A comment on “Expected Uncertain Utility”

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1 Introduction

In an innovative paper, replete with many important results and insights, Gul and Pesendorfer (2014) (hereafter GP) propose a novel model for choice under uncertainty. They consider a setting of purely subjective uncertainty in which the objects of choice are acts that, for each state of nature $\omega \in \Omega$, deliver a monetary prize x from a set of final prizes $X = [\ell, m]$, with $\ell < m$. We denote the set of acts by $\mathcal{F}$, and the decision-maker's preference relation defined over $\mathcal{F}$ by a weak order $\succsim$.

In GP's model the decision-maker (hereafter, DM) has a prior μ defined over $\mathcal{E}$, a σ-algebra of what they refer to as *ideal events*. GP interpret any ideal event E (in $\mathcal{E}$) as one for which the DM can precisely quantify that event's uncertainty by assigning it the probability $\mu(E)$. An event is deemed ideal by the DM if both it and its complement together satisfy a version of Savage's (1954) sure thing principle.

The utility of any act g that is adapted to the set of ideal events (what GP refer to as an *ideal* act) may be expressed as an expected utility:

$$V(g) = \int v(g)\, d\mu \tag{1}$$

for some Bernoulli utility v.

In evaluating any general act f in their model, the DM first forms an ideal (greatest) lower bound $[f]_1$ and an ideal (least) upper bound $[f]_2$ to represent the range of possible outcomes implied by the uncertainty that she cannot precisely quantify with her prior μ. The expected utility of f with which the DM will compare the desirability of f compared to other acts is then given by:

$$V(f) = \int u\left([f]_1, [f]_2\right) d\mu, \tag{2}$$

where $u(x, y)$ (with $x \leqslant y$) is the utility assigned by the DM to an unquantifiable uncertain prospect with prizes lying in the interval $[x, y]$. GP refer to such a DM as an *expected uncertain utility (EUU)*-maximizer and to the utility index u as an *interval utility*. As they note, when f is ideal its lower and upper bounds coincide and so expression (2) reduces to the expected utility formula in (1) for the Bernoulli utility $v(x) := u(x, x)$.

In order for a preference relation to admit an expected uncertain utility representation of the form given in (2), we require it to exhibit properties that ensure the existence of a rich σ-algebra of ideal events that enable us to associate with each act f its *envelope*, formally the mapping

$$[f]\colon \{[x, y] \in X \times X\colon x \leqslant y\} \to \mathbb{R},$$

defined by setting for each ω in Ω, $[f](\omega) := [[f]_1(\omega), [f]_2(\omega)]$. Furthermore, any pair of acts associated with the same envelope must reside in the same indifference class. This in turn allows us to derive from the original preference relation an induced preference relation over envelopes. The characterization of EUU maximization then boils down to establishing this induced relation over envelopes admits an SEU representation characterized by a prior μ defined over the set of ideal events and an interval utility $u(\cdot, \cdot)$.

Unfortunately, GP's characterization fails on two accounts as their axioms neither ensure

(i) the set of ideal events is a σ-algebra;

nor,

(ii) the interval utility is state-independent.

In this note we show strengthening one of GP's axioms along with a slight modification of their continuity axiom, provides a characterization of EUU-maximization. But first, we present in section 2 an example of an EUU-functional involving a state-dependent interval utility and show that the preferences generated by this example, despite satisfying all of GP's axioms, cannot be represented by an EUU function of the form in (2).

2 An example with a state-dependent interval utility.

Let the state space $\Omega = [0,1]$ be endowed with the Lebesgue measure μ. Let $\mathcal{E}_\mu$ denote the set of measurable events with respect to μ. Following GP, $[f]$ is the (interval) envelope of an act f; with $[f]_1$ (respectively, $[f]_2$) denoting the lower (respectively, upper) envelope.

Consider the preference relation $\succsim$ generated by the function

$$V(f) := \int_0^{\frac{1}{2}} \left(\frac{1}{2}[f]_1 + \frac{1}{2}[f]_2 \right) d\mu + \int_{\frac{1}{2}}^1 \left(\frac{2}{3}[f]_1 + \frac{1}{3}[f]_2 \right) d\mu \,. \tag{3}$$

Intuitively this is a "state-dependent" interval utility; however, for any ideal act f, since $[f]_1 = [f]_2$, V reduces to subjective expected utility with a linear Bernoulli utility.

We show that $\succsim$ satisfies GP's Axioms A1–A6 which we list here for the convenience of the reader. To state them we employ the following notation: for any pair of acts f and g and any event $C \subset \Omega$, fCg denotes the act that agrees with f on C and with g on the complement of C. We also require the following definitions.

An event E is *ideal* if $[fEh \succsim gEh$ and $hEf \succsim hEg]$ implies $[fEh' \succsim gEh'$ and $h'Ef \succsim h'Eg]$ for all acts f, g, h, and h'. An event A is *null* if $fAh \sim gAh$ for all acts f, g and h. An event D is *diffuse* if $E \cap D \neq \emptyset \neq E \cap D^c$ for every non-null ideal event E. Let $\mathcal{E}$ (respectively, $\mathcal{N}$, $\mathcal{D}$) be the set of all ideal (respectively, null, diffuse) events. Let $\mathcal{F}^e$ denote the set of *ideal simple* acts.[1]

[1] A simple act $f \in \mathcal{F}^o$ is one that has a finite range; hence for any $f \in \mathcal{F}^e$ we have $f^{-1}(x) \in \mathcal{E}$ for all x.

As in GP, we say an event E is *left* (respectively, *right*) ideal if $fEh \succsim gEh$ implies $fEh' \succsim gEh'$ (respectively $hEf \succsim hEg$ implies $h'Ef \succsim h'Eg$). Let $\mathcal{E}^l$ and $\mathcal{E}^r$ be the collection of left and right ideal sets respectively. GP's Lemma B0 establishes $\mathcal{E} = \mathcal{E}^l \cap \mathcal{E}^r$.

In line with GP's use of notation, events E, E', E_i et cetera denote ideal events while events D, D', D_i denote diffuse events.

A1 $\succsim$ is complete and transitive.

A2 $f > g$ implies $f \succ g$.

A3 $(yDx)Ex \sim (yD'x)Ex$ for all x, y, E, D and D'.

A4 If $y > x$ and $w > z$, then $yEx \succsim yE'x$ implies $wEz \succsim wE'z$

A5 If $f, g \in \mathcal{F}^e$ and $f \succ g$, then there is a partition $E_1, \cdots, E_n$ of Ω such that $\ell E_i f \succ mE_i g$ for all i.

A6 Let $g \succsim f_n \succsim h$ for all n. Then, (i) $f_n \in \mathcal{F}^e$ converges pointwise to f implies $g \succsim f \succsim h$. (ii) $f_n \in \mathcal{F}$ converges uniformly to f implies $g \succsim f \succsim h$.

To verify $\succsim$ satisfies the above six axioms, we utilize the fact that an event is deemed ideal by $\succsim$ if and only if it is measurable (that is, an element of $\mathcal{E}_\mu$).

Lemma 1. *For the relation $\succsim$ generated by V defined in (3) we have*

$$\mathcal{E} = \mathcal{E}_\mu \, .$$

Proof. See appendix.

Returning to the axioms we see each is verified as follows.

1. A1: Satisfied since $\succsim$ is generated by the real-valued function V defined in (3).

2. A2: Choose $f, g \in \mathcal{F}$ with $f > g$. If $f > g$ then $[f]_1(s) > [g]_1(s)$ and $[f]_2(s) > [g]_2(s)$ for all s. Applying Lemma 6:

$$V(f) - V(g) = \int_0^{0.5} \frac{1}{2}([f]_1 - [g]_1) + \frac{1}{2}([f]_2 - [g]_2) d\mu + \int_{0.5}^{1} \frac{2}{3}([f]_1 - [g]_1) + \frac{1}{3}([f]_2 - [g]_2) d\mu > 0\,.$$

3. A3: Without loss of generality assume $x \leq y$, then $[(yDx)Ex]_1 = x$ and $[(yDx)Ex]_2 = yEx$ which does not depend on D, when D is an diffuse event, so A3 holds.

4. A4 and A5: Trivially satisfied since V is SEU for ideal acts.

5. A6 (ii): As in GP (2014), if f^n converges to f uniformly then $[f^n]_1$ (respectively, $[f^n]_2$) converges to $[f]_1$ (respectively, $[f]_2$) pointwise. Since f^n is (zeroth-order) dominated by the constant act yielding the maximal payoff m for certain, invoking A2 we have $|u \circ [m](s)| \geq |u \circ [f^n](s)|$ for all s. Therefore the integral of $V(f^n)$ converges to the integral of $V(f)$ by the Dominance Convergence Theorem. A6 (i) is true by a similar argument.

Since the preference relation $\succsim$ generated by (3) satisfies GP's axioms 1 – 6, it follows from GP's Theorem 1 that it should admit an EUU representation with prior μ.[2]

To demonstrate the preference relation $\succsim$ generated by (3) is in fact not EUU, first notice for the (ideal) event $E = [0, 1/2]$ in $\mathcal{E}_\mu$, since $\mu(E) = 1/2$ it follows that $mE\ell \sim \ell Em$. Now fix a diffuse event D and consider the pair of acts f and g in which $f(E \cap D) = \{m\}$ with $f(\omega) = \ell$ otherwise, and $g(E^c \cap D) = \{m\}$ with $g(\omega) = \ell$, otherwise. For GP's EUU maximizer we must have $f \sim g$ since

$$\begin{aligned} EUU(f) &= \mu(E)u(\ell, m) + (1 - \mu(E))\, u(\ell, \ell) \\ &= \frac{1}{2}u(\ell, m) + \frac{1}{2}u(\ell, \ell) \\ &= \mu(E)u(\ell, \ell) + (1 - \mu(E))\, u(\ell, m) = EUU(g)\,. \end{aligned}$$

[2] This follows since two linear representations of the same preference relation must be affine transformations of each other.

However, since $[f]_1 = [g]_1 = \ell$, $[f]_2 = mE\ell$, and $[g]_2 = \ell Em$, for the function V defined in (3), we have

$$V(f) = \ell + \frac{1}{4}(m - \ell) > \ell + \frac{1}{6}(m - \ell) = V(g)\,,$$

that is, $f \succ g$. Thus, the preferences generated by V in (3) cannot be from the class of EUU maximizers.

3 Representation Theorem

We retain four of GP's axioms and propose strengthening A3 and modifying A6(i) while leaving the original A6(ii) unchanged. The strengthening of A3 ensures the constancy of conditional certainty equivalents of diffuse "bets" which rules out the (counter-)example from the previous section. The modification of A6(i) enables us to establish the set of ideal events is indeed a σ-algebra.

To see why we require a modification of GP's axiom A6(i), we point out in their proof of their Lemma B2 (which states the collection of ideal events is a σ-field), in the second paragraph on p28, they only establish

$$\begin{aligned} &[f \cup E_i h \succsim g \cup E_i h \text{ and } h \cup E_i f \succsim h \cup E_i g] \\ \implies &\left[f \cup E_i h' \succsim g \cup E_i h' \text{ and } h' \cup E_i f \succsim h' \cup E_i g\right] \text{ for all (ideal acts) } f, g, h, h' \in \mathcal{F}^e\,, \end{aligned}$$

and ***not*** for all (arbitrary acts) $f, g, h, h' \in \mathcal{F}$, as is required to establish an event is ideal. Now since their argument relies on their A6(i) which does not constrain non-ideal acts, without having first established the set of ideal events $\mathcal{E}$ is a σ-field, their earlier results (Lemmas A1 and A2 on p22) cannot establish the existence and uniqueness of the envelopes of acts which are needed to approximate non-ideal acts. Moreover, having failed to establish $\mathcal{E}$ is a $\sigma-$algebra, in turn means GP's Lemmas B4 and B5 on p28 cannot establish the existence and uniqueness of a countably additive probability measure μ on $\mathcal{E}$. We thus provide a stronger version of A6(i) based on Arrow (1974)'s (pp48) Monotone Continuity Axiom.

GP's axiom A6(i) implies a weaker version of Arrow's monotone continuity that applies to ideal acts and ideal events. Our new A6*(i) is the monotone continuity axiom applied to *all* acts and ideal events.

A6*

(i) Let $g \succsim fE_nf' \succsim h$ with $E_{n+1} \subset E_n$ for all n. Then $g \succsim f{\cap}E_nf' \succsim h$.

(ii) Let $g \succsim f_n \succsim h$ for all n. Then $f_n \in \mathcal{F}$ converges uniformly to f implies $g \succsim f \succsim h$.

It is straightforward to show that A6*(i) implies the countably additivity of ideal events and simplifies the proof of Theorem 1.

The next property ensures the conditional certainty equivalence between diffuse acts. Its role is similar to that of P3 in Savage's (1954) axiomatization of subjective expected utility.

A7 $xDy \succsim z$ implies $(xDy)Ef \succsim zEf$ for all x, y, z, f and D.

For simplicity, A7 can be combined with A3 into the following:[3]

A3* $yDx \succsim z \implies (yD'x)Ex \succsim zEx$ for all x, y, z, D, D', and E.

Lemma 2. *Assume A1. Then, A3* holds if and only if both A3 and A7 hold.*[4]

Theorem 1. *The relation $\succsim$ satisfies A1, A2, A3*, A4, A5, and A6* if and only if $\succsim$ admits an EUU representation.*

Proof. **Outline of sufficiency Proof:**
Notice it follows from Lemma 2 that A3* implies both A3 and A7 hold. The role of A7 will be elaborated later. Following the outline of GP's proof, we first observe that by standard arguments it follows that Axioms A1 – A5 plus our axiom A6* imply that the restriction of $\succsim$ to ideal acts yields an expected utility representation with a countably additive probability measure μ and a continuous Bernoulli utility $v\colon X \to \mathbb{R}$.

We turn now to general acts.

(i) The first step is to prove that $\mathcal{E}$ (the set of ideal events) is a σ-algebra. It uses our revised continuity axiom A6*, Fact 1, Fact 2 and other parts of Lemma B2 from

[3] We thank two referees for suggesting we consider incorporating the role A7 plays through a strengthening of A3.

[4] The proof is in Appendix A.

GP's Appendix B. Notice that the new continuity axiom is more than a technical tweak in this set-up; it ensures that $\mathcal{E}$ is not only an algebra but also a σ-algebra, as this guarantees the existence of the associated envelopes, and establishing the interval utility is well-defined also relies on the envelopes being well-defined.

(ii) The second step is to prove the existence and uniqueness (up to a measure zero set) of the envelope $[f]$. The argument follows the one in GP's Appendix A but we highlight that this step needs to use the result from step (i).

(iii) The third step is to prove that an EUU functional constructed using the prior μ from the SEU representation of the restriction of the preferences to ideal acts and an interval utility defined by setting $u(x\,,y) := v(z)$, where for any $x \leqslant y$, z is chosen such that $yDz \sim z$ for some diffuse D, represents $\succsim$. Axioms A2 and A6* together imply that $z \in [x\,,y]$, A3 means it does not matter which diffuse event D is used and Axiom A7 makes sure the constancy of the conditional certainty equivalents of diffuse acts, thereby ensuring this state independent interval utility is well-defined. It uses Lemma B3-B8, a modified version of Lemma B9, and Lemma B10.

As we noted in the introduction above, the expected uncertainty utility of an act f is intuitively the subjective expected utility of its envelope $[f]$: $EUU(f) = SEU([f])$. That is, Savage's P1 to P5 defined on the induced preferences over envelopes must be necessary, and our axioms must be sufficient to imply that the induced preferences over envelopes satisfy P1 to P5. Intuitively A1 implies P1; the definition of ideal events implies P2; A7 implies P3; A7 and A4 together imply P4; A2 implies P5.

The detailed sufficiency proof is presented here. The first few parts of GP's Lemma B2's proof demonstrate that $\mathcal{E}$ is an algebra. The remainder of step 1 is to prove that, with our revised continuity axiom, $\mathcal{E}$ is a $\sigma-$algebra but to do so we first require the following.

Lemma 3. *A null event $\hat{E}$ is ideal.*

Proof. A null event $\hat{E}$ is left ideal by definition. Then we will show a null event $\hat{E}$ is right ideal. Let $h\hat{E}f \succsim h\hat{E}g$. By definition of null event: $h\hat{E}f \sim h'\hat{E}f$ and $h\hat{E}g \sim h'\hat{E}g$.

By transitivity, $h'\hat{E}f \succsim h'\hat{E}g$, which finishes the proof. □

Lemma 4. *If $\{E_n\}$ is a sequence of null events, $\cap E_n$ is null.*

Proof. Assume *per contra*, $\cap E_n$ is non-null. There are f, g, h such that $f \cap E_n h \succ g \cap E_n h$, that is, $(f \cap E_n h)E_m h \succ (g \cap E_n h)E_m h$, which contradicts to the fact that E_m is null and so $\cap E_n$ is null. □

Lemma 5. *$\mathcal{E}$ is a σ−algebra.*

Proof. As $\mathcal{E}$ is an algebra already, we need only show the countable union of ideal events is ideal. We proceed by establishing the countable intersection of ideal events is ideal. Let $E_n \in \mathcal{E}$ and $E_{n+1} \subset E_n$. We first show $\cap E_n \in \mathcal{E}^l$. Assume *per contra*, there are f, g, h, h' such that $f \cap E_n h \succsim g \cap E_n h$ and $g \cap E_n h' \succ f \cap E_n h'$. By $g \cap E_n h' \succ f \cap E_n h'$, there is N such that for all $n > N$, $gE_n h' \succ fE_n h'$ and so $gE_n h \succ fE_n h$ and by Axiom 6*(i), $g \cap E_n h \succsim f \cap E_n h$.

We have $f \cap E_n h \sim g \cap E_n h$ and $g \cap E_n h' \succ f \cap E_n h'$. Since $f \cap E_n h = (f \cap E_n h)E_n h \sim g \cap E_n h = (g \cap E_n h)E_n h$, then $(f \cap E_n h)E_n h' \sim (g \cap E_n h)E_n h'$ and so $(f \cap E_n h) \cap E_n h' \sim (g \cap E_n h) \cap E_n h'$ by Axiom 6*(i), that is, $f \cap E_n h' \sim g \cap E_n h'$, which gives us a contradiction and $\cap E_n \in \mathcal{E}^l$.

We next show $\cap E_n \in \mathcal{E}^r$, that is, $h \cap E_n f \succsim h \cap E_n g$ implies $h' \cap E_n f \succsim h' \cap E_n g$ for all f, g, h, h'. It is enough to show that $(\cap E_n)^c$ is left ideal. We apply Theorem 1 of Gorman (1968) for this part. An event E is essential if for some $h \in \mathcal{F}$, there are $f, f' \in \mathcal{F}$ such that $fEh \succ f'Eh$, and is strictly essential if for all $h \in \mathcal{F}$, there are $f, f' \in \mathcal{F}$ such that $fEh \succ f'Eh$.

Since $\succsim$ is a weak order, Gorman's assumption (0) is satisfied. We let $A = \cap E_n \cup E_1^c$ and $B = E_1$ and so both events A and B are left ideal since $\cap E_n$, E_1 and E_1^c are left ideal. Thus, the required assumption (i) of Gorman's theorem is satisfied for A and B.[5] By the left idealness, the restriction of f on A is weakly ordered by $\succsim_A$, and the restriction of f on B is weakly ordered by $\succsim_B$. We also define $\cap E_n$, $B - A$ and E_1^c to be the three Gorman's sectors, which are groups of states. We next discuss the event essentiality of A and B and Gorman' P3.

[5] Gorman (1968) uses the name "separable event" for left ideal event in the sense of GP.

If $\cap E_n$ is null, $\cap E_n$ is ideal by Lemma 4. If $\cap E_n = \Omega$, $\cap E_n$ is also ideal. We will assume $\cap E_n$ nonnull and $E_1 \neq \Omega$ from now on. Then, we discuss three cases: case 1, $B - A$ is a null set; case 2, $B - A$ is essential but not strictly essential; case 3, $B - A$ is strictly essential. We use Gorman's definition for event essentiality.

Case 1: if $C = B - A$ is a null set, then $C = (\cap E_n)^c \setminus E_1^c$ must be ideal by Lemma 3 and so is $(\cap E_n)^c$.

Case 2: if $C = B - A$ is essential but not strictly essential, then there exists h such that $fCh \sim f'Ch$ for all f, f'. That is,

$$h\,(\cap E_n \cup E_1^c)\,f \sim h\,(\cap E_n \cup E_1^c)\,f' \quad \text{for all } f, f',$$

which implies

$$h'\,(E_n \cup E_1^c)\,f \sim h'\,(E_n \cup E_1^c)\,f' \quad \text{for all } f, f',$$

where $h'(s) = h(s)$ if $s \in \bigcap E_n \cup E_1^c$. Hence, $E_n \cup E_1^c$ is not strictly essential for each n.

Each event $E_n \cup E_1^c$ is ideal. Therefore, $(E_n \cup E_1^c)^c$ is either strictly essential or null. This leads to the conclusion that all $(E_n \cup E_1^c)^c$ must be null.

Given that C is essential, there exist h, f, and f' such that

$$fCh \succ f'Ch,$$

which is

$$h\,(\cap E_n \cup E_1^c)\,f \succ h\,(\cap E_n \cup E_1^c)\,f'.$$

By Axiom $A6^*(i)$, there exists $E_n \cup E_1^c$ such that

$$h\,(E_n \cup E_1^c)\,f \succ h\,(E_n \cup E_1^c)\,f'.$$

This implies that $(E_n \cup E_1^c)^c$ is non-null, and we have a contradiction. Therefore, C cannot be non-strict essential.

Case 3:: $C = B - A$ is strictly essential. Assume *per contra* C is not left ideal: there exist four acts f^1, f^2, f^3, f^4 such that $f^1Cf^3 \succsim f^2Cf^3$ and $f^1Cf^4 \prec f^2Cf^4$. We will utilize Gorman's Theorem 1. The main idea of this proof is in the spirit of Gul and Pesendorfer (2014) to transform the usual acts f, g into Gorman's acts on Gorman's outcome spaces $X_1 \times X_2 \times X_3$, defined as follows:

$$X_1 = [l,m]^{\cap E_n}, \quad X_2 = [0,1], \quad \text{and} \quad X_3 = [l,m]^{E_1^c}.$$

Recall the first sector is $S_1 = \cap E_n$, the second sector is $S_2 = C$, and the third sector is $S_3 = E_1^c$. The first and the third sectors are left ideal, that is, fS_1 and fS_3 are ranked with $\succsim_{S_1}$ and $\succsim_{S_3}$[6]. For this outcome space, Gorman's P2 is satisfied. For an act g, gS_1 denotes the restriction of g on S_1: $gS_1 \in X_1$.

Define Gorman's acts $\xi^1, \xi^2 \in X_1 \times X_2 \times X_3$ such that $\xi^i S_j \in X_j$ for $j \in \{1,2,3\}$. Define their preference relation

$$\xi^1 \succsim^* \xi^2 \iff g^1 \succsim g^2,$$

where $g^i S_1 = \xi^i S_1, \quad g^i S_3 = \xi^i S_3, \quad \text{and} \quad g^i S_2 = \xi^i S_2 \cdot f^3 + (1 - \xi^i S_2) \cdot f^4.$

The preference relation $\succsim^*$ is a weak ordering because $\succsim$ is. Gorman's P1 is implied. Gorman's P3 follows from S_1, S_2, and S_3 being all essential. Moreover, since $\succsim^*$ is a restricted preference from $\succsim$, S_2 is also strict essential.

All three required assumptions and three postulates of Gorman's Theorem 1 are satisfied for $\succsim^*$; hence C is a separable event with respect to $\succsim^*$.

Define four specific Gorman's acts $\{\zeta^i\}_{i\in\{1,2,3,4\}}$ such that $\zeta^i S_1 = f^i S_1$, $\zeta^i S_3 = f^i S_3$, $\zeta^1 S_2 = 1$, and $\zeta^2 S_2 = 0$. Since C is a separable event,

$$\zeta^1 C \zeta^3 \succsim^* \zeta^2 C \zeta^3 \iff \zeta^1 C \zeta^4 \succsim^* \zeta^2 C \zeta^4,$$

which implies

$$f^1 C f^3 \succsim f^2 C f^3 \iff f^1 C f^4 \succsim f^2 C f^4,$$

and we have a contradiction. Therefore, C must be left ideal. Since C is left ideal, $(\cap E_n)^c$ is left ideal and so $\cap E_n$ is right ideal.

Finally, suppose $E_n \in \mathcal{E}$ for each n, then both (E_n) and $(E_n)^c \in \mathcal{E}$ because $\mathcal{E}$ is an algebra. Then we have $\cap E_n^c \in \mathcal{E}$. By De Morgan's Law on countable union, $\cap E_n^c = (\cup E_n)^c \in \mathcal{E}$. Since $\mathcal{E}$ is an algebra, $\cup E_n \in \mathcal{E}$, which finishes the proof. □

Having established $\mathcal{E}$ is a $\sigma-$algebra, we can now apply GP's Lemmas A1, A2, and 1.

[6] $fS_1 \succsim_{S_1} f'S_1$ if and only if $fS_1 h \succsim f'S_1 h$ for all h and $fS_3 \succsim_{S_3} f'S_3$ if and only if $fS_3 h \succsim f'S_3 h$ for all h.

For the third step, GP's Lemma B3 ensures the existence of an SEU that represents the restriction of the preference relation to ideal acts. GP's Lemma B4 ensures that the probability measure μ of SEU characterized in B3 is also a prior. GP's Lemma B5 ensures that the utility function of SEU over ideal acts must be increasing and continuous. GP's Lemma B6 ensures that the set of all diffuse acts generated by the preference is same as the set of all diffuse acts generated by μ, and the certainty equivalent of xDy is unique. B6 ensures the interval utility is well-defined and GP's Lemma B8 ensures the monotonicity and continuity of the interval utility u. It is now enough to modify slightly GP's Lemma B9:

Lemma B9* The function V defined by (2) represents the restriction of $\succsim$ to the set of simple acts $\mathcal{F}^o$.

Proof. Before starting the proof, some notations are defined according to GP. $\mathcal{F}^e$ is the set of ideal simple acts. $\mathcal{F}^o$ is the set of simple acts. Let $E \in \mathcal{E}_\mu$, $N = \{1, \dots, n\}$, and $\{A_i\}_{i \in N}$ be a finite partition of E. Let $\mathcal{N}$ be the set of all non-empty subsets of N, and for $J \in \mathcal{N}$, let $\mathcal{N}(J) = \{L \in \mathcal{N} \mid L \subset J\}$. Let $A_J = \bigcup_{i \in J} A_i$, let C_J be the core of A_J, and let $C_N = E$. The ideal split $\{E^*_J\}_{J \in \mathcal{N}} \subset \mathcal{E}_\mu$ of $\{A_i\}_{i \in \mathcal{N}}$ is inductively defined as follows: $E^*_i := C_i$ for all $i \in \mathcal{N}$; for J such that $|J| > 1$,

$$E^*_J := C_J \setminus \bigcup_{L \in \mathcal{N}(J),\, L \neq J} E^*_L$$

Note that $\{E^*_J\}$ is a partition of E that satisfies

$$\bigcup_{L \in \mathcal{N}(J)} E^*_L \subset A_J \text{ for all } J \in \mathcal{N}$$

and $\mu^*(A_J) = \mu(C_J) = \sum_{L \in \mathcal{N}(J)} \mu(E^*_L)$. For any act $f \in \mathcal{F}_0$ with range $\{x_1, \dots, x_n\}$, let $\{E^*_J(f)\}$ be the ideal split of $\{f^{-1}(x_i)\}$.

Let $\{x_1, x_2, \dots, x_n\}$ be the range of simple act f, let $A_i = f^{-1}(x_i)$, and let $\{E^*_J(f)\}$ be an ideal split of $\{A_i\}$. GP's Lemma A2 implies that $\{E^*_J(f)\}$ exists and is unique up to measure zero. Define $N^+(f) = \{J \mid \mu(E^*_J(f)) > 0 \text{ and } |J| > 1\}$ and define $H_n = \{f \in \mathcal{F}_0 \mid n = |N^+(f)|\}$. The proof is by induction on H_n. Note that for $f \in H_0$, $V(f) = \int_X v(z)\mu(f^{-1}(z))dz = v(x)$ for x such that $\mu(\{x = f\}) = 1$. Hence, by

GP's Lemma B3(ii), V represents the restriction of $\succsim$ to H_0. Suppose V represents the restriction of $\succsim$ to H_n and choose $f \in H_{n+1}$. Define h_f as follows: if $f \in H_n$, then $h_f = f$; otherwise, choose $E_J^*(f)$ such that $|J| > 1$ and $\mu(E_J^*(f)) > 0$, choose $D \in \mathcal{D}$, and define f^* as follows:

$$f^*(\omega) = \begin{cases} f(\omega) & \text{if } \omega \notin E_J^*(f), \\ \max\{f(\omega), E_J^*(f)\} & \text{if } \omega \in D \cap E_J^*(f), \\ \min\{f(\omega), E_J^*(f)\} & \text{if } \omega \in D^c \cap E_J^*(f). \end{cases}$$

By GP's Lemma B7 and Axiom A3, $f^* \sim f$. Next, choose z such that $u(x,y) = v(z)$, where recall z is the certainty equivalent of xDy, for some diffuse D. Let $h_f(\omega) = f^*(\omega)$ for all $\omega \notin E_J^*(f)$ and $h_f(\omega) = z$ for all $\omega \in E_J^*(f)$. Axiom A7 ensures the constancy of the conditional certainty equivalents of diffuse acts: $h_f \sim f^* \sim f$. Notice that $h_f \in H_n$ and, by construction, $V(h_f) = V(f^*)$. By GP's Lemma A1, $[f^*] = [f]$ and, therefore, $V(f^*) = V(f)$. Thus, $V(f) = V(h_f)$ for some $h_f \in H_n$ such that $h_f \sim f$. Then, by induction, V represents $\succsim$ on H_{n+1}. □

As was the case in GP, the extension to all acts can be obtained using Axiom 6*(ii) and follows familiar arguments in GP's Lemma B10.

Necessity Proof: Fix V an EUU functional. GP prove that the relation $\succsim$ generated by V satisfies A1 – A5 and A6(ii). The rest of the proof will establish this relation also satisfies A6*(i) and A3*.

For A6*(i), when $n \to +\infty$, $E_n \to \cap E_n$. Without loss of generality assume $f \precsim f'$ when restricted to the event $E_n \setminus \cap E_n$. Because $\ell \leq f(\omega) \leq m$, we have $V(f \cap E_n f') - V(fE_n f') \leq \int_{E_n \setminus \cap E_n} \big(u(m,m) - u(\ell,\ell)\big) d\mu = \big(u(m,m) - u(\ell,\ell)\big)\mu(E_n \setminus \cap E_n)$. Since μ is a probability measure on a $\sigma-$ algebra, $E_n \to \cap E_n$ implies $\mu(E_n) \to \mu(\cap E_n)$ and $\mu(E_n \setminus \cap E_n) \to 0$. Therefore $V(f \cap E_n f') - V(fE_n f') \to 0$ as $n \to +\infty$.

Since Lemma 2 establishes A3* holds if A1, A3 and A7 all hold it is enough to show A7 holds. Suppose x, y, c are constant acts and $xDy \sim c$. We want to show that $(xDy)Ef \sim cEf$ for all E and f. When $x = y$ the axiom trivially holds. Without loss of generality, suppose $x < y$, then the interval utility $u(x,y) = u(c,c)$. Then

$\int_E u(x,y)d\mu = \int_E u(c,c)d\mu$, $\int_E u(x,y)d\mu + \int_{E^c} u([f]_1,[f]_2)d\mu = \int_E u(c,c)d\mu + \int_{E^c} u([f]_1,[f]_2)d\mu$, and so $V((xDy)Ef) = V(cEf)$. □

Appendix A Proofs of Lemma 1, Lemma 2 and Lemma 6

Proof of Lemma 1

$\mathcal{E}_\mu \subseteq \mathcal{E}$.

For any $E \in \mathcal{E}_\mu$ it is immediate from the representation in (3) that it satisfies the definition of an ideal event.

$\mathcal{E} \subseteq \mathcal{E}_\mu$.

It suffices to show that any event not in $\mathcal{E}_\mu$ will generate a violation of P2 for some acts. Fix $A \notin \mathcal{E}_\mu$ and let $E^{\{1\}}$ (respectively, $E^{\{2\}}$) denote the largest measurable subset of (respectively, the complement of) A.[7] Following GP, we consider the same events B_{11}, B_{12},B_{21} and B_{22}, all of which do not contain any element in $\mathcal{E}_\mu$ and for which $A = E^{\{1\}} \cup B_{11} \cup B_{12}$ and $A^c = E^{\{2\}} \cup B_{21} \cup B_{22}$. Let $E^{\{1,2\}} = B_{11} \cup B_{12} \cup B_{21} \cup B_{22} \in \mathcal{E}_\mu$.

For $x_1 < x_2 < x_3 \in X$ with $x_3 - x_2 > x_2 - x_1$, let $g = x_1$ and let

$$f = \begin{cases} x_3, & \text{if } x \in B_{11}, \\ x_2, & \text{if } x \in B_{12}, \\ x_1, & \text{otherwise.} \end{cases} \qquad h = \begin{cases} x_3, & \text{if } x \in B_{11}, \\ x_1, & \text{if } x \in B_{12}, \\ x_1, & \text{otherwise.} \end{cases} \qquad h' = \begin{cases} x_2, & \text{if } x \in B_{11}, \\ x_2, & \text{if } x \in B_{12}, \\ x_1, & \text{otherwise.} \end{cases}$$

So we have

$$[f_A h]_1 = x_1 \text{ and } [f_A h]_2 = x_3 E^{\{1,2\}} x_1$$
$$[f_A h']_1 = x_2 E^{\{1,2\}} x_1 \text{ and } [f_A h']_2 = x_3 E^{\{1,2\}} x_1$$
$$[g_A h]_1 = x_1 \text{ and } [g_A h]_2 = x_3 E^{\{1,2\}} x_1$$
$$[g_A h']_1 = x_1 \text{ and } [g_A h']_2 = x_2 E^{\{1,2\}} x_1$$

By the representation, we have $f_A h \sim g_A h$ but $f_A h' \succ g_A h'$, and so A is not a left ideal and A is not an ideal event. □

Proof of Lemma 2

[7] Since μ is countably additive such events exist and are unique up to zero measure sets.

(i) A3* implies A3. In A3*, setting $E = \Omega$, we have $yDx \succsim z \implies yD'x \succsim z$, and so $yD'x \succsim yDx$. Switching D and D', we have $yDx \succsim yD'x$. Thus, $yDx \sim yD'x$.

(ii) A3* implies A7. In A3*, setting $D' = D$, $yDx \succsim z \implies (yD'x)Ex \succsim zEx$, which implies $(yDx)Ef \succsim zEf$ for all f since E is an ideal event.

(iii) Conjugation of A3 and A7 imply A3*. Suppose $yDx \succsim z$, applying A7 we have $(yDx)Ex \succsim zEx$. Applying A3, we have $(yD'x)Ex \succsim zEx$ for any diffuse event D'.

□

We use the following lemma in the sufficiency proof.

Lemma 6. *Suppose that $f(\omega) > 0$ for all ω and $\mu(\Omega) > 0$, then $\int_\Omega f d\mu > 0$*

Proof. Define $\Omega_k = \Omega \cap \{f > \frac{1}{k}\}$. It follows that $\Omega = \cup\Omega_k$. Assume for each k, Ω_k is measure zero; then, Ω is measure zero, which contradicts to the fact that μ is a probability measure with $\mu(\Omega) = 1$. Therefore $\mu(\Omega_k) > 0$ for at least one k. So $\int_\Omega f \geq \int_{\Omega_k} f \geq \frac{1}{k}\mu(\Omega_k) > 0$. □

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