$$\begin{aligned} \min \ & \sum_j p_{ij} t_j \\ \text{s.t.} \ & \sum_j p_{ij} t_j - c_i \geq \sum_j p_{i'j} t_j - c_{i'} && \forall i' \neq i \\ & t_j \geq 0 && \forall j \end{aligned}$$

(a) Primal LP for action i.

$$\begin{aligned} \max \ & \sum_{i' \neq i} \lambda_{i'} (c_i - c_{i'}) \\ \text{s.t.} \ & \sum_{i' \neq i} \lambda_{i'} (p_{ij} - p_{i'j}) \leq p_{ij} && \forall j \\ & \lambda_{i'} \geq 0 && \forall i' \neq i \end{aligned}$$

(b) Dual LP for action i.

Figure 8: The minimum payment LP.

A Proof of Proposition 1

Proof of Proposition 1. Figure 8 shows the linear program determining the optimal classic contract implementing action i, and its dual. Consider the primal LP in Figure 8 (a) for action i, but with the objective $\min \sum_j p_{ij} t_j$ replaced with $\min 0$. Action i is implementable if and only if the resulting LP is feasible. The dual of this LP is:

$$\begin{aligned} \max \quad & \sum_{i' \neq i} \lambda_{i'} (c_i - c_{i'}) \\ & \sum_{i' \neq i} \lambda_{i'} (p_{ij} - p_{i'j}) \leq 0 && \forall \ j \\ & \lambda_{i'} \geq 0 && \forall \ i' \neq i \end{aligned}$$

By strong duality for a general primal-dual pair we can have the following four cases:

(1) The dual LP and the primal LP are both feasible.

(2) The dual LP is unbounded and the primal LP is infeasible.

(3) The dual LP is infeasible and the primal LP is unbounded.

(4) The dual LP and the primal LP are both infeasible.

In our case the dual LP is always feasible (we can choose $\lambda_{i'} = 0$ for all $i' \neq i$). This rules out cases (3) and (4). So in order to prove the claim it suffices to show that the dual LP is unbounded if and only if there exists a convex combination $(\gamma_{i'})_{i' \neq i}$ such that $\sum_{i' \neq i} \gamma_{i'} p_{i'j} = p_{ij}$ for all j and $\sum_{i' \neq i} \gamma_{i'} c_{i'} < c_i$.

We first show that if such a convex combination exists, then the dual LP is unbounded. Indeed, if such a convex combination exists, then it corresponds to a feasible solution to the dual LP because, for all j,

$$\sum_{i' \neq i} \gamma_{i'}(p_{ij} - p_{i'j}) = \left(\sum_{i' \neq i} \gamma_{i'} \right) p_{ij} - \left(\sum_{i' \neq i} \gamma_{i'} p_{i'j} \right) = 0,$$

where we used that $\sum_{i' \neq i} \gamma_{i'} = 1$ and $\sum_{i' \neq i} \gamma_{i'} p_{i'j} = p_{ij}$. Moreover, the objective value achieved by this solution is

$$\sum_{i' \neq i} \gamma_{i'}(c_i - c_{i'}) = \left(\sum_{i' \neq i} \gamma_{i'} \right) c_i - \left(\sum_{i' \neq i} \gamma_{i'} c_{i'} \right) = \delta$$

for some $\delta > 0$, where we used that $\sum_{i' \neq i} \gamma_{i'} = 1$ and $\sum_{i' \neq i} \gamma_{i'} c_{i'} < c_i$. But then for any $\kappa \geq 0$ setting the dual variables to $\kappa \cdot \gamma_{i'}$ for $i' \neq i$ results in a feasible solution whose objective value is equal to $\kappa \cdot \delta$. So the dual LP is unbounded.

We next show that if the dual LP is unbounded, then a convex combination with the desired properties must exist. Since the dual LP is unbounded, for any $\delta > 0$ there must be a feasible solution to the dual LP, $(\lambda_{i'})_{i' \neq i}$, such that $\sum_{i' \neq i} \lambda_{i'}(c_i - c_{i'}) \geq \delta$ and $\sum_{i' \neq i} \lambda_{i'}(p_{ij} - p_{i'j}) \leq 0$ for all j. Now consider $\gamma_{i'} = \lambda_{i'} / (\sum_{i' \neq i} \lambda_{i'})$ for all $i' \neq i$. We claim that $(\gamma_{i'})_{i' \neq i}$ is a convex combination with the desired properties. First note that $(\gamma_{i'})_{i' \neq i}$ is a convex combination, i.e., $\gamma_{i'} \in [0, 1]$ for all $i' \neq i$ and $\sum_{i' \neq i} \gamma_{i'} = 1$. Also note that

$$\sum_{i' \neq i} \gamma_{i'}(c_i - c_{i'}) = \frac{1}{\sum_{i' \neq i} \lambda_{i'}} \sum_{i' \neq i} \lambda_{i'}(c_i - c_{i'}) = \frac{1}{\sum_{i' \neq i} \lambda_{i'}} \cdot \delta > 0$$

and therefore $\sum_{i' \neq i} \gamma_{i'} c_{i'} < (\sum_{i' \neq i} \gamma_{i'}) c_i = c_i$. Moreover, for all j, using the fact that $\sum_{i' \neq i} \lambda_{i'} p_{i'j} \geq (\sum_{i' \neq i} \lambda_{i'}) p_{ij}$, we must have

$$\sum_{i' \neq i} \gamma_{i'} p_{i'j} = \frac{1}{\sum_{i' \neq i} \lambda_{i'}} \sum_{i' \neq i} \lambda_{i'} p_{i'j} \geq \frac{1}{\sum_{i' \neq i} \lambda_{i'}} \left(\sum_{i' \neq i} \lambda_{i'} \right) p_{ij} = p_{ij}.$$

So we know that for all j, $\sum_{i' \neq i} \gamma_{i'} p_{i'j} \geq p_{ij}$. We claim that, for all j, this inequality must hold with equality. Indeed, assume for contradiction that for some j' we have a

rewards:	$r_1 = 0$	$r_2 = \delta$	$r_3 = 2\delta$	$\dots$	$r_m = m-1$	costs
action 1:	0	$\frac{1}{m-1}$	$\frac{1}{m-1}$	$\dots$	$\frac{1}{m-1}$	$c_1 = 0$
action 2:	$\frac{1}{m-1}$	0	$\frac{1}{m-1}$	$\dots$	$\frac{1}{m-1}$	$c_2 = 0$
action 3:	$\frac{1}{m-1}$	$\frac{1}{m-1}$	0	$\dots$	$\frac{1}{m-1}$	$c_3 = 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
action m:	$\frac{1}{m-1}$	$\frac{1}{m-1}$	$\frac{1}{m-1}$	$\dots$	0	$c_m = 0$
action $m+1$:	$\frac{1}{(m-1)^2}$	$\frac{1}{(m-1)^2}$	$\frac{1}{(m-1)^2}$	$\dots$	$1 - \frac{1}{m-1}$	$c_{m+1} = 1$

Figure 9: Instance (c, r, p) used in the proof of Proposition 5.

strict inequality. By summing over all j, we then have

$$\sum_j \left(\sum_{i' \neq i} \gamma_{i'} p_{i'j} \right) > \sum_j p_{ij} = 1, \tag{8}$$

where we used that p_i is a probability distribution over outcomes j. On the other hand, we have that

$$\sum_j \left(\sum_{i' \neq i} \gamma_{i'} p_{i'j} \right) = \sum_{i' \neq i} \gamma_{i'} \left(\sum_j p_{i'j} \right) = \sum_{i' \neq i} \gamma_{i'} = 1, \tag{9}$$

where we used that the $p_{i'}$'s are also probability distributions over outcomes j and that $(\gamma_{i'})_{i' \neq i}$ is a convex combination. Combining (8) with (9) we get the desired contradiction. □

B Tightness of Theorem 1

Proposition 5. *Assume $m > 4$. There exist instances where the best ambiguous contract composed of $n - 1 = m$ payment functions is strictly better than any ambiguous contract that is composed of fewer payment functions.*

Proof. Consider the instance (c, r, p) depicted in Figure 9 with m outcomes and $n =$

$m+1$ actions, where δ is a vanishingly small positive number. First note that in this instance the expected reward R_i of any action $i \leq m$ is

$$R_i \leq \sum_{j=1}^{m} \frac{1}{m-1} r_j = \left(\sum_{j=1}^{m-1} \frac{(j-1)}{m-1} \cdot \delta \right) + \frac{m-1}{m-1} = 1 + \frac{1}{2}(m^2 - 3m + 2)\delta,$$

which can be made arbitrarily close to 1 by choosing δ small enough. On the other hand, the expected reward of action $m+1$ is

$$R_{m+1} = \left(\sum_{j=1}^{m-1} \frac{1}{(m-1)^2} \cdot (j-1)\delta \right) + \left(1 - \frac{1}{m-1} \right) \cdot (m-1) = (m-2) \left(1 + \frac{1}{2(m-1)} \delta \right).$$

Since δ is positive and $m > 4$, it follows that $R_{m+1} \geq m - 2$ and $W_{m+1} = R_{m+1} - c_{m+1} \geq m - 3 \geq 2$. To prove the claim it thus suffices to show that there exists a (consistent) IC ambiguous contract consisting of $n - 1 = m$ payment functions that implements action $m + 1$ with an expected payment equal to c_{m+1}; while any (consistent) IC ambiguous contract that consists of strictly fewer payment functions must pay strictly more in order to implement action $m + 1$.

We first show that we can indeed implement action $m + 1$ with an ambiguous contract consisting of $n - 1 = m$ payment functions with an expected payment equal to c_{m+1}. To this end consider the ambiguous contract $\langle \tau, m+1 \rangle$ consisting of $n-1 = m$ SOP payment functions:

$$\begin{aligned} \tau &= \{t^1, t^2, \ldots, t^m\} \quad s.t \\ t^j_j &= \frac{c_{m+1}}{p_{(m+1)j}} \quad \text{and} \quad t^j_{j'} = 0 \quad \forall j' \neq j \end{aligned}$$

It is then easy to verify, that for any payment function $t^j \in \tau$, $T_{m+1}(t^j) = c_{m+1}$, which shows that $\langle \tau, m + 1 \rangle$ is consistent. Since $T_{m+1}(\tau) = c_{m+1}$, the agent's utility for action $m + 1$ is $U_A(m + 1 \mid \tau) = 0$. On the other hand, for any action $i \neq m + 1$, there is an SOP contract $t^j \in \tau$ such that $T_i(t^j) = 0$. Thus, for any action $i \neq m + 1$, $U_A(i \mid \tau) \leq 0$, which shows that $\langle \tau, m + 1 \rangle$ is IC.

To complete the proof, assume by contradiction that there is a (consistent and) IC ambiguous contract $\langle \tau', m + 1 \rangle$ with $T_{m+1}(\tau') = c_{m+1}$ but $|\tau'| < n - 1 = m$. Since $U_A(m + 1 \mid \tau') = 0$, in order for $\langle \tau', m + 1 \rangle$ to implement action $m + 1$, we must have $U_A(i \mid \tau') \leq 0$ for each action $i \in [m]$. The only way to achieve this is to have, for each such action $i \in [m]$, a payment function $t' \in \tau'$ under which $T_{i'}(t') = 0$. Since for each action $i \in [m]$, there exist only one outcome, denoted by $j(i)$, for which it has 0 probability, and for all other j' we have that $p_{ij'} > 0$, we get that the only type of payment function for which its expected payment is 0, is an SOP payment function that pays for that specific outcome, $j(i)$. Since each action has a unique outcome for which it has 0 probability (i.e., for all $i, i' \in [m]$, $i \neq i'$, it holds that $j(i) \neq j(i')$), we get that for each action the SOP payment function for which its expected payment is 0, is unique as well. This shows that τ' must consist of at least $n - 1 = m$ payment functions, in contradiction to our assumption. $\square$

C Proof of Proposition 3

Proof of Proposition 3. By Proposition 2 action i is implementable if and only if there is no other action i' such that $p_{i'} = p_i$ and $c_{i'} < c_i$. Therefore, if $A = \emptyset$, then it must hold that $c_i = 0$. In this case, $\langle \{(0, \ldots, 0)\}, i \rangle$ implements action i, and clearly no other contract can do so with a lower expected payment. So suppose $A \neq \emptyset$, and consider the contract $\langle \tau, i \rangle$ for this case from the statement of the proposition. We have already argued that this contract is (consistent and) IC (in the proof of Proposition 2). It remains to show that it is optimal.

For a contradiction suppose that there exists a (consistent) IC contract $\langle \tau', i \rangle$, such that $T_i(\tau') < T_i(\tau)$. By Theorem 1, we can assume that $\langle \tau', i \rangle$ is an SOP contract.

Consider an action $i' \in A$ for which

$$\min \left\{ x \geq 0 \;\middle|\; p_{ij(i')} \cdot \frac{x}{p_{ij(i')}} - c_i \geq p_{i'j(i')} \cdot \frac{x}{p_{ij(i')}} - c_{i'} \right\} = T, \tag{10}$$

and recall that $T_i(\tau) = T$.

First note that since $\langle \tau, i \rangle$ is IC, we must have $T \geq c_i$. Moreover, we cannot have $T = c_i$, because then we would have $T_i(\tau') < T_i(\tau) = T = c_i$, which would contradict our assumption that $\langle \tau', i \rangle$ is IC.

So consider the case where $T > c_i$. In this case, since $p_{ij(i')}/p_{ij(i')} > p_{i'j(i')}/p_{ij(i')}$, Equation (10) implies that $c_i > c_{i'}$.

Since $\langle \tau', i \rangle$ is IC, there must be an SOP payment function $t' \in \tau'$ such that $U_A(i' \mid \tau') = U_A(i' \mid t') \leq U_A(i \mid t') = U_A(i \mid \tau')$. Since t' is SOP, there must be an outcome j' for which $t'_{j'} \geq 0$, while $t'_{j''} = 0$ for all $j'' \neq j'$.

If $j' = j(i')$, then

$$p_{i'j(i')} \cdot \frac{T_i(\tau')}{p_{ij(i')}} - c_{i'} = U_A(i' \mid t') \leq U_A(i \mid t') = p_{ij(i')} \cdot \frac{T_i(\tau')}{p_{ij(i')}} - c_i,$$

which contradicts the minimality of T.

Otherwise, $j' \neq j(i')$. In this case, since $U_A(i \mid t') \geq U_A(i' \mid t')$ and $c_i > c_{i'}$, we must have $p_{i'j'}/p_{ij'} < 1$. Moreover, by definition of $j(i')$, we must have $p_{i'j'}/p_{ij'} \geq p_{i'j(i')}/p_{ij(i')}$. So

$$T_i(\tau) = \frac{c_i - c_{i'}}{1 - \frac{p_{i'j(i')}}{p_{ij(i')}}} \leq \frac{c_i - c_{i'}}{1 - \frac{p_{i'j'}}{p_{ij'}}} \leq T_i(\tau'),$$

where the equality holds by definition of T, the first inequality uses that $c_i - c_{i'} > 0$, and the final inequality holds because $U_A(i \mid t') \geq U_A(i' \mid t')$. We obtain a contradiction to our assumption that $T_i(\tau') < T_i(\tau)$. □

D Proof of Theorem 2

Proof. Theorem 1 ensures that there exists an ambiguous contract $\langle \tau'', i \rangle$ consisting of the SOP payment functions $\tau'' = \{t^1, \ldots, t^k\}$ that implements action i with expected payment $T_i(\tau)$, and with t^1 and t^k specified as in point 2. To prove the result, it suffices to show that the ambiguous contract $\langle \tau', i \rangle$ with $\tau' = \{t^1, t^k\}$ also implements action i. In turn, it suffices for this result to note that

$$i' < i \implies T_{i'}(\tau'') = \min_{j=1,\ldots,k} p_{i'j} \frac{T_i(\tau)}{p_{ij}} \geq p_{i'h} \frac{T_i(\tau)}{p_{ih}} \geq \min_{j=1,k} p_{i'j} \frac{T_i(\tau)}{p_{ij}} = T_{i'}(\tau'), \quad (11)$$

$$i' > i \implies T_{i'}(\tau'') = \min_{j=1,\ldots k} p_{i'j} \frac{T_i(\tau)}{p_{ij}} \geq p_{i'\ell} \frac{T_i(\tau)}{p_{i\ell}} \geq \min_{j=1,k} p_{i'j} \frac{T_i(\tau)}{p_{ij}} = T_{i'}(\tau'). \quad (12)$$

The first inequality in each of these statements follows from the MLRP condition. In the case of (11), for example, this inequality follows from noting that if $i' < i$, then the MLRP condition implies that (since $h > j$) $p_{ih}p_{i'j} \geq p_{ij}p_{i'h}$. □

E Proof of Theorem 4

Proof of Theorem 4. By construction, the ambiguous contract $\langle \tau', i \rangle$ with $\tau' = \{t^1, t^k\}$ is consistent and $T_i(\tau') = T_i(\tau)$, so point 1 is satisfied. To show that $\langle \tau', i \rangle$ is incentive compatible, we note that since $\langle \tau, i \rangle$ implements action i, each action $i' \neq i$ has a monotone payment function $t^{(i')} \in \tau$ such that $U_A(i' \mid t^{(i')}) \leq U_A(i \mid \tau)$. It then suffices to show that each action $i' \neq i$ has a payment function $t \in \tau'$ for which $T_{i'}(t) \leq T_{i'}(t^{(i')})$.

For actions $i' \neq i$ such that $c_{i'} \leq c_i$, we claim that $T_{i'}(t^k) \leq T_{i'}(t^{(i')})$. Indeed, for any such action i',

$$T_{i'}(t^k) = \sum_{j=1}^{m} p_{i'j} \cdot t_j^k = \sum_{j=h}^{m} p_{i'j} \cdot t_j^k = p_{i'h} \cdot t_h^k,$$

where we used that $t_j^k = 0$ for $j < h$, and that $p_{i'j} = 0$ for $j > h$ by the MLRP condition. Substituting the definition of $t_h^k = T_i(\tau)/p_{ih}$ and using that $T_i(\tau) = T_i(t^{(i')})$ by consistency, we obtain

$$T_{i'}(t^k) = p_{i'h} \cdot \frac{T_i(t^{(i')})}{p_{ih}} = p_{i'h} \cdot \frac{\sum_{j \in [m]} p_{ij} \cdot t_j^{(i')}}{p_{ih}} = \sum_{j=1}^{h} \frac{p_{i'h} \cdot p_{ij}}{p_{ih}} \cdot t_j^{(i')},$$

where for the last step we used that $p_{ij} = 0$ for $j > h$ (by definition of h).

By the MLRP condition, since $c_{i'} \leq c_i$, for all $j \leq h$:

$$\frac{p_{i'h}}{p_{ih}} \leq \frac{p_{i'j}}{p_{ij}} \implies \frac{p_{i'h} \cdot p_{ij}}{p_{ih}} \leq p_{i'j}.$$

Using this we obtain

$$T_{i'}(t^k) \leq \sum_{j=1}^{h} p_{i'j} \cdot t_j^{(i')} = \sum_{j=1}^{m} p_{i'j} \cdot t_j^{(i')} = T_{i'}(t^{(i')}),$$

where we again used that $p_{i'j} = 0$ for $j > h$ by the MLRP condition.

For actions $i' \neq i$ such that $c_{i'} > c_i$ we claim that $T_{i'}(t^1) \leq T_{i'}(t^{(i')})$. In this case with $\ell' = \min\{j \in [m] \mid p_{i'j} > 0\}$, the expected payment for action i' under t^1 is

$$T_{i'}(t^1) = \sum_{j=1}^{m} p_{i'j} \cdot t_j^1 = \sum_{j=\ell'}^{m} p_{i'j} \cdot T_i(\tau) \leq T_i(\tau) = T_i(t^{(i')})$$

completing the proof. □

F Proof of Lemma 2

Proof of Lemma 2. If $W = 0$, then the linear contract $\langle (0, \ldots, 0), 1 \rangle$ is optimal and Equation (2) is immediate with $A = \{1\}$ and $\alpha_1 = 0$. So suppose $W > 0$.

Let $A \subseteq [n] \setminus \{1\}$ be a subset of actions formed by (i) excluding action 1 (the null action), (ii) excluding actions that cannot be implemented with a linear contract, and (iii) selecting a single action from each of the subsets (if any) of the remaining actions that have the same cost. Let $n' = |A|$. Note that since $W > 0$, it must be that $A \neq \emptyset$ and thus $1 \leq n' \leq n-1$. Relabel the actions $i \in A$ according to the smallest $\alpha \in [0, 1]$ such that $\langle t, i \rangle = \langle (\alpha r_1, \ldots, \alpha r_m), i \rangle$ is incentive compatible, and let α_i denote the corresponding α. The sequences $\{R_i\}$, $\{W_i\}$, $\{c_i\}$ for $i \in A$ are strictly increasing in i, with $R_1 > 0$ and $c_1 \geq 0$.

Observe that for $i = 1$ we have $\alpha_1 = c_1/R_1$, while for $i > 1$ we have

$$\alpha_i = \frac{c_i - c_{i-1}}{R_i - R_{i-1}}.$$

Using this notation, we have

$$\max_{\langle t,i \rangle \in \mathcal{L}(c,r',p)} U_P(\langle t, i \rangle) = \max_{i \in A} (1 - \alpha_i) R_i.$$

We next show an upper bound on the maximum social welfare. Note that

$$W = \max_{i \in A} W_i = W_{n'} = R_{n'} - c_{n'},$$

where we used that the highest-welfare action is among the actions in A. The upper bound follows from a lower bound on $(1 - \alpha_i)R_i$, summed up over all $i \in A$.

For $i = 1$, we use that, by the definition of α_1, it holds that $(1 - \alpha_1)R_1 = (1 - c_1/R_1)R_1 = R_1 - c_1$. For $i > 1$, we again use the formula for α_i, to obtain that

$$(1 - \alpha_i)R_i = \left(1 - \frac{c_i - c_{i-1}}{R_i - R_{i-1}}\right) R_i \geq (R_i - c_i) - (R_{i-1} - c_{i-1}),$$

where we additionally used that $R_i/(R_i - R_{i-1}) \geq 1$.

Hence, by a telescoping sum argument,

$$W = W_{n'} = (R_1 - c_1) + \sum_{i=2}^{n'} \Big((R_i - c_i) - (R_{i-1} - c_{i-1}) \Big) \le \sum_{i=1}^{n'} (1-\alpha_i) R_i.$$

Using this for the final inequality in the following, we thus obtain

$$\max_{\langle t,i\rangle \in \mathcal{L}(c,r,p)} U_P(\langle t,i\rangle) = \max_{i\in A} (1-\alpha_i)R_i \ge \frac{1}{n'} \sum_{i=1}^{n'} (1-\alpha_i) R_i \ge \frac{1}{n'} W,$$

giving the result. □

G Classic Contracts in Lower-Bound Instance

The following lemma shows that in the instance depicted in Figure 7, the principal cannot obtain a utility greater than 1 using a classic contract.

Lemma 3. *Let $n \ge 3$. Let $\gamma, \epsilon \in (0,1)$ and let $\delta = \epsilon \cdot \gamma^{n-2}$. Consider the parameterized instance (c,r,p) with n actions depicted in Figure 7. Then*

$$\max_{\langle t,i\rangle \in \mathcal{C}(c,r,p)} U_p(\langle t,i\rangle) \le 1.$$

Proof. We have $W_1 \le W_2 \le 1$, so there is nothing to show for these actions. Next consider any action i such that $3 \le i \le n-1$. Suppose $\langle t,i\rangle$ implements action i. Since action i puts zero probability on outcome 1, we can without loss of generality assume that $t_1 = 0$. We derive a lower bound on $T_i(t)$, by considering only the IC constraint that compares action i to action $i-1$. According to this constraint, we must have

$$(1-\gamma^{n-i})t_2 + \gamma^{n-i} t_3 - c_i \ge (1-\gamma^{n-i+1})t_2 + \gamma^{n-i+1} t_3 - c_{i-1}.$$

Since $(1-\gamma^{n-i}) < (1-\gamma^{n-i+1})$ and $\gamma^{n-i} > \gamma^{n-i+1}$, we obtain a lower bound on the expected payment $T_i(t)$ by setting $t_2 = 0$ and finding the smallest t_3 such that

$$\gamma^{n-i} t_3 - c_i \ge \gamma^{n-i+1} t_3 - c_{i-1}.$$

Rearranging and substituting c_i and c_{i-1}, this yields

$$t_3 \geq \frac{1}{\gamma^{n-i}} \cdot \frac{c_i - c_{i-1}}{1-\gamma} = \frac{1}{\gamma^{n-i}} \left(\frac{1}{\gamma^{i-2}} - 1 \right).$$

So the principal's utility from action i is at most

$$\gamma^{n-i} r_3 - \gamma^{n-i} t_3 \leq \frac{1}{\gamma^{i-2}} - \left(\frac{1}{\gamma^{i-2}} - 1 \right) = 1.$$

Next consider action n, and any IC contract $\langle t, n \rangle$. Since action n puts zero probability on outcome 2, we can without loss of generality assume $t_2 = 0$. To obtain an upper bound on the principal's utility, we proceed in a similar manner as before, except that now in addition to comparing to action $n-1$ we also compare to action 1. The comparison to action 1 gives

$$\delta t_1 + (1-\delta) t_3 - c_n \geq t_1,$$

or equivalently

$$t_3 - t_1 \geq \frac{c_n}{1-\delta}.$$

An important consequence of this is that $t_3 - t_1 \geq 0$. Combining this with the fact that the agent does not want to deviate to action $n-1$, we obtain

$$\gamma t_3 - c_{n-1} \leq \delta t_1 + (1-\delta) t_3 - c_n = t_3 - \delta(t_3 - t_1) - c_n \leq t_3 - c_n.$$

Rearranging and substituting c_n and c_{n-1}, this yields

$$t_3 \geq \frac{1}{1-\gamma}(c_n - c_{n-1}) = \frac{1}{\gamma^{n-2}} - 1.$$

Hence the principal's utility from action n is at most

$$(1-\delta) r_3 - (1-\delta) t_3 \leq 1 - \delta,$$

which completes the proof. □

H Algorithmic Implications

In this section, we show that the structural properties of optimal ambiguous contracts specified in Section 3 have important algorithmic implications. Specifically, they lead to polynomial-time algorithms for the optimal ambiguous contract problem.

Theorem 7 (Computation)**.** *There exists an algorithm that computes the optimal ambiguous contract in time $O(n^2m)$. If the instance satisfies MLRP, then the running time improves to $O(n^2 + m)$.*

Proof. We argue that for any given action $i \in [n]$ we can, in $O(nm)$ time, (1) decide whether it can be implemented by an ambiguous contract and (2) find the optimal contract $\langle \tau, i \rangle$ that implements it (if it is implementable). Applying this algorithm to all actions $i \in [n]$ and choosing the ambiguous contract that maximizes the principal's utility, we find the optimal ambiguous contract in $O(n^2m)$ time.

Fix any action $i \in [n]$. By Proposition 2 action i is implementable if there is no other action $i \neq i$ such that $p_{i'} = p_i$ and $c_{i'} < c_i$. For a given action $i' \neq i$ we can check whether $p_{i'} = p_i$ and $c_{i'} < c_i$ in $O(m)$ time. Applying this to all actions $i' \neq i$, we can test implementability in $O(nm)$ time.

For each action $i \in [n]$ that is implementable, we can determine an optimal ambiguous contract via Proposition 3. Note that we can determine the set A in $O(nm)$ time (simply by iterating over all actions $i' \neq i$ and checking whether $p_{i'} \neq p_i$). If $A = \emptyset$, then an optimal ambiguous contract is given by $\langle (0, \ldots, 0), i \rangle$, and outputting this contract requires $O(1)$ time. On the other hand, if $A \neq \emptyset$, then for each of the at most $n - 1$ actions in A we can determine the maximum likelihood ratio outcome $j(i')$ in $O(m)$ time. We can then compute action i''s contribution to T in $O(1)$ time. We can thus compute the maximum likelihood ratio outcomes for all $i' \in A$ and T in $O(nm)$ time. Outputting the optimal ambiguous contract described in Proposition 3 for this case takes $O(nm)$ time. We conclude that we can determine an optimal ambiguous contract for action i in $O(nm)$ time.

The improved running time for MLRP instances follows from the stronger characterization of optimal ambiguous contracts for such instances (Theorem 2). According to this characterization, to find the optimal ambiguous contract $\langle \tau, i \rangle$ that implements action i (if one exists), we can first determine the indices (ℓ_i, h_i) (defined in bullet 2). Then, for each action $i' \neq i$ such that $c_{i'} \leq c_i$ we can find the minimum payment on outcome h_i such that the agent prefers action i over action i'; while for actions

$i' \neq i$ such that $c_{i'} > c_i$ we can do the same with respect to outcome ℓ_i instead of h_i. Applying this procedure to all actions $i \in [n]$, we obtain an algorithm for finding the optimal ambiguous contract that runs in $O(n^2)$ time given access to the (ℓ_i, h_i) indices. The proof is completed by noting that for MLRP instances these indices are monotone (non-decreasing) in cost, and can thus be precomputed in $O(n+m)$ time. □

Notably, using similar ideas to the ones used in the proof of Theorem 7, one can show that for monotone contracts, there exists an algorithm that computes the optimal ambiguous contract in time $O(n^2 m)$. If the instance satisfies MLRP, then the running time improves to $O(n^2 + m)$.