega(x,y,p)$ that maximize the agent's own-consumption preference, given (x,y,p).

The equilibrium notion for production economies with emission tax is:

Definition 4.3. Let $\mathcal{F}=\{(X,\succ_\omega,P_\omega,e_\omega,\theta)_{\omega\in\Omega},(Y_j)_{j\in J},\mathcal{V},t,\lambda\}$ be an emission tax production economy. A $\mathcal{V}$-compliant emission tax equilibrium is a triple $(\bar{x},\bar{y},\bar{p})\in\mathcal{A}\times Y\times\Delta$ such that the following conditions are satisfied:

(1) $\mathrm{proj}_k(\bar{p})=-t$, i.e., the tax rate t specifies the emissions price of regulated commodities;

(2) $\bar{x}(\omega)\in D^t_\omega(\bar{x},\bar{y},\bar{p})$ for all $\omega\in\Omega$;

(3) $\bar{y}(j) \in S_j(\bar{p}) \equiv \underset{z \in Y_j}{\operatorname{argmax}}\, \bar{p} \cdot z$ for all $j \in J$. So every firm is profit maximizing given the price $\bar{p}$;

(4) $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{V}$.

Once the government sets the tax rate and the rebate scheme, the total net emissions of the regulated commodities are determined endogenously.

5. "Equivalence" and Inequivalence of Quota and Emission Tax Equilibria

In this section, we study the connection between quota and emission tax equilibrium. Both quota and emission tax equilibrium exhibit multiplicity, which eventually leads to *inequivalence* between quota and emission tax regulatory schemes. In particular, we demonstrate in Theorem 3, Theorem 4, Example 3, Example 4, Example 5 and Example 6 that:

(1) Every quota equilibrium can be realized as an emission tax equilibrium and vice versa. Despite this "equivalence" result, quota and emission tax equilibria are substantially different, due to the multiplicity of equilibria;

(2) For a given emission tax rate, there may be multiple equilibria with substantially different total net emissions associated with the same tax rate. Hence, if the tax rate is set in the hope of achieving the lowest total net emissions among these equilibria, the actual total net emissions could be the highest among these equilibria. Moreover, there may be no emission tax equilibrium for some emission tax rates;

(3) For a given quota, there may be multiple equilibria with substantially different quota prices associated with the same total net pollution emissions level;

(4) While we can always guarantee a desired total net pollution emissions target via setting a quota, it may be impossible to do so via emission tax.

5.1. "Equivalence" of Quota and Emission Tax Equilibria.

Our next theorem shows that every quota equilibrium can be realized as an emission tax equilibrium.

Theorem 3. *Let* $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (m^{(j)})_{j\in J}, (Y_j)_{j\in J}, \mathcal{Z}(m)\}$ *be a quota production economy as in Definition 3.2. Let* $(\bar{x}, \bar{y}, \bar{p})$ *be a* $\mathcal{Z}(m)$*-compliant quota equilibrium of* $\mathcal{E}$*. Let* $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J'}, \mathcal{V}, t, \lambda\}$ *be an emission tax production economy such that:*

(1) $\mathcal{V}_n = \mathbb{R}_{\leq 0}$ *for all* $n < k$ *and* $\mathcal{V}_n = \mathcal{Z}(m)_n$ *for all* $n \geq k$*;*

(2) J' *is the set of private firms in* J*;*

(3) $t = -\pi_k(\bar{p})$*;*

(4) $\lambda(\omega) = \frac{\sum_{j\in J} \theta_{\omega j} \pi_k(\bar{p}) \cdot m^{(j)}}{\pi_k(\bar{p}) \cdot m}$.

Then, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-compliant emission tax equilibrium of $\mathcal{F}$. Suppose, in addition, $k = 1$, i.e., there is one regulated commodity. Then $\lambda(\omega) = \frac{\sum_{j \in J} \theta_{\omega j} m^{(j)}}{m}$ is independent of the price.

The proof of Proposition 1 is given in Appendix A.5. Note that the government's rebate scheme depends on the equilibrium price in $\mathcal{E}$, and the excess demand is a different function of price in the two economies. Thus, different quota equilibria in $\mathcal{E}$ are emission tax equilibria in possibly different production economies with emission tax. However, when there is only one regulated commodity, excess demand is the same function of price in $\mathcal{E}$ and $\mathcal{F}$, and the government's rebate scheme is independent of the equilibrium price in $\mathcal{E}$. Theorem 3 does not rule out the possibility that multiple quota equilibria with different quotas are mapped to emission tax equilibria with the same tax rate.

Our next theorem shows that every emission tax equilibrium can be realized as a (global) quota equilibrium. Its proof is given in Appendix A.5.

Theorem 4. *Let $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega \in \Omega}, (Y_j)_{j \in J}, \mathcal{V}, t, \lambda\}$ be an emission tax production economy as in Definition 4.1. Let $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{V}$-compliant emission tax equilibrium. Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \tilde{\theta})_{\omega \in \Omega}, (m^{(j)})_{j \in J'}, (Y_j)_{j \in J'}, \mathcal{Z}(m)\}$ be a global quota economy where:*

(1) The set J' is the union of J and the government firm, denoted as firm 0, with the production set $Y_0 = \{0\}$;

(2) $m = -C(\bar{x}, \bar{y})$ and $\mathcal{Z}(m)_n = \mathcal{V}_n$ for all $n \geq k$;

(3) For every $\omega \in \Omega$, we have $\tilde{\theta}(\omega)(j) = \theta(\omega)(j)$ for all $j \in J$, and $\tilde{\theta}(\omega)(0)(p) = \lambda(\omega)$, i.e., the agents' shareholdings of the government firm equal the government's rebate scheme.

Then $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant global quota equilibrium for $\mathcal{E}$.

By Theorem 2 and Theorem 4, emission tax equilibrium is constrained weakly Pareto optimal, and is constrained Pareto optimal if agents' preferences are negatively transitive and locally nonsatiated. Theorem 4 does not rule out the possibility that multiple emission tax equilibria with different tax rates are mapped to quota equilibria with the same quota. One might think that Theorem 3 and Theorem 4 establish the interchangeability of emission taxes and quota. On the contrary, as we shall see, multiplicity of equilibria actually leads to *inequivalence* between quota and emission tax regulatory schemes.

5.2. **Multiplicity of Emission Tax Equilibria.** In this section, we focus on multiplicity of emission tax equilibrium. As we shall see, emission tax equilibrium specifies the price for emissions, but leaves uncertainty on emissions levels.

Example 3. Multiplicity of equilibria is a common phenomenon in exchange economies. The textbook example is Example 15.B.2 of Mas-Colell, Whinston, and Green (1995).[51] We extend this textbook example to show there may be multiple emission tax equilibria with different emissions levels, but the same tax rate.[52] In particular, define $\mathcal{F}$ to be the following emission tax economy:

(1) The economy $\mathcal{F}$ has commodities 0, 1 and 2. There are two agents with the same consumption set $\mathbb{R}^3_{\geq 0}$. The first agent's endowment is $(0.2, 2, r)$ and the second agent's endowment is $(0.2, r, 2)$, where $r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$;

(2) Commodity 0 is a regulated commodity. The compliance region is $\mathbb{R}_{\leq 0} \times \{0\}^2$. That is, we allow for free disposal of commodity 0 while requiring non-free-disposal at equilibrium of commodities 1 and 2. Any part of the endowments of commodity 0 that are not consumed are emitted;

(3) Both agents' preferences are characterized by utility functions. The first agent's utility function is $U_1(x_{01}, x_{11}, x_{21}) = \left(x_{11} - \frac{1}{8}x_{21}^{-8}\right) - \frac{1}{10}x_{01}^2$, where x_{01}, x_{11}, x_{21} are the first agent's consumption of commodities 0, 1, and 2. The second agent's utility function is $U_2(x_{02}, x_{12}, x_{22}) = \left(x_{22} - \frac{1}{8}x_{12}^{-8}\right) - \frac{1}{2}x_{02}^2$, where x_{02}, x_{12}, x_{22} are the second agent's consumption of commodities 0, 1, and 2. Both agents' utility functions are strictly increasing with respect to commodities 1 and 2, strictly decreasing with respect to commodity 0, and are strictly concave;

(4) The government imposes an emission tax on commodity 0. We compute emission tax equilibrium for different nonnegative tax rates. The government rebates the tax revenue equally to the two agents.

Apart from the addition of the regulated commodity 0, $\mathcal{F}$ is exactly the same exchange economy considered in Example 15.B.2 of Mas-Colell, Whinston, and Green (1995).

The government sets a tax rate t, hence the price of commodity 0, $p_0 = -t < 0$. By agents' utility functions, agents consume commodity 0 only to generate additional income to consume commodities 1 and 2. In this example, we normalize p_0, p_1, p_2 such that $p_1 + p_2 = 1$. By detailed computation in Section 3.3 of the Supplementary Material, agents' demands are

[51] The uniqueness of equilibria is assured only under special conditions. See Section 17.F of Mas-Colell, Whinston, and Green (1995).

[52] Note that the exchange economy $\mathcal{F}$ in Example 15.B.2 of Mas-Colell, Whinston, and Green (1995) is a *regular economy*. By the remark of Debreu (1970), there is an open set W of endowments such that, if the endowment of $\mathcal{F}$ is replaced by any element in W, the modified exchange economy still has the same number of multiple equilibria. This situation remains true for an open set of endowments in our extended exchange economy. Thus, our example is in no sense a “knife-edge” case.

functions of tax rates and price ratio $\frac{p_1}{p_2}$. Thus, given a tax rate t, we simply need to find zeros of a function (the excess demand function for commodities 1 and 2) of a single variable $\frac{p_1}{p_2}$ in order to determine the equilibria corresponding to this tax rate.[53] For a wide range of tax rates, we numerically compute equilibrium price ratio $\frac{p_1}{p_2}$ and percentage reduction of commodity 0 at each of the equilibria. We present our findings in the following table:

Table 1: Equilibrium Price Ratio $\frac{p_1}{p_2}$ and Percentage Reduction of Commodity 0 Emissions for Various Tax Rates

	Equilibrium 1		Equilibrium 2		Equilibrium 3	
Tax Rates	$\frac{p_1}{p_2}$	% Reduction	$\frac{p_1}{p_2}$	% Reduction	$\frac{p_1}{p_2}$	% Reduction
0.000	2.000	0.00	1.000	0.00	0.500	0.00
0.002	2.003	5.25	0.995	6.01	0.502	8.23
0.004	2.012	10.50	0.979	12.09	0.510	16.32
0.006	2.027	15.74	0.952	18.31	0.523	24.13
0.008	2.047	20.98	0.912	24.79	0.544	31.48
0.010	2.071	26.21	0.856	31.75	0.578	38.09
0.012	2.099	31.44	0.754	40.16	0.653	42.92
0.0122827	2.103	32.18	0.700	42.51	0.700	42.51
0.012283	2.103	32.18	/	/	/	/
0.0380099	2.602	100.00	/	/	/	/
$\geq$ 0.03800995	/	/	/	/	/	/

TABLE 1. This table presents the price ratio $\frac{p_1}{p_2}$ and the percentage reduction of commodity 0 emissions at equilibrium for a wide range of tax rates. The percentage reduction of commodity 0 emissions is the aggregate amount of commodity 0 consumed, and hence not emitted, over the aggregate endowment of commodity 0. The price ratio and percentage reduction are rounded as indicated. There are three distinct equilibria for each of the first seven tax rates. For the tax rate 0.0122827, Equilibria 2 and 3 merge into one equilibrium. For tax rates 0.012283 and 0.0380099, there is a unique equilibrium. For tax rates greater than or equal to 0.03800995, there is no equilibrium.

The following two figures provide further illustrations of equilibrium price ratios $\frac{p_1}{p_2}$ and percentage reduction of commodity 0 as functions of tax rates.

[53]By **??**, if the excess demand for commodity 1 is zero, so is the excess demand for commodity 2.

Equilibrium Price Ratios as Functions of Tax Rates

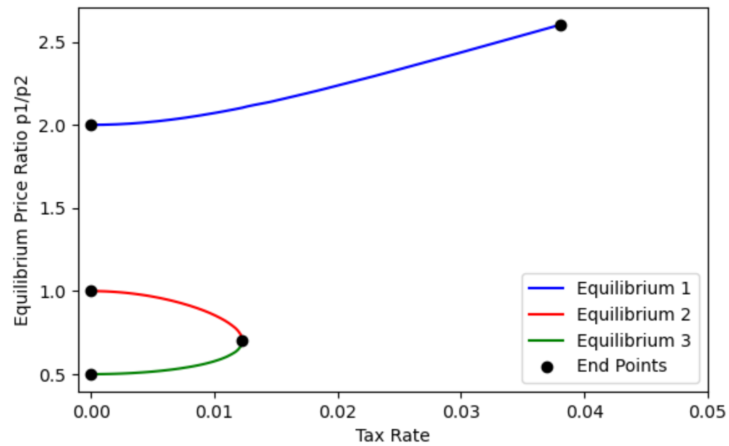

FIGURE 1. This figure plots the equilibrium price ratios $\frac{p_1}{p_2}$ as functions of tax rates. The price ratios for the second and the third equilibria merge at the tax rate 0.0122827, and there is a unique equilibrium beyond this tax rate. The price ratio of the first equilibrium is an increasing function of tax rates, and there is no equilibrium for tax rates greater than or equal to 0.03800995.

Emissions Percentage Reduction as Functions of Tax Rates

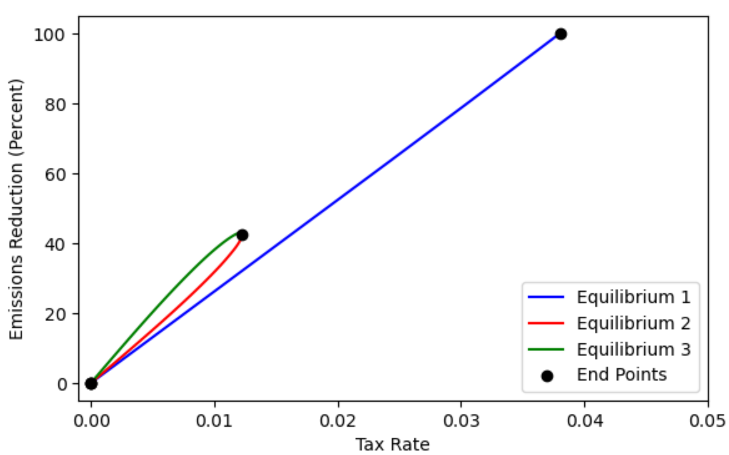

FIGURE 2. This figure plots the percentage reduction of emissions of commodity 0, as functions of tax rates. The percentage reductions are increasing functions of tax rates for each equilibrium. The percentage reductions for the second and the third equilibria merge at the tax rate 0.0122827, and there is a unique equilibrium beyond this tax rate. There is no equilibrium for tax rates greater or equal to 0.03800995.

We now summarize our findings from Table 1 and the above two figures:

(1) Agents consume commodity 0 only to generate additional income for consuming commodities 1 and 2. Thus, the three equilibria associated with the tax rate 0 have identical price ratios $\frac{p_1}{p_2}$ and consumptions of commodities 1 and 2 to the three equilibria in Example 15.B.2 of Mas-Colell, Whinston, and Green (1995);

(2) As we can see from Table 1 and Fig. 2, for each tax rate in the interval $[0, 0.0122827)$, there are three equilibria with substantially different reductions of commodity 0 emissions. Hence, if the tax rate is set in the hope of achieving the maximum reduction of commodity 0 emissions among these three equilibria, the actual reduction could easily be the minimum among these three equilibria;

(3) The emissions percentage reduction of the first equilibrium is almost linear as a function of tax rates. This almost linearity is specific to this example and not generic;

(4) The second and the third equilibria merge at the tax rate 0.0122827, and then disappear. There is a unique equilibrium for tax rates in the interval $(0.0122827, 0.0380099]$. Moreover, within the interval $(0.0122827, 0.0380099]$, the percentage reduction of commodity 0 emissions increases as tax rates increase;

(5) One might hope that higher tax rates always lead to lower emissions of the regulated commodity. However, tax rate increases also shift the equilibrium price ratio, which may offset the effect created by higher tax rates. In this example, suppose that the social planner sets the tax rate at 0.0122827 and the economy settles at the merged equilibria 2 and 3. Then, as illustrated in Fig. 2, any small increase in the tax rate will force the economy to shift to equilibrium 1, resulting in an increase in the emissions of commodity 0. We suspect there are examples in which the relationship between tax rates and emissions reduction is not monotonic, even along a single branch of the equilibrium correspondence;

(6) When the tax rate is greater than or equal to 0.03800995, agents' aggregate demand for commodity 0 exceeds the aggregate endowment, so there is no equilibrium.

In this example, there are multiple emission tax equilibria with the same tax rates but substantially different total net emissions of the regulated commodity. Thus, while the total net emissions of regulated commodities of a quota equilibrium are fully specified by the quota, it is entirely possible that the total net emissions of regulated commodities are substantially larger than the desired total net emissions when the social planner sets the tax rate.

Multiplicity of emission tax equilibrium can also arise from linear production technology. In particular, given any prespecified emissions target below the highest possible emissions level, there may be no tax rate that will guarantee that the target is achieved.

Example 4. Let $\mathcal{F} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, \mathcal{V}, t, \lambda\}$ be an emission tax production economy:

(1) $\mathcal{F}$ has three commodities CO_2, coal and electricity. CO_2 is the regulated commodity;
(2) There is a single agent with consumption set $\{0\} \times \mathbb{R}^2_{\geq 0}$ and endowment $e = (0, 1, 0)$. The agent derives utility from electricity, is indifferent over coal consumption, and suffers from a negative externality arising from the total net CO_2 emissions;
(3) There are two producers with production sets $Y_1 = \{(r, -r, r) : r \in \mathbb{R}_{\geq 0}\}$ and $Y_2 = \{(-2r, 0, -r) : r \in \mathbb{R}_{\geq 0}\}$. The first producer has the production technology to burn r units of coal to generate r units of electricity and r units of CO_2. The second producer has the production technology to use r units of electricity to sequester $2r$ units of CO_2. Both producers distribute all profit to the agent;
(4) The compliance region $\mathcal{V}$ is $\mathbb{R}_{\leq 0} \times \{0\}^2$. We allow for free disposal of CO_2 while requiring non-free-disposal at equilibrium for coal and electricity;
(5) We let the tax rate $t \geq 0$ vary to compute emission tax equilibrium associated with different tax rates. The government rebates all its tax revenue to the agent.

Claim 5.1. *There is a $\mathcal{V}$-compliant emission tax equilibrium if and only if $t \leq \frac{1}{4}$.*[54]

We now investigate the possibility of achieving an emissions target via emission taxation.

Claim 5.2. *If the emission tax rate $t < \frac{1}{4}$, then the total net emissions of CO_2 at every $\mathcal{V}$-compliant emission tax equilibrium is 1.*

By Claim 5.2, if the government sets the emission tax rate to be less than $\frac{1}{4}$, then the total net CO_2 emissions is 1 unit, which is the highest possible CO_2 emissions. We now consider the case where the emission tax rate is $\frac{1}{4}$.

Claim 5.3. *For every $0 \leq v \leq 1$, there is a $\mathcal{V}$-compliant emission tax equilibrium associated with the emission tax rate $\frac{1}{4}$ such that the total net emissions of CO_2 is v.*

[54]Bonnisseau, del Mercato, and Siconolfi (2023) provide examples of the nonexistence of equilibria for economies with externalities. The nonexistence of emission tax equilibria is driven by the fact that exogeneous tax rates limit the ability to adjust prices to achieve equilibrium, which does not seem related to the source of non-existence in Bonnisseau, del Mercato, and Siconolfi (2023).

Conclusions of Claim 5.1, Claim 5.2, and Claim 5.3 are illustrated in Fig. 3, and their proofs are presented in Section 3.4 of the Supplementary Material. By these claims, we conclude that, in this example, it is impossible to set a tax rate to ensure that the CO_2 emissions are strictly less than 1.

Emission Tax with Linear Technology (Example 4)

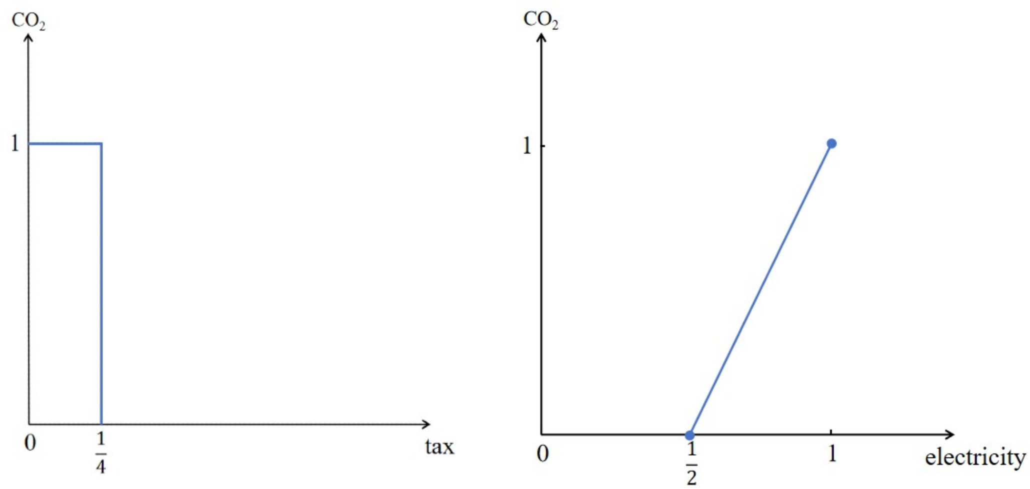

FIGURE 3. The left figure plots the equilibrium total net CO_2 emissions as a correspondence of the tax rate $t \in [0, \frac{1}{4}]$. The right figure plots the equilibrium electricity consumption/CO_2 emissions pairs corresponding to $t = \frac{1}{4}$. Note that it is impossible to set an emission tax rate to ensure that the CO_2 emissions is strictly less than 1.

In the above example, linear production technology is the driving force behind multiplicity of emission tax equilibria. When a firm's production technology is linear, its profit-maximizing production correspondence is an unbounded ray for certain prices, and empty for other prices. In the case of an unbounded ray, this often results in a continuum of equilibria with the same tax rate but different total net emissions. Linear production technology is commonly assumed in models addressing Pigouvian taxation in the presence of externalities.[55] A successful general equilibrium theory of climate change must either eschew linear production technology, or take seriously the problems arising from the resulting multiplicity of equilibria.

We have so far focused on multiplicity of emission tax equilibria. We will see, in Section 5.3, that quota equilibrium may also exhibit multiplicity. In fact, quota equilibrium precisely achieves the emissions target, but leaves uncertainty on quota prices.

[55] See, e.g. Sandmo (1975), Bovenberg and van der Ploeg (1994), Bovenberg and Mooij (1994), Bovenberg and Goulder (1996), and Goulder and Williams III (2003).

5.3. **Inequivalence of Quota and Emission Tax Equilibria.** In this section, we present two examples of the inequivalence between quota and emission tax regulatory schemes. These examples have the same emission tax production economies as in Example 3 and Example 4, respectively. As we shall see, the inequivalence between quota and emission tax regulatory schemes is a direct consequence of multiplicity of equilibria.

Example 5. Let $\mathcal{F}$ be the same emission tax production economy as in Example 3. Recall that any endowments of commodity 0 that are not consumed are emitted. Suppose that the government has chosen a 30% emissions reduction target for commodity 0, so that 70% of the aggregate endowment of commodity 0 is emitted. Let us compare the options (quota versus emission tax) for achieving this reduction. As in most of this paper, all of our analysis remains valid regardless of how that emissions reduction was chosen.

We first focus on emission tax. The first problem for the government is to determine a tax rate that will hopefully lead to the 30% reduction. To do this, the government needs to know agents' demand functions.[56] From this information, the government can compute tax rates which are consistent with the 30% reduction. The following table presents the emission percentage reduction corresponding to tax rates that are consistent with the 30% reduction.

Table 2: Percentage Reduction of Commodity 0 Emissions at Tax Rates for which one Equilibrium achieves 30% Reduction

Tax Rates	Equilibrium 1 % Reduction	Equilibrium 2 % Reduction	Equilibrium 3 % Reduction
0.011448	30.00	37.49	41.97
0.00951705	24.95	30.00	36.59
0.0075832	19.89	23.41	30.00

Table 2. This table presents tax rates and percentage reduction of commodity 0 emissions when the percentage reduction of commodity 0 emissions is 30% for at least one equilibrium. The emissions percentage reductions are rounded as indicated. At the tax rate 0.011448, the percentage reduction of emissions is 30% at Equilibrium 1. At the tax rate 0.00951705, the percentage reduction of emissions is 30% at Equilibrium 2. At the tax rate 0.0075832, the percentage reduction of emissions of is 30% at Equilibrium 3.

As the diagonal of Table 2 illustrates, for each of the three equilibrium branches, there is a tax rate that achieves the 30% reduction. Unfortunately, for each of these tax rates, there are

[56] Note that the government does not need to know agents' utility functions, which are much harder to observe than demand functions.

two equilibria with reduction substantially different from 30%. Thus, none of these tax rates necessarily guarantees a 30% reduction. Hence, it is impossible for the government to achieve the 30% reduction via emission tax, even if the government has full information about agents' utility functions. The tax rate is fixed, but the emissions reduction is uncertain.

We now turn our attention to the quota regulatory scheme. Formally, we consider the following quota production economy $\mathcal{E}$:

(1) The economy $\mathcal{E}$ has the same set of commodities and agents as $\mathcal{F}$. The agents' endowments, consumption sets and utility functions are also the same for $\mathcal{E}$ and $\mathcal{F}$;
(2) There is only a government firm with production set $Y_0 = \{0\}$. The government firm distributes profit equally to the two agents;
(3) Let m be the negative of the quota on commodity 0. The quota-compliance region $\mathcal{Z}(m)$ is $\{m\} \times \{0\}^2$, i.e., we impose a total quota $-m$ on commodity 0 and require non-free-disposal at equilibrium for commodities 1 and 2. We let m vary to compute quota equilibrium associated with different quota levels.

$\mathcal{F}$ and $\mathcal{E}$ are essentially the same economy except:

- In $\mathcal{F}$, the government sets the tax rate and rebates the tax revenue equally to agents;
- In $\mathcal{E}$, the government sets the quota and rebates the quota revenue equally to agents.

In order to obtain quota equilibria with emissions equal to 70% of the aggregate endowment of commodity 0, by Theorem 3 and Theorem 4, we simply need to identify all emission tax equilibria that result in a 30% reduction of commodity 0 emissions. The following table presents the quota prices corresponding to this quota level at each of the three equilibria.[57]

Table 3: Quota Prices for which Quota Equilibria achieve 30% Reduction			
	Equilibrium 1	Equilibrium 2	Equilibrium 3
% Reduction	Quota Price	Quota Price	Quota Price
30.00	0.011448	0.00951705	0.0075832

Table 3. This table presents quota prices for all equilibria when the percentage reduction of commodity 0 is set to be 30%. The quota price is the negative of the price of the regulated commodity 0, and represents the price of per unit emission rights of the regulated commodity 0. There are three quota equilibria associated with a 30% emissions reduction. The emission rights prices are 0.011448, 0.00951705, 0.0075832 for Equilibrium 1, 2, 3, respectively.

[57]Note that, as expected from Theorem 3 and Theorem 4, the equilibrium quota prices agree with those tax rates in Table 2 for which one emission tax equilibrium achieves a 30% emissions reduction.

As Table 3 illustrates, there are three equilibria associated with the 70% of the aggregate endowment level of the commodity 0 quota, and each equilibrium exactly achieves the 30% reduction target. Note that the government requires no information about agents' preferences in order to guarantee the 30% reduction via quota. The emissions reduction is fixed, but the quota price is uncertain.

Example 6. Let $\mathcal{F}$ be the same emission tax production economy as in Example 4. Suppose the government has chosen a CO_2 emissions level $\hat{v} < 1$.[58] Let us compare the options (quota versus emission tax) for achieving this desired emissions level. As in most of this paper, all of our analysis remains valid regardless of how that emissions level was chosen. By Claim 5.1, Claim 5.2, and Claim 5.3, it is impossible to guarantee the desired emissions level via emission tax, even if the government has full information about agents' demand functions.

We now turn our attention to the quota regulatory scheme. Formally, we consider the following quota production economy $\mathcal{E}$:

(1) The economy $\mathcal{E}$ has the same agent and set of commodities as $\mathcal{F}$. The agent's endowment, consumption set and preference are also the same for $\mathcal{E}$ and $\mathcal{F}$;
(2) $\mathcal{E}$ and $\mathcal{F}$ have the same private firms. In addition, $\mathcal{E}$ has a government firm with production set $Y_0 = \{0\}$. Since there is only one agent, all firms distribute profit to the single agent;
(3) Let m be the negative of the quota on CO_2. All quotas are allocated to the government firm. The quota-compliance region $\mathcal{Z}(m)$ is $\{m\} \times \{0\}^2$, i.e., we impose a total quota $-m$ on CO_2 and require non-free-disposal at equilibrium for coal and electricity. We let m vary to compute quota equilibrium associated with different quota levels.

$\mathcal{F}$ and $\mathcal{E}$ are essentially the same economy except:

- In $\mathcal{F}$, the government sets the tax rate and rebates the tax revenue to the agent;
- In $\mathcal{E}$, the government sets the quota and rebates the quota revenue to the agent.

By Claim 5.3, for any $0 \leq v \leq 1$, there is a $\mathcal{V}$-compliant emission tax equilibrium such that the total net CO_2 emissions is v. By Theorem 4, for any quota level $0 \leq -m \leq 1$, $\mathcal{E}$ has a $\mathcal{Z}(m)$-compliant quota equilibrium. Hence, the government simply sets the quota level to be $\hat{v}$, then every quota equilibrium associated with this quota level exactly achieves the desired total net CO_2 emissions level.

[58] Note that the largest possible CO_2 emissions level is 1.

6. Concluding Remarks and Future Work

The main contributions of this paper are:

(1) We formulate the quota equilibrium model, which includes two polar cases: cap-and-trade equilibrium (in which the emission property rights are vested in the private firms), and global quota equilibrium (in which the emission property rights are vested in the government, which redistributes the proceeds to the agents according to the rebate scheme). In Example 1, we compare quota equilibrium with equilibrium of the associated unregulated economy. Example 2 shows that the property rights specified in quota equilibrium have a major impact on the distribution of welfare among agents;

(2) The government's choice of quota is exogeneous to the quota equilibrium model. Theorem 1 shows that quota equilibrium exists for all feasible quota. Moreover, Theorem 2 shows that every quota equilibrium consumption-production pair is constrained Pareto optimal, i.e., Pareto optimal among all consumption-production pairs with the same total net emissions of the regulated commodities. Both Theorem 1 and Theorem 2 remain valid regardless of how the quota was chosen;

(3) We formulate the emission tax equilibrium model in which the government allows for arbitrary net emissions of regulated commodities, but charges a tax on the emissions. The government then rebates its tax revenue to agents according to a rebate scheme. In contrast to the case of quota equilibrium, Example 3 and Example 4 illustrate that emission tax equilibrium need not exist for certain tax rates;

(4) The emission tax equilibrium model and the global quota equilibrium model allocate property rights to agents in identical ways, namely, through the government rebate scheme. In particular, as shown in Theorem 3 and Theorem 4, every quota equilibrium can be realized as an emission tax equilibrium with a corresponding tax rate and government rebate scheme, and every emission tax equilibrium can be realized as a global quota equilibrium. One might think that Theorem 3 and Theorem 4 establish the equivalence between emission taxes and quota. However, as shown in Example 3, Example 4, Example 5 and Example 6, different quota prices may arise in equilibrium from a single quota, and different emissions levels may arise in equilibrium from a single tax rate. Thus, quota equilibria exhibit uncertainty in quota prices, while emission tax equilibria exhibit uncertainty in total net emissions;

(5) Multiplicity of equilibria actually lead to inequivalence between quota and emission tax equilibrium. Given an emissions target, the government can guarantee achieving

the target by setting it as the quota; no information is required on agents' preferences or firms' production technology. On the other hand, the government needs extensive knowledge of agents' demands and firms' supply correspondences in order to determine a tax rate that is consistent with the emissions target. Even with full information, as Example 5 and Example 6 illustrate, multiplicity of equilibria may prevent the government from achieving the emissions target via emission tax.

The paper also suggests the following promising directions for future work:

(1) The analysis of inequality requires the notion of quota equilibrium with tranfers. In a separate paper Anderson and Duanmu (2024), we address the extent to which the Second Welfare Theorem holds for the quota equilibrium model. We introduce the notion of *fixed-quota equilibrium* which is widely used for the regulation of local externalities. For example, the externality generated by the release of toxic chemicals into River A affects different people than the release into River B. Thus, the externality depends on the whole vector of firms' emissions of regulated commodities, rather than just on the sum of the emissions of firms. Such firm-specific externalities are typically addressed by binding, not tradable, quotas. Fixed-quota equilibrium constrains the production technology of each firm, just as the typical regulation of chemical discharges into rivers constrains the production choices of each firm. We show that every constrained Pareto optimum is a fixed-quota equilibrium with transfers;

(2) The Arrow-Debreu model is inherently static.[59] It would be desirable to extend our equilibrium model to multiple time periods.

(3) There is a single government in our quota and emission tax models. Hence, our models do not allow for international trade of commodities. It would be desirable to develop quota and emission tax models with multiple governments, for example the Unites States, China and the European Union, in order to model the bargaining among countries over their emissions.

A. Appendix

This appendix provides complete proofs of Theorem 1, Theorem 2, Theorem 3, and Theorem 4. We first, in Appendix A.2, prove a special case of Theorem 1, where the quota-compliance region is a cone. We then complete the proof of Theorem 1 in Appendix A.3.

[59]The addition of Arrow securities allows consideration of multiple time periods in the Arrow-Debreu model, but the markets meet only once. Golosov et al. (2014) analyze a dynamic stochastic general equilibrium (DSGE) model with climate change as an externality.

The proof of Theorem 2 is in Appendix A.4. Finally, the proofs of Theorem 3 and Theorem 4 are given in Appendix A.5.

We first introduce the concept of quota quasi-equilibrium. Let

$$\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$$

be a quota production economy as in Definition 3.2. Given $(x, y, p) \in \mathcal{A} \times Y \times \Delta$, the *quota quasi-demand set* $\bar{D}^m_\omega(x, y, p)$ is defined as:

$$\{z \in B^m_\omega(y, p) : v \succ_{x_{-\omega},y,\omega,p} z \implies p \cdot v \geq p \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j}\big(p \cdot y(j) + \mathrm{proj}_k(p) \cdot m^{(j)}\big)\}.$$

A $\mathcal{Z}(m)$*-compliant quota quasi-equilibrium* is $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that:

(1) $\bar{x}(\omega) \in \bar{D}^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$;
(2) $\bar{y}(j) \in S^m_j(\bar{p})$ for all $j \in J$;
(3) $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m)$.

Note that quasi-equilibrium is not stable since agents could, in principle, do better within their budget sets. Thus, the interest of the quasi-equilibrium concept is purely mathematical:

(1) In Appendix A.1, we show that, under Assumption 1, every quota quasi-equilibrium is a quota equilibrium;
(2) In Appendix A.2, we prove the special case of Theorem 1 by first establishing the existence of a quota quasi-equilibrium, then applying the result mentioned in the previous item to show that the quota quasi-equilibrium is a quota equilibrium;
(3) In Appendix A.3, we complete the proof of Theorem 1.

A.1. **From Quasi-Equilibrium to Equilibrium.** Under Assumption 1 and zero quota, every $\mathcal{Z}(m)$-compliant quota quasi-equilibrium is a $\mathcal{Z}(m)$-compliant quota equilibrium.

Lemma A.1. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ be a quota production economy satisfying Assumption 1. Suppose $m^{(j)} = 0$ for all $j \in J$. Then every $\mathcal{Z}(m)$-compliant quota quasi-equilibrium is a $\mathcal{Z}(m)$-compliant quota equilibrium.*

Proof. Let $(\bar{x}, \bar{y}, \bar{p})$ be a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium. For each consumer ω, define $\delta_\omega : \Delta \twoheadrightarrow X_\omega$ as $\delta_\omega(p) = \{x_\omega \in X_\omega : p \cdot x_\omega < p \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j} \sup\{p \cdot y : y \in Y_j\}\}$. We start by establishing the following claim:

Claim A.2. $\delta_{\omega_0}(\bar{p}) \neq \emptyset$.

Proof. By Assumption 1, the set $X_{\omega_0} - \sum_{j\in J}\theta_{\omega_0 j}Y_j$ has an empty interior U_{ω_0} and $e(\omega_0) \in U_{\omega_0}$. Hence, we can pick $u_{\omega_0} \in \mathbb{R}^\ell$ such that $\bar{p}\cdot u_{\omega_0} < 0$ and that $(e(\omega_0)+u_{\omega_0}) \in (X_{\omega_0} - \sum_{j\in J}\theta_{\omega_0 j}Y_j)$. As $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium, we thus have $\bar{p}\cdot\tilde{x}_{\omega_0} < \bar{p}\cdot e(\omega_0) + \sum_{j\in J}\theta_{\omega_0 j}\bar{p}\cdot\bar{y}(j)$ for some $\tilde{x}_{\omega_0} \in X_{\omega_0}$. So we have $\delta_{\omega_0}(\bar{p}) \neq \emptyset$. □

Claim A.2 leads to the following result:

Claim A.3. *If $\hat{x} \in X_{\omega_0}$ with $(\hat{x},\bar{x}(\omega_0)) \in P_{\omega_0}(\bar{x}_{-\omega_0},\bar{y},\bar{p})$, then $\bar{p}\cdot\hat{x} > \bar{p}\cdot e(\omega_0)+\sum_{j\in J}\theta_{\omega_0 j}\bar{p}\cdot\bar{y}(j)$, i.e., $\hat{x}$ is not in agent ω_0's budget set given $\bar{p}$ and $\bar{y}$.*

Proof. Let $\hat{x} \in X_{\omega_0}$ be such that $(\hat{x},\bar{x}(\omega_0)) \in P_{\omega_0}(\bar{x}_{-\omega_0},\bar{y},\bar{p})$. By Claim A.2, pick $z_{\omega_0} \in \delta_{\omega_0}(\bar{p})$. As $m^{(j)} = 0$ for all $j \in J$, we have $\bar{p}\cdot\hat{x} \geq \bar{p}\cdot e(\omega_0) + \sum_{j\in J}\theta_{\omega_0 j}\bar{p}\cdot\bar{y}(j)$ since $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium. As $P_{\omega_0}(\bar{x}_{-\omega_0},\bar{y},\bar{p})$ is continuous, there exists $\lambda \in (0,1)$ such that $(\lambda z_{\omega_0} + (1-\lambda)\hat{x},\bar{x}(\omega_0)) \in P_{\omega_0}(\bar{x}_{-\omega_0},\bar{y},\bar{p})$.

Assume that $\bar{p}\cdot\hat{x} = \bar{p}\cdot e(\omega_0) + \sum_{j\in J}\theta_{\omega_0 j}\bar{p}\cdot\bar{y}(j)$. Then we have $(\lambda z_{\omega_0} + (1-\lambda)\hat{x},\bar{x}(\omega_0)) \in P_{\omega_0}(\bar{x}_{-\omega_0},\bar{y},\bar{p})$ and $\lambda z_{\omega_0}+(1-\lambda)\hat{x} \in \delta_{\omega_0}(\bar{p})$. This furnishes us with a contradiction since $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium. So, we have $\bar{p}\cdot\hat{x} > \bar{p}\cdot e(\omega_0)+\sum_{j\in J}\theta_{\omega_0 j}\bar{p}\cdot\bar{y}(j)$. □

As the agent ω_0 has a strongly monotone own-consumption preference on the commodity s and the projection $\pi_s(X_{\omega_0})$ is unbounded, by Claim A.3, we conclude that $\bar{p}_s > 0$.

Claim A.4. *For every $\omega \in \Omega$, $\delta_\omega(\bar{p}) \neq \emptyset$.*

Proof. Note that, for every $\omega \in \Omega$, there is an open set V_ω containing the s-th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j\in J}\theta_{\omega j}Y_j$ for all $v \in V_\omega$. As $\bar{p}_s > 0$, we can pick $u_\omega \in \mathbb{R}^\ell$ such that $\bar{p}\cdot u_\omega < 0$ and that $(e(\omega)+u_\omega) \in (X_\omega - \sum_{j\in J}\theta_{\omega j}Y_j)$. As $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium, we have $\bar{p}\cdot\tilde{x}_\omega < \bar{p}\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j)$ for some $\tilde{x}_\omega \in X_\omega$. So, we have $\delta_\omega(\bar{p}) \neq \emptyset$. □

We now show that $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium. For each $\omega \in \Omega$, by Claim A.4, pick $z_\omega \in \delta_\omega(\bar{p})$ and $\hat{x}_\omega \in X_\omega$ such that $(\hat{x}_\omega,\bar{x}(\omega)) \in P_\omega(\bar{x}_{-\omega},\bar{y},\bar{p})$. As $m^{(j)} = 0$ for all $j \in J$, we have $\bar{p}\cdot\hat{x}_\omega \geq \bar{p}\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j)$ since $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium. As $P_\omega(\bar{x}_{-\omega},\bar{y},\bar{p})$ is continuous, there exists $\lambda \in (0,1)$ such that $(\lambda z_\omega + (1-\lambda)\hat{x}_\omega,\bar{x}(\omega)) \in P_\omega(\bar{x}_{-\omega},\bar{y},\bar{p})$. Assume that $\bar{p}\cdot\hat{x}_\omega = \bar{p}\cdot e(\omega) + \sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j)$. Then we have $(\lambda z_\omega + (1-\lambda)\hat{x}_\omega,\bar{x}(\omega)) \in P_\omega(\bar{x}_{-\omega},\bar{y},\bar{p})$ and $\lambda z_\omega + (1-\lambda)\hat{x}_\omega \in \delta_\omega(\bar{p})$. This is a contradiction since $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium. Therefore, we have $\bar{p}\cdot\hat{x}_\omega > \bar{p}\cdot e(\omega)+\sum_{j\in J}\theta_{\omega j}\bar{p}\cdot\bar{y}(j)$. Hence, $(\bar{x},\bar{y},\bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium. □

A.2. **Existence of Quota Equilibrium with Quota-Compliance Cone.** In this section, we prove a special case of Theorem 1 when the quota-compliance region is a cone.

Theorem A.5. *Let $\mathcal{E} = \{(X, \succ_\omega, P_\omega, e_\omega, \theta_\omega)_{\omega\in\Omega}, (Y_j)_{j\in J}, (m^{(j)})_{j\in J}, \mathcal{Z}(m)\}$ be a quota production economy as in Definition 3.2. Let $\mathcal{Z}^0 = \{p \in \Delta : (\forall z \in \mathcal{Z}(m))(p \cdot z \leq 0)\}$ be the polar cone of $\mathcal{Z}(m)$. Suppose $\mathcal{E}$ satisfies Assumption 1, and the following conditions:*

(1) $m^{(j)} = 0$ for all $j \in J$;

(2) for all $\omega \in \Omega$, we have $0 \in X_\omega$, $\hat{X}_\omega$ being compact and the preference map P_ω taking value in $\mathcal{P}_H$;

(3) for all $\omega \in \Omega$, for each $(x, y) \in \mathcal{O}$, there exists $u \in X_\omega$ such that $(u, x_\omega) \in \bigcap_{p\in\Delta\cap\mathcal{Z}^0} P_\omega(x_{-\omega}, y, p)$;

(4) the aggregate production set $\bar{Y} = \left\{\sum_{j\in J} y(j) : y \in Y\right\}$ is closed and convex, and for each $j \in J$, the set $\hat{Y}_j$ is relatively compact;

(5) $\bar{Y} + \mathcal{Z}(m)$ is closed.

Then, there exists $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$ such that:

(1) $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium;

(2) we have $\bar{p} \in \mathcal{Z}^0$ and $\bar{p} \cdot \left(\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j)\right) = 0$.

Proof. As $m = \sum_{j\in J} m^{(j)} = 0$, the quota-compliance region $\mathcal{Z}(m)$ is a cone, and neither the government firm nor the private firms have any quota. Thus, Theorem A.5 is similar to Proposition 3.2.3 in Florenzano (2003). For $\omega \in \Omega$, define the correspondence $Q_\omega : \mathcal{A} \times Y \times \Delta \twoheadrightarrow X_\omega$ by $Q_\omega(x, y, p) = \{a \in X_\omega | (a, x_\omega) \in P_\omega(x_{-\omega}, y, p)\}$. Note that Q_ω is lower hemicontinuous since the preference map P_ω is continuous. As P_ω takes value in $\mathcal{P}_H$ and is irreflexive, $x_\omega \notin \mathrm{conv}(Q_\omega(x, y, p))$ for all $(x, y, p) \in \mathcal{A} \times Y \times \Delta$ and all $\omega \in \Omega$. By Item 3, we have $\bigcap_{p\in\Delta\cap\mathcal{Z}^0} Q_\omega(x, y, p) \neq \emptyset$ for all $(x, y) \in \mathcal{O}$. By Assumption 1, we have $e(\omega) \in X_\omega - \sum_{j\in J} \theta_{\omega j} Y_j$ for all $\omega \in \Omega$. As $\hat{X}_\omega$ is compact for every $\omega \in \Omega$, $\bar{Y}$ is closed and convex, $\hat{Y}_j$ is relatively compact for every $j \in J$ and $\bar{Y} + \mathcal{Z}(m)$ is closed, by Proposition 3.2.3 in Florenzano (2003), we conclude that $\mathcal{E}$ has a $\mathcal{Z}(m)$-compliant quota quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p}) \in \mathcal{A} \times Y \times \Delta$. Moreover, we have $\bar{p} \in \mathcal{Z}^0$ and $\bar{p} \cdot \left(\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j)\right) = 0$. As $m^{(j)} = 0$ for all $j \in J$, by Lemma A.1, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium. □

A.3. **Proof of Theorem 1.**

Proof of Theorem 1. By Item 1 in Assumption 1 and the second bullet of Item 2 in Assumption 1, we have, for all $\omega \in \Omega$, $e(\omega) \in X_\omega - \sum_{j\in J} \theta_{\omega j}(Y_j + \{E(m^{(j)})\})$. So the set $\mathcal{O}$ of quota-compliant consumption-production pairs is nonempty, hence is feasible to achieve the quota. Let $\mathcal{E}' = \{(X, \succ'_\omega, P'_\omega, e_\omega, \theta)_{\omega\in\Omega}, (Y'_j)_{j\in J}, (\tilde{m}^{(j)}), \mathcal{Z}'\}$ be a quota production economy:

(1) $Y'_j = Y_j + \{E(m^{(j)})\}$ for all $j \in J$. Let $Y' = \prod_{j\in J'} Y'_j$;
(2) Each firm's quota $\tilde{m}^{(j)}$ is 0;
(3) Let $\mathcal{Z}' = \{v \in \mathbb{R}^\ell : (\forall n \le k)(v_n = 0) \wedge (\forall k < n \le \ell)(v_n \in \mathcal{Z}(m)_n)\}$;
(4) We only define the induced preference map P'_ω [60]. For $y \in Y'$ and $j \in J$, let $y(\mathcal{E}) \in Y$ be such that $y(\mathcal{E})_j = y_j - E(m^{(j)})$. For $\omega \in \Omega$, the preference map $P'_\omega : \mathcal{A}_{-\omega} \times Y' \times \Delta \to \mathcal{P}$ is $P'_\omega(x, y, p) = (X_\omega, \{(a, b) \in X_\omega \times X_\omega | (x_{+\omega}(a), y(\mathcal{E}), p) \succ_\omega (x_{+\omega}(b), y(\mathcal{E}), p)\})$.

To show that the derived economy $\mathcal{E}'$ has a $\mathcal{Z}'$-compliant quota equilibrium, we must verify that $\mathcal{E}'$ satisfies the assumptions of Theorem A.5. It is easy to see that:

(1) By the construction of P'_ω, it takes value in $\mathcal{P}_H$ for all $\omega \in \Omega$. By Assumption 1, we have $e_\omega \in X_\omega - \sum_{j\in J} \theta_{\omega j}(Y_j + \{E(m^{(j)})\})$. Hence, we have $e(\omega) \in X_\omega - \sum_{j\in J} \theta_{\omega j} Y'_j$ for all $\omega \in \Omega$;
(2) there exists an agent $\omega_0 \in \Omega$ such that the set $X_{\omega_0} - \sum_{j\in J} \theta_{\omega_0 j} Y'_j$ has non-empty interior $U_{\omega_0} \subset \mathbb{R}^\ell$ and $e(\omega_0) \in U_{\omega_0}$;
(3) there exists a commodity $s \in \{1, 2, \ldots, \ell\}$ such that:
   - the projection $\pi_s(X_{\omega_0})$ is unbounded, and the agent ω_0 has a strongly monotone own-consumption preference on the commodity s;
   - for every $\omega \in \Omega$, there is an open set V_ω containing the s-th coordinate $e(\omega)_s$ of $e(\omega)$ such that $(e(\omega)_{-s}, v) \in X_\omega - \sum_{j\in J} \theta_{\omega j} Y'_j$ for all $v \in V_\omega$;
(4) The projection of $\mathcal{Z}'$ to the first k-th coordinates is $\{0\}$.

For every $i \in \Omega$, let $\hat{X}'_i$ be the set of quota-compliant consumption for the i-th agent of the economy $\mathcal{E}'$. In particular, $\hat{X}'_i$ takes the form of:

$$\left\{ x_i \in X_i : \exists(\tilde{x}, y) \in \prod_{\omega\neq i} X_\omega \times \prod_{j\in J} Y'_j, x_i + \sum_{\omega\neq i} \tilde{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} y(j) \in \mathcal{Z}' \right\}.$$

It is straightforward to verify that $\hat{X}'_i$ is the same set as $\hat{X}_i$. Hence, $\hat{X}'_\omega$ is compact for all $\omega \in \Omega$ since $\hat{X}_\omega$ is compact for all $\omega \in \Omega$.

Claim A.6. *For all $\omega \in \Omega$ and all quota-compliant consumption-production pairs (x, y) of $\mathcal{E}'$, there exists $u \in X_\omega$ so that $(u, x(\omega)) \in \bigcap_{p\in\Delta\cap(\mathcal{Z}')^0} P'_\omega(x_{-\omega}, y, p)$.*

[60] The agent's global preference relation $\succ'_\omega$ is defined similarly. To establish the existence of a quota equilibrium, one only needs to work with the preference map P'_ω.

Proof. Fix $\omega \in \Omega$. Let (x, y) be a quota-compliant consumption-production pair of $\mathcal{E}'$. Then $(x, y(\mathcal{E}))$ is a quota-compliant consumption-production pair of $\mathcal{E}$. So there exists $u \in X_\omega$ such that $(u, x(\omega)) \in \bigcap_{p \in \Delta \cap (\mathcal{Z}')^0} P_\omega(x_{-\omega}, y(\mathcal{E}), p)$. As $P'_\omega(x_{-\omega}, y, p) = P_\omega(x_{-\omega}, y(\mathcal{E}), p)$ for all $p \in \Delta$, we have $(u, x(\omega)) \in \bigcap_{p \in \Delta \cap (\mathcal{Z}')^0} P'_\omega(x_{-\omega}, y, p)$. □

Claim A.7. *The aggregate production set $\bar{Y}' = \sum_{j \in J} Y'_j$ is closed and convex. For each $j \in J$, the set $\hat{Y}'_j$ of quota-compliant production plans for the j-th producer for the economy $\mathcal{E}'$ is relatively compact. Finally, the set $\bar{Y}' + \mathcal{Z}'$ is closed.*

Proof. Note that $\bar{Y}' = \bar{Y} + \{E(m)\}$, where $E(m) \in \mathbb{R}^\ell_{\leq 0}$ is the vector such that its projection to the first kth coordinates is m and its rest coordinates are 0. As $\{E(m)\}$ is a singleton and $\bar{Y}$ is closed and convex, $\bar{Y}'$ is closed and convex. Note that $\bar{Y}' + \mathcal{Z}' = \bar{Y} + \{E(m)\} + \mathcal{Z}' = \bar{Y} + \mathcal{Z}(m)$. Hence, we know that $\bar{Y}' + \mathcal{Z}'$ is closed. For every $j \in J$, the set $\hat{Y}'_j$ of quota-compliant production plans for the j-th producer for the economy $\mathcal{E}'$ is:

$$\left\{ y_j \in Y'_j : \exists (x, \tilde{y}) \in \mathcal{A} \times \prod_{i \neq j} Y'_i, \sum_{\omega \in \Omega} x(\omega) - \sum_{\omega \in \Omega} e(\omega) - y_j - \sum_{i \neq j} \tilde{y}(i) \in \mathcal{Z}' \right\}.$$

Note that an element $y_j \in \hat{Y}'_j$ if and only if $y_j - E(m^{(j)}) \in \hat{Y}_j$. As $\hat{Y}_j$ is relatively compact for every $j \in J$, $\hat{Y}'_j$ is relatively compact for every $j \in J$. □

By Theorem A.5, there is a $\mathcal{Z}'$-compliant quota equilibrium $(\bar{x}, \bar{y}, \bar{p})$ for $\mathcal{E}'$ where $\bar{p} \in (\mathcal{Z}')^0$. We now show that $(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ is a quota equilibrium for $\mathcal{E}$:

(1) Note that we have $\bar{p} \cdot \bar{y}(j) = \bar{p} \cdot \bar{y}(\mathcal{E})(j) + \text{proj}_k(\bar{p}) \cdot m^{(j)}$. For every $j \in J$, we have $\bar{y}(j) \in \operatorname*{argmax}_{z \in Y'_j} \bar{p} \cdot z$. As $\text{proj}_k(\bar{p}) \cdot m^{(j)}$ is a constant over Y_j, we have $\bar{y}(\mathcal{E})(j) \in S^m_j(\bar{p})$;

(2) As $\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) \in \mathcal{Z}'$, we have

$$\sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(\mathcal{E})(j) = \sum_{\omega \in \Omega} \bar{x}(\omega) - \sum_{\omega \in \Omega} e(\omega) - \sum_{j \in J} \bar{y}(j) + E(m) \in \mathcal{Z}(m).$$

Claim A.8. *$\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ for all $\omega \in \Omega$.*

Proof. Note that $\bar{p} \cdot \bar{y}(j) = \bar{p} \cdot \bar{y}(\mathcal{E})(j) + \text{proj}_k(\bar{p}) \cdot m^{(j)}$ for all $j \in J$. Thus, for all $\omega \in \Omega$, the quota budget set $B'_\omega(\bar{y}, \bar{p})$ for agent ω of the economy $\mathcal{E}'$ can be written as:

$$\left\{ z \in X_\omega : \bar{p} \cdot z \leq \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j} \big(\bar{p} \cdot \bar{y}(\mathcal{E})(j) + \text{proj}_k(\bar{p}) \cdot m^{(j)} \big) \right\},$$

which is the same as the quota budget set $B^m_\omega(\bar{y}(\mathcal{E}), \bar{p})$ of the economy $\mathcal{E}$. As $P_\omega(\bar{x}_{-\omega}, \bar{y}(\mathcal{E}), \bar{p}) = P'_\omega(\bar{x}_{-\omega}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$, the quota demand set $D'_\omega(\bar{x}, \bar{y}, \bar{p})$ for agent ω of the economy $\mathcal{E}'$

is the same as the quota demand set $D^m_\omega(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ of the economy $\mathcal{E}$. Thus, we conclude that $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ for all $\omega \in \Omega$. □

By Claim A.8, $(\bar{x}, \bar{y}(\mathcal{E}), \bar{p})$ is a quota equilibrium for $\mathcal{E}$ where $\bar{p} \in (\mathcal{Z}')^0$. □

A.4. **Proof of Theorem 2.**

Proof of Theorem 2. Suppose that there is a quota-compliant consumption-production pair $(\hat{x}, \hat{y})$ such that $C(\hat{x}, \hat{y}) = C(\bar{x}, \bar{y})$ and strongly Pareto dominates $(\bar{x}, \bar{y})$. For all $\omega \in \Omega$, $\big(\hat{x}(\omega), \bar{x}(\omega)\big) \in P_\omega(\bar{x}_{-\omega}, \bar{y}, \bar{p})$. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium, for all $\omega \in \Omega$, we have:

$$\bar{p} \cdot \hat{x}(\omega) > \bar{p} \cdot e(\omega) + \sum_{j \in J} \theta_{\omega j}\big(\bar{p} \cdot \hat{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)}\big).$$

Thus, we have $\bar{p}\big(\sum_{\omega\in\Omega} \hat{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \hat{y}(j)\big) > \mathrm{proj}_k(\bar{p}) \cdot m$. As $\mathcal{Z}(m)_n = \{0\}$ for all $k < n \leq \ell$ and $C(\hat{x}, \hat{y}) = C(\bar{x}, \bar{y})$,[61] we have $\bar{p}\big(\sum_{\omega\in\Omega} \hat{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \hat{y}(j)\big) = \mathrm{proj}_k(\bar{p}) \cdot m$, which leads to a contradiction. So $(\bar{x}, \bar{y})$ is constrained weakly Pareto optimal.

We now show that $(\bar{x}, \bar{y})$ is constrained Pareto optimal if $P_\omega(\bar{x}_{-\omega}, \bar{y}, \bar{p})$ is negatively transitive and locally non-satiated for all $\omega \in \Omega$. Suppose that there is a quota-compliant consumption-production pair $(\hat{x}, \hat{y})$ such that $C(\hat{x}, \hat{y}) = C(\bar{x}, \bar{y})$ and Pareto dominates $(\bar{x}, \bar{y})$. Then there is some $\omega_0 \in \Omega$ such that $\big(\hat{x}(\omega_0), \bar{x}(\omega_0)\big) \in P_{\omega_0}(\bar{x}_{-\omega_0}, \bar{y}, \bar{p})$. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium, we have $\bar{p} \cdot \hat{x}(\omega_0) > \bar{p} \cdot e(\omega_0) + \sum_{j\in J} \theta_{\omega_0 j}\big(\bar{p} \cdot \hat{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)}\big)$.

Claim A.9. *For every $\omega \in \Omega$, we have $\bar{p} \cdot \hat{x}(\omega) \geq \bar{p} \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j}\big(\bar{p} \cdot \bar{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)}\big)$.*

Proof. Suppose there is a $\omega_1 \in \Omega$ so that $\bar{p} \cdot \hat{x}(\omega_1) < \bar{p} \cdot e(\omega_1) + \sum_{j\in J} \theta_{\omega_1 j}\big(\bar{p} \cdot \bar{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)}\big)$. As $P_{\omega_1}(\bar{x}, \bar{y}, \bar{p})$ is locally non-satiated, there is a $u \in X_{\omega_1}$ such that $\big(u, \hat{x}(\omega_1)\big) \in P_{\omega_1}(\bar{x}_{-\omega_1}, \bar{y}, \bar{p})$ and $\bar{p} \cdot u < \bar{p} \cdot e(\omega_1) + \sum_{j\in J} \theta_{\omega_1 j}\big(\bar{p} \cdot \bar{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)}\big)$. As $P_{\omega_1}(\bar{x}_{-\omega_1}, \bar{y}, \bar{p})$ is negatively transitive, we have $\big(u, \bar{x}(\omega_1)\big) \in P_{\omega_1}(\bar{x}_{-\omega_1}, \bar{y}, \bar{p})$. This leads to a contradiction since $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium. □

By Claim A.9, $\bar{p} \cdot \hat{x}(\omega) \geq \bar{p} \cdot e(\omega) + \sum_{j\in J} \theta_{\omega j}\big(\bar{p} \cdot \hat{y}(j) + \mathrm{proj}_k(\bar{p}) \cdot m^{(j)}\big)$ for all $\omega \in \Omega$. So, we have $\bar{p}\big(\sum_{\omega\in\Omega} \hat{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \hat{y}(j)\big) > \mathrm{proj}_k(\bar{p}) \cdot m$. By the same argument as in the first paragraph, this leads to a contradiction. So $(\bar{x}, \bar{y})$ is constrained Pareto optimal. □

[61]If we know that the equilibrium prices for unregulated commodities are non-negative, then the conclusion of the theorem remains valid without assuming $\mathcal{Z}(m)_n = \{0\}$ for all $k < n \leq \ell$.

A.5. **Proofs of Theorem 3 and Theorem 4.**

Proof of Theorem 3. We first consider the special case where $\mathcal{E}$ is a global quota economy, i.e., $m^{(j)} = 0$ for every private firm $j \in J$. In this case, for all $\omega \in \Omega$, we have $\lambda(\omega) = \theta_{\omega 0}$. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant quota equilibrium, we have $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$. As $\mathcal{V} \supset \mathcal{Z}(m)$, we have $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{V}$. As $t = -\pi_k(\bar{p})$ and $C(\bar{x}, \bar{y}) = -m$, we have $B^m_\omega(\bar{y}, \bar{p}) = B^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. As $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$, we have $\bar{x}(\omega) \in D^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. Hence, $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-compliant emission tax equilibrium for the economy $\mathcal{F}$. For general production economies with quota, the conclusions follow from Proposition 1 and Proposition 2. □

Proof of Theorem 4. As $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{V}$-compliant emission tax equilibrium, we have $-C(\bar{x}, \bar{y}) \leq 0$. By choosing $m = -C(\bar{x}, \bar{y})$, we have $\sum_{\omega\in\Omega} \bar{x}(\omega) - \sum_{\omega\in\Omega} e(\omega) - \sum_{j\in J} \bar{y}(j) \in \mathcal{Z}(m)$. As $\bar{y}(j) \in S_j(\bar{p})$ for all $j \in J$, we have $\bar{y}(j) \in S^m_j(\bar{p})$ for all $j \in J$. As $t = -\mathrm{proj}_k(\bar{p})$, $m = -C(\bar{x}, \bar{y})$, $\tilde{\theta}(\omega)(0)(p) = \lambda(\omega)$ for all $\omega \in \Omega$, and $\mathcal{E}$ is a global quota economy, we have $B^m_\omega(\bar{y}, \bar{p}) = B^t_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. Hence, we have $\bar{x}(\omega) \in D^m_\omega(\bar{x}, \bar{y}, \bar{p})$ for all $\omega \in \Omega$. So $(\bar{x}, \bar{y}, \bar{p})$ is a $\mathcal{Z}(m)$-compliant global quota equilibrium of $\mathcal{E}$. □

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