BARGAINING AND EXCLUSION WITH MULTIPLE BUYERS

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Abstract. A seller trades with q out of n buyers who have valuations $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ via sequential bilateral bargaining. When $q < n$, buyer payoffs vary across equilibria in the patient limit, but seller payoffs do not, and converge to

$$\max_{l \leq q+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{q+1} \right].$$

If l^* is the (generically unique) maximizer of this optimization problem, then each buyer $i < l^*$ trades with probability 1 at the fair price $a_i/2$, while buyers $i \geq l^*$ are excluded from trade with positive probability. Bargaining with buyers who face the threat of exclusion is driven by a *sequential outside option principle*: the seller can sequentially exercise the outside option of trading with the extra marginal buyer $q + 1$, then with the new extra marginal buyer q, and so on, extracting full surplus from each buyer in this sequence and enhancing the outside option at every stage. A seller who can serve all buyers ($q = n$) may benefit from creating scarcity by committing to exclude some remaining buyers as negotiations proceed. An *optimal exclusion commitment*, within a general class, excludes a single buyer but maintains flexibility about which buyer is excluded. Results apply symmetrically to a buyer bargaining with multiple sellers.

1. Introduction

Consider a seller whose supply is valuable to multiple buyers. If the seller is a monopolist, this is a classical setting, which is well understood under various assumptions regarding information and price discrimination. Under complete information and perfect price discrimination, the monopolist extracts all surplus from every buyer. We investigate what happens in the complete information setting when the terms of trade are determined by *bargaining* between the seller and each individual buyer. What profits does the seller earn and which buyers does she trade with in a bargaining game with fixed supply? What payoffs do buyers get? If there is no scarcity and the seller serves all buyers, then the standard equal ("fair") division of surplus between the seller and each buyer should be expected. However, if there is scarcity and some buyers are necessarily "excluded," then the seller should be able to exploit competition among buyers and obtain higher than fair prices. This suggests that the

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seller may benefit from limiting supply, and leads to a related question: if the seller may reduce supply or place more general restrictions on the sets of buyers she transacts with, what restrictions will be most profitable and what outcomes will emerge?[1]

We consider a market in which a seller contracts independently with q out of n individual buyers with respective values (net of seller cost) $a_1 \geq a_2 \geq \ldots \geq a_n > 0$. For convenience, we use language suggesting that the seller is offering q units of the same "good" for sale, and each buyer has unit demand. However, the seller's transactions with each buyer may be idiosyncratic; the main restriction we impose is that there are no externalities between buyer valuations. We study the following bargaining game, which we refer to as *the game with supply* q. Negotiations occur over time, and players have a common discount factor $\delta \in (0, 1)$. In each round, the seller strategically picks a buyer to bargain with, and with equal probability each of the two players proposes a price to the other. If the proposal is accepted, then the seller trades with the buyer at the proposed price, the buyer exits the game, and the seller continues to bargain with the remaining buyers in the next round. If the proposal is rejected, then bargaining proceeds with the same set of buyers in the next round. The game ends when the seller has traded with q buyers.

We analyze Markov perfect equilibria (MPEs) of the game with supply q—subgame perfect equilibria (SPEs) in which each player's strategy in a round depends only on the set of buyers with whom the seller has not yet traded, and actions taken within that round.[2] Our main results concern limit MPE outcomes as δ goes to 1. We will frequently affix the qualifiers "limit" and "asymptotic" to describe limit outcomes as $\delta \to 1$ in a collection of MPEs for discount factors $\delta \in (0, 1)$ (but drop qualifiers for brevity in some cases).

If $q = n$, so all buyers can be served, then the seller splits the surplus equally with each individual buyer, and her profits converge to $a_1/2 + a_2/2 + \ldots + a_n/2$ as $\delta \to 1$. This is closely related to the classic result on convergence of (symmetric) non-cooperative bargaining in the style of Rubinstein (1982) to the Nash (1950) bargaining solution (Binmore 1980; Binmore, Rubinstein and Wolinsky 1986).

Suppose next that supply is smaller than the number of buyers ($q < n$). For the remainder of the introduction (but not in the formal treatment), we assume for simplicity that buyer values are distinct. Consider first the case in which the seller has unit supply ($q = 1$). Proposition 1 in Manea (2018) characterizes MPEs in this simple case. If $a_2 \leq a_1/2$, then the seller bargains exclusively with buyer 1, and the two players split the gains equally,

[1]The importance of exclusion restrictions in the context of individually negotiated agreements with multiple agents has been examined in applied work. Gal-Or (1997) emphasizes the power of exclusion in an early paper. In the health economics literature, it has been widely noted that insurance companies offer "narrow" hospital networks (e.g., Howard 2014; Liebman 2018; Ho and Lee 2019; Ghili 2022).

[2]SPEs usually have little predictive power in bargaining games with more than two players (e.g., Herrero 1985, Rubinstein and Wolinsky 1990, Abreu and Manea 2012a, Manea 2018, Elliott and Nava 2019), and MPE is frequently invoked as an equilibrium refinement in such settings. In Abreu and Manea (2022), we show that there is extreme variation in SPE outcomes even in the simple version of the model considered here where $n = 2$ and $q = 1$.

trading at (average) price $a_1/2$. In this case, the outside option of trading with buyer 2 is too weak to enhance the seller's bargaining power in negotiations with buyer 1. If $a_2 > a_1/2$, then for high δ, the seller randomizes between buyers 1 and 2 in equilibrium, and each buyer trades at prices converging to a_2, but the probability of bargaining (and trading) with buyer 2 converges to 0 as δ goes to 1. Now, the outside option of trading with buyer 2 is *binding*, and the seller *exercises* it with positive but vanishing probability as $\delta \to 1$. An *outside option principle* emerges from this analysis of MPEs: the seller trades with buyer 1 with limit probability 1 at limit price $\max(a_1/2, a_2)$.[3] Therefore, when $q = 1$ trade is asymptotically efficient, and buyer 2 provides an *endogenous* outside option that has a limit equilibrium value of a_2.[4]

By analogy with the unit supply case, one might conjecture that when $q > 1$ the seller should attain asymptotic profits of

$$\sum_{i=1}^{q} \max\left(\frac{a_i}{2}, a_{q+1}\right). \tag{1}$$

However, this conjecture is incorrect. Formula (1) may be rationalized in terms of the following presumptions: (i) the seller trades efficiently (with limit probability 1) with buyers $1, \ldots, q$; (ii) bargaining with each of the buyers $1, \ldots, q$ is driven by a fixed outside option provided by the *extra marginal* buyer $q + 1$; (iii) the value of the outside option provided by buyer $q + 1$ in equilibrium is a_{q+1} (i.e., buyer $q + 1$ has zero limit payoff). The first two presumptions turn out to be incorrect, as they fail to take into account the dynamic nature of outside options under sequential bargaining. For instance, consider a setting with $n = 3, q = 2$ and suppose that $a_3 > a_1/2$, so that both buyers 2 and 3 constitute binding outside options in bargaining with buyer 1 in subgames where the seller has a single unit left. In this case, trading with buyer 2 in the first round at the highest (individually rational) price of a_2 is *not* (asymptotically) more profitable than trading with buyer 3 at a price of a_3. Indeed, in the next round, when bargaining with buyer 1, the seller obtains a price of a_2 if buyer 2 is available as an outside option, but a lower price of a_3 if buyer 3 is the outside option. In either case, the seller's profit would be $a_2 + a_3$. Hence, buyer 2 is valuable to the

[3]The assumption of Markov behavior is important for this conclusion. In Abreu and Manea (2022), we show that SPEs in the setting with $n = 2, q = 1$ are very permissive (in part expanding on a point made by Rubinstein and Wolinsky (1990))—the price may be above or below the outside option price, and the allocation may be asymptotically inefficient in either case. We proceed to propose refinements that are behaviorally plausible in the context of this bargaining environment and yield the intuitive predictions of the outside option principle. Although these refinements do not imply Markov behavior, they provide support for MPE predictions in the bargaining game considered here. See also Maskin and Tirole (2001) and Bhaskar, Mailath and Morris (2013) for alternative foundations for the Markov equilibrium assumption.

[4]In the original treatment (Binmore 1985; Binmore, Rubinstein and Wolinsky 1986; Sutton 1986; Binmore, Shaked and Sutton 1989), outside options were assumed to have *exogenous* values that can be obtained by traders without bargaining with third parties. Subsequent research on search and matching in labor markets and bargaining in markets with multiple buyers and sellers emphasized the endogeneity of outside options derived from bargaining with several parties.

seller both directly as a trading partner and indirectly as an outside option when bargaining with buyer 1 in the event that the seller trades with buyer 3 first. Thus, buyer 2 might not necessarily manage to "outbid" buyer 3 in the first round. This suggests that trade need not be asymptotically efficient when $q > 1$, which we confirm in examples with $n = 3, q = 2$.

Although the extra marginal buyer $q+1$ may trade with positive limit probability as $\delta \to 1$ in a collection of MPEs, we prove that the seller is always able to extract full surplus from buyer $q + 1$ (hence, the third presumption above is correct). This property of MPEs allows us to replace the outside option principle for the case $q = 1$ with a *sequential outside option principle* for the case $q > 1$. The seller can sequentially *exercise* outside options by trading with the extra marginal buyer $q + 1$ at limit price a_{q+1}, then trading with the new extra marginal buyer q at limit price a_q (buyer q becomes extra marginal in the subgame with supply $q - 1$), and so on, thereby *enhancing* the outside option at every round. Since some buyers may be too valuable to be excluded, it may be beneficial for the seller to exclude a buyer $l > 1$ and include all lower index buyers. When buyer l is available, the threat of replacing buyer $i \leq l - 1$ with some higher value buyer is blunted, and the seller may be unable to extract full surplus from buyer i. Nevertheless, we show that each buyer i must pay at least a fair limit price of $a_i/2$ as $\delta \to 1$ in any collection of MPEs. This leads to the following *lower bound* on the seller's asymptotic MPE profits:

$$(2) \qquad M^{*q} := \max_{l \leq q+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{q+1} \right].$$

A polar argument leads to the surprising conclusion that M^{*q} also constitutes an *upper bound* on the seller's asymptotic MPE profits. Therefore, the seller's profits converge to M^{*q} in any collection of MPEs as $\delta \to 1$.

The static optimization problem displayed in (2) yields the seller's *payoffs* in the dynamic bargaining game with supply $q < n$. The optimization problem is also informative about the seller's *behavior*, in particular about which buyers get to trade with certainty and which buyers face the threat of exclusion in equilibrium. In the generic case in which the static optimization problem has a unique maximizer l^*, we establish that for sufficiently high δ, in any MPE, buyers $i < l^*$ are guaranteed to be included—and trade at the fair price $a_i/2$—while buyers $i \geq l^*$ are excluded with positive probability. Furthermore, if $l^* \neq q + 1$, then buyer l^* is included with limit probability 1 as $\delta \to 1$.

While MPEs are (asymptotically) payoff equivalent for the seller, each buyer's payoff and probability of trade can vary across convergent sequences of MPEs. We develop a partial characterization of buyer payoffs that leverages the formula for seller profits in every subgame.

We also consider a strategic situation in which the seller has unconstrained supply ($q = n$), but can sharpen competition by excluding some buyers in the course of negotiations. An *exclusion commitment* specifies a subset of buyers to be excluded from future negotiations depending on the set of buyers who have already traded. This general formulation allows for elaborate patterns of exclusion. Despite the potential multiplicity of MPEs in the bargaining game induced by some exclusion commitments, we find that an *optimal exclusion commitment* can be defined unambiguously and takes a simple form: no buyer is excluded from bargaining until $n-1$ units are sold, and then the remaining buyer is excluded (this commitment leads to the game with supply $n-1$). Under this commitment, the seller excludes a single buyer, but decides flexibly who to include at every stage. The result implies that maintaining one unit of shortage allows the seller to extract the full benefits of exclusion, and creating more scarcity or treating buyers asymmetrically cannot increase profits.

Finally, we discuss exclusion commitments in settings in which the seller has an exogenous supply constraint $q < n$. In this case, in line with the intuition above, the seller does not benefit from making commitments to exclude buyers before all available q units are sold. In particular, a reduction in supply is detrimental to the seller.

We contrast our findings with those of Ho and Lee (2019), who were the first to analyze exclusion commitments in a model of network formation via bargaining.[5] In their model, a "seller" who commits to form q links delegates q independent "representatives" to each bargain over the formation of one link. When specialized to our setting, this delegated-agent bargaining protocol delivers formula (1) for seller profits, and implies that the seller may benefit from reducing supply. Ho and Lee's representatives are compartmentalized and cannot effectively coordinate to maximize joint profit, whereas in our model the seller internalizes the dynamic implications of *sequential* trades with individual buyers.[6] Our more conventional bargaining protocol enables the seller to extract higher profits via the sequential outside option principle embodied in formula (2).

The paper is organized as follows. Section 2 introduces the bargaining model, and Section 3 provides a preliminary lemma and an example. In Section 4, we develop bargaining theoretic principles that we use in Section 5 to obtain the formula for seller profits. Section 6 characterizes included and excluded buyers. Sections 7 and 8 formalize our notion of exclusion commitments and identify the optimal commitment. Section 9 concludes. Proofs omitted in the main body of the paper appear in the Appendix.

[5]Also motivated by the questions of network endogeneity and optimal exclusion, Liebman (2018) considers a bargaining model between a health insurer and several hospitals in which the insurer commits to a network size and then bargains with randomly selected hospitals. His analysis restricts attention to equilibria with immediate agreement, but such equilibria do not exist under random matching when hospitals are heterogeneous and bargaining frictions are small. As this is the case we are primarily interested in, a direct comparison with his results is not possible. Taking a cooperative approach, Ghili (2022) studies network formation in the pairwise stability framework of Jackson and Wolinsky (1996) assuming that payoffs are determined by Nash bargaining.

[6]Stole and Zwiebel (1996) and Arie, Grieco and Rachmilevitch (2018) analyze bargaining models in which a player signs bilateral contracts with several others in sequence, but in their models the order of negotiations is exogenous.

2. Model

Consider a market where an agent, player 0, signs bilateral contracts with q out of n players from the set $N = \{1, 2, \ldots, n\}$. To fix terminology, we refer to player 0 as the *seller*, to the players in N as *buyers*, and to the bilateral contracts as *goods*. In this language, the seller has $q \leq n$ units of a good, and each of the n buyers has unit demand.[7] Assume that buyer i's *value* for the good (net of seller cost) is a_i, where $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and these values are common knowledge. There are no externalities: buyer values are independent of who else gets a unit of the good.

The seller trades with individual buyers sequentially. In every round $t = 0, 1, \ldots$, the seller (strategically) selects a buyer i to bargain with (among those who have not yet traded). Bargaining between the seller and buyer i in round t proceeds via the random-proposer protocol: with probability $1/2$ each of the two players proposes a price, and the other decides whether to accept or reject the proposal. If the proposal is accepted, the seller trades with buyer i at the proposed price, buyer i exits the game, and the seller continues to bargain with the remaining buyers in round $t+1$. Otherwise, bargaining proceeds with the same set of buyers in round $t + 1$. The game ends when the seller trades all q units.[8] Players have a common discount factor $\delta \in (0, 1)$: payoffs obtained in round t are discounted by δ^t. The game has perfect information.

We call this the *bargaining game with exogenous supply* q, or the *game with supply* q for short. We will also be interested in situations in which there is no inherent scarcity, i.e., $q = n$, but the seller may strategically commit to exclude buyers in order to enhance competition. The model with exclusion commitments is analyzed in Section 7.

We analyze *Markov perfect equilibria (MPEs)* of the game with supply q, which are subgame perfect equilibria in which each player's strategy in every round depends only on the *state* S—the set of buyers with whom the seller has not already traded—and the actions taken within the round (including nature's random selection of proposer). By definition, in an MPE, behavior in any subgame that starts at the beginning of a bargaining round (before the seller's selection of a bargaining partner) in state S does not depend on the history of play prior to that round. We refer to any such subgame as *subgame* S.

For any MPE of the game with supply q, let $u_i(S)$ denote the expected payoff of player $i \in S \cup \{0\}$ in state S, and $\pi_i(S)$ the probability that the seller chooses to bargain with buyer i in state S. Our main results apply to *collections of MPEs for discount factors* $\delta \in (0, 1)$ in

[7]The seller may customize the "good" for each buyer upon purchase; the setting with multiple units of a homogenous good is a special case.

[8]Proposition 4.ii in Rubinstein and Wolinsky (1990) introduced this "voluntary matching" bargaining protocol (their wording emphasizes the seller's strategic selection of bargaining partner, in contrast to random matching) in a setting with unit supply. We employed similar bargaining protocols in Abreu and Manea (2012b, 2022) and Manea (2018). This bargaining protocol is distinct from the "random proposer" protocol of Elliott and Nava (2019) and Talamas (2019, 2020) whereby a "proposer" is randomly recognized in every round, and the proposer strategically selects a bargaining partner but also makes the offer.

the game with supply q, which for every δ in $(0, 1)$ specify an MPE σ^δ of the game with supply q in which players have discount factor δ. When the variables $u_i(S)$ and $\pi_i(S)$ associated with a collection of MPEs $(\sigma^\delta)_{\delta\in(0,1)}$ converge as $\delta \to 1$, we will denote the corresponding limits by $\bar{u}_i(S)$ and $\bar{\pi}_i(S)$, respectively. We will also simplify notation by writing $u_i, \pi_i, \bar{u}_i, \bar{\pi}_i$ for the variables $u_i(N), \pi_i(N), \bar{u}_i(N), \bar{\pi}_i(N)$ associated with the initial state N, respectively.

3. A Preliminary Lemma and an Example

Lemma 1 provides basic scaffolding for the arguments that follow. It first establishes that in any MPE there is *trade in every round*, that is, if in some round, the seller bargains with buyer i with positive probability in equilibrium, then conditional on approaching buyer i, agreement is reached with probability 1. Hence, the game with supply q ends in q rounds.

The lemma also states that MPE variables satisfy the following conditions:

$$(3) \quad u_0(S) \geq \frac{1}{2}\left(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)\right) + \frac{1}{2}\delta u_0(S), \text{ with equality if } \pi_i(S) > 0$$

$$(4) \quad u_i(S) = \pi_i(S)\left(\frac{1}{2}\left(a_i + \delta u_0(S \setminus \{i\}) - \delta u_0(S)\right) + \frac{1}{2}\delta u_i(S)\right) + \sum_{k\in S\setminus\{i\}} \pi_k(S)\delta u_i(S \setminus \{k\}),$$

where $u_0(S \setminus \{i\}) = u_i(S \setminus \{k\}) = 0$ if $|S| = n - q + 1$ and $i, k \in S$.

Finally, the lemma shows that if the seller bargains with buyer i in state S in equilibrium (and relevant variables converge), the price that buyer i pays in state S converges to $a_i - \bar{u}_i(S)$ regardless of whether the seller or buyer i is chosen to make the offer.

Lemma 1. *In any MPE of the game with supply q, there is trade in every round, and payoffs and the seller's mixing probabilities satisfy conditions* (3) *and* (4) *for every state S. If $\pi_i(S) > 0$ along a sequence of MPEs associated with a sequence of discount factors going to 1, and $u_i(S)$ converges to $\bar{u}_i(S)$, then both the price the seller offers to buyer i and the price buyer i offers to the seller in state S converge to $a_i - \bar{u}_i(S)$.*

The formal proof of Lemma 1 and other proofs omitted in the main body of the paper appear in the Appendix. To understand condition (3), note that the seller may select buyer i for bargaining in state S, and if chosen to propose, can offer a price arbitrarily close to $a_i - \delta u_i(S)$ that i will accept; following an agreement with buyer i, the seller obtains a continuation equilibrium payoff of $\delta u_0(S \setminus \{i\})$. When buyer i is chosen to propose, the seller may at worst reject i's offer and enjoy a continuation payoff of $\delta u_0(S)$; in equilibrium, buyer i will make an offer that makes the seller indifferent between accepting and rejecting. If the seller bargains with buyer i with positive probability in state S, then her realized payoff from trading with i should be equal to her equilibrium payoff $u_0(S)$. The buyer payoff equation (4) has a similar interpretation.

An example. With the preliminary analysis in place, we are able to solve simple examples. This exercise illustrates how equilibria "work" and highlights a distinctive feature of our bargaining game—the dynamic equilibrium evolution of outside options. It is also helpful in developing appropriate conjectures. We are interested in the following questions, which concern limit equilibrium outcomes as $\delta \to 1$: Is the MPE unique? If not, does each buyer trade with the same probability in all MPEs? Are buyer payoffs constant across MPEs? Are seller payoffs constant across MPEs?

We consider an example in which a seller with supply $q = 2$ bargains with three buyers who have values $a_1 = 4, a_2 = 3, a_3 = 1$. This example demonstrates that the answer to each of the first three questions is negative. The negative answer to the second question implies that MPEs are not always asymptotically efficient. Interestingly, the example is consistent with the answer to the fourth question being positive.[9]

In any MPE for high δ, the seller must approach at least two buyers with positive probability in the initial state.[10] For δ sufficiently close to 1, the example admits three classes of MPEs that are distinguished by the set of buyers with whom the seller may trade in the initial state. In one class, the seller trades with every buyer in the initial state. In the other two classes, the seller trades with buyer 3 and only one of buyers 1 and 2, respectively, in the initial state. Here, we derive the limit structure of each class of MPEs as $\delta \to 1$. In the Appendix, we prove that each type of MPE exists for high δ.

We analyze the game from the "back," starting with the simple subgames in which the seller has a single unit remaining (after having traded with one buyer). Proposition 1 of Manea (2018) characterizes the unique MPE outcomes for such subgames. In states $\{i, 3\}$ $(i = 1, 2)$, the outside option of trading with buyer 3 is not sufficiently valuable to improve the seller's bargaining position with buyer i, and the seller sells the remaining unit with probability 1 to buyer i at expected price $a_i/2$: $\pi_i(\{i, 3\}) = 1, u_0(\{i, 3\}) = u_i(\{i, 3\}) = a_i/2, u_3(\{i, 3\}) = 0$. In state $\{1, 2\}$, the outside option of trading with buyer 2 is binding, and the seller randomizes between buyers 1 and 2 in equilibrium, but the probability of choosing buyer 2 converges to 0 as δ goes to 1; buyer 1 trades with limit probability 1 at limit price a_2: $\bar{\pi}_1(\{1, 2\}) = 1, \bar{u}_0(\{1, 2\}) = 3, \bar{u}_1(\{1, 2\}) = 1, \bar{u}_2(\{1, 2\}) = 0$.

What about play in the initial state $\{1, 2, 3\}$, before any trade has happened? Lemma 1 implies that when the seller chooses to bargain with buyer i along a convergent sequence of MPEs, trade takes place at the common limit price $a_i - \bar{u}_i$ regardless of whether the seller

[9]These qualitative findings are robust to perturbations in the specified buyer values.

[10]More generally, in states where the seller has more than one good left, randomization between trading with multiple buyers is a necessary feature of the seller's strategy in any MPE for high δ. If the seller chose to bargain with a single buyer in such a state in a proposed MPE, then that buyer would "hold up" the seller for half of her gains from future trades. As we argue in the context of Lemma 2 in the next section, the seller would then have a profitable deviation that involves changing the order of trades, thereby undermining the putative MPE.

or buyer i is picked to propose. It follows that

$$\bar{u}_i = \bar{\pi}_i \bar{u}_i + \sum_{k \in N \setminus \{i\}} \bar{\pi}_k \bar{u}_i(N \setminus \{k\}). \tag{5}$$

Since the seller must obtain her equilibrium payoff regardless of which buyer she trades with in the initial state of the MPE, we have that

$$\pi_i > 0 \text{ for all } \delta \implies \bar{u}_0 = a_i - \bar{u}_i + \bar{u}_0(N \setminus \{i\}). \tag{6}$$

Note that we already know the limit equilibrium values $\bar{u}_i(N \setminus \{k\})$ and $\bar{u}_0(N \setminus \{i\})$ for subgames following the first trade.

How does equilibrium multiplicity arise? At a high level, the seller's randomization in the initial state determines buyer equilibrium payoffs via (5), and in turn buyer payoffs have to be compatible with the support of the seller's randomization via (6). This system allows for three consistent solutions with distinct buyer payoffs.

We next confirm the intuition that given the scarcity, the lowest valuation buyer must get zero limit payoff in every sequence of MPEs. Since buyer 3 obtains zero payoff in states $\{1, 3\}$ and $\{2, 3\}$, equation (5) implies that $\bar{u}_3 > 0$ only if $\bar{\pi}_3 = 1$. However, if $\bar{\pi}_3 = 1$, then (5) leads to $\bar{u}_2 = \bar{u}_2(\{1, 2\}) = 0$, and (6) (for $i = 3$) implies that $\bar{u}_0 = 1 - \bar{u}_3 + \bar{u}_0(\{1, 2\}) < 1 + 3 = 4$. Then, taking the limit $\delta \to 1$ in (3) (for $i = 2$) leads to $\bar{u}_0 \geq 3 - \bar{u}_2 + \bar{u}_0(\{1, 3\}) = 3 + 4/2 = 5$, a contradiction. We conclude that $\bar{u}_3 = 0$.

Consider now the class of MPEs in which for high δ, the seller approaches all three buyers with positive probability in the initial state. Since $\pi_1, \pi_2, \pi_3 > 0$, (6) implies that

$$\bar{u}_0 = a_1 - \bar{u}_1 + \bar{u}_0(\{2, 3\}) = a_2 - \bar{u}_2 + \bar{u}_0(\{1, 3\}) = a_3 - \bar{u}_3 + \bar{u}_0(\{1, 2\}).$$

As $\bar{u}_0(\{2, 3\}) = 1.5, \bar{u}_0(\{1, 3\}) = 2, \bar{u}_0(\{1, 2\}) = 3$ and $\bar{u}_3 = 0$, we have that $\bar{u}_0 = 4, \bar{u}_1 = 1.5$ and $\bar{u}_2 = 1$. The required limit mixing probabilities are obtained by plugging these limit payoffs in formula (5) for buyers $i = 1, 2$:[11]

$$\begin{aligned} 1.5 &= \bar{\pi}_1 \times 1.5 + \bar{\pi}_2 \times 2 + \bar{\pi}_3 \times 1 \\ 1 &= \bar{\pi}_1 \times 1.5 + \bar{\pi}_2 \times 1 + \bar{\pi}_3 \times 0. \end{aligned}$$

Combining these equations with $\bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 = 1$ leads to the unique solution $\bar{\pi}_1 = 0.5, \bar{\pi}_2 = \bar{\pi}_3 = 0.25$. In this class of MPEs, trade is inefficient with limit probability $\bar{\pi}_3 = 0.25$.

We now turn to a second class of MPEs, in which $\pi_1 = 0$ and $\pi_2, \pi_3 > 0$ for high δ. In this case, (6) implies that $\bar{u}_0 = a_2 - \bar{u}_2 + \bar{u}_0(\{1, 3\}) = a_3 - \bar{u}_3 + \bar{u}_0(\{1, 2\})$. As $\bar{u}_0(\{1, 3\}) = 2, \bar{u}_0(\{1, 2\}) = 3$ and $\bar{u}_3 = 0$, we obtain that $\bar{u}_0 = 4$ and $\bar{u}_2 = 1$. Noting that $\bar{u}_2(\{1, 2\}) = 0$, formula (5) (for $i = 2$) and $\pi_1 = 0$ imply that $\bar{u}_2 = 0$ if $\bar{\pi}_2 < 1$. It follows that $\bar{\pi}_2 = 1$. Using

[11] Equation (5) for buyer 3 does not create any restriction on limit mixing probabilities because buyer 3 gets limit payoff 0 in every state.

(5) again (for $i = 1$) yields $\bar{u}_1 = 2$. It is easy to verify the optimality of choosing $\pi_1 = 0$ for the seller.[12] In this class of MPEs, trade is asymptotically efficient.[13]

The third class of MPEs is similar to the second, with the roles of buyers 1 and 2 interchanged. Analogous arguments imply that $\bar{\pi}_1 = 1$, and yield the limit payoffs for this class: $\bar{u}_0 = 4, \bar{u}_1 = 1.5, \bar{u}_2 = 1.5, \bar{u}_3 = 0$.[14]

The following table summarizes each player's limit payoffs and the seller's first-round mixing probabilities in the three classes of MPEs.

$a_1 = 4, a_2 = 3, a_3 = 1$	$\bar{u}_0$	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$(\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$
$\pi_1, \pi_2, \pi_3 > 0$	4	1.5	1	0	$(0.5, 0.25, 0.25)$
$\pi_1 = 0; \pi_2, \pi_3 > 0$	4	2	1	0	$(0, 1, 0)$
$\pi_2 = 0; \pi_1, \pi_3 > 0$	4	1.5	1.5	0	$(1, 0, 0)$

In the first two classes of MPEs, the seller is indifferent between trading with buyer 2 at (limit) price 2 and with buyer 3 at the *lower* price 1 in the first round. This is explained by the impact of the first trade on the equilibrium value of the outside option—and thus the bargaining power of buyer 1—in the second round. If the seller trades with buyer 2 first, then she can obtain only the "fair" price $a_1/2 = 2$ from buyer 1 due to buyer 3's uncompetitiveness, while if she trades with buyer 3 first, then she can demand the higher price $a_2 = 3$ from buyer 1 due to buyer 2's competitiveness. Despite the risk of being excluded, buyer 2 does not have an incentive to (further) "outbid" buyer 3 by agreeing to pay a price above 2 in equilibrium. If the seller were to make a more aggressive demand to buyer 2, buyer 2 would decline, preferring to gamble on the MPE probability that the seller will not trade with buyer 3 in the next round. We comment more on variation in buyer strengths across the three classes of MPEs in the Appendix.

Although limit buyer payoffs and probabilities of trade vary across the three classes of MPEs for this example, limit seller payoffs do not, and are equal to 4 in all MPEs. In Section 5, we prove that limit MPE seller payoffs are unique in general, and derive a formula for their value, which in this example reduces to $\bar{u}_0 = a_2 + a_3$. Other common features of MPEs in this example, which will also be explained by our results, are that buyer 1 trades with limit probability 1 and that buyer 3 gets zero limit payoff.

[12] Given buyer 1's equilibrium expectations, a first trade with buyer 1 would generate limit price $a_1 - \bar{u}_1 = 2$, and would be followed by a trade with buyer 2 at expected price $a_2/2 = 1.5$. This would yield limit profit 3.5 for the seller, which is smaller than $\bar{u}_0 = 4$.

[13] The existence of an asymptotically efficient MPE is not guaranteed in general. In the Online Appendix, we discuss an example with $n = 3, q = 2$ and buyer values $a_1 = 5, a_2 = 4, a_3 = 3$ in which asymptotically efficient MPEs do not exist.

[14] Similar limit arguments establish that for high δ, there are no MPEs in which the seller chooses $\pi_3 = 0$ and $\pi_1, \pi_2 > 0$. Since the seller needs to randomize among at least two buyers in the first round of every MPE for high δ (see footnote 10), it follows that the three asymptotic structures described above constitute the only potential limit points of MPEs as $\delta \to 1$.

4. Key Lemmas

We now develop some core results upon which our subsequent analysis builds. These results are intuitive, and indeed familiar in the case $q = 1$, but their complete proofs for the case $q > 1$ are not straightforward. We present proof sketches at the end of the section.

Lemma 2 shows that in any collection of MPEs of the game with supply q for $\delta \in (0, 1)$, no buyer i can acquire the good for less than the "fair" price $a_i/2$ in the limit as $\delta \to 1$. This is intuitive because within each round in which the seller bargains with buyer i, the seller and buyer i make offers with equal probability, but the seller has the additional advantage of choosing her bargaining partner and possibly trading with other buyers if agreement is not reached in the current round.

Lemma 2 (Buyers pay at least fair prices)**.** *In any collection of MPEs of the game with supply q for discount factors $\delta \in (0, 1)$,*

$$\limsup_{\delta \to 1} u_i \leq \frac{a_i}{2}.$$

Lemma 3 establishes that in the game with supply $q < n$, the payoffs of buyers $q+1, \ldots, n$ converge to 0 as $\delta \to 1$.[15] To get some perspective on this result, assume that buyer values are distinct. For $q = 1$, the result asserts that all buyers other than the buyer with the highest value have zero limit payoffs. This is an implication of Proposition 1 of Manea (2018). In this case, the highest valuation buyer trades with limit probability 1, and all other buyers with limit probability 0. The case $q > 1$ is more subtle: with sequential trade, a high value buyer is valuable to the seller both as a direct trading partner in the current round and as a better outside option when trading with other buyers in the future, and therefore might not necessarily manage to "outbid" a lower valuation buyer.

Lemma 3 (Buyers $q+1, \ldots, n$ get zero payoffs under supply q)**.** *In any collection of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0, 1)$, the payoffs of buyers $q+1, \ldots, n$ converge to 0 as $\delta \to 1$.*

Lemma 4 below establishes that a buyer i who trades *with probability 1* in a sequence of MPEs for $\delta \to 1$—even when this occurs with some delay and perhaps stochastically in any given round—pays at most the fair price $a_i/2$ in the limit. This result may be viewed as a counterpoint to the outside option principle—a buyer who is never under the threat of exclusion in equilibrium cannot be exploited (relative to fair pricing) by the seller.

Lemma 4 (Buyers sure to trade pay at most fair prices)**.** *Let $(\sigma^{\delta_z})_{z \geq 0}$ be a sequence of MPEs for the game with supply q in which the discount factors δ_z converge to 1 as $z \to \infty$. If the*

[15] The result implies that every buyer $i \leq q$ with $a_i = a_{q+1}$ also gets a zero limit payoff (via an argument that exchanges the labels of buyers i and $q + 1$). Hence, buyers with values that do not exceed the extra marginal value get zero limit payoffs.

seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$, then

$$\liminf_{z\to\infty} u_i \geq \frac{a_i}{2}.$$

We emphasize that "sure trade" in the naming of Lemma 4 refers to trade with *exact* probability 1 in a sequence of MPEs associated with a sequence of discount factors converging to 1. As discussed in the context of subgames in the example from the previous section, when $a_1 > a_2 > a_1/2$ in the setting with unit supply, trade with buyer 1 takes place with *limit probability 1 as* $\delta \to 1$, but in this case the outside option of trading with buyer 2 is binding, and buyer 1 pays a limit price of a_2, which is above the fair price $a_1/2$.

Lemmata 2 and 4 have the following corollary.

Corollary 1 (Fair pricing with sure trade). *Let $(\sigma^{\delta_z})_{z\geq 0}$ be a sequence of MPEs for the game with supply q in which the discount factors δ_z converge to 1 as $z \to \infty$. If the seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$, then the expected payoff of buyer i converges to $a_i/2$ as $z \to \infty$.*

While this result echoes classic results on convergence to the Nash bargaining solution in Rubinstein-style alternating-offer bargaining (Binmore 1980; Binmore, Rubinstein and Wolinsky 1986), the argument here is more involved due to the presence of other buyers, the seller's strategic (and typically stochastic) selection of bargaining partner at every stage, and the resultant non-stationary interaction between the seller and each buyer. In general, the exact price a buyer pays in MPEs for a fixed δ depends on the state in which the buyer trades, but the result shows that if the buyer is certain to trade, then all these prices converge to the fair price as $\delta \to 1$.[16]

The example from the previous section demonstrates that although trading with exact probability 1 is a sufficient condition, it is not a necessary condition for fair pricing in the limit. Indeed, in the second class of MPEs in the example, buyer 1 trades with probability smaller than 1 but converging to 1 for $\delta \to 1$ and obtains a limit payoff of $a_1/2$ (buyer 2 is in an analogous situation in the third class of MPEs).

We briefly turn to the game with unconstrained supply, i.e., $q = n$. By Lemma 1, in every MPE of the game with supply $q = n$, there is trade in every round. It follows that the seller trades with each buyer i with probability 1 in one of the first n rounds, and Corollary 1 and Lemma 1 imply that trade takes place at an expected discounted price converging to $a_i/2$

[16] We establish a result of a similar flavor for a network setting in earlier work (Abreu and Manea 2012b). In that model, every link generates a unit surplus and each player needs to trade with a neighbor. We show that every player who is guaranteed to trade in equilibrium—even when trade occurs in an evolving network and potentially with different neighbors—obtains asymptotic payoffs of at least 1/2. Elliott and Nava (2019) also obtain a related result in a network setting with heterogeneous link values. In the efficient MPEs they analyze, every pair of players who trade with each other with probability 1 face a stationary environment of trading opportunities with other neighbors, but these outside options cannot be binding. Consequently, each such pair effectively trades in a stationary two-player bargaining game, and agreements reflect "Rubinstein payoffs" independent of the state of the network.

as $\delta \to 1$ (regardless of the timing of the agreement and nature's selection of proposer in the seller's interaction with buyer i). We established the following result.

Corollary 2. *In any collection of MPEs of the game with supply $q = n$ for discount factors $\delta \in (0,1)$, each buyer i's payoff converges to a_i, and the seller's profit converges to $\sum_{i \in N} a_i/2$ as $\delta \to 1$.*

We conclude the section by sketching some key steps in the proofs of Lemmata 2-4. Readers satisfied with the intuitions provided above may proceed to the next section. Consider an MPE for the game with discount factor δ. An important implication of Lemma 1 that the proofs rely on is that

$$u_i = \frac{2\pi_i(1-\delta)}{2-\delta-\delta\pi_i} \times \frac{a_i + \delta u_0(N \setminus \{i\})}{2} + \sum_{k \in N \setminus \{i\}} \frac{\pi_k(2-\delta)}{2-\delta-\delta\pi_i} \times \delta u_i(N \setminus \{k\}). \tag{7}$$

Moreover, we have that

$$\frac{2\pi_i(1-\delta)}{2-\delta-\delta\pi_i} + \sum_{k \in N \setminus \{i\}} \frac{\pi_k(2-\delta)}{2-\delta-\delta\pi_i} = 1. \tag{8}$$

Therefore, formula (7) expresses buyer i's MPE payoff as a convex combination of half of the gains $a_i + \delta u_0(N \setminus \{i\})$ generated by a trade between the seller and buyer i in the initial state, and buyer i's continuation payoffs $\delta u_i(N \setminus \{k\})$ after the seller trades with other buyers k in the initial state. If $\pi_i = 1$, then the weight $2\pi_i(1-\delta)/(2-\delta-\delta\pi_i)$ on the first term equals 1, and the two players share the gains from trade $a_i + \delta u_0(N \setminus \{i\})$ equally. In this case, buyer i becomes a "bottleneck" for the seller's access to gains from future trades, which enables him to "hold up" the seller for half of those gains. Lemma 2 shows that the seller is able to avoid such hold-ups in equilibrium whenever her continuation profits have a positive limit. More generally, it is possible that $\lim_{\delta \to 1} \pi_i = 1$ in a sequence of MPEs for $\delta \to 1$, and the weight $2\pi_i(1-\delta)/(2-\delta-\delta\pi_i)$ has a positive limit, which depends on π_i's rate of convergence to 1 as $\delta \to 1$. For instance, in the second class of MPEs for the example in the previous section, the weight corresponding to buyer 2 converges to 2/5 as $\delta \to 1$. By contrast, if $\lim_{\delta \to 1} \pi_i < 1$, then the weight converges to 0. In this case, buyer i's asymptotic payoffs are driven exclusively by his payoffs in subgames following trades with other buyers. Taking the limit $\delta \to 1$ in (7) for a sequence of MPEs in which all state variables converge, we obtain

$$\bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{1-\bar{\pi}_i} \bar{u}_i(N \setminus \{k\}). \tag{9}$$

This formula facilitates inductive arguments in the proofs of Lemmata 2-4, with the case $\bar{\pi}_i = 1$ requiring separate treatment. While in the latter case, formula (7) is not informative about buyer i's limit payoff without knowledge of π_i's rate of convergence to 1 as δ goes to 1, it carries the information that $\bar{u}_k = \bar{u}_k(N \setminus \{i\})$ when applied for buyers $k \neq i$, which we leverage in the proofs.

The proof of Lemma 2 proceeds by induction on q (with base case $q = 0$). For the inductive step, it is sufficient to establish that $\bar{u}_i \leq a_i/2$ for all $i \in N$ for any sequence of MPEs in which the state variables converge as $\delta \to 1$. From the induction hypothesis, we know that $\bar{u}_i(N \setminus \{k\}) \leq a_i/2$ for all $k \neq i$. If $\bar{u}_0(N \setminus \{i\}) = 0$, then it is easy to reach the conclusion from (7) and (8): each term in the convex combination describing buyer i's payoff, including $(a_i + \delta u_0(N \setminus \{i\}))/2$, is asymptotically bounded above by $a_i/2$. If $\bar{\pi}_i < 1$, then the conclusion follows directly from (9). We are left with the case $\bar{\pi}_i = 1$ and $\bar{u}_0(N \setminus \{i\}) > 0$ (which, as noted earlier, arises for $i = 2$ in the second class of MPEs in the example from the previous section). The latter inequality implies that the seller trades with some buyer $k \in N \setminus \{i\}$ with positive limit probability in the second round of the game after an agreement with i, i.e., $\bar{\pi}_k(N \setminus \{i\}) > 0$. As $\bar{\pi}_i = 1$, the arguments above imply that $\bar{u}_k = \bar{u}_k(N \setminus \{i\})$. It follows $\bar{u}_0 = a_i - \bar{u}_i + a_k - \bar{u}_k + \bar{u}_0(N \setminus \{i, k\})$. The seller may deviate to first trading with buyer k at a price converging to $a_k - \bar{u}_k$, and then trading with buyer i at a price converging to $a_i - \bar{u}_i(N \setminus \{k\})$ to obtain a limit profit of $a_k - \bar{u}_k + a_i - \bar{u}_i(N \setminus \{k\}) + \bar{u}_0(N \setminus \{i, k\})$. For this deviation not to be profitable for the seller for high δ in the sequence of MPEs, it must be that $\bar{u}_i \leq \bar{u}_i(N \setminus \{k\})$, which proves the inductive step via the induction hypothesis.

The proof of Lemma 3 also proceeds by induction on q. For the inductive step, consider a buyer $i \geq q + 1$. We need to argue that $\bar{u}_i = 0$. As in the case of Lemma 2, it is sufficient to establish this for a sequence of MPEs in which state variables converge as $\delta \to 1$. A trade with any buyer $k \neq i$ leads to a game with supply $q - 1$ in which the induction hypothesis implies that $\bar{u}_i(N \setminus \{k\}) = 0$. If $\bar{\pi}_i < 1$, then (9) leads to $\bar{u}_i = 0$. To deal with the delicate case in which $\bar{\pi}_i = 1$, we consider a deviation whereby the seller switches the order of trades with buyer i and another buyer k if $q > 1$ like in the proof of Lemma 2 (or trades with another buyer j for which $a_j \geq a_i$ at limit price a_j if $q = 1$).

For Lemma 4, we argue inductively that $\bar{u}_i \geq a_i/2$ for every buyer i that trades with probability 1 in a sequence of MPEs with $\delta \to 1$. Consider such a buyer i. If $\pi_k > 0$ along a subsequence, then buyer i must trade with probability 1 in subgame $N \setminus \{k\}$, which by the induction hypothesis implies that $\bar{u}_i(N \setminus \{k\}) \geq a_i/2$. The inductive step follows from noting that the payoffs $(a_i + \delta u_0(N \setminus \{i\}))/2$ and $\delta u_i(N \setminus \{k\})$ in the convex combination (7) are asymptotically bounded below by $a_i/2$.

5. Seller Profits

The main result of this section establishes that the seller's MPE payoffs are essentially unique for δ close to 1, and provides a simple formula for the seller's limit profit as δ goes to 1. The uniqueness of asymptotic seller payoffs is unexpected in light of the example discussed in Section 3, which showcases multiple MPEs that are not asymptotically equivalent in terms of buyer payoffs or trading probabilities.

Theorem 1 (Seller profits). *In any collection of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0,1)$, the seller's expected profit converges as $\delta \to 1$ to*

$$M^{*q} := \max_{l \le q+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{q+1} \right]. \tag{10}$$

To prove this theorem, we argue that M^{*q} constitutes both an upper and a lower bound on the seller's asymptotic profit in every sequence of MPEs for the game with supply q for $\delta \to 1$. The first result establishes the upper bound.

Lemma 5 (Upper bound on seller profits). *In any collection of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0,1)$,*

$$\limsup_{\delta \to 1} u_0 \le M^{*q}.$$

We sketch the proof of Lemma 5 here. Consider an MPE of the game with supply $q < n$. Let l be the smallest index of a buyer who trades with probability smaller than 1 in the MPE. We have that $l \le q + 1$. By Lemma 1, the MPE generates a probability distribution over sequences of q distinct buyers with whom the seller trades in the first q rounds of the game. By definition, there exists at least one such sequence $\mathbb{S}$ that excludes buyer l but includes buyers $1, 2, \ldots, l-1$. Since choosing to bargain with buyers in the sequence $\mathbb{S}$ is optimal for the seller, it must be that the seller's MPE payoff is equal to her expected payoff from trading over $\mathbb{S}$. As $\mathbb{S}$ arises with positive probability in equilibrium, each buyer $j < l$ trades with probability 1 in the subgame following agreements with his predecessors in $\mathbb{S}$. Lemma 4 implies that the (limit) expected discounted price the seller collects from buyer j in the subgame is at most $a_j/2$. Hence, the seller's limit payoff from trading with buyers $1, \ldots, l-1$ over $\mathbb{S}$ does not exceed $a_1/2 + \ldots + a_{l-1}/2$. The seller receives no payment from buyer l along $\mathbb{S}$, and can at most extract all surplus from the remaining $q - l + 1$ buyers with the highest valuations. If follows that the seller's limit profit is bounded above by M^{*q}.

Remarkably, it is also the case that the seemingly coarse upper bound M^{*q} constitutes a lower bound on the seller's asymptotic profits in MPEs for the game with supply q as $\delta \to 1$.

Lemma 6 (Lower bound on seller profits). *In any collection of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0,1)$,*

$$\liminf_{\delta \to 1} u_0 \ge M^{*q}.$$

To prove this result, let l^* be a maximizer in the optimization problem defining M^{*q}, and consider a collection of MPEs of the game with supply $q < n$ for $\delta \in (0,1)$. The seller may deviate from her equilibrium strategy to a strategy that generates trades with buyers in the sequence $q+1, q, \ldots, l^*+1, l^*-1, \ldots, 1$ over a fixed but long enough time horizon with probability arbitrarily close to 1. Under this deviation, the seller bargains successively with each buyer in the sequence, rejecting all offers and waiting to become the proposer. Upon being selected to propose to buyer i, the seller makes an offer that buyer i accepts in equilibrium. By Lemma 3, for high enough δ, buyer $i = q+1, q, \ldots, l^*+1$ will accept price offers arbitrarily close to a_i when it is his turn to trade. Similarly, by Lemma 2, buyer $i = l^*-1, \ldots, 1$ will accept price offers arbitrarily close to $a_i/2$. Over a long enough time horizon, the seller will win the coin toss against all buyers in the sequence with probability arbitrarily close to 1, and the deviation secures seller profits arbitrarily close to M^{*q} for high δ. We conclude that the seller's asymptotic profits in the collection of MPEs are bounded below by M^{*q}.

Since the two bounds on the seller's asymptotic payoffs in the game with supply q delivered by Lemmata 5 and 6 coincide, they must be tight. Therefore, in any collection of MPEs for the game with supply q, the seller's profits converge to M^{*q} as $\delta \to 1$, which proves Theorem 1.

We remark that while the strategy underlying the proof of Lemma 6 enables the seller to achieve her limit MPE payoff M^{*q} asymptotically in the game with supply q, it does not necessarily describe the seller's behavior in any MPE, and may even be played with limit probability 0 as $\delta \to 1$. Indeed, when the maximizer in (10) is unique and different from $q+1$, this is an implication of forthcoming Theorem 2.

Sequential outside option principle. Theorem 1 yields a *sequential outside option principle* for settings in which a seller trades sequentially with several, but not all, potential buyers. Recall that the standard outside option principle implies that if the seller has one unit for sale and there are multiple buyers, the second highest valuation is a lower bound on the price the seller can extract from the highest-value buyer. Similarly, if there are q units for sale and n buyers, if we think of the extra marginal buyer $q+1$ as a *static* outside option, $q \cdot a_{q+1}$ should be a lower bound on seller profits.[17] In our *dynamic* bargaining process, the seller can sequentially *exercise* the outside option by trading with the extra marginal buyer $q+1$ first, the new extra marginal buyer q next, and so on; the outside option provided by the extra marginal buyer improves every round. In particular, this argument implies that the seller can extract a profit of $a_2 + \ldots + a_{q+1}$ by trading in sequence with buyers $q+1, q, \ldots, 2$. This is the value of the maximand in (10) for $l = 1$. Our formula for seller profits (10) recognizes that it might be too costly to exclude buyers with high valuations, and combines Lemma 3 with Lemma 2. The latter implies that the seller can trade with buyers from a top interval of valuations at fair (or better) prices.

For another perspective on the sequential exercise of outside options, we revisit the example from the introduction in which $n = 3, q = 2$ and $a_3 > a_1/2$. As argued there, trading with buyer 2 in the first round even at the highest possible price of a_2 is not more valuable than trading with buyer 3 at a price of a_3 (which is feasible in the limit for $\delta \to 1$ by Lemma 3). This is because in the next round, when bargaining with buyer 1, the seller can demand a

[17] The model of Ho and Lee (2019) applied to our setting actually predicts limit seller payoffs of $q \cdot a_{q+1}$ when the outside option provided by buyer $q+1$ is binding for buyers $1, \ldots, q$. See Section 8 for further discussion.

price of a_2 if buyer 2 is available as an outside option, but a lower price of a_3 if buyer 3 is the outside option. In either case, the seller's limit profit is $a_2 + a_3$. This example shows that buyers who are more valuable for inclusion may also be more valuable for exclusion when additional units remain to be sold to even more valuable buyers.

Prices and buyer payoffs. In the example from Section 3, we have seen that limit prices and payoffs for the buyers, unlike limit profits for the seller, may vary across MPEs of the game with supply q when $1 < q < n$. What can then be said about a buyer's limit prices and payoffs along a sequence of MPEs? Frequently quite a bit, even with relatively coarse information about the seller's mixing probabilities in the class of MPEs in question.

A *sequence of MPEs* for the game with supply q associated with a sequence of discount factors δ going to 1 is said to be *convergent* if the corresponding variables $u_i(S)$ and $\pi_i(S)$ as well as the support of $\pi(S)$ converge along the sequence. Since there is a finite set of possible supports for the seller's randomization among buyers in every state S, convergence of the support of $\pi(S)$ is equivalent to the support being constant far enough in the sequence; hence, for each $i \in S$, either $\pi_i(S) = 0$ or $\pi_i(S) > 0$ after a point in the sequence. Every sequence of MPEs contains a subsequence that is convergent according to this definition.

Consider a convergent sequence of MPEs for the game with supply $q < n$, and fix a state S and a buyer $i \in S$ such that $\pi_i(S) > 0$ for δ near 1. Lemma 1 implies that trade with buyer i in state S takes place at limit *price* $\bar{u}_0(S) - \bar{u}_0(S \setminus \{i\})$.[18] Note that we know $\bar{u}_0(S)$ and $\bar{u}_0(S \setminus \{i\})$: they can be computed explicitly by applying Theorem 1 to subgames S and $S \setminus \{i\}$, respectively. More importantly, Lemma 1 implies that $\bar{u}_0(S) = a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_i(S)$, which leads to

$$\bar{u}_i(S) = a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S).$$

Hence, buyer i's limit payoff in state S can be determined without knowledge of the exact probability $\pi_i(S)$ (as long as it is positive) or granular details of the different paths of trade with buyer i starting from state S.

We classify buyer i's trades in the overall game based on sequences of trades with other buyers $i_1, \ldots, i_k$ that lead to buyer i's *first chance to trade* with positive probability in state $S = N \setminus \{i_1, \ldots, i_k\}$, i.e., $\pi_i(N \setminus \{i_1, \ldots, i_k\}) > 0$, and $\pi_i(N \setminus \{i_1, \ldots, i_{k'}\}) = 0$ for $k' < k$. We then use the formula for $\bar{u}_i(S)$ above to account for trades made by buyer i immediately after such sequences (in state S) or following intermediate trades with other buyers (in states $S' \subset S$). This leads to the following result, which expresses buyer i's limit payoff in the overall game as a weighted sum of terms $a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S)$, where the weight $\bar{\theta}_i(S)$ assigned to state S is derived from the limit equilibrium probability of trade with sequences of buyers $i_1, \ldots, i_k$ that have the above properties (the formal definition of $\bar{\theta}_i(S)$ can be found in the Appendix).

[18] As earlier, we use bar notation for the limits of equilibrium variables along the sequence of MPEs.

Proposition 1 (Buyer payoffs)**.** *For any convergent sequence of MPEs of the game with supply $q < n$, we have that*

$$\bar{u}_i = \sum_{S \ni i} \bar{\theta}_i(S)(a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S)).$$

If $\pi_i(N) > 0$ for δ close to 1, then $\bar{\theta}_i(N) = 1$ and $\bar{\theta}_i(S) = 0$ for all other S containing i. Proposition 1 then implies that $\bar{u}_i = a_i + \bar{u}_0(N \setminus \{i\}) - \bar{u}_0(N)$. The result summarizes what can be said more generally about $\bar{u}_i$. The computation of $\bar{\theta}_i(S)$ requires knowledge of the seller's mixing probabilities for *other* buyers who get opportunities to trade before buyer i has a chance, but not of the probabilities with which the seller bargains with i in different states. In some cases, the seller's mixing probabilities for those other buyers may be inferred from their limit payoffs, which in turn can be determined from Proposition 1. In the Online Appendix, we show how this type of exercise leads to a quick derivation of buyers' limit payoffs and trading probabilities in the example from Section 3.[19]

Generalization to heterogeneous proposal probabilities. Consider a more general model in which when bargaining with buyer i, the seller gets the opportunity to make an offer with probability $p_i \in (0, 1)$ and buyer i with complementary probability. In this version of the model, fair pricing for buyer i corresponds to the price $p_i a_i$, and the formula for limit seller profits generalizes to[20]

$$\max_{l \leq q+1} \left[p_1 a_1 + p_2 a_2 + \ldots + p_{l-1} a_{l-1} + a_{l+1} + \ldots + a_{q+1} \right]. \tag{11}$$

We comment on some intriguing implications for which buyers are included and excluded under this bargaining protocol in Section 6.

Extension to random matching. Our bargaining protocol allows the seller to strategically choose which buyer she bargains with in every round. An alternative protocol entails random matching between the seller and individual buyers according to exogenously given probabilities. The protocol with strategic choice of bargaining partner is easier to work with and also seems more natural in our setting, in which the seller with multiple units may wish to trade only with a particular subset of buyers. An awkwardness of the random matching protocol is that the seller gets matched to bargain with buyers that she does not have an incentive to trade with, and such matches lead to delay in equilibrium. Nevertheless, our results extend: the seller can replicate strategic choice of bargaining partners simply by waiting to be matched with a desired buyer at an expected cost of delay that vanishes as

[19] Nonetheless, limit buyer payoff equations do not always carry sufficient information about limit mixing probabilities, as we discuss in the Online Appendix in the context of the example mentioned in footnote 13.

[20] Note that $p_i a_i$ may not be decreasing in i. Indeed, the sequence $(p_i a_i)_{i=1}^n$ may even be increasing, and one might conjecture an analogue of the profit formula based on reindexing the buyers in terms of the decreasing order of $p_i a_i$. Interestingly, the straightforward generalization is the correct one, and our upper and lower bound arguments extend directly to this case.

$\delta \to 1$. At a high level, this is why Theorem 1 and the supporting lemmata extend with minor modifications.[21] We provide details in the Appendix.

In the Online Appendix, we revisit the example from Section 3 in the context of random matching, and argue that there are exact analogues to each of the three classes of MPEs we derived for the benchmark model. Hence, asymptotic inefficiency and multiplicity of MPEs persist in this alternative model.

6. Included and Excluded Buyers

Theorem 1 reveals a close connection between the maximum M^{*q} in the simple static optimization problem displayed in (10) and the seller's profits in the complex dynamic bargaining game with supply $q < n$. As we have seen concretely in the example from Section 3, the seller can attain the total profits M^{*q} in a variety of ways and from different sets of buyers in equilibrium. Nevertheless, Theorem 2 below shows that the optimization problem is also informative—via its maximizers l—about which buyers are certain to trade and which buyers face the threat of "exclusion" in the game.

Generically, the static optimization problem has a unique maximizer l^*. For this generic case, we show that every buyer $i < l^*$ trades with probability 1 in any MPE for high enough δ. The converse is also true: every buyer $i \geq l^*$ trades with probability less than 1 in MPEs for high δ. Thus, buyers $i < l^*$ are guaranteed to be "included"—and hence by Corollary 1 trade at the fair price $a_i/2$ in the limit as $\delta \to 1$—while buyers $i \geq l^*$ are "excluded" with positive probability in equilibrium for high δ. We establish that if $l^* \neq q+1$ and $a_{l^*} > a_{l^*+1}$, then in any collection of MPEs for $\delta \in (0,1)$, buyer l^* trades with limit probability 1 as $\delta \to 1$. In this case, l^* is the buyer with the highest value that is excluded with positive probability in equilibrium, but the probability of excluding l^* vanishes as $\delta \to 1$. However, if $l^* = q+1$, then in MPEs for high δ, the seller trades with the top q buyers with probability 1, and hence trades with buyer l^* with probability 0. We also prove that the seller trades only with buyers with the top $q+1$ valuations, extending the logic of "two is enough for competition" to situations with multiple transactions: an extra buyer is enough for competition. In the Appendix, we state and prove a general version of the theorem that also deals with non-generic cases in which the static optimization problem (10) has multiple maximizers. The proofs of the claims track the evolution of the formula for seller profits in subgames as trade takes place (and involve further use of the supporting lemmata).

Theorem 2 (Included and excluded buyers)**.** *Fix $q < n$, and suppose that the optimization problem displayed in* (10) *has a unique maximizer l^*. Then, there exists $\underline{\delta} < 1$ such that*

[21]We conjecture that the generalization of our results as embodied in (11) extends to any sequential bilateral bargaining protocol that allows the seller to mimic strategic choice of partners at an expected cost of delay going to zero for $\delta \to 1$ (as long as in every bargaining round with buyer i, the seller and buyer i make offers with probabilities $(p_i, 1 - p_i)$).

the following statements hold for every MPE of the game with supply q and discount factor $\delta > \underline{\delta}$.

- *The seller trades with buyer i with probability 1 if and only if $i < l^*$.*
- *If $l^* = q+1$, then the seller trades exclusively with buyers $1, \ldots, q$.*
- *The seller trades with probability 0 with any buyer i for which $a_i < a_{q+1}$.*

If $l^* \neq q+1$ and $a_{l^*} > a_{l^*+1}$, then in any collection of MPEs of the game with supply q for $\delta \in (0,1)$, the probability that the seller trades with buyer l^* converges to 1 as $\delta \to 1$.

The result also highlights subtle differences between the static optimization problem defining M^{*q} and the equilibrium of the dynamic bargaining game: the missing term corresponding to the value of buyer l^* in the formula for M^{*q} does not translate into buyer l^* carrying all the burden of exclusion in the game. Indeed, buyer l^* is almost certain to be included in the limit $\delta \to 1$. In particular, this means that the strategy delivering the lower bound on limit seller profits in the proof of Theorem 1 is played with limit probability 0 in MPEs for $\delta \to 1$.

An example with $n = 3, a_1 = a_2 = 3, a_3 = 1$ shows that weakening the hypothesis $a_{l^*} > a_{l^*+1}$ to require that $a_{l^*} > a_n$ in Theorem 2 does not guarantee the conclusion that buyer l^* trades with limit probability 1. In this example, we have that $l^* = 1$ and $a_1 > a_3$, but there exists a class of MPEs with $\bar{\pi}_1 = \bar{\pi}_2 = 1/4$ and $\bar{\pi}_3 = 1/2$. In this class of MPEs, the seller trades with buyer $l^* = 1$ with limit probability $3/4 < 1$ as $\delta \to 1$.[22]

Theorem 2 generalizes to the version of the model in which the seller gets the opportunity to make offers with probability $p_i \in (0,1)$ when bargaining with buyer i. The corresponding l^* solves the optimization problem (11). The conclusion that a buyer $i < l^*$ for whom p_i is relatively small is included with probability 1 is counterintuitive. However, note that $l^* > i$ implies that $p_i a_i \geq a_{l^*}$, so p_i cannot be arbitrarily low when $i < l^*$. Another implication of the result is that when $l^* = 1$ and $a_1 > a_2$, even if p_1 is relatively low and $p_1 a_1 < p_2 a_2$, the seller trades with buyer 1 with limit probability 1 in MPEs for $\delta \to 1$. This is in sharp contrast with the fact that in a market with $q = 1$ and $p_1 a_1 < p_2 a_2$, if the seller had to commit to bargain exclusively with *either* buyer 1 or 2, she would choose buyer 2.

7. Optimal Exclusion Commitments When $q = n$

We now turn to a strategic situation in which the seller has unconstrained supply $q = n$, but might find it profitable to increase competition between buyers via exclusion commitments. We model such commitments as follows. An *exclusion commitment* $\mathcal{E}$ is a function from the set of all subsets of N to itself such that $\mathcal{E}(S) \subseteq S$, $\mathcal{E}(\{i\}) = \{i\}$ for all $i \in N$, and $\mathcal{E}(S) \subseteq \mathcal{E}(S \setminus \{i\})$ for all $i \in S \setminus \mathcal{E}(S)$. In the *game with exclusion commitment* $\mathcal{E}$, bargaining proceeds like in the *game with supply* q, but trade is restricted by $\mathcal{E}$: after a history in

[22] This example admits two other classes of MPEs with $\pi_1 = 0$ and $\pi_2 = 0$, respectively, similarly to the example from Section 3.

which the seller has not yet traded with a subset of buyers S, she *excludes* the buyers in $\mathcal{E}(S)$, and may only bargain with buyers in $S \setminus \mathcal{E}(S)$; the game ends when $\mathcal{E}(S) = S$. The condition $\mathcal{E}(\{i\}) = \{i\}$ for $i \in N$ ensures that the seller ultimately excludes at least one buyer from trade. The condition $\mathcal{E}(S) \subseteq \mathcal{E}(S \setminus \{i\})$ for $i \in S \setminus \mathcal{E}(S)$ requires that exclusions be irreversible: if the seller is committed to exclude a buyer at a given stage, she eliminates that buyer from all future negotiations.[23] As in the case of the game with exogenous supply, the payoff relevant state for the definition of MPEs in the game with exclusion commitment $\mathcal{E}$ is given by S and the actions in the current round.

A salient class of exclusion commitments, which treats buyers symmetrically, is the $\tilde{q}$-*supply commitment* for $\tilde{q} < n$. This commitment, denoted by $\mathcal{E}^{\tilde{q}}$, is specified by $\mathcal{E}^{\tilde{q}}(S) = S$ if $|S| > n - \tilde{q}$, and $\mathcal{E}^{\tilde{q}}(S) = \emptyset$ otherwise. This means that the game ends exactly after $\tilde{q}$ trades. Hence, the game with $\tilde{q}$-supply commitment is identical to the game with supply $\tilde{q}$.

We seek to derive optimal exclusion commitments for the seller under the least and the most favorable selection of MPEs asymptotically as $\delta \to 1$. Let $\Sigma^\delta(\mathcal{E})$ denote the set of MPEs in the game with an exclusion commitment $\mathcal{E}$ in which players have a common discount factor δ, and $u_0(\sigma, \delta)$ denote the seller's expected payoff under a strategy profile σ. We investigate the following bounds and their associated optimal exclusion commitments $\mathcal{E}$:

$$\begin{aligned} \underline{M} &= \max_{\mathcal{E}} \liminf_{\delta \to 1} \inf_{\sigma \in \Sigma^\delta(\mathcal{E})} u_0(\sigma, \delta) \\ \overline{M} &= \max_{\mathcal{E}} \limsup_{\delta \to 1} \sup_{\sigma \in \Sigma^\delta(\mathcal{E})} u_0(\sigma, \delta). \end{aligned}$$

Our main result about optimal exclusion commitments shows that the two bounds coincide, and are achieved by the same exclusion commitment: the $(n-1)$-supply commitment.[24] As the game with the $(n-1)$-supply commitment is identical to the game with supply $n-1$, Theorem 1 implies that the common value of the bounds is $M^{*(n-1)}$. This exclusion commitment entails that the seller commits to exclude a single buyer but allows her the flexibility to decide dynamically which buyer is excluded. Therefore, maintaining a single unit of shortage at every stage allows the seller to extract all potential benefits of exclusion,

[23] If buyer j is excluded in state S but not in state $S \setminus \{i\}$ for some buyer i with whom trade is allowed in state S, then the potential competition offered by buyer j when bargaining with buyer i in state S is unnecessarily lost. For instance, in a situation where $\mathcal{E}(S) = S \setminus \{i\}$ and $j \in S \setminus \mathcal{E}(S \setminus \{i\})$, buyer i would be a "gateway" to accessing buyer j from state S and could "hold up" the seller for half of the profits she later collects from buyer j. Our formulation of exclusion commitments precludes such hold-ups (but allows for others; see footnote 26).

[24] This is not always the only optimal commitment. For instance, if the optimization problem defining $M^{*(n-1)}$ has a maximizer $l^* > 1$, then modifying the $(n-1)$-supply commitment to rule out paths of trade that exclude buyer 1 generates another optimal exclusion commitment $\mathcal{E}$ ($\mathcal{E}$ differs from $\mathcal{E}^{n-1}$ only in that $\mathcal{E}(\{1, i\}) = \{i\}$ for $i \neq 1$). To achieve the asymptotic bound $M^{*(n-1)}$ in the game with exclusion commitment $\mathcal{E}$, the seller can first trade with buyer 1 at a limit price of at least $a_1/2$, which is feasible by the extension of Lemma 2 to path independent exclusion commitments (such as $\mathcal{E}$) mentioned in footnote 26, and then reach a subgame in which $\mathcal{E}$ reduces to a $(n-2)$-supply commitment, in which we know from Theorem 1 that the seller can obtain an asymptotic payoff of $a_2/2 + \ldots + a_{l^*-1}/2 + a_{l^*+1} + \ldots + a_n$.

and the seller does not benefit from exclusion commitments that treat buyers asymmetrically or create additional scarcity.[25]

Theorem 3 ($(n-1)$-supply commitment is optimal)**.** *The $(n-1)$-supply commitment solves the maximization problems associated with both $\underline{M}$ and $\overline{M}$, and furthermore $\underline{M} = \overline{M} = M^{*(n-1)}$.*

Our permissive formulation of exclusion commitments implies that the conclusion of Theorem 3 is correspondingly strong, while the optimal commitment we identify is simple and does not exploit the permitted complexity. Thus, skeptics who feel that complex commitments are implausible may be reassured by the simplicity of the result, and others need not be concerned that allowing for additional complexity might lead to higher seller profits.

The proof leverages the body of results developed thus far. Since the $(n-1)$-supply commitment is one of the exclusion commitments $\mathcal{E}$ allowed in the optimization problem defining $\underline{M}$, and by Theorem 1, the seller's profit in any collection of MPEs for the game with supply $n-1$ converges to $M^{*(n-1)}$ for $\delta \to 1$, it follows that $\underline{M} \geq M^{*(n-1)}$.

Lemmata 1 and 4 generalize to the game with any exclusion commitment without substantial changes in the proofs.[26] Then, a straightforward adaptation of the argument for Lemma 5 implies that in every collection of MPEs for the game with any exclusion commitment $\mathcal{E}$ for discount factors $\delta \in (0,1)$, the limit superior of the seller's expected profit as $\delta \to 1$ does not exceed $M^{*(n-1)}$. Hence, $\overline{M} \leq M^{*(n-1)}$. As $\overline{M} \geq \underline{M}$, we conclude that $\underline{M} = \overline{M} = M^{*(n-1)}$, which means that the $(n-1)$-supply commitment is optimal for both optimization problems.

Theorem 2 implies that the (generically unique) maximizer l in the optimization problem defining $M^{*(n-1)}$ represents a cutoff for the buyers who are included with certainty in MPEs under the optimal exclusion commitment for high δ. By Corollary 1, these buyers must trade at fair prices in the limit $\delta \to 1$. The other buyers face the risk of exclusion and may have to pay higher than fair prices (as discussed in the context of Corollary 1, some of these buyers can also trade at fair prices).

By definition, an exclusion commitment requires that at least one buyer does not trade. It is possible that the seller attains higher profits without excluding any buyer: formally,

[25] This conclusion is somewhat counterintuitive. Consider an example with $n = 30$ and $a_1 = \ldots = a_{10} = 100, a_{11} = \ldots = a_{20} = 10, a_{21} = \ldots = a_{30} = 1$. It may be tempting to conjecture that in this market the seller should optimally commit to exclude one buyer of each of the three types thereby extracting full surplus from all but one buyer of every type.

[26] While Lemma 2 is not directly needed for the arguments here, we note parenthetically that it extends to the game with exclusion commitment $\mathcal{E}$ with straightforward proof modifications if $\mathcal{E}$ is *path independent*, that is, for every state S that can be reached in the game and all $i \neq j \in S$, we have that $j \in (S \setminus \{i\}) \setminus \mathcal{E}(S \setminus \{i\})$ if and only if $i \in (S \setminus \{j\}) \setminus \mathcal{E}(S \setminus \{j\})$ (a key step in the argument for Lemma 2 concerns a deviation by the seller to a strategy that changes the order of trade for a pair of buyers). An example of an exclusion commitment that violates path independence for which Lemma 2 does not hold is given by $\mathcal{E}(\{1,2,3\}) = \{3\}, \mathcal{E}(\{1,3\}) = \{1,3\}, \mathcal{E}(\{2,3\}) = \{3\}$ in a setting with $n = 3, q = 2$. Under this commitment, buyer 3 is always excluded, and the seller can trade with buyer 2 after buyer 1, but not the other way around. This game has MPEs in which buyer 1 holds up the seller and gets limit payoff $a_1/2 + a_2/4$.

this corresponds to the game with supply $q = n$, in which the seller obtains limit profits $\sum_{i \in N} a_i/2$ by Corollary 2. Theorem 3 implies that the seller is better off with an optimal exclusion commitment whenever $M^{*(n-1)} > \sum_{i \in N} a_i/2$.[27] Note that this is often the case. The condition $M^{*(n-1)} \leq \sum_{i \in N} a_i/2$ is equivalent to $a_l \geq a_{l+1} + \ldots + a_n$ for all $l \leq n-1$, which in turn implies that $a_l \geq 2a_{l+2}$ for all $l \leq n-2$. This requires extreme differences in valuations be maintained consistently through the sequence of buyers: if there exist three consecutive buyers whose valuations do not drop by half, optimal commitments would strictly dominate having no commitments.

Similarly, the condition $l^* \neq n$ invoked in Theorem 2 for the game with supply $n-1$ is likely to be satisfied: $l^* = n$ implies that $M^{*(n-1)} = \sum_{i \in N \setminus \{n\}} a_i/2 < \sum_{i \in N} a_i/2$. When $l^* = n$, buyers $1, \ldots, n-1$ are served with certainty in the game with $(n-1)$-supply commitment. In this case the seller would be better off in the game without exclusion, in which she trades with all buyers with certainty.

When bargaining with an optimal commitment dominates bargaining without commitment, the threat of exclusion enables the seller to extract higher payoffs by flexibly serving $n-1$ of the group of n buyers than she would by serving any subset of $n-1$ buyers with certainty, and indeed by serving *all* n buyers with certainty. It follows directly that one or more buyers must trade with positive probability at higher than fair prices.

We conclude this section with a general MPE existence result.

Proposition 2 (Existence)**.** *An MPE exists for the game with any exogenous supply and for the game with any exclusion commitment.*[28]

8. Optimal Exclusion in the Game with Supply $q < n$

Does a seller with supply $q < n$ benefit from making exclusion commitments stricter than her exogenous supply constraint? An exclusion commitment $\mathcal{E}$ is *more restrictive* than the q-supply commitment $\mathcal{E}^q$ if $\mathcal{E}(S) = S$ whenever $|S| = n - q$ and, furthermore, $\mathcal{E}(S) = S$ for some S with $|S| > n - q$. Again, the argument for Lemma 5 can be easily adapted to show that M^{*q} is an upper bound on limit profits the seller can obtain using any exclusion commitment that is more restrictive than $\mathcal{E}^q$. On the other hand, Theorem 1 shows that the seller's limit profit in the game with supply q is M^{*q}. It follows that in the setting with supply $q < n$, the seller does not benefit from making commitments to exclude buyers at any stage before all available q units are sold (in the language of footnote 27, the seller

[27] An implication of Theorem 3 is that when this inequality is satisfied, a seller who owns $q \geq n$ units and has the option to "burn" some units before bargaining proceeds would optimally burn $q-n+1$ units. Relatedly, Manea (2021) discusses an example in which if buyers make offers more frequently than the seller, the seller is better off supplying a single buyer instead of all.

[28] For $q < n$, the game with supply q is identical to the game with q-supply commitment, so the only game with exogenous supply outside the class of games with exclusion commitments is the game with supply $q = n$.

does not have an incentive to "burn" any of the q units).[29] In particular, for any $\tilde{q} < q$, the $\tilde{q}$-supply exclusion commitment is detrimental to a seller with supply q (this follows directly from noting that $M^{\tilde{q}} < M^q$). This conclusion echoes the intuition from the case with unconstrained supply that any scarcity persisting through the trading process ($q < n$) induces sufficient competition among buyers to deliver the gains of the sequential outside option principle, and further exclusion does not benefit the seller.

This result does not hold in Ho and Lee's (2019) delegated-agent model of bargaining with threat of replacement. In that model, the seller announces a set of buyers ("network") she will "target." The network consists of the most valuable $\tilde{q} \leq q$ buyers. The seller then assigns a representative to each buyer in the announced network, and instructs each representative to bargain only with her assigned buyer and any buyer outside the network. Ho and Lee show that the announced network forms in equilibrium with limit probability 1 as $\delta \to 1$, and the seller's limit profit is $\sum_{i=1}^{\tilde{q}} \max(a_i/2, a_{\tilde{q}+1})$. This expression may be rewritten as

$$\max_{l \leq \tilde{q}+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + (\tilde{q} - l + 1) a_{\tilde{q}+1} \right].$$

Observe that

$$\max_{l \leq \tilde{q}+1} \left[\frac{a_1 + \ldots + a_{l-1}}{2} + (\tilde{q} - l + 1) a_{\tilde{q}+1} \right] \leq \max_{l \leq \tilde{q}+1} \left[\frac{a_1 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{\tilde{q}+1} \right] = M^{*\tilde{q}}.$$

The difference $a_{l+1} + \ldots + a_{\tilde{q}+1} - (\tilde{q} - l + 1) a_{\tilde{q}+1} \geq 0$ in the expressions being maximized in the two optimization problems above is due to the fact that under Ho and Lee's bargaining protocol, every representative relies on the outside option provided by the extra marginal buyer $q+1$ when bargaining with her assigned buyer. In particular, if a representative exercises the outside option of trading with buyer $q+1$, her assigned buyer does not become available to the other representatives as a more valuable outside option. In other words, the protocol followed by the seller's representatives rules out the strategy underlying our sequential outside option principle. For a fixed $\tilde{q}$, the total profits the seller achieves in the setting of Ho and Lee are lower than $M^{*\tilde{q}}$ in general due to both the difference in the maximand for every $l \leq \tilde{q} - 1$ and the possibility that the two optimization problems have different maximizers l.

A seller with supply $q < n$ may benefit from reducing supply to some $\tilde{q} < q$ in the setting of Ho and Lee. As noted above, the resulting total profits in this case are smaller than or equal to $M^{*\tilde{q}}$. In our setting, the seller cannot benefit from restricting supply because $M^{*\tilde{q}} < M^{*q}$. For a concrete example, suppose that $q = 4$ in a market with $n = 5, a_1 = a_2 = a_3 = a_4 = 3, a_5 = 2$. Under the protocol of Ho and Lee, a commitment to supply only three of the four units ("burn" one unit before bargaining) increases seller profits from 8 to

[29] Note, however, that there are exclusion commitments $\mathcal{E}$ more restrictive than $\mathcal{E}^q$ that generate the same limit profits as $\mathcal{E}^q$. This is the case, for instance, if $\mathcal{E}(\{1,3\}) = \{3\}$ and $\mathcal{E}(S) = \mathcal{E}^q(S)$ for all other states S in the example from Section 3. If $q \leq n-2$, this is also the case if $\mathcal{E}(S) = \mathcal{E}^q(S) \cup \{q+2, \ldots, n\}$ for all S.

9. In our model, the optimal exclusion commitment does not require a supply reduction and generates profits of 11 in this example.

9. Conclusion

This paper studies bilateral bargaining between a seller and multiple buyers. The results are most interesting when the seller is unable to serve all buyers either because supply is limited or because the seller commits to excluding some potential buyers. Our analysis reveals that commitments to contract with fewer than the available number of buyers could be a highly effective bargaining tool. We quantify the resultant benefits to the seller. The theory applies symmetrically to the case of a buyer negotiating with multiple sellers.

Our main results characterize seller profits as well as prices, payoffs and trading probabilities for individual buyers under exogenous supply constraints. We also investigate optimal exclusion commitments in the absence of supply constraints. In the process, we formalize exclusion commitments in a general way. The analysis uncovers some key bargaining theoretic principles for the environments considered. On the one hand, buyers cannot hold up the seller in the sense of paying less than fair prices. On the other hand, buyers who are included with certainty must trade at exactly fair prices. Our theory yields a novel sequential outside option principle that captures the role of scarcity in inducing competition between buyers when several successive transactions are possible. With sequential trade, the outside option changes dynamically, and in particular may become increasingly more attractive, enabling a seller who contracts with multiple buyers to extract more surplus than if she were to threaten buyers with a static outside option, as assumed in preceding research on exclusion. We show that in equilibrium the seller optimally chooses a top segment of buyers to include with certainty at fair prices, and exploits the others via the sequential outside option principle.

In many applications, there are externalities between buyers. A buyer's marginal value may depend on the set of buyers that the seller ultimately contracts with. In future research, we seek to address this generalization. We also hope to explore extensions to settings with multiple sellers and multiple buyers.

Appendix

Proof of Lemma 1. Consider an MPE for the game with supply q. In subgame S, the seller can trade only with buyers in S. It follows that the total surplus created in subgame S is bounded above by $\sum_{i \in S} a_i$. As $u_0(S) \geq 0$, we have that

$$\sum_{i \in S} u_i(S) \leq \sum_{i \in S \cup \{0\}} u_i(S) \leq \sum_{i \in S} a_i. \tag{12}$$

Hence, there exists $i \in S$ such that $u_i(S) \leq a_i$.[30] Since the seller has the option to bargain with buyer i in the first period of subgame S and make an acceptable offer that leaves buyer i with utility arbitrarily close to $\delta u_i(S)$, but otherwise demand positive prices and refuse all offers in the future, we have that

$$u_0(S) \geq \frac{1}{2}(a_i - \delta u_i(S)) > 0.$$

As every buyer $i \in S$ will reject offers that yield utility smaller than $\delta u_i(S)$ in state S of the MPE, the payoff the seller receives when making an offer is bounded above by $\max_{i \in S}(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S))$. Standard arguments demonstrate that the seller expects a payoff of $\delta u_0(S)$ in the event the buyer chosen for bargaining is selected to be the proposer (regardless of whether the offer is accepted or rejected). Then, $u_0(S) > 0$ implies that $\max_{i \in S}(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)) > \delta u_0(S)$. As the seller can obtain a payoff arbitrarily close to $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$ by making an acceptable offer to buyer i, it must be that $\pi_i(S) > 0$ only if i maximizes the expression $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$. For such i, we know that $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S) > \delta u_0(S)$.

Optimality of MPE strategies requires that if $\pi_i(S) > 0$, and the seller is selected to be the proposer, then she makes an offer that yields utility $\delta u_i(S)$ for buyer i and utility $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S) > \delta u_0(S)$ for the seller, and buyer i must accept the offer with probability 1 in equilibrium. Similarly, if buyer i is the proposer, he makes an offer that yields utility $\delta u_0(S)$ for the seller, and the seller accepts it with probability 1. The payoff equations follow.

Finally, we prove the statement regarding limit prices. Consider a sequence of MPEs associated with a sequence of discount factors $(\delta_z)_{z \geq 0}$ under which $\pi_i(S) > 0$ and $\lim_{z \to \infty} u_i(S) = \bar{u}_i(S)$. For all $z \geq 0$, the arguments above lead to

$$u_0(S) = \frac{1}{2}(a_i + \delta_z u_0(S \setminus \{i\}) - \delta_z u_i(S)) + \frac{1}{2}\delta_z u_0(S),$$

which implies that $\delta_z u_0(S) - \delta_z u_0(S \setminus \{i\})$ converges to $a_i - \bar{u}_i(S)$ for $z \to \infty$. When the seller makes an offer to buyer i in subgame S, the equilibrium price $a_i - \delta_z u_i(S)$ converges to $a_i - \bar{u}_i(S)$ as $z \to \infty$. If instead buyer i is the proposer, then the equilibrium price $\delta_z u_0(S) - \delta_z u_0(S \setminus \{i\})$ converges to the same limit. □

MPEs in subgames with two buyers and one good. In a subgame $\{i, j\}$ in which buyers i and j with $i < j$ are competing for a single unit, the unique MPE outcomes can be derived from the proof of Proposition 1 in Manea (2018).[31] If $a_j \leq a_i/2$, then for any $\delta \in (0, 1)$, the

[30] The only change necessary to extend this proof to the game with an exclusion commitment $\mathcal{E}$ involves replacing the set S with the set of buyers $S \setminus \mathcal{E}(S)$ who are still permitted to trade in state S in this sequence of arguments. By definition, under any exclusion commitment $\mathcal{E}$, only buyers in $S \setminus \mathcal{E}(S)$ can trade in subgame S.

[31] See also Abreu and Manea (2022) for an extensive discussion of the structure of MPEs in this case.

outside option of trading with buyer j is not binding in the MPE ($\pi_j(\{i,j\}) = 0$), and

$$u_0(\{i,j\}) = u_i(\{i,j\}) = a_i/2, u_j(\{i,j\}) = 0.$$

If $a_j > a_i/2$, then for δ in the non-empty interval $(2(1-a_j/a_i), 1)$, the outside option is binding ($\pi_j(\{i,j\}) > 0$), and the unique MPE payoffs are given by

$$\begin{aligned} u_0(\{i,j\}) &= \frac{(3-2\delta)(a_i+a_j)-\Delta}{2(4-3\delta)} \\ u_i(\{i,j\}) &= \frac{(2+\delta-2\delta^2)a_i-(2-\delta)(3-2\delta)a_j+(2-\delta)\Delta}{2\delta(4-3\delta)} \\ u_j(\{i,j\}) &= \frac{(2+\delta-2\delta^2)a_j-(2-\delta)(3-2\delta)a_i+(2-\delta)\Delta}{2\delta(4-3\delta)}, \end{aligned}$$

where $\Delta := \sqrt{(3-2\delta)^2(a_i^2+a_j^2)-2(7-8\delta+2\delta^2)a_ia_j}$. □

Existence of the MPEs described in the example from Section 3. Define $A_i := a_i + \delta u_0(\{1,2,3\} \setminus \{i\})$ for every buyer $i = 1,2,3$ (with $u_0(\{1,2,3\} \setminus \{i\})$ specified in the previous section).

To establish the existence of the first class of MPEs, we restrict π_3 to the interval $[3/16, 5/16]$, and derive values for π_2 from MPE conditions as a function of δ and π_3, with the understanding that π_1 is given by $1-\pi_2-\pi_3$ in all expressions. Applying (7) for $i = 1,2$ and letting $j = 3-i$, we obtain

$$u_i = \frac{2\pi_i(1-\delta)}{2-\delta-\delta\pi_i}\frac{A_i}{2} + \frac{\pi_j(2-\delta)}{2-\delta-\delta\pi_i}\delta u_i(\{i,3\}) + \frac{\pi_3(2-\delta)}{2-\delta-\delta\pi_i}\delta u_i(\{1,2\}). \tag{13}$$

Furthermore, equation (7) for $i = 3$ reduces to

$$u_3 = \frac{2\pi_3(1-\delta)}{2-\delta-\delta\pi_3}\frac{A_3}{2} \tag{14}$$

because $u_3(\{1,3\}) = u_3(\{2,3\}) = 0$ (as implied by the discussion in the previous section). The seller's indifference between trading with each buyer $i = 1,2$ and buyer 3 leads to

$$A_i - \delta u_i = A_3 - \delta u_3. \tag{15}$$

For $i = 1,2$, we plug the formulae for u_i and u_3 from (13) and (14) in (15), multiply the resulting equality by $2(2-\delta-\delta\pi_i)$ using the substitution $\pi_1 = 1-\pi_2-\pi_3$, and collect the terms containing π_2 to obtain a linear equation in π_2 of the form $x_i(\delta,\pi_3)\pi_2 = y_i(\delta,\pi_3)$. Even though we do not consider the case $\delta = 1$ in our analysis of the game, note that all terms above are well defined for $\delta = 1$ (when $\pi_3 \in [3/16, 5/16]$), and x_i and y_i are continuous functions on the compact domain $\{(\delta,\pi_3) | \delta \in [0,1], \pi_3 \in [3/16, 5/16]\}$. By the Heine-Cantor theorem, x_i and y_i are uniformly continuous over the domain. It follows that for every $\varepsilon > 0$, there exists $\underline{\delta}(\varepsilon) \in [0,1)$ such that the value of each of these functions for any $(\delta,\pi_3) \in D(\varepsilon) := [\underline{\delta}(\varepsilon), 1] \times [3/16, 5/16]$ is within ε of its corresponding value at $(1,\pi_3)$. We find that for all $\pi_3 \in [3/16, 5/16]$,

$$x_1(\delta=1,\pi_3) = 1,\ y_1(\delta=1,\pi_3) = \pi_3,\ x_2(\delta=1,\pi_3) = 1,\ y_2(\delta=1,\pi_3) = 1-3\pi_3.$$

If $\varepsilon < 1$, then $x_i(\delta, \pi_3) > 0$ for all $(\delta, \pi_3) \in D(\varepsilon)$. Consider the function $f : D(1/8) \to \mathbb{R}$ defined by $f(\delta, \pi_3) = y_1(\delta, \pi_3)/x_1(\delta, \pi_3) - y_2(\delta, \pi_3)/x_2(\delta, \pi_3)$. We have that $f(\delta = 1, \pi_3) = 4\pi_3 - 1$. Then, $f(\delta, \pi_3 = 3/16) < 0 < f(\delta, \pi_3 = 5/16)$ for all $\delta \in [\underline{\delta}(1/8), 1]$. As f is continuous in its second argument, the intermediate value theorem implies that for every $\delta \in [\underline{\delta}(1/8), 1]$, there exists $\pi_3(\delta) \in (3/16, 5/16)$ such that $f(\delta, \pi_3(\delta)) = 0$. Define $\pi_2(\delta) = y_1(\delta, \pi_3(\delta))/x_1(\delta, \pi_3(\delta))$. For sufficiently small ε, if $\delta \in [\underline{\delta}(\varepsilon), 1)$, then $\pi_2(\delta) \in (1/8, 3/8)$.

We conclude that for small enough ε, the game with any discount factor $\delta \in [\underline{\delta}(\varepsilon), 1)$ has an MPE in which $(\pi_1, \pi_2, \pi_3) = (1 - \pi_2(\delta) - \pi_3(\delta), \pi_2(\delta), \pi_3(\delta))$. Buyer payoffs in this MPE are obtained by substituting (π_1, π_2, π_3) in (13) and (14); the obtained values are clearly positive. The seller's payoff is given by the common value of $2/(2-\delta)(A_i - \delta u_i)$ for $i = 1, 2, 3$ (consequence of (15)), and is positive because $A_3 > 0$ and (14) implies that $u_3 \leq A_3/2$. The MPE must have the asymptotic structure derived in Section 3 for $\delta \to 1$.

We next establish the existence of the second class of MPEs. For $\pi_1 = 0, \pi_2 \in [0, 1]$ and $\delta \in (0, 1)$, let $u_i(\delta, \pi_2)$ denote the expression on the right-hand side of formula (7), in which we substitute $\pi_3 = 1 - \pi_2$. Define

$$f(\delta, \pi_2) = A_2 - \delta u_2(\delta, \pi_2) - (A_3 - \delta u_3(\delta, \pi_2)).$$

Note that $\lim_{\delta \to 1} A_2 = 5, \lim_{\delta \to 1} A_3 = 4$, and $\lim_{\delta \to 1} u_2(\delta, \pi_2 = 0) = 0, u_3(\delta, \pi_2 = 0) = A_3/2$, while $u_2(\delta, \pi_2 = 1) = A_2/2, \lim_{\delta \to 1} u_3(\delta, \pi_2 = 1) = 0$. It follows that $\lim_{\delta \to 1} f(\delta, \pi_2 = 0) = 3$ and $\lim_{\delta \to 1} f(\delta, \pi_2 = 1) = -1.5$. Since f is continuous in δ, there exists $\underline{\delta} \in [0, 1)$ such that $f(\delta, \pi_2 = 0) > 0 > f(\delta, \pi_2 = 1)$ for $\delta \in (\underline{\delta}, 1)$. Then, the continuity of f in its second argument implies the existence of $\pi_2(\delta) \in (0, 1)$ such that $f(\delta, \pi_2(\delta)) = 0$. The corresponding steps in Section 3 demonstrate that $\lim_{\delta \to 1} \pi_2(\delta) = 1$, and that for δ close to 1 there exists an MPE for the game with discount factor δ in which $(\pi_1, \pi_2, \pi_3) = (0, \pi_2(\delta), \pi_3(\delta))$.

The existence proof for the third class of MPEs is analogous to that for the second class. □

Buyer strengths in the three classes of MPEs for the example. Each buyer $i = 1, 2$ is in the most favorable position if the seller trades first with the other buyer $j = 3 - i$ because buyer 3 provides a weak (and non-binding) outside option following a trade with j. The second and the third classes of MPEs showcase these dynamic equilibrium forces vividly as they involve respective trades with buyers 2 and 1 in the first round with limit probability 1. Consequently, buyers 1 and 2 receive their highest possible asymptotic MPE payoffs—half of their values—in the second and the third class of MPEs, respectively (see Lemma 2).

The first class of MPEs yields the lowest asymptotic payoffs for both buyers 1 and 2 because it entails trading with buyer 3 with limit probability 0.25. In this event, buyers 1 and 2 compete for the remaining unit, and the outside option provided by buyer 2 is binding when the seller is bargaining with buyer 1. This leads to low payoffs for both buyers in the subgame following a trade with buyer 3, $\bar{u}_1(\{1, 2\}) = 1$ and $\bar{u}_2(\{1, 2\}) = 0$, and in the overall game, $\bar{u}_1 = 1.5$ and $\bar{u}_2 = 1$. In the second class of MPEs, buyer 2 also has a low payoff of $\bar{u}_2 = 1$. In this case, a positive limit probability of trading with buyer 3 would be an extremely powerful threat to buyer 2, leading via (5) to $\bar{u}_2 = 0$. Implementing the threat with small but vanishing probability as $\delta \to 1$ ($\pi_3 > 0$, but $\bar{\pi}_3 = 0$) is sufficient to exploit buyer 2 and drive his limit payoff down to 1, thereby maintaining the seller's indifference between trading with buyers 2 and 3 in the initial state. Similar equilibrium forces determine buyer 1's weak position in the third class of MPEs. □

Proof of Lemma 2. We establish the result for all games with supply q by induction on q. The base case $q = 0$ is trivial as all buyers receive zero payoffs in a degenerate game in which no trade is possible.

For the inductive step, consider the game with supply q, and fix a corresponding collection of MPEs $(\sigma^\delta)_{\delta \in (0,1)}$. It is sufficient to show that if u_i converges over a sequence of δ's going to 1, then its limit is at most $a_i/2$ for every buyer i. We can assume by passing to a subsequence $(\delta_z)_{z \geq 0} \to 1$ that all equilibrium variables $u_j, u_j(S), \pi_j, \pi_j(S)$ converge as $z \to \infty$ to limits denoted by $\bar{u}_j, \bar{u}_j(S), \bar{\pi}_j, \bar{\pi}_j(S)$. We need to prove that $\bar{u}_i \leq a_i/2$ for all $i \in N$.

Following an agreement with buyer k, players reach subgame $N \setminus \{k\}$—a game with supply $q-1$, in which the induction hypothesis applies. Hence, $\bar{u}_i(N \setminus \{k\}) \leq a_i/2$ for all $k \neq i$.

Fix a discount factor δ belonging to the sequence (δ_z) and a buyer $i \in N$ such that $\pi_i > 0$ under σ^δ. By Lemma 1, we have that

$$u_0 = \frac{1}{2}\left(a_i + \delta u_0(N \setminus \{i\}) - \delta u_i\right) + \frac{1}{2}\delta u_0 \tag{16}$$

$$u_i = \pi_i \left(\frac{1}{2}\left(a_i + \delta u_0(N \setminus \{i\}) - \delta u_0\right) + \frac{1}{2}\delta u_i\right) + \sum_{k \in N \setminus \{i\}} \pi_k \delta u_i(N \setminus \{k\}). \tag{17}$$

Solving the pair of equations (16) and (17) with unknowns u_0 and u_i and reorganizing terms, we obtain formula (7) from Section 4 (when $\pi_i = 0$, this formula follows directly from (17) even though (16) is not valid in this case). The identities (8) and (9) from Section 4 will also be useful.

If $\bar{\pi}_i < 1$, then (9) leads to

$$\bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{1 - \bar{\pi}_i} \bar{u}_i(N \setminus \{k\}) \leq \frac{a_i}{2}.$$

If $\bar{u}_0(N \setminus \{i\}) = 0$, then for any $\varepsilon > 0$, there exists $\underline{z}$ such that if $z \geq \underline{z}$, then $\delta_z u_0(N \setminus \{i\}) \leq 2\varepsilon$ and $\delta_z u_i(N \setminus \{k\}) \leq a_i/2 + \varepsilon$ for all $k \in N \setminus \{i\}$. Equations (7) and (8) then lead to $u_i \leq a_i/2 + \varepsilon$ for all $z \geq \underline{z}$. Hence, $\bar{u}_i \leq a_i/2$.

For the rest of the proof, assume that $\bar{\pi}_i = 1$ and $\bar{u}_0(N \setminus \{i\}) > 0$. The latter inequality implies that the seller trades with some buyer $k \in N \setminus \{i\}$ with positive limit probability in the second round after reaching the agreement with i under σ^{δ_z}. Hence, $q \geq 2$ and $\bar{\pi}_k(N \setminus \{i\}) > 0$.

Since $\bar{\pi}_i = 1 > 0$, taking the limit $z \to \infty$ for in equation (16) for $\delta = \delta_z$ we obtain

$$\bar{u}_0 = a_i + \bar{u}_0(N \setminus \{i\}) - \bar{u}_i. \tag{18}$$

Similarly, $\bar{\pi}_k(N \setminus \{i\}) > 0$ implies that

$$\bar{u}_0(N \setminus \{i\}) = a_k + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_k(N \setminus \{i\}). \tag{19}$$

As $\bar{\pi}_i = 1$, it must be that

$$\bar{u}_k = \bar{u}_k(N \setminus \{i\}). \tag{20}$$

Putting equalities (18)-(20) together, we obtain

$$\bar{u}_0 = a_i + a_k + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_i - \bar{u}_k. \tag{21}$$

Since the seller may bargain with buyer k in state N and with buyer i in state $N \setminus \{k\}$, we have that

$$\begin{aligned} \bar{u}_0 &\geq a_k + \bar{u}_0(N \setminus \{k\}) - \bar{u}_k \\ \bar{u}_0(N \setminus \{k\}) &\geq a_i + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_i(N \setminus \{k\}), \end{aligned}$$

and hence

$$\bar{u}_0 \geq a_i + a_k + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_i(N \setminus \{k\}) - \bar{u}_k. \tag{22}$$

Then, (21) and (22) imply that $\bar{u}_i \leq \bar{u}_i(N \setminus \{k\})$. Since $\bar{u}_i(N \setminus \{k\}) \leq a_i/2$, we conclude that $\bar{u}_i \leq a_i/2$. □

Proof of Lemma 3. We prove the claim for all games with supply q and any number of buyers $n > q$ by induction on q, with the base case $q = 0$ being trivial like in the proof of Lemma 2 (applying the inductive hypothesis requires a reindexing of the buyers in decreasing order of valuations in subgames). For the inductive step, fix $q \geq 1$, and consider a collection of MPEs $(\sigma^\delta)_{\delta \in (0,1)}$ for the game with supply q, and a buyer $i \geq q+1$. If buyer i's payoff under σ^δ does not converge to 0 for $\delta \to 1$, then there exists a sequence of discount factors going to 1 for which i's payoff converges to a positive limit. By passing to a subsequence, we can assume that the other equilibrium variables also converge. We will establish that $\bar{u}_i = 0$, contradicting the hypothesis above.

For any $k \in N \setminus \{i\}$, buyer i's value is among the highest q in subgame $N \setminus \{k\}$. Since subgame $N \setminus \{k\}$ is a game with supply $q - 1$, the induction hypothesis implies that

$$\bar{u}_i(N \setminus \{k\}) = 0, \forall k \in N \setminus \{i\}. \tag{23}$$

If $\bar{\pi}_i < 1$, then (9) implies that

$$\bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{1 - \bar{\pi}_i} \bar{u}_i(N \setminus \{k\}).$$

Using (23), we conclude that $\bar{u}_i = 0$.

Consider now the case $\bar{\pi}_i = 1$. Applying (9) for buyers $j \neq i$, we obtain $\bar{u}_j = \bar{u}_j(N \setminus \{i\})$.

If $q = 1$,[32] then $\bar{u}_0 = a_i - \bar{u}_i \leq a_i \leq a_2$. As $n \geq 2$, there exists $j \in \{1, 2\} \setminus \{i\}$ for which $\bar{u}_j = u_j(N \setminus \{i\}) = 0$. Since the seller may deviate to trading with such a buyer j at a limit price of a_j, it follows that $\bar{u}_0 \geq a_j$. We conclude that $a_2 \leq a_j \leq \bar{u}_0 = a_i - \bar{u}_i \leq a_i \leq a_2$, which is possible only if all weak inequalities hold with equality. In particular, $a_i - \bar{u}_i = a_i$ leads to $\bar{u}_i = 0$, as claimed.

Now suppose that $q \geq 2$. Then the game with supply q does not end after the seller trades with buyer i in the first round. In subgame $N \setminus \{i\}$, there exists a fixed $j \neq i$ such that $\bar{\pi}_j(N \setminus \{i\}) > 0$. The conditions $\bar{\pi}_i > 0$ and $\bar{\pi}_j(N \setminus \{i\}) > 0$ along with Lemma 1 lead to

$$\bar{u}_0 = a_i - \bar{u}_i + \bar{u}_0(N \setminus \{i\}) = a_i - \bar{u}_i + a_j - \bar{u}_j(N \setminus \{i\}) + \bar{u}_0(N \setminus \{i, j\}).$$

As $\bar{u}_j = \bar{u}_j(N \setminus \{i\})$, we obtain

$$\bar{u}_0 = a_i - \bar{u}_i + a_j - \bar{u}_j + \bar{u}_0(N \setminus \{i, j\}). \tag{24}$$

The seller has the option to deviate and trade with buyer j first at a limit price of $a_j - \bar{u}_j$, and with i second at a price of $a_i - \bar{u}_i(N \setminus \{j\}) = a_i$ (by (23), we have that $\bar{u}_i(N \setminus \{j\}) = 0$). Optimality of the seller's strategy in the sequence of MPEs requires that this deviation does not generate a higher limit profit for the seller:

$$\bar{u}_0 \geq a_j - \bar{u}_j + a_i + \bar{u}_0(N \setminus \{i, j\}). \tag{25}$$

Formula (24) and inequality (25) imply that $\bar{u}_i \leq 0$, and hence $\bar{u}_i = 0$. □

Proof of Lemma 4. We prove the result by induction on q, with the base case $q = 0$ being trivial as all buyers trade with probability 0, not 1, in a degenerate game. Following an agreement with buyer k, players reach subgame $N \setminus \{k\}$—a game with supply $q - 1$, in which the induction hypothesis applies.

For the inductive step, consider a sequence of discount factors $(\delta_z)_{z \geq 0}$ converging to 1 and an associated sequence of MPEs $(\sigma^{\delta_z})_{z \geq 0}$ for the game with supply q such that the seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$.

It is sufficient to prove that if u_i converges along a subsequence of $(\delta_z)_{z \geq 0}$, then its limit is at least $a_i/2$. We can assume by passing to a subsequence that all equilibrium variables $u_k, u_k(S), \pi_k, \pi_k(S)$ converge as $z \to \infty$ to limits denoted by $\bar{u}_k, \bar{u}_k(S), \bar{\pi}_k, \bar{\pi}_k(S)$, and furthermore that the set $K = \{k \in N | \pi_k > 0 \text{ under } \sigma^{\delta_z}\}$ is constant for all $z \geq 0$.[33] We need to show that $\bar{u}_i \geq a_i/2$.

Fix $\varepsilon > 0$. For $k \in K$, we have that $\pi_k > 0$, and the assumption that the seller trades with buyer i with probability 1 under σ^{δ_z} for all z implies that the seller trades with buyer i with probability 1 in subgame $N \setminus \{k\}$ under σ^{δ_z} for all z. The induction hypothesis then

[32] The case $q = 1$ follows from Manea's (2018) Proposition 1. Here we provide a self-contained treatment.

[33] The sequence $((u_k(S), \pi_k(S))_{k,S}, K)_{z \geq 0}$ derived from the sequence of MPEs $(\sigma^{\delta_z})_{z \geq 0}$ is contained in a compact subset of a Euclidean space, so by the Bolzano-Weierstrass theorem it admits a convergent subsequence. Since K can take only a finite set of values, convergence on component K of the subsequence is equivalent to K being constant starting at some point in the subsequence.

shows that $\bar{u}_i(N \setminus \{k\}) \geq a_i/2$ for all $k \in K \setminus \{i\}$. Hence, there exists $\underline{z}$ such that if $z \geq \underline{z}$, then $\delta_z u_i(N \setminus \{k\}) \geq a_i/2 - \varepsilon$ for all $k \in K$. Given the definition of K, note that the range $N \setminus \{i\}$ can be replaced by $K \setminus \{i\}$ in the summations from equations (7) and (8). Then,

$$\frac{a_i + \delta u_0(N \setminus \{i\})}{2} \geq \frac{a_i}{2} \text{ and } \delta_z u_i(N \setminus \{k\}) \geq a_i/2 - \varepsilon, \forall k \in K \setminus \{i\},$$

imply that $u_i \geq a_i/2 - \varepsilon$ for all $z \geq \underline{z}$. As $\varepsilon > 0$ was chosen arbitrarily, it follows that $\bar{u}_i \geq a_i/2$, as asserted. □

Proof of Lemma 5. Fix a collection of MPEs $(\sigma^\delta)_{\delta \in (0,1)}$ for the game with supply $q < n$. For every $\delta \in (0, 1)$, there exists at least one buyer with whom the seller trades with probability smaller than 1 under σ^δ; let $l(\sigma^\delta) \in N$ be the smallest index among buyers with this property. Clearly, $l(\sigma^\delta) \leq q + 1$.

It is sufficient to prove that if u_0 converges along a sequence $(\delta_z)_{z \geq 0}$ going to 1, then its limit does not exceed M^{*q}. We can assume by passing to a subsequence that all equilibrium variables $u_k, u_k(S), \pi_k, \pi_k(S)$ converge as $z \to \infty$ (to limits denoted by $\bar{u}_k, \bar{u}_k(S), \bar{\pi}_k, \bar{\pi}_k(S)$). Since N is finite, the subsequence can be selected to additionally satisfy $l(\sigma^{\delta_z}) = i$ for a fixed $i \leq q + 1$ and all $z \geq 0$. We need to establish that $\bar{u}_0 \leq M^{*q}$.

By Lemma 1, for every $z \geq 0$, the MPE σ^{δ_z} generates a probability distribution over sequences of q different buyers that the seller approaches for bargaining in the first q rounds (with each approach resulting in an agreement). As $l(\sigma^{\delta_z}) = i$, there exists one such sequence $\mathbb{S}$ that arises with positive probability under σ^{δ_z} and excludes buyer i. By passing to a subsequence of $(\delta_z)_{z \geq 0}$ if necessary, we can assume that $\mathbb{S}$ is the same for all z. Since trading over $\mathbb{S}$ is a best response for the seller under the MPE σ^{δ_z}, the seller's equilibrium payoff is equal to her expected payoff from selecting bargaining partners in the sequence $\mathbb{S}$.[34]

As $\mathbb{S}$ arises with positive probability under σ^{δ_z} and $l(\sigma^{\delta_z}) = i$, each buyer $j < i$ is guaranteed to trade under σ^{δ_z} in the subgame following agreements with his predecessors in the sequence $\mathbb{S}$. Lemma 4 implies that the expected discounted price in the agreement with buyer j along $\mathbb{S}$ converges to a limit less than or equal to $a_j/2$ as $z \to \infty$.

Clearly, the seller cannot extract a price greater than a_j from any buyer $j > i$ in the sequence $\mathbb{S}$. Since the seller does not trade with buyer i over $\mathbb{S}$, and there are q buyers in $\mathbb{S}$, we have that

$$\bar{u}_0 \leq \frac{a_1 + a_2 + \ldots + a_{i-1}}{2} + a_{i+1} + \ldots + a_{q+1} \leq M^{*q}.$$

□

[34] To better understand this claim, note that every Markov behavior strategy of the seller can be decomposed into two dimensions: mixing probabilities between buyers in every state at the beginning of a round, and proposal and acceptance decisions at every state within a round. In an MPE, the seller's strategy must be optimal against buyer strategies (and moves by nature), and hence the seller's decisions on the first dimension should also be optimal when we fix her play on the second dimension and the others' strategies. This implies that the seller should be indifferent between all sequences of buyers that occur in equilibrium (given the expected payoffs derived from bargaining with each buyer over each such sequence).

Definition of $\bar{\theta}(S)$ and proof of Proposition 1. Fix a convergent sequence of MPEs, and consider a (possibly empty) sequence of trades with buyers $i_1, \ldots, i_k$ distinct from i such that $\pi_i(N \setminus \{i_1, \ldots, i_k\}) > 0$ and $\pi_i(N \setminus \{i_1, \ldots, i_{k'}\}) = 0$ for $k' < k$. Let $\mathcal{I}_i$ denote the set of sequences $(i_1, \ldots, i_k)$ with this property. Note that every trade of buyer i occurs after one and only one sequence in $\mathcal{I}_i$, either immediately or following intermediate trades with other buyers. Hence, buyer i's limit payoff in the overall game can be expressed as an expected value of the payoffs

$$\bar{u}_i(S) = a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S)$$

over sequences $(i_1, \ldots, i_k)$ in $\mathcal{I}_i$ such that $\{i_1, \ldots, i_k\} = N \setminus S$.

Let $\bar{\pi}_{i_1,\ldots,i_k} = \bar{\pi}_{i_1}(N)\bar{\pi}_{i_2}(N \setminus \{i_1\}) \ldots \bar{\pi}_{i_k}(N \setminus \{i_1, \ldots, i_{k-1}\})$ denote the probability that the seller trades in sequence with buyers $i_1, \ldots, i_k$ (with the value corresponding to the empty sequence understood to be 1), and define

$$\bar{\theta}_i(S) = \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_i:\{i_1,\ldots,i_k\}=N\setminus S} \bar{\pi}_{i_1,\ldots,i_k}.$$

We have that

$$\begin{aligned}\bar{u}_i = \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_i} \bar{\pi}_{i_1,\ldots,i_k}\bar{u}_i(N \setminus \{i_1, \ldots, i_k\}) &= \sum_{S\ni i} \bar{u}_i(S) \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_i:\{i_1,\ldots,i_k\}=N\setminus S} \bar{\pi}_{i_1,\ldots,i_k} \\ &= \sum_{S\ni i} \bar{\theta}_i(S)\bar{u}_i(S) = \sum_{S\ni i} \bar{\theta}_i(S)(a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S)).\end{aligned}$$

This establishes the desired result. □

Proof modifications for the game with random matching. Suppose that in every state S, each buyer $i \in S$ is randomly matched to bargain with the seller with probability $p_i(S) > 0$. Let $u_i(S)$ denote the expected payoff of player $i \in S \cup \{0\}$ in subgame S, and $\pi_i(S)$ be the probability that the seller trades with buyer i in state S (conditional on reaching state S, but not conditional on buyer i being randomly matched with the seller in state S; thus, $\pi_i(S) \leq p_i(S)$). As in the benchmark model, it is sufficient to consider sequences of MPEs for discount factors $\delta \to 1$ in which the variables $u_i(S)$ and $\pi_i(S)$ converge. It is useful to focus on subsequences of MPEs with the additional property that the support of $\pi(S)$ is constant for every state S, so that for any fixed pair $i \in S$, either $\pi_i(S) > 0$ or $\pi_i(S) = 0$ uniformly in the subsequence. With random matching, the seller may be matched with a buyer with whom agreement is not incentive compatible, and this will cause trading delay. The analogue of the "trade in every round" property from Lemma 1 in the model with random matching is that in every state there is a buyer with whom the seller trades with probability 1 conditional on being matched: for every S, there exists $i \in S$ such that $\pi_i(S) = p_i(S)$.

The payoff equations under random matching can be written as follows:

$$\begin{aligned} u_0(S) &= \sum_{k\in S} \pi_k(S)\left(\frac{1}{2}(a_k + \delta u_0(S\setminus\{k\}) - \delta u_k(S)) + \frac{1}{2}\delta u_0(S)\right) + \left(1 - \sum_{k\in S}\pi_k(S)\right)\delta u_0(S) \\ u_i(S) &= \pi_i(S)\left(\frac{1}{2}\left(a_i + \delta u_0(S\setminus\{i\}) - \delta u_0(S)\right) + \frac{1}{2}\delta u_i(S)\right) \\ &+ \sum_{k\in S\setminus\{i\}} \pi_k(S)\delta u_i(S\setminus\{k\}) + \left(1 - \sum_{k\in S}\pi_k(S)\right)\delta u_i(S). \end{aligned}$$

While the seller is no longer indifferent between trading with every buyer $i \in S$ for which $\pi_i(S) > 0$, optimality of the seller's strategy implies that in every state S the seller should be indifferent between all buyers in the support of $\pi(S)$ in the patient limit. For a sequence of MPEs in which $\pi_i(S) > 0$ and state variables converge (to limits denoted by a bar), this means that

$$\bar{u}_0(S) = a_i - \bar{u}_i(S) + \bar{u}_0(S\setminus\{i\}).$$

As in the case of the game with strategic choice of bargaining partner, in state S buyer i trades at an asymptotic price of $a_i - \bar{u}_i(S)$ regardless of whether he wins the coin toss to propose when getting matched. Taking the limit $\delta \to 1$ in buyer i's payoff equation for the initial state N, the asymptotic indifference property leads to the following counterpart to (9):

(26) $$\sum_{j\in N\setminus\{i\}} \bar{\pi}_j > 0 \implies \bar{u}_i = \sum_{k\in N\setminus\{i\}} \frac{\bar{\pi}_k}{\sum_{j\in N\setminus\{i\}}\bar{\pi}_j}\bar{u}_i(N\setminus\{k\}).$$

This condition plays a key role in extending the proofs of Lemmata 2-6 to the model with random matching.

Formulae (7) and (8) rely on the seller's exact indifference when mixing between buyers and do no have immediate analogues in the setting with random matching. The use of these formulae in the treatment of the case $\bar{u}_0(N\setminus\{i\}) = 0$ in the proof of Lemma 2 can be circumvented by noting that $\bar{u}_0(N\setminus\{i\}) = 0$ implies that $q = 1$. The game with random matching for $q = 1$ can be analyzed separately to argue that $\bar{u}_i \le a_i/2$.

The proof of Lemma 4 relies more extensively on (7) and (8). We can deal with the case $\sum_{j\in N\setminus\{i\}} \bar{\pi}_j > 0$ via (26). Consider now the case $\sum_{j\in N\setminus\{i\}} \bar{\pi}_j = 0$. It must be that for high enough δ, we have that $\pi_i = p_i(N)$ and $\pi_j < p_j(N)$ for $j \neq i$. It follows that

$$a_i + \delta u_0(N\setminus\{i\}) - \delta u_i \ge \delta u_0 \ge a_j + \delta u_0(N\setminus\{j\}) - \delta u_j, \forall j \neq i.$$

Then, the seller's payoff equation leads to

$$u_0 \le \sum_{k\in N} \pi_k\left(\frac{1}{2}(a_i + \delta u_0(N\setminus\{i\}) - \delta u_i) + \frac{1}{2}\delta u_0\right) + \left(1 - \sum_{k\in N}\pi_k\right)\delta u_0.$$

This leads to an upper bound for u_0 that depends on u_i, which can be substituted in buyer i's payoff equation to obtain a lower bound on u_i similar to the right hand-side of (7):

$$u_i \geq \frac{2\pi_i(1-\delta)}{(1-\delta+\delta\sum_{j\in N}\pi_j)(2-2\delta+\delta\sum_{j\in N\setminus\{i\}}\pi_j)} \times \frac{a_i+\delta u_0(N\setminus\{i\})}{2} + \sum_{k\in N\setminus\{i\}} \frac{\pi_k(2-2\delta+\delta\sum_{j\in N}\pi_j)}{(1-\delta+\delta\sum_{j\in N}\pi_j)(2-2\delta+\delta\sum_{j\in N\setminus\{i\}}\pi_j)} \times \delta u_i(N\setminus\{k\}).$$

The sum of the coefficients in the equation above simplifies to

$$\frac{\sum_{j\in N}\pi_j}{1-\delta+\delta\sum_{j\in N}\pi_j},$$

which converges to 1 as $\delta \to 1$ (both the numerator and the denominator converge to $\bar{\pi}_i > 0$).[35] This makes it possible to proceed with the inductive proof of Lemma 4 as in the benchmark model.

For the game with random matching, the crucial step identifying the sequence of buyers $\mathbb{S}$ in Lemma 5 does not rely on exact indifference for the seller, but instead uses the seller's asymptotic indifference. We can construct a sequence over which trade occurs with positive probability (this can be defined based solely on the support of every $\pi(S)$, which is constant in the subsequence of MPEs under consideration)—and hence generates the seller's asymptotic MPE payoff—which excludes buyer l and includes buyers $1, \dots, l-1$. This allows us to extend Theorem 1 to the model with random matching. □

Theorem 2 (General version). *Let $\underline{l}^*$ and $\bar{l}^*$ be the smallest and the largest indices l that achieve the maximum in* (10)*, respectively, and let $l(\sigma)$ denote the lowest index of a buyer with whom the seller trades with probability less than 1 under strategy profile σ. There exists $\underline{\delta} < 1$ such that every MPE σ of the game with supply q for any discount factor $\delta > \underline{\delta}$ satisfies the following properties:*

- *$l(\sigma)$ is a maximizer in the optimization problem* (10)*.*
- *If $i < \underline{l}^*$, then the seller trades with buyer i with exact probability 1.*
- *If $i \geq \bar{l}^*$, then the seller trades with buyer i with probability smaller than 1.*
- *If $a_i < a_{q+1}$, then the seller trades with buyer i with probability 0.*

Furthermore, every collection of MPEs $(\sigma^\delta)_{\delta\in(0,1)}$ of the game with supply q for discount factors $\delta \in (0,1)$ has the following asymptotic properties for $\delta \to 1$:

- *If $i < \bar{l}^*$, then the probability that the seller trades with buyer i converges to 1, and the expected payoff of buyer i under σ^δ converges to $a_i/2$.*

[35] This expression can be interpreted as the present value of a prize of 1 received at a stochastic time in an environment where the probability of getting the prize at a given date conditional on not having received it earlier is $\sum_{j\in N}\pi_j$, which reflects the fact that the first trade takes place with probability $\sum_{j\in N}\pi_j$ in the game with random matching.

- *If $\bar{l}^* \neq q+1$ and $a_{\bar{l}^*} > a_{\bar{l}^*+1}$, then the probability that the seller trades with buyer $\bar{l}^*$ converges to 1.*
- *If $\bar{l}^* = q+1$, then the seller trades with buyers $1, \ldots, q$ with probability converging to 1.*

Proof of general version of Theorem 2. We prove the first claim of the result by contradiction. If the claim is not true, then there exist a sequence of discount factors $\delta_z \to 1$ and associated equilibria σ^{δ_z} such that $l(\sigma^{\delta_z})$ is not a maximizer in the optimization problem (10). Moreover, the sequence may be selected such that $l(\sigma^{\delta_z}) = j$ for some fixed j and all $z \geq 0$. Then, the argument from Lemma 5 shows that

$$\bar{u}_0 \leq \frac{a_1 + a_2 + \ldots + a_{j-1}}{2} + a_{j+1} + \ldots + a_n \leq M^{*q}.$$

Since $\bar{u}_0 = M^{*q}$ by Theorem 3, it follows that j achieves the maximum M^{*q} in the optimization problem (10), contradicting the assumption $l(\sigma^{\delta_z}) = j$ is not a maximizer in (10).

The second claim of the result follows from the first. Since $l(\sigma)$ is a maximizer in (10) for every MPE σ when $\delta > \underline{\delta}$, the definition of $\underline{l}^*$ implies that $\underline{l}^* \leq l(\sigma)$.

The proof of the third claim proceeds by contradiction similarly to the first. If the claim is not true, then there exists a buyer $i \geq \bar{l}^*$ and a sequence of discount factors $(\delta_z)_{z\geq 0}$ such that the seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$. As above, $(\delta_z)_{z\geq 0}$ can be selected so that $l(\sigma^{\delta_z}) = j$ for a fixed j and all z. Since i trades with probability 1 under σ^{δ_z}, we have that $j \neq i$, and hence $i > j$. Moreover, each buyer in the set $K = \{1, 2, \ldots, j-1, i\}$ trades with probability 1 under σ^{δ_z} for all z.

Following steps analogous to the proof of Lemma 5, the seller's payoff under σ^{δ_z} is equal to her expected payoff from selecting bargaining partners in a fixed sequence that excludes buyer j and includes each buyer $k \in K$ at a limit (discounted) price of at most $a_k/2$. This means that the seller obtains at most fair prices from buyers $1, 2, \ldots, j-1$ and can extract at most full surplus from a set of $q - j + 1$ buyers different from buyer j, with strictly less than full surplus extraction from buyer i. We conclude that

$$\bar{u}_0 < \frac{a_1 + \ldots + a_{j-1}}{2} + a_{j+1} + \ldots + a_{q+1} \leq M^{*q},$$

which contradicts Theorem 3.

For the fourth claim, we argue by induction on q that for all $q \geq 0$, in every MPE of the game with supply q for high enough δ, any buyer i for which $a_i < a_{q+1}$ trades with probability 0 (applying the inductive hypothesis requires a reindexing of buyers in subgames as in Lemma 3). The base case $q = 0$ is trivial.

To prove the inductive step, assume that $q \geq 1$, and fix a buyer i for which $a_i < a_{q+1}$. Suppose that there exists a sequence of MPEs of the game with supply q for discount factors $(\delta_z)_{z\geq 0}$ converging to 1 along which the seller trades with buyer i with positive probability in the first period of the game. By passing to a subsequence of $(\delta_z)_{z\geq 0}$ along which all relevant

MPE variables converge, we have that

$$\bar{u}_0 \leq a_i + \bar{u}_0(N \setminus \{i\}).$$

Since the seller has the option to first trade with buyer $q+1$ at a limit price of a_{q+1} by Lemma 3, the optimality of her equilibrium strategy implies that

$$\bar{u}_0 \geq a_{q+1} + \bar{u}_0(N \setminus \{q+1\}).$$

Note that both subgames $N \setminus \{i\}$ and $N \setminus \{q+1\}$ have supply $q-1$. When applied to each subgame, Theorem 1 implies that the seller's limit payoff depends only on the top q buyer values. Since $i > q+1$, buyers $k \leq q$ have the top q valuations in either subgame, and hence $\bar{u}_0(N \setminus \{i\}) = \bar{u}_0(N \setminus \{q+1\})$. However, $a_{q+1} > a_i$ generates a contradiction with the inequalities above. This argument establishes that for sufficiently high δ, the seller does not trade with buyer i in the first period of any MPE.

As buyer i does not have one of the top q values in subgame $N \setminus \{j\}$ for any $j \neq i$, the induction hypothesis implies that in all MPEs for high enough δ, the seller should trade with buyer i with probability 0 in every such subgame. Therefore, the seller trades with buyer i with probability 0 in any MPE for high enough δ.

For the second half of the result, fix a collection of MPEs $(\sigma^\delta)_{\delta \in (0,1)}$ of the game with supply q for discount factors $\delta \in (0,1)$.

We first prove the claim regarding payoffs in the first statement. For an argument by contradiction, assume that the expected payoff of buyer $i < \bar{l}^*$ does not converge to $a_i/2$ as $\delta \to 1$. Consider a sequence of discount factors $\delta_z \to 1$ such that buyer i's payoff under σ^{δ_z} converges to a different limit $\bar{u}_i$. By Lemma 2, $\bar{u}_i \leq a_i/2$, so it must be that $\bar{u}_i < a_i/2$. As $z \to \infty$, the seller can deviate from $\sigma_0^{\delta_z}$ to successively trade with buyer i at a limit price of $a_i - \bar{u}_i$, then with each buyer $j = q+1, q, \ldots, \bar{l}^*+1$ at limit price a_j by Lemma 3, and then with each buyer $j = 1, \ldots, \bar{l}^*-1$ different from i at a limit price of at least $a_j/2$ by Lemma 2. This deviation delivers the following lower bound on the seller's limit profit:

$$\bar{u}_0 \geq a_i - \bar{u}_i + a_{\bar{l}^*+1} + \ldots + a_{q+1} + \sum_{j=1, j\neq i}^{\bar{l}^*-1} \frac{a_j}{2} > \frac{a_1 + a_2 + \ldots + a_{\bar{l}^*-1}}{2} + a_{\bar{l}^*+1} + \ldots + a_{q+1} = M^{*q},$$

where the strict inequality is a consequence of $a_i - \bar{u}_i > a_i/2$, and the equality follows from the definition of $\bar{l}^*$. Thus, $\bar{u}_0 > M^{*q}$, contradicting Theorem 3.

We established that $\bar{u}_i = a_i/2$ for every buyer $i < \bar{l}^*$. Lemma 2 then implies that every such buyer trades with limit probability 1 under σ^δ as $\delta \to 1$.

We prove the second claim of the second half of the result also by contradiction. Assume that $\bar{l}^* \neq q+1$ and $a_{\bar{l}^*} > a_{\bar{l}^*+1}$, and suppose that the probability that the seller trades with buyer $\bar{l}^*$ under σ^δ does not converge to 1 for $\delta \to 1$. Then, there exists a sequence of discount factors $\delta_z \to 1$ such that the probability that the seller trades with buyer $\bar{l}^*$ under σ^{δ_z} converges to a limit less than 1, and MPE variables converge. It follows that there exists a path over which the seller trades under σ^{δ_z} with limit probability greater than 0 as $z \to \infty$ with a sequence of buyers $(i_1, \ldots, i_q)$ that does not include $\bar{l}^*$.

As argued above, each buyer $i < \bar{l}^*$ obtains a limit payoff of $a_i/2$ under σ^{δ_z} as $z \to \infty$. Since Lemma 2 implies that buyer i pays a limit price of at least $a_i/2$ in every state he trades with the seller, buyer i should pay a price that converges to $a_i/2$ in every subgame that arises with positive limit probability under σ^{δ_z} as $z \to \infty$.

Let k be the largest index such that $i_k > \bar{l}^*$. Note that k is well defined given the assumption that $\bar{l}^* \neq q+1$. Consider the subgame $S := N \setminus \{i_1, \ldots i_{k-1}\}$, which has supply $q-k+1$. It must be that $\bar{\pi}_{i_k}(S) > 0$.

Define $J = S \setminus \{\bar{l}^*, i_k\}$. For $j \in J$, we have that $j < \bar{l}^*$, so the seller obtains a limit price of exactly $a_j/2$ when trading with buyer j as argued above. The seller can extract a price of at most a_{i_k} from buyer i_k, so her limit profit in subgame S does not exceed

$$M(S) := \frac{a_{i_{k+1}} + \ldots + a_{i_q}}{2} + a_{i_k}.$$

Applying Theorem 1 to subgame S, we get that $\bar{u}_0(S) \geq M(S)$. Hence, $\bar{u}_0(S) = M(S)$.

Since $\bar{\pi}_{i_k}(S) > 0$, we have that $\bar{\pi}_{\bar{l}^*}(S) < 1$, and a version of formula (9) leads to

$$\bar{u}_{\bar{l}^*}(S) = \sum_{j \in S \setminus \{\bar{l}^*\}} \frac{\bar{\pi}_j(S)}{1 - \bar{\pi}_{\bar{l}^*}(S)} \bar{u}_{\bar{l}^*}(S \setminus \{j\}). \tag{27}$$

Subgame $S \setminus \{i_k\}$ has supply $q-k$, and contains $q-k$ buyers $i_{k+1}, \ldots, i_q > \bar{l}^*$. Lemma 3 implies that $\bar{u}_{\bar{l}^*}(S \setminus \{i_k\}) = 0$.

Consider now any $j \in J$ with $\bar{\pi}_j(S) > 0$, so that subgame $S \setminus \{j\}$ is reached with positive limit probability under σ^{δ_z} as $z \to \infty$. As argued above, the seller trades with buyer j at limit price $a_j/2$ in subgame S, and has to trade with every other buyer $j' \in J \setminus \{j\}$ with limit probability 1 at limit price $a_{j'}/2$ in subgame $S \setminus \{j\}$. Hence,

$$\bar{u}_0(S \setminus \{j\}) = M(S) - a_j/2 \quad \& \quad \bar{u}_{j'}(S \setminus \{j\}) = a_{j'}/2, \forall j' \in J \setminus \{j\}.$$

Since the maximum total surplus achievable in subgame $S \setminus \{j\}$ is $\sum_{j' \in J \setminus \{j\}} a_{j'} + a_{\bar{l}^*}$, it follows that

$$\bar{u}_{\bar{l}^*}(S \setminus \{j\}) \leq \sum_{j' \in J \setminus \{j\}} a_{j'} + a_{\bar{l}^*} - \bar{u}_0(S \setminus \{j\}) - \sum_{j' \in J \setminus \{j\}} \bar{u}_{j'}(S \setminus \{j\}) = a_{\bar{l}^*} - a_{i_k}.$$

As $\bar{\pi}_{i_k}(S) > 0$, $\bar{u}_{\bar{l}^*}(S \setminus \{i_k\}) = 0, \bar{u}_{\bar{l}^*}(S \setminus \{j\}) \leq a_{\bar{l}^*} - a_{i_k}$ for all $j \in J$ with $\bar{\pi}_j(S) > 0$, and $a_{\bar{l}^*} - a_{i_k} \geq a_{\bar{l}^*} - a_{\bar{l}^*+1} > 0$, equation (27) implies that

$$\bar{u}_{\bar{l}^*}(S) < a_{\bar{l}^*} - a_{i_k}.$$

However, in subgame S, the seller can deviate from $\sigma_0^{\delta_z}$ to bargain with $\bar{l}^*$ and trade at limit price $a_{\bar{l}^*} - \bar{u}_{\bar{l}^*}(S) > a_{i_k}$, and then bargain with each buyer $j \in J$ and trade at a limit price of at least $a_j/2$ by Lemma 2 for $z \to \infty$. This deviation generates a limit profit greater than $M(S)$ for the seller, contradicting $\bar{u}_0(S) = M(S)$.

Finally, if $\bar{l}^* = q+1$, then the first claim of the second half of the result implies that the seller trades with each of the q buyers $i < \bar{l}^*$ with limit probability 1. If $\bar{l}^* = \underline{l}^* = q+1$, then the second claim of the first half of the result implies that the seller must trade with buyers $1, \ldots, q$ with exact probability 1 for sufficiently high δ. □

Proof of Proposition 2. We establish the existence of an MPE for the bargaining game with exclusion commitment. The proof for the bargaining game with exogenous supply is analogous.

Consider the game with an exclusion commitment $\mathcal{E}$. It will be convenient to use the notation $\mathcal{I}(S) := S \setminus \mathcal{E}(S)$. We inductively construct MPE expected payoffs and bargaining probabilities for all players working backward from terminal states. Let m be the maximum number of trades possible under $\mathcal{E}$. In every subgame in which the seller has traded with exactly m buyers (terminal nodes), the payoffs of all players are 0. Assuming that we specified MPE strategies for subgames in which the seller has traded with at least $m'+1$ buyers, we next construct MPE expected payoffs and bargaining probabilities for subgames in which the seller has traded with exactly m' buyers. Consider such a subgame S. We will argue that the constructed payoffs satisfy

$$u_i(S) \geq 0, \forall i \in \mathcal{I}(S) \cup \{0\} \tag{28}$$

$$\sum_{i \in \mathcal{I}(S) \cup \{0\}} u_i(S) \leq \sum_{i \in \mathcal{I}(S)} a_i. \tag{29}$$

Consider a candidate payoff profile $(u_i(S))_{i \in \mathcal{I}(S)}$ for the "active" buyers in state S contained in the simplex

$$\mathcal{U} = \{(u_i(S))_{i \in \mathcal{I}(S)} | u_i(S) \geq 0, \forall i \in \mathcal{I}(S); \sum_{i \in \mathcal{I}(S)} u_i(S) \leq \sum_{i \in \mathcal{I}(S)} a_i\}.$$

We construct a correspondence $F : \mathcal{U} \rightrightarrows \mathcal{U}$ as follows. For every $(u_i(S))_{i \in \mathcal{I}(S)} \in \mathcal{U}$, let $u_0'(S)$ be the payoff the seller can attain by making acceptable offers to optimally selected buyers and $\Pi(S) \subseteq \Delta(\mathcal{I}(S))$ the set of optimal bargaining probabilities for the seller in state S:

$$u_0'(S) = \frac{1}{2-\delta} \max_{i \in \mathcal{I}(S)} (a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)) \tag{30}$$

$$\Pi(S) = \Delta\Big(\arg\max_{i \in \mathcal{I}(S)} (a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S))\Big). \tag{31}$$

The correspondence F maps $(u_i(S))_{i \in \mathcal{I}(S)}$ to the set of profiles $(u_i'(S))_{i \in \mathcal{I}(S)}$ given by

$$u_i'(S) = \pi_i(S) \left(\frac{1}{2} (a_i + \delta u_0(S \setminus \{i\}) - \delta u_0'(S)) + \frac{1}{2} \delta u_i(S) \right) + \sum_{k \in \mathcal{I}(S) \setminus \{i\}} \pi_k(S) \delta u_i(S \setminus \{k\}) \tag{32}$$

for any selection of bargaining probabilities $\pi(S) \in \Pi(S)$.

F is convex-valued because $\Pi(S)$ is a convex set for every element of $\mathcal{U}$.

We next argue that the range of F is indeed included in $\mathcal{U}$. For any $(u_i(S))_{i\in\mathcal{I}(S)} \in \mathcal{U}$, there exists $i \in \mathcal{I}(S)$ such that $a_i > \delta u_i(S)$. Otherwise, $\sum_{i\in\mathcal{I}(S)} u_i(S) \geq (\sum_{i\in\mathcal{I}(S)} a_i)/\delta > \sum_{i\in\mathcal{I}(S)} a_i$. It follows that there exists $i \in \mathcal{I}(S)$ such that $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S) > 0$, and hence $u_0'(S) > 0$.

Then, for any $\pi(S) \in \Pi(S)$, the condition $\pi_i(S) > 0$ implies that $u_0'(S) < a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$, which leads to $\delta u_0'(S) < a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$. Therefore, $a_i + \delta u_0(S \setminus \{i\}) - \delta u_0'(S) > \delta u_i(S) \geq 0$. Since all other terms appearing on the right-hand side of (32) are non-negative, we conclude that $u_i'(S) \geq 0$ for all $i \in \mathcal{I}(S)$.

We are left to show that $\sum_{i\in\mathcal{I}(S)} u_i'(S) \leq \sum_{i\in\mathcal{I}(S)} a_i$. Given conditions (30) and (31), $u_0'(S)$ solves the following equation for any $\pi(S) \in \Pi(S)$:

$$u_0'(S) = \sum_{i\in\mathcal{I}(S)} \pi_i(S) \left(\frac{1}{2} \left(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)\right) + \frac{1}{2} \delta u_0'(S) \right). \tag{33}$$

Summing up equations (32) over all $i \in \mathcal{I}(S)$ and equation (33), we obtain

$$\begin{aligned}\sum_{i\in\mathcal{I}(S)\cup\{0\}} u_i'(S) &= \sum_{i\in\mathcal{I}(S)} \pi_i(S)(a_i + \delta \sum_{k\in(\mathcal{I}(S)\setminus\{i\})\cup\{0\}} u_k(S \setminus \{i\})) \\ &\leq \max_{i\in\mathcal{I}(S)} \left(a_i + \delta \sum_{k\in\mathcal{I}(S\setminus\{i\})} a_k \right) \leq \sum_{i\in\mathcal{I}(S)} a_i.\end{aligned}$$

The first inequality follows from the fact that condition (29) holds for subgame $S \setminus \{i\}$ (formally, we set $u_k(S \setminus \{i\}) = 0$ for $k \in \mathcal{E}(S \setminus \{i\})$), and the second from the requirement that $\mathcal{E}$ satisfies $\mathcal{E}(S) \subseteq \mathcal{E}(S \setminus \{i\})$ for $i \in S \setminus \mathcal{E}(S) = \mathcal{I}(S)$, and hence $\mathcal{I}(S \setminus \{i\}) \subseteq \mathcal{I}(S) \setminus \{i\}$.

Since $u_0'(S)$ varies continuously with $(u_i(S))_{i\in\mathcal{I}(S)}$, and $\Pi(S)$ has closed graph as a correspondence defined on $\mathcal{U}$, it follows that F has closed graph. Kakutani's fixed-point theorem then implies that F has a fixed point $(u_i(S))_{i\in\mathcal{I}(S)}$. We then define $u_0(S)$ to be the corresponding $u_0'(S)$ and recover the probabilities $(\pi_i(S))_{i\in\mathcal{I}(S)}$ associated with the fixed point.

We can now construct an MPE. In state S, the seller chooses to bargain with buyer i with probability $\pi_i(S)$. When the seller bargains with buyer i, if the seller is selected to be the proposer, she offers an acceptable price that gives the buyer utility $\delta u_i(S)$, and similarly the buyer makes an acceptable offer that gives the seller utility $\delta u_0(S)$. A simple inductive argument combined with the payoff equations above proves that the constructed strategies generate the expected payoffs given by u. By the single-deviation principle, the specification of bargaining probabilities and offers in state S, in conjunction with the assumed behavior in subgames following a trade in state S, induces an MPE. □

References

1. Abreu, D. and M. Manea (2012a) Markov equilibria in a model of bargaining in networks, *Games Econ. Behav.*, **75**, 1-16.
2. Abreu, D. and M. Manea (2012b) Bargaining and Efficiency in Networks, *J. Econ. Theory* **147**, 43-70.
3. Abreu, D. and M. Manea (2022) Bargaining Foundations for the Outside Option Principle, mimeo.

4. Arie, G., P.L.E. Grieco and S. Rachmilevitch (2018) Nash Overpricing and Sequential Negotiations, mimeo.
5. Binmore, K. (1980) Nash Bargaining Theory II, ICERD D.P. 80/14, *Econ. Bargaining*, Oxford.
6. Binmore, K. (1985) Bargaining and Coalitions, *Game Theoretic Models of Bargaining*, (A. Roth, ed.), Cambridge University Press.
7. Binmore, K., A. Rubinstein and A. Wolinsky (1986) The Nash Bargaining Solution in Economic Modelling, *RAND J. Econ.* **17**, 176.
8. Binmore, K., A. Shaked and J. Sutton (1989) An Outside Option Experiment, *Q. J. Econ.*, **104**, 753-770.
9. Bhaskar, V., G.J. Mailath, S. Morris (2013) A Foundation for Markov Equilibria in Sequential Games with Finite Social Memory, *Rev. Econ. Stud.*, **80**, 925-948.
10. Elliott, M. and F. Nava (2019) Decentralized Bargaining in Matching Markets: Efficient Stationary Equilibria and the Core, *Theoretical Econ.*, **14**, 211-251.
11. Gal-Or, E. (1997) Exclusionary Equilibria in Health-Care Markets, *J. Econ. and Manag. Strategy*, **6**, 5-43.
12. Ghili, S. (2022) Network Formation and Bargaining in Vertical Markets: The Case of Narrow Networks in Health Insurance, *Mark. Sci.*, **41**, 501-527.
13. Herrero, M. (1985) A Strategic Bargaining Approach to Market Institutions, Ph.D. thesis, London School of Economics.
14. Ho, K. and R.S. Lee (2019) Equilibrium Provider Networks: Bargaining and Exclusion in Health Care Markets, *Amer. Econ. Rev.*, **109**, 473-522.
15. Howard, D.H. (2014) Adverse Effects of Prohibiting Narrow Provider Networks, *N. Engl. J. Med.*, **371**, 591-593.
16. Jackson, M.O. and A. Wolinsky (1996) A Strategic Model of Social and Economic Networks, *J. Econ. Theory*, **71**, 44-74.
17. Liebman, E. (2018) Bargaining in Markets with Exclusion: An Analysis on Health Insurance Networks, mimeo.
18. Manea, M. (2018) Intermediation and Resale in Networks, *J. Pol. Econ.*, **126**, 1250-1301.
19. Manea, M. (2021) Bottleneck Links, Essential Intermediaries, and Competing Paths of Diffusion in Networks, *Theoretical Econ.*, **16**, 1017-1053.
20. Maskin, M. and J. Tirole (2001) Markov Perfect Equilibrium, I. Observable Actions, *J. Econ. Theory*, *100*, 191-219.
21. Nash, J. (1950) The Bargaining Problem, *Econometrica*, **18**, 155-162.
22. Rubinstein, A. (1982) Perfect equilibria of a bargaining model, *Econometrica*, **50**, 97-110.
23. Rubinstein, A. and A. Wolinsky (1990) Decentralized Trading, Strategic Behaviour and the Walrasian Outcome, *Rev. Econ. Stud.*, **57**, 63-78.
24. Sutton, J. (1986) Non-Cooperative Bargaining Theory: An Introduction, *Rev. Econ. Stud.*, **53**, 709-724.
25. Stole, L.A. and J. Zwiebel (1996) Intra-Firm Bargaining under Non-Binding Contracts, *Rev. Econ. Stud.*, **63**, 375-410.
26. Talamas, E. (2019) Price Dispersion In Stationary Networked Markets, *Games and Econ. Behav.*, **115**, 247-264.
27. Talamas, E. (2020) Nash bargaining with endogenous outside options, mimeo.