Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL DYNAMIC INFORMATION ACQUISITION"

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This file contains omitted proofs from the paper "Optimal Dynamic Information Acquisition."

S1. LEMMAS FOR THEOREM 1

S1.1. *Lemmas for Lemma A.1*

IN THIS SECTION, I prove a few auxiliary results for Lemma A.1. Lemma S.1 (Lemma S.2) shows that any information structure can be decomposed into a continuous-time (discrete-time) belief process such that the flow reduction of uncertainty is constant. Lemma S.3 shows that V_{dt} converges uniformly.

LEMMA S.1: *$H \in C(\Delta(X))$ is strictly concave. $\forall \pi \in \Delta^2(X)$, let $\mu = \mathbb{E}_\pi[\nu]$. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and stochastic process $\langle \mu_t \rangle_{t\in[0,1]}$ such that*:

(i) *$\langle \mu_t \rangle$ is a martingale.*
(ii) *$\mu_0 = \mu$, $\mu_1 \sim \pi$.*
(iii) *$\forall t_1, t_2 \in [0, 1]$ and $t_1 < t_2$, $E[H(\mu_{t_1}) - H(\mu_{t_2})|\mathcal{F}_{t_1}] = (t_2 - t_1)E[H(\mu_0) - H(\mu_1)]$.*

PROOF: The proof takes three steps. Let $M = \mathbb{E}_\pi[(H(\mu) - H(\nu))]$.

Step 1. Discretize $\Delta(X)$. Since $H(\mu)$ is a continuous function on $\Delta(X)$, by the Heine–Cantor theorem, $H(\mu)$ is uniformly continuous. $\forall k \in \mathbb{N}$, let $\varepsilon_k = \frac{M}{2^k}$ and δ_k be the corresponding continuity parameter for ε_k. Partition $\Delta(X)$ into a set of cubes of size $d_k \le \delta_k$.

Now, consider all d_1 cubes with nonzero measure under π. Denote them by $\{D_i\}_{i\in I}$. $\forall i \in I$, let $\mu_i = \mathbb{E}_\pi[\nu|\nu \in D_i]$, $\pi_i(\nu) = \pi(\nu|\nu \in D_i)$, and $q_i = \pi(D_i)$. Let $\mu_i(\lambda) = \lambda\mu + (1-\lambda)\mu_i$. Then

$$H\big(\mu_i(\lambda)\big) - \lambda \sum q_j \mathbb{E}_{\pi_j}\big[H(\nu)\big] - (1-\lambda)\mathbb{E}_{\pi_i}\big[H(\nu)\big] \tag{S.1}$$

is a continuous function of λ, equals $H(\mathbb{E}_{\pi_i}[\nu]) - \mathbb{E}_{\pi_i}[H(\nu)] \le \varepsilon_1$ when $\lambda = 0$, and equals M when $\lambda = 0$. Then, by the intermediate value theorem there exists λ_i such that equation (S.1)$= \frac{M}{2}$. Let $\widehat{q}_i = \sum_i \frac{q_i/(1-\lambda_i)}{\sum_j q_j/(1-\lambda_j)}$. Define $\widehat{\pi}_0(\nu) = \sum \widehat{q}_i \delta_{\mu_i(\lambda_i)}(\nu)$, $\widehat{\pi}_i = (1-\lambda_i)\pi_i(\nu) + \lambda \sum q_j \pi_j(\nu)$. It can be verified that: (i) $\mathbb{E}_{\widehat{\pi}_0}[\nu] = \sum \widehat{q}_i \mu_i(\lambda_i) = \mu$, (ii) $\mathbb{E}_{\widehat{\pi}_i}[\nu] = (1-\lambda_i)\mu_i + \lambda \sum_j q_j \mu_i = \mu_i(\lambda_i)$, and (iii) $H(\mu_i(\lambda_i)) - \mathbb{E}_{\widehat{\pi}_i}[H(\nu)] = \frac{M}{2}$, $\forall i \in I \cup \{0\}$. In plain words, in this step, I decompose π into two stages $\widehat{\pi}_0$ and $\widehat{\pi}_i$. $\widehat{\pi}_0$ has a finite support $\{\mu_i(\lambda_i)\}$, and in each step, the expected reduction of H is $\frac{M}{2}$.

Step 2. Define the continuous time process for $t \in [0, \frac{1}{2}]$. Let the finite support distribution $\widehat{\pi}_0$ be denoted by $\sum p_i \delta_{\nu_i}(\mu)$. Let $\nu_i(\lambda) = \lambda\mu + (1-\lambda)\nu_i$. Then, by the same argument as in step 1, $\forall i$, there exists $\lambda_i(t) \in [0, 1]$ such that

$$H\big(\nu_i\big(\lambda_i(t)\big)\big) - \lambda_i(t) \sum p_j H(\nu_j) - \big(1 - \lambda_i(t)\big)H(\nu_i) = \left(\frac{1}{2} - t\right)M. \tag{S.2}$$

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Note that since H is strictly concave, the LHS of equation (S.2) is strictly concave in λ_i; hence, $\lambda_i(t)$ can be chosen to be strictly decreasing. Define $\Xi(t) \triangleq \{\nu_i(\lambda_i(t))\}_{i\in I}$. It can be verified that $\Xi(t)$ are disjoint for different t.[S.1]

Define $p_i(t) = \frac{p_i/(1-\lambda_i(t))}{\sum_j p_j(1-\lambda_j(t))}$. Then $\mu = \sum p_i(t)\nu_i(\lambda_i(t))$. $\forall \frac{1}{2} \geq t' > t \geq 0$, define

$$p(\nu_j(\lambda_j(t'))|\nu_i(\lambda_i(t))) = \begin{cases} p_j(t')\dfrac{\lambda_i(t)-\lambda_i(t')}{1-\lambda_i(t')} & \text{when } j \neq i, \\ p_j(t')\dfrac{\lambda_i(t)-\lambda_i(t')}{1-\lambda_i(t')} + \dfrac{1-\lambda_i(t)}{1-\lambda_i(t')} & \text{when } j = i. \end{cases}$$

Note that the Markov kernel p is well-defined because $\Xi(t)$ are disjoint. It is easy to verify that (i) $\sum_j p(\nu_j(\lambda_j(t'))|\nu_i(\lambda_i(t))) = 1$, (ii) $\sum_j p(\nu_j(\lambda_j(t'))|\nu_i(\lambda_i(t)))\nu_j(\lambda_j(t')) = \nu_i(\lambda_i(t))$, (iii) $H(\nu_i(\lambda_i(t))) - \sum_j p(\nu_j(\lambda_j(t'))|\nu_i(\lambda_i(t)))H(\nu_j(\lambda_j(t'))) = (t'-t)M$, and (iv)

$$\begin{aligned} &\sum_i p_i(t)p(\nu_j(\lambda_j(t'))|\nu_i(\lambda_i(t))) \\ &= \sum_i p_i(t)p_j(t')\frac{\lambda_i(t)-\lambda_i(t')}{1-\lambda_i(t')} + p_j(t)\frac{1-\lambda_j(t)}{1-\lambda_j(t')} \\ &= p_j(t')\frac{\sum_i p_i\left(\frac{1}{1-\lambda_i(t)} - \frac{1}{1-\lambda_i(t')}\right)}{\sum p_i/(1-\lambda_i(t))} + p_j(t')\frac{\sum_i p_i/(1-\lambda_i(t'))}{\sum_i p_i/(1-\lambda_i)(t)} \\ &= p_j(t'). \end{aligned}$$

Let the joint distribution of a finite collection of $\mu_{t_1}, \dots, \mu_{t_k}$ for an increasing sequence $\{t_1, \dots, t_k\} \in [0, \frac{1}{2}]$ be defined by

$$\text{Prob}(\mu_{t_1}, \dots, \mu_{t_k}) = p(\mu_{t_2}|\mu_{t_1})p(\mu_{t_3}|\mu_{t_2})\cdots p(\mu_{t_k}|\mu_{t_{k-1}}). \tag{S.3}$$

The joint probability satisfies the condition for the Daniell–Kolmororov theorem if the Chapman–Kolmogorov equation is satisfied: $\forall 0 \leq t < t' < t'' \leq \frac{1}{2}$, $\forall i, l$,

$$\begin{aligned} &\sum_j p(\nu_l(\lambda_l(t''))|\nu_j(\lambda_j(t')))p(\nu_j(\lambda_j(t'))|\nu_i(\lambda_i(t))) \\ &\quad= \sum_j p(\nu_l(\lambda_l(t''))|\nu_j(\lambda_j(t')))p_j(t')\frac{\lambda_i(t)-\lambda_i(t')}{1-\lambda_i(t')} + p(\nu_l(\lambda_l(t''))|\nu_i(\lambda_i(t')))\frac{1-\lambda_i(t)}{1-\lambda_i(t')} \\ &\quad= p_l(t'')\frac{\lambda_i(t)-\lambda_i(t')}{1-\lambda_i(t')} + p_l(t'')\frac{\lambda_i(t')-\lambda_i(t'')}{1-\lambda_i(t'')}\frac{1-\lambda_i(t)}{1-\lambda_i(t')} + \mathbf{1}_{l=i}\frac{1-\lambda_i(t)}{1-\lambda_i(t'')} \end{aligned}$$

[S.1]Since the paths $\nu_i(\lambda_i(t))$ are linear in $\Delta(X)$, two paths indexed by i, j intersect only if ν_i, ν_j, and μ are linearly dependent. This implies that when two measures with support (μ, ν_i) and (μ, ν_j) have the same mean, they are ordered by mean preserving spread order, which violates equation (S.2) if they corresponds to the same t.

$$= p_l(t'')\frac{\lambda_i(t)-\lambda_i(t'')}{1-\lambda_i(t'')} + \mathbf{1}_{l=i}\frac{1-\lambda_i(t)}{1-\lambda_i(t'')}$$
$$= p\big(\nu_l(\lambda_l(t''))|\nu_i(\lambda_i(t))\big).$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Markov martingale $\langle\mu_t\rangle$ such that its finite dimensional marginal distributions are given by equation (S.3). In particular, $\mu_{0.5} \sim \widehat{\pi}_0$ and

$$\mathbb{E}\big[H(\mu_t) - H(\mu_{t+s})|\mathcal{F}_t\big] = \mathbb{E}\big[H(\mu_t) - H(\mu_{t+s})|\mu_t\big] = s \cdot M.$$

Step 3. $\forall\mu$ in the (finite) support of $\widehat{\pi}_0$, it corresponds to some $\mathbb{E}_{\widehat{\pi}_i}[\nu]$. Since the distribution $\widehat{\pi}_i$ satisfy $H(\mu) - \mathbb{E}_{\widehat{\pi}_i}[H(\nu)] = \frac{M}{2}$, we can apply step 2 and construct $\langle\mu_t\rangle|_{\mu_{\frac{1}{2}}=\mu}$ for $t \in [\frac{1}{2}, \frac{3}{4}]$ such that it is a Markov martingale that satisfies $\mathbb{E}[H(\mu_t) - H(\mu_{t+s})|\mu_t] = s \cdot M$. By recursively applying step 2, we construct a martingale process $\langle\mu_t\rangle$ for $t \in [0, 1)$ satisfying $\mathbb{E}[H(\mu_t) - H(\mu_{t+s})|\mathcal{F}_t] = s \cdot M$, $\forall t + s < 1$. By the martingale convergence theorem, $\mu_t \xrightarrow{P} \pi$. Therefore, the definition of $\langle\mu_t\rangle$ can be extended continuously to [0, 1], and the three properties in Lemma S.1 are satisfied. *Q.E.D.*

LEMMA S.2: *$H \in C(\Delta X)$ is strictly concave. $\forall\pi \in \Delta^2(X)$, let $\mu = \mathbb{E}_\pi[\nu]$. $\forall T \in \mathbb{N}$. There exists a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ and stochastic process $\langle\widehat{\mu}_t\rangle_{t\in 1}^T$ such that:*
(i) *$\langle\widehat{\mu}_t\rangle$ is a martingale.*
(ii) *$\widehat{\mu}_0 = \mu$ and $\widehat{\mu}_T \sim \pi$.*
(iii) $\mathbb{E}[H(\widehat{\mu}_{t+1} - \widehat{\mu}_t)|\widehat{\mathcal{F}}_t] = \frac{1}{T}\mathbb{E}_\pi[H(\mu) - H(\nu)]$.

PROOF: Lemma S.2 is a direct corollary of Lemma S.1. Construct $\langle\mu_t\rangle$ according to Lemma S.1. Then define $\pi(\widehat{\mu}_{t+1}|\widehat{\mu}_{1,\dots,t}) = \pi(\mu_{\frac{t+1}{T}}|\mu_0, \dots, \mu_{\frac{t}{T}})$. Let $\widehat{\mathcal{F}}$ be the natural filtration of $\langle\widehat{\mu}_t\rangle$. All three properties are straightforward. *Q.E.D.*

LEMMA S.3: *Given Assumption* 2, *let* $\widehat{V}(\mu) = \overline{\lim}_{dt\to 0} V_{dt}(\mu)$. *Then* $\lim_{dt\to 0}\|V_{dt}(\mu) - \widehat{V}(\mu)\|_{l_\infty} = 0$.

PROOF: The proof of Lemma S.3 takes three steps:

Step 1. $\forall dt > 0$, let $dt_n = \frac{dt}{2^n}$. Then V_{dt_n} is an increasing sequence. $\forall$ strategy $(\langle\widehat{\mu}_t\rangle, \widehat{\tau})$ associated with dt_n, define $\widetilde{\mu}_{2t} = \widehat{\mu}_t$. $\widetilde{\mu}_{2t+1}$ is defined according to Lemma S.2 such that $\mathbb{E}[H(\widetilde{\mu}_{2t+1}) - H(\widetilde{\mu}_{2t})|\widehat{\mathcal{F}}_t] = \mathbb{E}[H(\widetilde{\mu}_{2t+2}) - H(\widetilde{\mu}_{2t+1})|\widetilde{\mu}_{2t+1}] = \frac{1}{2}\mathbb{E}[H(\widehat{\mu}_{t+1}) - H(\widehat{\mu}_t)|\widehat{\mathcal{F}}_t]$. $\widetilde{\tau} = 2\widehat{\tau}$. Clearly, $\widetilde{\tau}$ is measurable to $\langle\widetilde{\mu}_t\rangle$'s natural filtration. Then

$$V_{dt_{n+1}}(\mu)$$
$$\geq \mathbb{E}\left[e^{-\rho\, dt_{n+1}\widetilde{\tau}}F(\widetilde{\mu}_{\widetilde{\tau}}) - \sum_{t=0}^{\widetilde{\tau}-1} e^{-\rho\, dt_{n+1}t}C\left(\frac{\mathbb{E}[H(\widetilde{\mu}_{t+1}) - H(\widetilde{\mu}_t)|\widetilde{\mathcal{F}}_t]}{dt_{n+1}}\right)dt_{n+1}\right]$$
$$= \mathbb{E}\left[e^{-\rho\, dt_n\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}}) - \sum_{t=0}^{\widehat{\tau}-1} e^{-\rho\, dt_{n+1}2t}(1 + e^{-\rho\, dt_{n+1}})C\left(\frac{\mathbb{E}[H(\widehat{\mu}_{t+1}) - H(\widehat{\mu}_t)|\widehat{\mathcal{F}}_t]}{2\, dt_{n+1}}\right)dt_{n+1}\right]$$
$$\geq \mathbb{E}\left[e^{-\rho\, dt_n\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}}) - \sum_{t=0}^{\widehat{\tau}-1} e^{-\rho\, dt_n t}C\left(\frac{\mathbb{E}[H(\widehat{\mu}_{t+1}) - H(\widehat{\mu}_t)|\widehat{\mathcal{F}}_t]}{dt_n}\right)dt_n\right].$$

The first inequality is from $(\langle\widetilde{\mu}_t,\widetilde{\tau}\rangle)$ being an admissible strategy. The second inequality is from $e^{-\rho\, dt_{n+1}}<1$. Since the inequality holds for all strategies, $V_{dt_{n+1}}\geq V_{dt_n}$.

Step 2. $\forall\, dt>0$, $V_{2dt}\geq V_{dt}-\rho\, dt\sup F$. $\forall$ strategy $(\langle\widehat{\mu}_t\rangle,\widehat{\tau})$ associated with dt, define $\check{\mu}_t=\widehat{\mu}_t$, $\check{\tau}=\widehat{\tau}$ when $\widehat{\tau}$ is even, and $\check{\tau}=\widehat{\tau}+1$ when $\widehat{\tau}$ is odd. Define $\widetilde{\mu}_t=\check{\mu}_{2t}$ and $\widetilde{\tau}=\frac{1}{2}\check{\tau}$. Then

$$
\begin{aligned}
V_{2dt}(\mu)&\geq\mathbb{E}\left[e^{-\rho 2dt\widetilde{\tau}}F(\widetilde{\mu}_{\widetilde{\tau}})-\sum_{t=0}^{\widetilde{\tau}-1}e^{-\rho 2dtt}C\Big(\frac{\mathbb{E}\big[H(\widetilde{\mu}_{t+1})-H(\widetilde{\mu}_t)|\widetilde{\mathcal{F}}_t\big]}{2\,dt}\Big)2\,dt\right]\\
&=\mathbb{E}\Bigg[e^{-\rho\, dt\check{\tau}}F(\check{\mu}_{\check{\tau}})\\
&\quad-\sum_{t=0,t\text{ even}}^{\check{\tau}-2}e^{-\rho\, dtt}C\Big(\frac{\mathbb{E}\big[H(\check{\mu}_t)-H(\check{\mu}_{t+1})+H(\check{\mu}_{t+1})-H(\check{\mu}_{t+2})|\check{\mathcal{F}}_t\big]}{2\,dt}\Big)2\,dt\Bigg]\\
&\geq\mathbb{E}\Bigg[e^{-\rho\, dt\check{\tau}}F(\check{\mu}_{\check{\tau}})-\sum_{t=0,t\text{ even}}^{\check{\tau}-2}e^{-\rho\, dtt}\bigg(C\Big(\frac{\mathbb{E}\big[H(\check{\mu}_t)-H(\check{\mu}_{t+1})|\check{\mathcal{F}}_t\big]}{dt}\Big)\,dt\\
&\quad+\mathbb{E}\Big[C\Big(\frac{\mathbb{E}\big[H(\check{\mu}_{t+1})-H(\check{\mu}_{t+2})|\check{\mathcal{F}}_{t+1}\big]}{dt}\Big)\,dt\Big]\bigg)\Bigg]\\
&\geq\mathbb{E}\left[e^{-\rho\, dt\check{\tau}}F(\check{\mu}_{\check{\tau}})-\sum_{t=0}^{\widehat{\tau}-1}e^{-\rho\, dtt}C\Big(\frac{\mathbb{E}\big[H(\check{\mu}_t)-H(\check{\mu}_{t+1})|\check{\mathcal{F}}_t\big]}{dt}\Big)\,dt\right]\\
&\quad-\big(1-e^{-\rho\, dt}\big)\mathbb{E}\left[\sum_{t=0}^{\widehat{\tau}-1}e^{-\rho\, dtt}C\Big(\frac{\mathbb{E}\big[H(\check{\mu}_t)-H(\check{\mu}_{t+1})|\check{\mathcal{F}}_t\big]}{dt}\Big)\,dt\right]\\
&\geq\mathbb{E}\left[e^{-\rho\, dt\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=0}^{\widehat{\tau}-1}e^{-\rho\, dtt}C\Big(\frac{\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]}{dt}\Big)\,dt\right]\\
&\quad-\big(1-e^{-\rho\, dt}\big)\sup F.
\end{aligned}
$$

The first inequality is from $(\langle\widetilde{\mu}_t,\widetilde{\tau}\rangle)$ being an admissible strategy. The second inequality is from the convexity of C. The third inequality is from C being nonnegative. The last inequality is from the fact that it is without loss of optimality to assume that the total cost associated with strategy $(\langle\widehat{\mu}_t\rangle,\widehat{\tau})$ is less than $\sup F$. Since the inequality holds for all strategies, $V_{2dt}\geq V_{dt}-dt\sup F$. This together with step 1 shows that V_{dt_n} converges and $\|V_{dt}-\lim V_{dt_n}\|\leq 2\,dt\sup F$.

Step 3. $\forall\, dt,\, dt'>0$, $V_{dt'}\leq\lim V_{dt_n}$. $\forall\, dt'>dt>0$ let $N\in\mathbb{N}$ satisfy $N\,dt\leq dt'<(N+1)\,dt$. $\forall$ strategy $(\langle\widehat{\mu}_t\rangle,\widehat{\tau})$ associated with dt', define $\widetilde{\mu}_{t(N+1)+n}=\widehat{\mu}_t$ for all $n\in[0,N]$ and $\widetilde{\tau}=(N+1)\widehat{\tau}$. Then

$$
V_{dt}(\nu)\geq\mathbb{E}\left[e^{-\rho\, dt\widetilde{\tau}}F(\widetilde{\mu}_{\widetilde{\tau}})-\sum_{t=0}^{\widetilde{\tau}-1}e^{-\rho\, dtt}C\Big(\frac{\mathbb{E}\big[H(\widetilde{\mu}_{t+1})-H(\widetilde{\mu}_t)|\widetilde{\mathcal{F}}_t\big]}{dt}\Big)\,dt\right]
$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg[e^{-\rho(N+1)\,dt\widehat{\tau}} F(\widehat{\mu}_{\widehat{\tau}}) \\
&\quad - \sum_{t=0}^{\widehat{\tau}} e^{-\rho\,dt(N+1)t} \sum_{n=0}^{N} e^{-\rho\,dtn} C\Big(\frac{(\mathbb{E}[H(\widehat{\mu}_{t+1}) - H(\widehat{\mu}_t)|\widehat{\mathcal{F}}_t]}{(N+1)}\,dt\Big)\,dt\Bigg] \\
&\geq \mathbb{E}\Bigg[e^{-\rho\,dt'\widehat{\tau}} e^{-\rho\,dt\widehat{\tau}} F(\widehat{\mu}_{\widehat{\tau}}) - \sum_{t=0}^{\widehat{\tau}} e^{-\rho\,dt't} C\Big(\frac{(\mathbb{E}[H(\widehat{\mu}_{t+1}) - H(\widehat{\mu}_t)|\widehat{\mathcal{F}}_t]}{(N+1)\,dt}\Big)(N+1)\,dt\Bigg] \\
&\geq \mathbb{E}\Bigg[e^{-\rho\,dt'\widehat{\tau}} F(\widehat{\mu}_{\widehat{\tau}}) - \sum_{t=0}^{\widehat{\tau}} e^{-\rho\,dt't} C\Big(\frac{(\mathbb{E}[H(\widehat{\mu}_{t+1}) - H(\widehat{\mu}_t)|\widehat{\mathcal{F}}_t]}{dt'}\Big)\,dt'\Bigg] \\
&\quad - \mathbb{E}\big[\big(e^{-\rho\,dt'\widehat{\tau}} - e^{-\rho(dt'+dt)\widehat{\tau}}\big) F(\widehat{\mu}_{\widehat{\tau}})\big].
\end{aligned}
$$

The first inequality is from $(\langle \widetilde{\mu}_t, \widetilde{\tau}\rangle)$ being an admissible strategy. The second inequality is from $e_{-\rho dtn} < 1$. The last inequality is from the convexity of C. Note that $\mathbb{E}[(e^{-\rho\,dt'\widehat{\tau}} - e^{-\rho(dt'+dt)\widehat{\tau}}) F(\widehat{\mu}_{\widehat{\tau}})] \leq \sum_{\tau=0}^{\infty}(e^{-\rho\,dt'\tau} - e^{-\rho(dt+\tau')\tau}) \sup F \leq \frac{\rho\,dt \sup F}{(1-e^{-\rho(dt'+dt)})^2}$. Therefore, $\lim V_{dt_n} \geq V_{dt'}$. By symmetry, $\lim V_{dt'_n} \geq V_{dt}$; hence, $\forall\, dt$, $\lim V_{dt_n}$ is identical (denoted by $\widehat{V}$). Step 2 has already shown that $\|V_{dt} - \widehat{V}\| \leq 2\,dt \sup F$. Then Lemma S.3 is proven. *Q.E.D.*

S1.2. *Lemmas for Lemma A.2*

LEMMA S.4: *X is finite. $V, H \in C(\Delta X)$ and H is concave. $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is continuous, increasing, and convex. Then $\forall \mu \in \Delta(X), \exists \pi^*$ such that $|\mathrm{supp}(\pi^*)| \leq 2|X|$ and solves*

$$
\sup_{\substack{\pi\in\Delta^2 X,\\ \mathbb{E}_\pi[\nu]=\mu}} E_\pi\big[V(\nu)\big] - f\big(H(\mu) - E_\pi\big[H(\nu)\big]\big). \tag{S.4}
$$

There exists $\lambda \in \mathrm{d}f(H(\mu) - E_{\pi^*}[H(\nu)])$ such that $E_{\pi^*}[(V + \lambda H)(\nu)] = \mathrm{co}(V + \lambda H)(\mu)$.

PROOF: *Existence*: Define $\mathcal{V} = \{(\mathbb{E}_\pi[V(\nu)], \mathbb{E}_\pi[H(\mu) - H(\nu)])|\pi \in \Delta^2(X)\ \&\ \mathbb{E}_\pi[\nu] = \mu\}$. $\mathcal{V}$ is closed since $\Delta^2(X)$ has bounded support and both V and H are continuous. Therefore, the function $v - f(I)$ defined on $\mathcal{V}$ has a maximizer (v^*, I^*).

Lagrangian: Define $\mathcal{U} = \{(v, I) \in \mathbb{R}^2 | v - f(I) > v^* - f(I^*))\}$. It is easy to verify that both $\mathcal{V}$ and $\mathcal{U}$ are convex (by the linearity of the expectation operator, convexity of $\Delta^2(X)$, and convexity of f). $\mathcal{U}$ is open. By the optimality of (v^*, I^*), $\mathcal{U} \cap \mathcal{V} = \emptyset$. Then, by the supporting hyperplane theorem, there exists λ_1, λ_2 such that

$$
\lambda_1 v + \lambda_2 I \leq 0, \quad \forall (v, h) \in \mathcal{V}; \tag{S.5}
$$

$$
\lambda_1 v + \lambda_2 I > 0, \quad \forall (v, h) \in \mathcal{U}. \tag{S.6}
$$

Note that $(v^*, I^*) \in \mathcal{V} \cap \bar{\mathcal{U}}$. Then equation (S.6) implies

$$
\begin{aligned}
&(v^*, I^*) \in \arg\min \lambda_1 v + \lambda_2 I \\
&\quad \text{s.t.} \quad v - f(I) \geq v^* - f(I^*).
\end{aligned}
$$

Since the objective function is linear and the constraint is convex, the Kuhn–Tucker condition (generalized to subgradients) implies that $\exists\lambda \in \partial f(I^*)$ and $\beta \leq 0$ such that $\lambda_1 = \beta$, $\lambda_2 = -\lambda\beta$. Since $(\lambda_1, \lambda_2) \neq 0$, $\beta > 0$ and it is wlog to assume that $\lambda_1 = 1$, $\lambda_2 = -\lambda$.

Equation (S.5) implies $(v^*, I^*) \in \arg\max_{(v,I)\in\mathcal{V}} v - \lambda I$. Let $(v^*, I^*) = (\mathbb{E}_{\pi^*}[V(\nu)], H(\mu) - \mathbb{E}_{\pi^*}[H(\nu)])$. Then

$$\pi^* \in \arg\max_{\mathbb{E}_\pi[\nu]=\mu} \mathbb{E}_\pi\big[V(\nu)\big] - \lambda\mathbb{E}_\pi\big[H(\mu) - H(\nu)\big].$$

Meanwhile, π^* solves equation (S.4) since $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Support size: Now I show that π^* can be chosen that $|\mathrm{supp}(\pi)| \leq 2|X|$. $\mathcal{V}$ is a convex and compact set. Since (v^*, I^*) maximizes $v - \lambda I$ on $\mathcal{V}$, it is an exterior point of $\mathcal{V}$. Then by the Krein–Milman theorem, $(v^*, I^*) \in \mathrm{conv}(\mathrm{ext}(\mathcal{V}))$.[S.2] By Caratheodory's theorem, (v^*, I^*) is a convex combination of $s_1, s_2 \in \mathrm{ext}(\mathcal{V})$: $(v^*, I^*) = \alpha s_1 + (1 - \alpha)s_2$.

By Straszewicz's theorem, each extreme point s_i is the limit of exposed points: $s_i = \lim_{n\to\infty} s_i^n$, $\{s_i^n\} \subset \exp(\mathcal{V})$. By the definition of exposed points, $\forall i, n$, there exists $\lambda_1^{i,n}$ and $\lambda_2^{i,n}$ such that s_i^n is the unique maximizer of $\boldsymbol{\lambda}^{i,n} \cdot s$ for $s \in \mathcal{V}$. By Caratheodory's theorem, there exists $\pi_{i,n}$ with support size $|X|$ and maximizes $\lambda_1^{i,n}\mathbb{E}_\pi[V(\nu)] + \lambda_2^{i,n}\mathbb{E}_\pi[H(\mu) - H(\nu)]$. Since s_i^n is unique, $s_i^n = (\mathbb{E}_{\pi_{i,n}} V(\nu), E_{\pi_{i,n}}[H(\mu) - H(\nu)])$. Since $\{\pi_{i,n}\}$ have finite support size $|X|$, there exists a converging subsequence $\pi_{i,n} \to \pi_i^*$ (converges in each mass point and its probability). Then $|\mathrm{Supp}(\pi_i^*)| = |X|$.

Let $\pi^* = \alpha\pi_1^* + (1 - \alpha)\pi_2^*$. Then by continuity of the expectation operator, $(\mathbb{E}_{\pi^*}[V(\nu)], H(\mu) - \mathbb{E}_{\pi^*}[H(\nu)]) = (v^*, I^*)$. $\mathrm{Supp}(\pi^*) \leq 2|X|$. As I have argued, π^* solves both equation (S.4) and the Lagrangian. Note that the Lagrangian has an equivalent convex hull characterization:

$$\sup_{\mathbb{E}_\pi[\nu]=\mu} \mathbb{E}_\pi\big[V(\nu) + \lambda H(\nu)\big] = \mathrm{co}(V + \lambda H)(\mu).$$

Q.E.D.

S1.3. *Lemmas for Lemma A.3*

LEMMA S.5: *Given Assumption* 2, *let* $\widehat{V}(\mu) = \lim_{dt\to 0} V_{dt}(\mu)$. *Then* $\widehat{V} \in \mathcal{L}$.

PROOF: I prove by induction on the dimensionality of μ. When $\mu = \delta_x$, $\mathrm{supp}(\mu)$ is a singleton. So Lemma S.5 trivially holds. Now it is sufficient to prove that $\widehat{V}$ is pointwise Lipschitz at any interior μ.

First, since $\widehat{V}$ is the uniform limit of continuous V_{dt}'s, $\widehat{V}$ is continuous. $\forall\mu \in \Delta X^{\mathrm{o}}$, suppose for the sake of contradiction that $\widehat{V}$ is not pointwise Lipschitz. Then $\exists\ \mu_n \to \mu$, $\frac{|\widehat{V}(\mu_n) - \widehat{V}(\mu)|}{\|\mu_n - \mu\|} \geq n$. There are two possible cases:

- $\frac{\widehat{V}(\mu_n) - \widehat{V}(\mu)}{\|\mu_n - \mu\|} \geq n$. Let ν_n be a point in $\partial\Delta X$ such that μ_n, μ, ν_n are three ordered points on a straight line. Let p_n, q_n be such that $p_n + q_n = 1$, $p_n\mu_n + q_n\nu_n = \mu$. Pick any I

[S.2] $\mathrm{ext}(\mathcal{V})$ denotes the extreme points of $\mathcal{V}$.

such that $C(I) < \infty$. Then

$$I\frac{\widehat{V}(\nu_n)-\widehat{V}(\mu)+\dfrac{\widehat{V}(\mu_n)-\widehat{V}(\mu)}{\|\mu_n-\mu\|}\|\nu_n-\mu\|}{H(\mu)-H(\nu_n)-\dfrac{H(\mu_n)-H(\mu)}{\|\mu_n-\mu\|}\|\nu_n-\mu\|}$$

$$\geq I\frac{\widehat{V}(\nu_n)-\widehat{V}(\mu)+n\|\nu_n-\mu\|}{H(\mu)-H(\nu_n)-\dfrac{H(\mu_n)-H(\mu)}{\|\mu_n-\mu\|}\|\nu_n-\mu\|}.$$

Since the nominator is bounded, μ being interior implies that $\|\nu_n-\mu\|$ is strictly positive in the limit. Take $n\to\infty$ on the RHS, the RHS goes to infinity. Therefore, there exists N such that $\forall n\geq N$, RHS is larger than $3\rho\sup F+2C(I)$:

$$I\frac{\widehat{V}(\nu_n)-\widehat{V}(\mu)+\dfrac{\widehat{V}(\mu_n)-\widehat{V}(\mu)}{\|\mu_n-\mu\|}\|\nu_n-\mu\|}{H(\mu)-H(\nu_n)-\dfrac{H(\mu_n)-H(\mu)}{\|\mu_n-\mu\|}\|\nu_n-\mu\|}\geq 3\rho\sup F+2C(I)$$

$$\Longrightarrow \quad \frac{\big(\|\mu_n-\mu\|\big)\widehat{V}(\nu_n)+\|\nu_n-\mu\|\widehat{V}(\mu_n)-\big(\|\mu_n-\mu\|+\|\nu_n-\mu\|\big)\widehat{V}(\mu)}{-\|\mu_n-\mu\|H(\mu_n)-\|\nu_n-\mu\|H(\mu_n)+\big(\|\mu_n-\mu\|+\|\nu_n-\mu\|\big)H(\mu)}$$

$$\geq\frac{3\rho}{I}\sup F+\frac{2C(I)}{I}$$

$$\Longrightarrow \quad \frac{p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)-\widehat{V}(\mu)}{-p_nH(\mu_n)-q_nH(\nu_n)+H(\mu)}\geq\frac{3\rho}{I}\sup F+\frac{2C(I)}{I}$$

$$\Longrightarrow \quad \frac{p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)-\widehat{V}(\mu)}{I(\mu_n,\nu_n|\mu)}\geq\frac{3\rho}{I}\sup F+\frac{2C(I)}{I}$$

$$\Longrightarrow \quad p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)-\widehat{V}(\mu)\geq\frac{3\rho}{I}\sup FI(\mu_n,\nu_n|\mu)+2C(I)\frac{I(\mu_n,\nu_n|\mu)}{I}$$

$$\Longrightarrow \quad p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)-2C(I)\frac{I(\mu_n,\nu_n|\mu)}{I}$$

$$\geq\widehat{V}(\mu)\Big(1+2\frac{\rho}{I}I(\mu_n,\nu_n|\mu)\Big)+\sup F\frac{\rho}{I}I(\mu_n,\nu_n|\mu)$$

$$\Longrightarrow \quad p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)-2C(I)\frac{I(\mu_n,\nu_n|\mu)}{I}$$

$$\geq\widehat{V}(\mu)e^{\frac{\rho}{I}I(\mu_n,\nu_n|\mu)}+\sup F\frac{\rho}{I}I(\mu_n,\nu_n|\mu).$$

The last inequality comes from $\forall x>0$, $1+2x>e^x$. Now we have

$$e^{-\rho\frac{I(\mu_n,\nu_n|\mu)}{I}}\big(p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)\big)-2e^{-\rho\frac{I(\mu_n,\nu_n|\mu)}{I}}C(I)\frac{I(\mu_n,\nu_n|\mu)}{I}$$

$$\geq\widehat{V}(\mu)+e^{-\rho\frac{I(\mu_n,\nu_n|\mu)}{I}}\sup F\frac{\rho}{I}I(\mu_n,\nu_n|\mu).$$

Since $\mu_n \to \mu$, $\lim_{n\to\infty} I(\mu_n,\nu_n|\mu)=0$. Then pick N sufficiently large that $\forall n\geq N$:

$$e^{-\rho\frac{I(\mu_n,\nu_n|\mu)}{I}}\left(p_n\widehat{V}(\mu_n)+q_n\widehat{V}(\nu_n)\right)-\frac{I(\mu_n,\nu_n|\mu)}{I}C(I)\geq\widehat{V}(\mu)+\frac{\rho I(\mu_n,\nu_n|\mu)}{2I}\sup F.$$

From now on, keep n fixed, and pick $dt=\frac{I(\mu_n,\nu_n|\mu)}{I}$ and $dt_m=\frac{dt}{2^m}$. m is chosen sufficiently large that $\left|\widehat{V}-V_{dt_m}\right|e^{\frac{\rho}{I}I(\mu_n,\nu_n|\mu)}<\frac{\rho I(\mu_n,\nu_n|\mu)}{8c}\sup F$, then

$$e^{-\rho\frac{I(\mu_n,\nu_n|\mu)}{I}}\left(p_nV_{dt_m}(\mu_n)+q_nV_{dt_m}(\nu_n)\right)-dtC\Big(\frac{I(\mu_n,\nu_n|\mu)}{dt}\Big)\geq V_{dt_m}(\mu)+\frac{\rho\,dt}{4}\sup F.$$

Consider a strategy that divides $I(\mu_n,\nu_n|\mu)$ into 2^m periods uniformly (based on Lemma S.2), and follows the optimal strategy of V_{dt_m} at the end of the 2^m periods. The payoff is

$$\begin{aligned}
&e^{-\rho\,dt}\left(p_nV_{dt_m}(\mu_n)+q_nV_{dt_m}(\nu_n)\right)-\sum_{t=0}^{2^m-1}e^{-\rho t\,dt_m}\,dt_m\cdot C\Big(\frac{I(\mu_n,\nu_n|\mu)/2^m}{dt_m}\Big)\\
&>e^{-\rho\,dt}\left(p_nV_{dt_m}(\mu_n)+q_nV_{dt_m}(\nu_n)\right)-\sum_{t=0}^{2^m-1}e^{-\rho\,dt}\,dt_m\cdot C\Big(\frac{I(\mu_n,\nu_n|\mu)/2^m}{dt_m}\Big)\\
&=e^{-\rho\,dt}\Big(p_nV_{dt_m}(\mu_n)+q_nV_{dt_m}(\nu_n)-dt\cdot C\Big(\frac{I(\mu_n,\nu_n|\mu)}{dt}\Big)\Big)\\
&\geq V_{dt_m}(\mu)+\frac{\rho\,dt}{4}\sup F.
\end{aligned}$$

The second line scales all the nonnegative costs with a term larger than 1. Taking m sufficiently large, the last line is strictly larger than $V_{dt_m}(\mu)$, contradiction.

- $\frac{\widehat{V}(\mu_n)-\widehat{V}(\mu)}{\|\mu_n-\mu\|}\leq -n$. Then pick $\nu_n\in\partial\Delta X$ such that μ,μ_n,ν_n are three ordered points on a straight line. Let p_n,q_n be such that $p_n+q_n=1$, $p_n\mu+q_n\nu_n=\mu_n$. Pick any I such that $C(I)<\infty$. We have

$$\begin{aligned}
&I\frac{\widehat{V}(\nu_n)-\widehat{V}(\mu_n)+\dfrac{\widehat{V}(\mu)-\widehat{V}(\mu_n)}{\|\mu_n-\mu\|}\|\nu_n-\mu_n\|}{H(\mu_n)-H(\nu_n)-\dfrac{H(\mu)-H(\mu_n)}{\|\mu_n-\mu\|}\|\nu_n-\mu_n\|}\\
&\geq I\frac{\widehat{V}(\nu_n)-\widehat{V}(\mu_n)+n\|\nu_n-\mu_n\|}{H(\mu_n)-H(\nu_n)-\dfrac{H(\mu)-H(\mu_n)}{\|\mu_n-\mu\|}\|\nu_n-\mu_n\|}.
\end{aligned}$$

Take $n\to\infty$ on RHS, we observe that RHS goes to infinity. Therefore, there exists N such that $\forall n\geq N$, RHS is larger than $3\rho\sup F+2C(I)$:

$$\begin{aligned}
\Longrightarrow\ p_n\widehat{V}(\mu)+q_n\widehat{V}(\nu_n)-2C(I)\frac{I(\mu,\nu_n|\mu_n)}{I}&\geq\widehat{V}(\mu_n)+3\frac{\rho I(\mu,\nu_n|\mu_n)}{I}\sup F\\
&\geq e^{\rho\frac{I(\mu,\nu_n|\mu_n)}{I}}\widehat{V}(\mu_n)+\frac{\rho I(\mu,\nu_n|\mu_n)}{I}\sup F.
\end{aligned}$$

Similar to the last part, N can be chosen sufficiently large that

$$e^{-\rho\frac{I(\mu,\nu_n|\mu_n)}{I}}\left(p_n\widehat{V}(\mu)+q_n\widehat{V}(\nu_n)\right)-\frac{I(\mu,\nu_n|\mu_n)}{I}C(I)\geq\widehat{V}(\mu_n)+\frac{\rho I(\mu,\nu_n|\mu_n)}{I}\sup F.$$

Then pick $dt=\frac{I(\mu,\nu_n|\mu_n)}{I}$ and $dt_m=\frac{dt}{2^m}$. m can be chosen sufficiently large that

$$e^{-\rho\,dt}\left(p_nV_{dt_m}(\mu)+q_nV_{dt_m}(\nu_n)\right)-dtC(I)\geq V_{dt_m}(\mu_n)+\frac{\rho\,dt}{2}\sup F.$$

Consider a similar strategy as before that divides experiment uniformly:

$$\begin{aligned}&e^{-\rho\,dt}\left(p_nV_{dt_m}(\mu)+q_nV_{dt_m}(\nu_n)\right)-\sum_{t=0}^{2^m-1}e^{-\rho t\,dt_m}\,dt_m\cdot C\left(\frac{I(\mu,\nu_n|\mu_n)/2^m}{dt_m}\right)\\&\geq V_{dt_m}(\mu_n)+\frac{\rho\,dt}{4}\sup F.\end{aligned}$$

RHS is strictly larger than $V_{dt_m}(\mu_n)$. This experiment dominates the optimal experiment of the dt_m problem at μ_n, contradiction. *Q.E.D.*

LEMMA S.6: *$\forall f(x)$ differentiable on (a,b). $\forall x,y\in(a,b)$,*

$$\frac{1}{2}\inf_{z\in(x,y)}D^2f(z,y)\left|y-x\right|^2\leq f(y)-f(x)-f'(x)(y-x)\leq\frac{1}{2}\sup_{z\in(x,y)}D^2f(z,y)\left|y-x\right|^2.$$

PROOF:

- First inequality: let $\underline{D}=\inf_{z\in(x,y)}D^2f(z,y)$. Suppose by contradiction the statement is not true, then there exists $\varepsilon>0$ such that $\frac{\underline{D}-\varepsilon}{2}\left|y-x\right|^2>f(y)-f(x)-f'(x)(y-x)$. Let $h(w)=f(w)-f(x)-f'(x)(w-x)-\frac{\underline{D}-\varepsilon}{2}(w-x)^2$. Then $h(x)=0$, $h'(x)=0$ and $h(y)<0$. Now consider $\max_z h(z)-\frac{h(y)}{y-x}(z-x)$. By continuity of h, maximizer z^* exists in $[x,y]$. FOC implies $h'(z^*)=\frac{h(y)}{y-x}$ so $z^*\neq x$. The objective function is 0 at both x,y so $z^*\neq y$. Then optimality of z^* implies $\forall dz$ sufficiently small:

$$\begin{aligned}&h(z^*+dz)-\frac{h(y)}{y-x}(z^*+dz-x)\leq h(z^*)-\frac{h(y)}{y-x}(z^*-x)\\\Longrightarrow\quad&f(z^*+dz)-f(z^*)-f'(x)\,dz-\frac{\underline{D}-\varepsilon}{2}(2z^*-2x+dz)\,dz\\&\quad\leq dz\left(f'(z^*)-f'(x)-(\underline{D}-\varepsilon)(z^*-x)\right)\\\Longrightarrow\quad&\frac{f(z^*+dz)-f(z^*)-f'(z^*)\,dz}{dz^2}\leq\frac{\underline{D}-\varepsilon}{2}\\\Longrightarrow\quad&D^2f(z^*,y)<\underline{D}.\end{aligned}$$

  This contradiction shows that the first inequality holds.

- Second inequality: let $\bar{D}=\sup_{z\in(x,y)}D^2(z,y)$. Suppose by contradiction the statement is not true, then there exists $\varepsilon>0$ such that $\frac{\bar{D}+\varepsilon}{2}\left|y-x\right|^2<f(y)-f(x)-f'(x)(y-z)$.

Let $h(w) = f(w) - f(x) - f'(x)(w-x) - \frac{\bar{D}+\varepsilon}{2}(w-x)^2$. Then $h(x)=0, h'(x)=0$ and $h(y)>0$. Now consider $\min_z h(z) - \frac{h(y)}{y-x}(z-x)$. By continuity of h, minimizer z^* exists in $[x,y]$. FOC implies $h'(z^*) = \frac{h(y)}{y-z}$ so $z^* \neq x$. Then optimality of z^* implies $\forall dz$ sufficiently small:

$$\begin{aligned}
h(z^*+dz) - \frac{h(y)}{y-x}(z^*+dz-x) &\geq h(z^*) - \frac{h(y)}{y-x}(z^*-x) \\
\implies \quad & f(z^*+dz) - f(z^*) - f'(x)\,dz - \frac{\bar{D}+\varepsilon}{2}(2z^*-2x+dz)\,dz \\
& \geq dz\big(f'(z^*) - f'(x) - (\bar{D}+\varepsilon)(z^*-x)\big) \\
\implies \quad & \frac{f(z^*+dz) - f(z^*) - f'(z^*)\,dz}{dz^2} \geq \frac{\bar{D}+\varepsilon}{2} \\
\implies \quad & D^2 f(z^*, y) > \bar{D}.
\end{aligned}$$

This contradiction shows that the second inequality holds. *Q.E.D.*

S2. OMITTED PROOFS AND LEMMAS FOR THEOREM 2

Section S2.1 proves a technical lemma Lemma S.7 that verifies equation (31). Section S2.2 verifies that $V(\mu)$ defined by equation (33) in the proof of Theorem 2 is a C^1 function.

S2.1. *Lemmas for Theorem 2*

LEMMA S.7: *Suppose V_m solves equation* (44) *on $[\mu_0, \mu_m)$ and satisfies*

$$V_m(\mu_0) \geq \sup_{\nu \geq \mu_0, I} \frac{I}{\rho} \frac{F_{m'}(\nu) - V_m(\mu_0) - V_m'(\mu_0)(\nu-\mu_0)}{J(\mu_0,\nu)} - \frac{C(I)}{\rho}$$

for $m' > m$. Then $\forall \mu \in [\mu_0, \mu_m)$:

$$V_m(\mu) \geq \max_{\nu \geq \mu, I} \frac{I}{\rho} \frac{F_{m'}(\nu) - V_m(\mu) - V_m'(\mu)(\nu-\mu)}{J(\mu,\nu)} - \frac{C(I)}{\rho}.$$

Lemma S.7 shows that the constructed value function in Lemma B.3 satisfies equation (31): Suppose that at $\widehat{\mu}_k$, all $F_{k'}$ with $k' \geq k$ are suboptimal given V_k, then applying Lemma S.7 to V_{k-1} implies that for $\mu \geq \widehat{\mu}_k$, all $F_{k'}$ with $k' \geq k$ are suboptimal. Moreover, $\widehat{\mu}_{k-1}$ is chosen that F_{k-1} is suboptimal given V_{k-1}. Therefore, by induction on k, $\forall k$, when $\mu \geq \widehat{\mu}_k$, $F_{k'}$ with $k' \geq k$ is suboptimal.

PROOF: Let (ν_0, I_0) be the optimal policy at μ_0. The optimality condition implies

$$\begin{cases} C'(I_0) = \dfrac{F_m(\nu_0) - V_m(\mu_0) - V_m'(\mu_0)(\nu_0-\mu_0)}{J(\mu_0,\nu_0)}, \\ C'(I_0) = \dfrac{F_m' - V_m'(\mu_0)}{H'(\mu_0) - H'(\nu_0)}. \end{cases} \tag{S.7}$$

Therefore, $\forall \nu \geq \nu_0$,

$$\begin{aligned}
&\frac{F_{m'}(\nu) - F_m(\nu_0) - F_m'(\nu - \nu_0)}{J(\nu_0, \nu)} \\
&= \frac{\left[F_{m'}(\nu) - V_m(\mu_0) - V_m'(\mu_0)(\nu - \mu_0)\right] - \left[F_m(\nu_0) - V_m(\mu_0) - V_m'(\mu_0)(\nu_0) - \mu_0\right] - \left[\left(V_m'(\nu_0) - V_m'(\mu_0)\right)(\nu - \nu_0)\right]}{J(\mu_0, \nu) - J(\mu_0, \nu_0) - \left[\left(H'(\mu_0) - H'(\nu_0)\right)(\nu - \nu_0)\right]} \\
&\leq C'(I_0). \qquad \text{(S.8)}
\end{aligned}$$

The inequality is from the fact that the ratio of the first terms is less than $C'(I_0)$ (the assumption of the lemma) and the ratios of the rest of the terms are $C'(I_0)$ (equation (S.7)). Now, let (ν, I) be the optimal policy at μ. Similarly, the optimality condition implies

$$\begin{cases}
C'(I) = \dfrac{F_m(\nu) - V_m(\mu) - V_m'(\mu)(\nu - \mu)}{J(\mu, \nu)} \geq \dfrac{F_m(\nu') - V_m(\mu) - V_m'(\mu)(\nu' - \mu)}{J(\mu, \nu')}, \\
C'(I) = \dfrac{F_m' - V_m'(\mu)}{H'(\mu) - H'(\nu)}.
\end{cases} \qquad \text{(S.9)}$$

Since $\mu \geq \mu_0$, $C'(I) \geq C'(I_0)$. $\forall \nu' > \nu_0$,

$$\begin{aligned}
&\frac{F_{m'}(\nu') - V_m(\mu) - V_m'(\mu)(\nu' - \mu)}{J(\mu, \nu')} \\
&= \frac{\left[F_{m'}(\nu') - F_m(\nu_0) - F_m'(\nu' - \nu_0)\right] + \left[F_m(\nu_0) - V_m(\mu) - F_m'(\nu_0 - \mu)\right] + \left[\left(F_m' - V_m'(\mu)\right)(\nu' - \nu_0)\right]}{J(\nu_0, \nu') + J(\mu, \nu_0) + \left[\left(H'(\mu) - H'(\nu_0)\right)(\nu' - \nu_0)\right]} \\
&\leq \frac{\left[F_{m'}(\nu') - F_m(\nu_0) - F_m'(\nu' - \nu_0)\right] + \left[F_m(\nu_0) - V_m(\mu) - F_m'(\nu_0 - \mu)\right] + \left[\left(F_m' - V_m'(\mu)\right)(\nu' - \nu_0)\right]}{J(\nu_0, \nu') + J(\mu, \nu_0) + \left[\left(H'(\mu) - H'(\nu)\right)(\nu' - \nu_0)\right]} \\
&\leq C'(I).
\end{aligned}$$

The first inequality is from H being concave. The second inequality is from the ratio of the first terms being less than $C'(I_0)$ (equation (S.8)), the ration of the second terms being less than $C'(I)$, and the ratio of the third terms being $C'(I)$ (equation (S.9)). Note that for $\nu < \nu_0$, $F_{m'}(\nu) \leq F_m(\nu)$ so the inequality holds automatically. Therefore, Lemma S.7 is proved. *Q.E.D.*

S2.2. *Proof of Smoothness*

PROOF: I prove by showing that on $[\mu^{**}, 1]$, $V(\mu)$ is piecewise defined as $V_{\mu^\diamond}$ for $\mu^\diamond \in \Omega$. I first show three auxiliary results.

LEMMA S.8: $\forall \underline{\mu}_k \geq \mu^{**}$, *there exists* $\mu^\diamond \in \Omega$ *such that* $V_{\mu^\diamond}(\underline{\mu}_k) > F(\underline{\mu}_k)$.

Lemma S.8 shows that $V > F$ wherever F has a kink.

PROOF: $\forall k$ such that $\underline{\mu}_{k-1} > \mu^{**}$, $\lim_{\mu \to \underline{\mu}_k} U(\mu) = \infty$. So $U(\mu) > F(\mu)$ in a left neighborhood of $\underline{\mu}_k$. Let $\mu^\diamond = \inf\{\mu \geq \mu^{**} | \forall \mu' \in (\mu, \underline{\mu}_k),\ U(\mu') > F(\mu')\}$. Then $\mu^\diamond$ exists since the set is nonempty. Then consider $V_{\mu^\diamond}(\mu)$. I claim that $V_{\mu^\diamond}(\mu) > F(\mu)$, $\forall \mu \in (\mu^\diamond, \underline{\mu}_k)$. Suppose not, then by the intermediate value theorem, there exists μ' such that $V_{\mu^\diamond}(\mu') \leq F(\mu)$ and $V'_{\mu^\diamond}(\mu') \leq F(\mu)$. However, this implies $V_{\mu^\diamond}(\mu') = \max_{\nu \geq \mu, I} \frac{I}{\rho} \frac{F(\nu) - V_{\mu^\diamond}(\mu') - V'_{\mu^\diamond}(\nu - \mu')}{J(\mu', \nu)} - C(I) \geq U(\mu') > F(\mu')$, contradiction. Now, suppose $V_{\mu^\diamond}(\underline{\mu}_k) = F(\underline{\mu}_k)$, by the construction of $V_{\mu^\diamond}$, $\lim_{\mu \to \underline{\mu}_k^-} V'_{\mu^\diamond}(\mu) = F'^-(\underline{\mu}_k)$. Then $\frac{F(\nu) - V_{\mu^\diamond}(\mu) - V'_{\mu^\diamond}(\nu - \mu)}{J(\mu, \nu)} \to \frac{\Delta F'(\underline{\mu}_k)(\nu - \underline{\mu}_k)}{J(\underline{\mu}_k, \nu)}$ when $\mu \to \underline{\mu}_k$, and $J(\underline{\mu}_k, \nu) = O((\nu - \underline{\mu}_k)^2)$. Therefore, $\lim_{\mu \to \underline{\mu}_k} V_{\mu^\diamond}(\mu) = \infty$, contradiction. *Q.E.D.*

LEMMA S.9: *$\forall \mu_0 \leq \mu_1 \in \Omega$, let $I_i = \{\mu | V_{\mu_i}(\mu) > F(\mu)\}$. $I_1 \cap I_0 \neq \emptyset \implies I_1 \subset I_0$ and $V_{\mu_0} \geq V_{\mu_1}$.*

Lemma S.9 shows that any two V_{μ_0} and V_{μ_1} do not cross when they are above F.

PROOF: The only possible violation of Lemma S.9 is that $\exists \mu' \in I_0 \bigcap I_1$ such that $V_{\mu_1}(\mu') > V_{\mu_0}(\mu')$. Since at μ_1, $V_{\mu_0}(\mu_1) > V_{\mu_1}(\mu_1) = F(\mu_1)$, by intermediate value theorem, there exists $\xi \in (\mu_1, \mu')$ such that $V_{\mu_1}(\xi) > V_{\mu_0}(\xi)$ and $V'(\mu_1)(\xi) > V'(\mu_0)(\xi)$. Since $\xi \in I_1$, there exists ν, I solving equation (31) for $V_{\mu_1}(\xi)$:

$$
\begin{aligned}
V_{\mu_0}(\xi) &\geq \frac{I}{\rho} \frac{F(\nu) - V_{\mu_0}(\xi) - V'_{\mu_0}(\xi)(\nu - \xi)}{J(\xi, \nu)} - C(I) \\
&> \frac{I}{\rho} \frac{F(\nu) - V_{\mu_1}(\xi) - V'_{\mu_1}(\xi)(\nu - \xi)}{J(\xi, \nu)} - C(I) \\
&= V_{\mu_1}(\xi) > V_{\mu_0}(\xi);
\end{aligned}
$$

contradiction. *Q.E.D.*

LEMMA S.10: *$\exists \Delta$ such that $\forall \mu^\diamond \in \Omega$, on $\{\mu | V_{\mu^\diamond}(\mu) > F(\mu)\}$, $V'_{\mu^\diamond}(\mu)$ has Lipschitz parameter Δ.*

PROOF: I first argue that $\{\mu | V(\mu) > F(\mu)\}$ is bounded away from 1. $\forall \mu$ in the set, there exists $\mu^\diamond$ such that $V_{\mu^\diamond}(\mu) > F(\mu)$. By the construction of $V_{\mu^\diamond}$, let $\nu(\mu)$ be the optimal posterior, $\mu < \nu(\mu) < \nu(\mu^\diamond)$. The FOC of ν at $\mu^\diamond$ implies $\frac{F'(\nu(\mu^\diamond)) - F'(\mu^\diamond)}{H'(\mu^\diamond) - H'(\nu(\mu^\diamond))} = C'(I(\mu^\diamond))$. Since F is piecewise linear and $C'(I(\mu^\diamond)) > 0$, $\mu^\diamond$ is bounded above by the last kink of F. Since $\rho F(\mu^\diamond) = C'(I(\mu^\diamond)) - C(I(\mu^\diamond))$, $C'(I(\mu^\diamond))$ is bounded above. Then, since $\lim_{\mu \to 1} |H'(\mu)| = \infty$, $\nu(\mu^\diamond)$ is bounded away from 1; hence μ is bounded away from 1. In each smooth region of ν, by the envelope theorem,

$$
\begin{aligned}
& V'_{\mu^\diamond}(\mu) = -\frac{I}{\rho} \frac{\nu - \mu}{J(\mu, \nu)} \left(V''_{\mu^\diamond}(\mu) + C'(I) H''(\mu) \right) > 0 \\
\implies \quad & V''_{\mu^\diamond}(\mu) + C'(I) H''(\mu) < 0.
\end{aligned}
$$

$C'(I)$ is bounded since $C(I)$ is bounded by $\sup F$. Assumption 3 implies that $-H''(\mu)$ is bounded. Therefore, there exists Δ such that $V''_{\mu^\diamond}(\mu) \leq \Delta$. On the other hand,

$$
-V''_{\mu^\diamond}(\mu) \leq \frac{\rho}{I} \frac{J(\mu, \nu)}{\nu - \mu} V'_{\mu^\diamond}(\mu).
$$

The RHS is bounded since μ is bounded, and V' is uniformly bounded by the analysis of ODE equation (50) in Lemma B.4. Therefore, Δ can be chosen that $|V''_{\mu^\diamond}(\mu)| \leq \Delta$. Since the discrete points of ν is finite, V'_{μ_0} is an integral of V''_{μ_0}; hence, V'_{μ_0} has Lipschitz parameter Δ. *Q.E.D.*

Now, I return to the proof of smoothness.

- *Step 1*: By Lemma S.10, $\{V_{\mu^\diamond}\}$ is a family of totally bounded and equicontinuous functions. Therefore, $V(\mu) = \sup_{\mu^\diamond \in \Omega} V_{\mu^\diamond}(\mu)$ is continuous and $\{\mu \geq \mu^{**} | V(\mu) > F(\mu)\}$ can be written as $\bigcup I_m$ where I_m are disjoint open intervals.
- *Step 2*: $\forall I_m$, pick an arbitrary $\mu \in I_m$. Let $\Omega(\mu) = \{\mu^\diamond \in \Omega | V_{\mu^\diamond}(\mu) > F(\mu)\}$. Lemma S.9 implies that the regions $\{\mu' | V_{\mu^\diamond}(\mu') > F(\mu')\}$ are nested open intervals for $\mu^\diamond \in \Omega(\mu)$. Define $\widetilde{V}(\mu') = \sup_{\mu^\diamond \in \Omega(\mu)} V_{\mu^\diamond}(\mu')$. Then $\widetilde{I} = \{\mu' | \widetilde{V}(\mu') > F(\mu')\}$ is an open interval containing μ. $\forall \mu' \in \widetilde{I}$, $V(\mu') = \widetilde{V}(\mu')$, because otherwise there exists $\mu^\diamond$ such that $V_{\mu^\diamond}(\mu') > \widetilde{V}(\mu') > F(\mu')$. Lemma S.9 implies that $\mu^\diamond \in \Omega(\mu)$ and contradicts the definition of $\widetilde{V}$. Since $\widetilde{V} = V$ on $\widetilde{I}$ and $\widetilde{V}$ is continuous (by Lemma S.10), $\widetilde{I} = I_m$. Therefore, $V(\mu') = \widetilde{V}(\mu')$ on I_m.
- *Step 3*: Let $\mu_m = \inf I_m$. I claim that $\mu_m \in \Omega(\mu)$ and $V(\mu') = V_{\mu_m}(\mu')$ on I_m. Suppose not, $\exists$ a decreasing sequence $\{\mu_k\} \subset \Omega(\mu)$ such that $\mu_m = \lim \mu_k$. By Lemma S.9, V_{μ_k} is an increasing sequence and $\lim V_{\mu_k} = V$ on I_m. Since $\{V'_{\mu_k}\}$ are equicontinuous and totally bounded on I_m (Lemma S.10), a subsequence converges uniformly. As a result, V is differentiable on I_m and $V' = \lim V'_{\mu_k}$. $\forall \mu' \in I_m$, let (ν_k, I_k) be the optimizer of V_{μ_k}. Then since $V'_{\mu_k}(\mu')$ and $V_{\mu_k}(\mu')$ converges to $V'(\mu')$ and $V(\mu')$, respectively, the limit point of (ν_k, I_k) is an optimizer of V. As a result, $V(\mu)$ solves equation (31) on I. Since the solution to equation (31) is unique (Lemma B.4), $V(\mu) = V_{\mu_m}(\mu)$ on I_m.

To sum up, V is defined in the region $[\mu^*, 1]$ as

$$V(\mu) = \begin{cases} V_{\mu^*}(\mu) & \text{if } \mu \in [\mu^*, \mu^{**}], \\ V_{\mu^m}(\mu) & \text{if } \mu \in I^m, \\ F(\mu) & \text{otherwise.} \end{cases}$$

Now, I prove $V(\mu) \in C^1[\mu^*, 1]$. Define

$$V_n(\mu) = \begin{cases} V_{\mu^m}(\mu) & \text{if } \mu \in \bigcup_{m \leq n} I_m, \\ F(\mu) & \text{otherwise.} \end{cases}$$

Then $V_n(\mu) \to V(\mu)$. By Lemma S.8, we can without loss assume that the first n I_m's cover all $\underline{\mu}_m$'s. Fix n, $\forall \mu$, $\forall l \geq n$, if $\mu \in \bigcup_{m \leq n} I_m$ or $\mu \notin \cup I_m$, then $V'_n(\mu) = V'_l(\mu)$, else if $\mu \in \bigcup_{m > n} I_m$, then $|V'_n(\mu) - F'(\mu)| = 0$ and $|V'_l(\mu) - F'(\mu)|$ is bounded by $\Delta \sup_{m \geq n} |I_m|$ (Lemma S.10). Therefore, $V'_n(\mu)$ is a Cauchy sequence. Since each V'_n is continuous, V' is continuous, V is a C^1 function, and $V' = F'$ when $V = F$. *Q.E.D.*

S3. PROOFS IN SECTION 5

S3.1. *Proof of Theorem 3*

PROOF: *Sufficiency*: suppose f is UPS, let H be the potential function defining it. Then

$$\begin{aligned}\frac{1}{\mathbb{E}[\widehat{\tau}]}\mathbb{E}\left[\sum_{t=1}^{\widehat{\tau}}\mathbb{E}\left[H(\widehat{\mu}_{t-1})-H(\widehat{\mu}_t|\widehat{\mathcal{F}}_{t-1})\right]\right] &= \frac{1}{\mathbb{E}[\widehat{\tau}]}\mathbb{E}\left[\mathbb{E}\left[\sum_{t=1}^{\widehat{\tau}}\left(H(\widehat{\mu}_{t-1})-H(\widehat{\mu}_t)\right)|\widehat{F}_{\widehat{\tau}}\right]\right]\\ &= \frac{1}{\mathbb{E}[\widehat{\tau}]}\mathbb{E}\left[H(\widehat{\mu}_0)-H(\widehat{\mu}_{\widehat{\tau}})\right]\end{aligned}$$

is a function of $(\widehat{\mu}_{\widehat{\tau}}, \mathbb{E}[\widehat{\tau}])$.

Necessity: $\forall \pi \in \Delta^2(X)$ and $T \in [1, \infty)$, define $\langle\widehat{\mu}_t\rangle$, $\widehat{\tau}$ as follows: $\widehat{\mu}_0 = \mathbb{E}_\pi[\nu]$, $\widehat{\mu}_1 \sim \pi$ and $\widehat{\mu}_t \equiv \widehat{\mu}_1$ for $t \geq 1$. Define $\text{prob}(\widehat{\tau} = t) = (1 - \frac{1}{T})^t \frac{1}{T}$ for $t \geq 1$ (independent to $\langle\widehat{\mu}_t\rangle$). Then $\mathbb{E}[\widehat{\tau}] = T$ and $\widehat{\mu}_{\widehat{\tau}} \sim \pi$. Equation (9) implies that $g(\pi, T) \cdot T$ is constant for all T; hence, it is wlog to define $g(\pi, T) = k(\pi)\frac{1}{T}$, where $k(\pi)$ satisfies the following identity:

$$\mathbb{E}\left[\sum_{t=1}^{\widehat{\tau}} f(\widehat{\mu}_t|\widehat{\mathcal{F}}_{t-1})\right] = k(\widehat{\mu}_{\widehat{\tau}}).$$

Define $\bar{\pi}_\mu$ as the fully revealing information structure with prior μ and $H(\mu) = k(\bar{\pi}_\mu)$. Now, $\forall \pi \in \Delta^2(X)$, define $\langle\widehat{\mu}_t\rangle$, $\widehat{\tau}$ as follows: $\widehat{\mu}_1 \sim \pi$, $\widehat{\mu}_2|\widehat{\mu}_1 \sim \bar{\pi}_{\widehat{\mu}_1}$, $\widehat{\tau} \equiv 2$. Therefore, equation (9) implies that

$$\begin{aligned} & f(\pi) + \mathbb{E}_\pi\left[f(\bar{\pi}_\nu)\right] = k(\bar{\pi}_{\mu_0})\\ \Longrightarrow \quad & f(\pi) = H(\mu_0) - \mathbb{E}_\pi\left[H(\nu)\right].\end{aligned}$$

$f(\pi) \geq 0$ for all $\pi \in \Delta^2(X)$ implies that H is a concave function. *Q.E.D.*

S3.2. *Proof of Proposition 1*

PROOF: Let $h(\cdot)$ be defined by $C'(\frac{1}{\rho}(h(x)C'(h(x)) - C(h(x)))) = x$. Let $\Delta = \frac{h(\sup F)}{h(\inf F)}$. Suppose $(p > 0, \nu)$ is the optimal policy at μ. Then the optimality condition implies

$$\begin{aligned} h(V(\mu)) &= \frac{V(\nu) - V(\mu) - V'(\mu)(\nu - \mu)}{J(\mu, \nu)}\\ &= \frac{\int_\mu^\nu \int_\mu^\eta V''(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta}{J(\mu, \nu)}\\ &\leq \frac{\int_\mu^\nu \int_\mu^\eta h(V(\xi)) \cdot J''_{\nu\nu}(\xi, \xi)\,\mathrm{d}\xi\,\mathrm{d}\eta}{J(\mu, \nu)}\\ \Longrightarrow \quad J(\mu, \nu) &\leq \Delta \int_\mu^\nu \int_\mu^\eta \frac{g''(1)}{(\xi - \xi^2)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Note that $\frac{1}{(\xi-\xi^2)^2}$ is the second derivative of $H^*(\xi)=(2\xi-1)\log(\frac{\xi}{1-\xi})$. Therefore,

$$J(\mu,\nu)\le\Delta\cdot g''(1)\big(H^*(\nu)-H^*(\mu)-H^{*'}(\mu)(\nu-\mu)\big)$$
$$\implies\quad g\left(\frac{\nu}{1-\nu}\frac{1-\mu}{\mu}\right)\le\Delta g''(1)\big(H^*(\nu)-H^*(\mu)-H^{*'}(\mu)(\nu-\mu)\big).$$

The second inequality is from $\frac{\nu}{\mu}+\frac{1-\nu}{1-\mu}\ge\min\{\frac{1}{\mu},\frac{1}{1-\mu}\}\ge1$. The contraposition proves Proposition 1. *Q.E.D.*

S3.3. *Proof of Proposition 2*

PROOF: $\forall\mu\in E$, let $I=p\cdot\frac{\nu-\mu}{\mu(1-\mu)}$ the optimality condition is

$$\rho V(\mu)=\sup_{I,\nu}I\mu(1-\mu)\frac{V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{|\nu-\mu|}-C(I).$$

Suppose for contradiction that the optimal ν is interior and $\nu>\mu$, then the FOC for ν is

$$\big(V'(\nu)-V'(\mu)\big)(\nu-\mu)-\big(V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)\big)=0$$
$$\iff\quad V'(\nu)=\frac{V(\nu)-V(\mu)}{\nu-\mu}=\frac{\int_\mu^\nu V'(\xi)\,\mathrm{d}\xi}{\nu-\mu}\le V'(\nu).$$

The inequality is by V begin convex. Equality holds only if $V'(\xi)\equiv V'(\nu)$ for $\xi\in[\mu,\nu]$, which is impossible because this implies $V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)=0$. The strict inequality leads to a contradiction. A symmetric argument rules out the $\nu<\mu$ case as well. Therefore, $\nu\in\{0,1\}\subset E^C$. *Q.E.D.*

REMARK 1: Restricting attention to convex value functions when the cost is prior independent is justified by the following fact: The DM can always choose a prior independent strategy (specifying the history-dependent choices of experiments and actions), which yields an expected utility function that is linear in the prior belief. The optimal value is then the upper envelope of all these linear functions, and hence is convex.

S3.4. *Proof of Theorem 4*

PROOF: Suppose $\mu\in D$ and $\rho V(\mu)=\max_\sigma\frac{1}{2}\sigma^2V''(\mu)-C(\kappa(\mu,\sigma))$. Assumption 4 states that $\kappa(\mu,\sigma)=\frac{1}{2}\sigma^2J''_{\nu\nu}(\mu,\mu)$. Then the optimality condition implies $V''(\mu)=C'(\frac{1}{2}\times\sigma^2J''_{\nu\nu}(\mu,\mu))J''_{\nu\nu}(\mu,\mu)$ for optimal $\sigma\implies\frac{1}{2}\sigma(\mu)^2=C'^{-1}(\frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)})\frac{1}{J''_{\nu\nu}(\mu,\mu)}\implies\rho V(\mu)=\frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}\cdot C'^{-1}(\frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)})-C(C'^{-1}(\frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}))$. In other words, $h(\rho V(\mu))=\frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}$.

Since Gaussian learning is optimal at μ, this suggests that $\sup_\nu\frac{V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{J(\mu,\nu)}$ is locally maximized at $\nu\to\mu$. Therefore, the FOC

$$\frac{V'(\nu)-V'(\mu)}{J(\mu,\nu)}-\frac{J'_\nu(\mu,\nu)}{J(\mu,\nu)^2}\big(V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)\big)$$

must be nonnegative when $\nu \to \mu^-$ and nonpositive when $\nu \to \mu^+$. Consider the limit of the FOC when $\nu \to \mu$. The L'Hospital's rule implies

$$\begin{aligned}
\lim_{\nu\to\mu} \mathrm{FOC} &= \frac{\lim_{\nu\to\mu}\Big(V''(\nu) - J''_{\nu\nu}(\mu,\nu)\dfrac{V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{J(\mu,\nu)} - J'_{\nu}(\mu,\nu)\cdot \mathrm{FOC}\Big)}{\lim_{\nu\to\mu} J'_{\nu}(\mu,\nu)} \\
&= \frac{1}{2}\frac{\Big(V''(\nu) - J''_{\nu\nu}(\mu,\nu)\dfrac{V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{J(\mu,\nu)}\Big)}{\lim_{\nu\to\mu} J'_{\nu}(\mu,\nu)} \\
&= \frac{1}{2}\frac{\lim_{\nu\to\mu}\Big(V^{(3)}(\nu) - J^{(3)}_{\nu\nu\nu}(\mu,\nu)\dfrac{V(\nu)-V(\mu)(\nu-\mu)}{J(\mu,\nu)} - J''_{\nu\nu}(\mu,\mu)\cdot \mathrm{FOC}\Big)}{\lim_{\nu\to\mu} J''_{\nu\nu}(\mu,\mu)} \\
&= \frac{1}{3}\frac{V^{(3)}(\mu) - J^{(3)}_{\nu\nu\nu}(\mu,\mu)\dfrac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}}{J''_{\nu\nu}(\mu,\mu)} = 0. \qquad \text{(S.10)}
\end{aligned}$$

Now consider $h(\rho V(\mu)) - \frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}$. By assumption, it is nonnegative and achieves 0 at μ. Thus, it is locally minimized at μ, which implies FOC:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mu}\Big(h(\rho V(\mu)) - \frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}\Big) = 0 \\
\implies\quad & \frac{\mathrm{d}h(\rho V(\mu))}{d\mu} - \frac{V^{(3)}(\mu)}{J''_{\nu\nu}(\mu,\mu)} + \frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)^2}\big(J^{(3)}_{\nu\nu\nu}(\mu,\mu) + J^{(3)}_{\nu\nu\mu}(\mu,\mu)\big) = 0 \\
\implies\quad & \frac{\mathrm{d}h(\rho V(\mu))}{\mathrm{d}\mu} + h(\rho V(\mu))\frac{J^{(3)}_{\nu\nu\mu}(\mu,\mu)}{J''_{\nu\nu}(\mu,\mu)^2} \\
& = \frac{V^{(3)}(\mu) - J^{(3)}_{\nu\nu\nu}(\mu,\mu)\dfrac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}}{J''_{\nu\nu}(\mu,\mu)} = 0. \qquad \text{(S.11)}
\end{aligned}$$

The last equality is implied by equation (S.10). Now suppose for the purpose of contradiction that there exists $\mu_n \to \mu$ such that $h(\rho V(\mu_0)) = \frac{V''(\mu_n)}{J''_{\nu\nu}(\mu_n,\mu_n)}$. This implies that equation (S.11) holds for each μ_n:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\mu}\Big(\frac{\mathrm{d}h(\rho V(\mu))}{\mathrm{d}\mu} J''_{\nu\nu}(\mu,\mu) + h(\rho V(\mu)) J^{(3)}_{\nu\nu\mu}(\mu,\mu)\Big) = 0 \\
\implies\quad & \frac{\mathrm{d}^2 h(\rho V(\mu))}{\mathrm{d}\mu^2} J''_{\nu\nu}(\mu,\mu) + \frac{\mathrm{d}h(\rho V(\mu))}{\mathrm{d}\mu}\big(2J^{(3)}_{\nu\nu\mu}(\mu,\mu) + J^{(3)}_{\nu\nu\nu}(\mu,\mu)\big) \\
& + h(\rho V(\mu))\big(J^{(4)}_{\nu\nu\nu\mu}(\mu,\mu) + J^{(4)}_{\nu\nu\mu\mu}(\mu,\mu)\big) = 0
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & \left(\rho h'(\rho V(\mu))V''(\mu)+\rho^2 h''(\rho V(\mu))V'(\mu)^2\right)J''_{\nu\nu}(\mu,\mu) \\
& - h(\rho V(\mu))\frac{J^{(3)}_{\nu\nu\mu}(\mu,\mu)}{J''_{\nu\nu}(\mu,\mu)^2}\left(2J^{(3)}_{\nu\nu\mu}(\mu,\mu)+J^{(3)}_{\nu\nu\nu}(\mu,\mu)\right) \\
& + h(\rho V(\mu))\left(J^{(4)}_{\nu\nu\nu\mu}(\mu,\mu)+J^{(4)}_{\nu\nu\mu\mu}(\mu,\mu)\right)=0 \\
\Longrightarrow \quad & \rho h'(\rho V(\mu))J''_{\nu\nu}(\mu,\mu)^2 - \frac{h''(\rho V(\mu))h(\rho V(\mu))}{h'(\rho V(\mu))^2}\frac{J^{(3)}_{\nu\nu\mu}(\mu,\mu)^2}{J''_{\nu\nu}(\mu,\mu)^2} \\
& - \frac{J^{(3)}_{\nu\nu\mu}(\mu,\mu)}{J''_{\nu\nu}(\mu,\mu)}\left(2J^{(3)}_{\nu\nu\mu}(\mu,\mu)+J^{(3)}_{\nu\nu\nu}(\mu,\mu)\right) \\
& + \left(J^{(4)}_{\nu\nu\nu\mu}(\mu,\mu)+J^{(4)}_{\nu\nu\mu\mu}(\mu,\mu)\right)=0. \qquad \text{(S.12)}
\end{aligned}$$

By assumption, $\mu \in D$. Since $V(\mu) \in [\inf F, \sup F]$, equation (S.12) cannot hold at μ. So $K=\{\mu \in D | \rho V(\mu)=\max_\sigma \frac{1}{2}\sigma^2 V''(\mu)-C(\kappa(\mu,\sigma))\}$ contains no limiting point. Take any compact subset M of D, $M\cap K$ is finite, and hence of zero measure. $\forall$ sequence of compact subset M_n satisfying $\bigcup M_n = D$,[S.3] K's measure $m(K)\le \sum m(M_n\cap K)=0$. *Q.E.D.*

Proof of Corollary 1

PROOF: When $C(I)=I^\alpha$, $h(x)$ is defined by $h(x)(\frac{h(x)}{\alpha})^{\frac{1}{\alpha-1}}-(\frac{h(x)}{\alpha})^{\frac{\alpha}{\alpha-1}}=x \Longrightarrow h(x)=(\frac{x}{\alpha^{\frac{1}{1-\alpha}}-\alpha^{\frac{\alpha}{1-\alpha}}})^{\frac{\alpha-1}{\alpha}}$. Since $h(x)$ is a power function, $\frac{h''(x)h(x)}{h'(x)^2}$ is a constant (that depends only on α). $\rho h'(\rho x)=\rho^{\frac{\alpha-1}{\alpha}}\cdot x^{-\frac{1}{\alpha}}\cdot\frac{\alpha-1}{\alpha}\cdot(\alpha^{\frac{1}{1-\alpha}}-\alpha^{\frac{\alpha}{1-\alpha}})^{\frac{1-\alpha}{\alpha}}$. Therefore, $J''_{\nu\nu}(\mu,\mu)$ is bounded away from 0 since $J''_{\nu\nu}(\mu,\mu)>0$ and is continuous on compact set $[0,1]$, $x^{-\frac{1}{\alpha}}$ is bounded below by $\sup F^{-\frac{1}{\alpha}}>0$. All other terms in $L(\mu,x)$ are bounded since $J\in C^4[0,1]$. Therefore, for ρ large enough, $L(x,\mu)>0$ and $E=D$. The rest follows from Theorem 4. *Q.E.D.*

Proof of Corollary 2

PROOF: Note that $J''_{\nu\nu}(\mu,\mu)=\frac{\kappa(\mu,\sigma)}{\sigma^2}$ is fixed, and hence is bounded away from 0, independent of the choice of J. When $\varepsilon\to 0$, all other terms except the first positive term in $L(\mu,x)$ converges to 0. Therefore, for ε sufficiently small, $L(\mu,x)>0$ and $E=D$. The rest follows from Theorem 4. *Q.E.D.*

S3.5. *Proof of Theorem 5*

PROOF: Consider the discrete time problem:

$$V_{dt}(\mu)=\sup_{\langle\widehat{\mu}_t\rangle\in\widehat{\mathbb{M}},\widehat{\tau}}\mathbb{E}\left[e^{-\rho\,dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=0}^{\widehat{\tau}-1}e^{-\rho\,dt\cdot t}\lambda\mathbb{E}\left[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\right]\right]. \qquad \text{(S.13)}$$

Equation (S.13) is defined analogously to equation (12), as the discretization of equation (11). Then the value in equation (11) is bounded above by $\overline{\lim}_{dt\to 0}V_{dt}$. Take any fea-

[S.3] Since $J\in C^{(4)}$, D is open; hence, such sequence $\{M_n\}$ exists.

sible strategy $(\langle\widehat{\mu}_t\rangle, \widehat{\tau})$ of equation (S.13):

$$\begin{aligned}&\mathbb{E}\left[e^{-\rho dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=0}^{\widehat{\tau}-1}e^{-\rho dt\cdot t}\lambda\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]\right]\\&\quad=\operatorname{Prob}(\widehat{\tau}=0)E\big[F(\widehat{\mu}_0)-\lambda\big(H(\widehat{\mu}_1)-H(\widehat{\mu}_0)\big)\big]\\&\qquad+\operatorname{Prob}(\widehat{\tau}\geq 1)\mathbb{E}\big[e^{-\rho dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\lambda\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]|\widehat{\tau}\geq 1\big].\end{aligned}\tag{S.14}$$

If equation (S.14) is negative, then stopping at $t=0$ is a strict improvement. Therefore, I assume wlog all subsequent continuation payoffs are nonnegative. As a result,

$$\begin{aligned}&\mathbb{E}\left[e^{-\rho dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=0}^{\widehat{\tau}-1}e^{-\rho dt\cdot t}\lambda\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]\right]\\&\leq\operatorname{Prob}(\widehat{\tau}=0)E\big[F(\widehat{\mu}_0)-\lambda\big(H(\widehat{\mu}_1)-H(\widehat{\mu}_0)\big)\big]\\&\quad+e^{\rho dt}\cdot\operatorname{Prob}(\widehat{\tau}\geq 1)\mathbb{E}\left[e^{-\rho dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=1}^{\widehat{\tau}}\lambda e^{-\rho dt\cdot t}\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]|\widehat{\tau}\geq 1\right]\\&=\operatorname{Prob}(\widehat{\tau}\leq 1)\mathbb{E}\big[F(\widehat{\mu}_{\widehat{\tau}})-\lambda\big(H(\widehat{\mu}_2)-H(\widehat{\mu}_0)\big)\big]\\&\quad+e^{\rho dt}\cdot\operatorname{Prob}(\widehat{\tau}\geq 2)\mathbb{E}\left[e^{-\rho dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=2}^{\widehat{\tau}}\lambda e^{-\rho dt\cdot t}\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]|\widehat{\tau}\geq 2\right]\\&\ \vdots\\&=\lim_{T\to\infty}\Bigg(\operatorname{Prob}(\widehat{\tau}\leq T)\mathbb{E}\big[F(\widehat{\mu}_{\widehat{\tau}})-\lambda\big(H(\widehat{\mu}_T)-H(\widehat{\mu}_0)\big)\big]\\&\quad+e^{T\rho dt}\cdot\operatorname{Prob}(\widehat{\tau}\geq T)\mathbb{E}\left[e^{-\rho dt\cdot\widehat{\tau}}F(\widehat{\mu}_{\widehat{\tau}})-\sum_{t=T}^{\widehat{\tau}}\lambda e^{-\rho dt\cdot t}\mathbb{E}\big[H(\widehat{\mu}_t)-H(\widehat{\mu}_{t+1})|\widehat{\mathcal{F}}_t\big]|\widehat{\tau}\geq T\right]\Bigg)\\&=\mathbb{E}\big[F(\widehat{\mu}_{\widehat{\tau}})-\lambda\big(H(\widehat{\mu}_{\widehat{\tau}})-H(\widehat{\mu}_0)\big)\big]\\&\leq\sup_{\mathbb{E}_\pi[\nu]=\widehat{\mu_0}}\mathbb{E}_\pi\big[F(\nu)-\lambda\big(H(\mu)-H(\nu)\big)\big].\end{aligned}$$

Clearly, the uniform upper bound $\sup_{\mathbb{E}_\pi[\nu]=\widehat{\mu_0}}\mathbb{E}_\pi[F(\nu)-\lambda(H(\mu)-H(\nu))]$ does not depend on dt. Thus,

$$V(\mu)\leq\sup_{\pi\in\Delta^2(X)}E_\pi\big[F(\nu)-\lambda\big(H(\mu)-H(\nu)\big)\big].\tag{S.15}$$

On the other hand, take any $\pi\in\Delta^2(X)$ and $p>0$. Let J_t be a Poisson counting process with parameter p. Define a compound Poisson process: $\mu_t=\mu$ if $J_t=0$; $\mu_t\sim\pi$ if $J_t=1$. Let $\tau=\{t|J_t=1\}$. Then the expected payoff from strategy $(\langle\mu_t\rangle,\tau)$ can be solved by the following HJB:

$$\rho\widehat{V}(\mu) = p\big(\mathbb{E}_\pi\big[F(\nu)\big] - V(\mu)\big) - \lambda\big(p\big(\mathbb{E}_\pi\big[H(\mu) - H(\nu)\big]\big)\big)$$
$$\implies \quad \widehat{V}(\mu) = \frac{p}{\rho + p}\big(E_\pi\big[F(\nu) - \lambda\big(H(\mu) - H(\nu)\big)\big]\big).$$

Since $V(\mu) \geq \widehat{V}(\mu)$ regardless of π and p,

$$V(\mu) \geq \sup_{\pi\in\Delta^2(X)} E_\pi\big[F(\nu) - \lambda\big(H(\mu) - H(\nu)\big)\big]. \tag{S.16}$$

Combine equations (S.15) and (S.16), equation (11) is proved. *Q.E.D.*

S3.6. *Proof of Theorem 6*

PROOF: $\forall \mu \in E$, it is WLOG to assume $\text{Supp}(\mu) = X$ since states with zero prior probability are irrelevant. Let $(p(\mu), \nu(\mu), \sigma(\mu))$ denote the optimal policy solving equation (3).

Step 1. Derive optimality condition. Suppose $\sigma(\mu) = 0$ and let $I(\mu) = -p(\mu) \times (H(\nu(\mu)) - H(\mu) - \nabla H(\mu)(\nu(\mu) - \mu))$. Equation (3) implies

$$\rho V(\mu) = I(\mu)\frac{V\big(\nu(\mu)\big) - V(\mu) - \nabla V(\mu)\big(\nu(\mu) - \mu\big)}{H(\mu) - H\big(\nu(\mu)\big) + \nabla H(\mu)\big(\nu(\mu) - \mu\big)} - C\big(I(\mu)\big)$$
$$= \sup_{I,\nu} I\frac{V(\nu) - V(\mu) - \nabla V(\mu)(\nu - \mu)}{H(\mu) - H(\nu) + \nabla H(\mu)(\nu - \mu)} - C(I). \tag{S.17}$$

By assumption, $C(I)$ is strictly convex. $I(\mu)$ is the unique solution to $C'(I(\mu)) = \frac{V(\nu(\mu))-V(\mu)-\nabla V(\mu)(\nu(\mu)-\mu)}{H(\mu)-H(\nu(\mu))+\nabla H(\mu)(\nu(\mu)-\mu)}$; hence, $\rho V(\mu) = I(\mu)C'(I(\mu)) - C(I(\mu))$. Note that the function $f(x) = \frac{1}{\rho}(xC'(x) - C(x))$ is a C^1 and strictly increasing mapping from $\mathbb{R}^+$ to $\mathbb{R}^+$. Thus, it has a C^1 and strictly increasing inverse. Let $h(x) = f^{-1}(C'^{-1}(x))$. Then $h(x)$ is a C^1 and strictly increasing function and

$$h\big(V(\mu)\big) = \frac{V\big(\nu(\mu)\big) - V(\mu) - \nabla V(\mu)\big(\nu(\mu) - \mu\big)}{H(\mu) - H\big(\nu(\mu)\big) + \nabla H(\mu)\big(\nu(\mu) - \mu\big)}$$
$$= \sup_{\nu} \frac{V(\nu) - V(\mu) - \nabla V(\mu)(\nu - \mu)}{H(\mu) - H(\nu) + \nabla H(\mu)(\nu - \mu)}. \tag{S.18}$$

The optimality condition implies

$$\begin{cases} G(\nu') - G(\mu) - \nabla G(\mu)(\nu' - \mu) \leq 0, \\ G\big(\nu(\mu)\big) - G(\mu) - \nabla G(\mu)\big(\nu(\mu) - \mu\big) = 0, \end{cases} \tag{S.19}$$

where $G(\nu) = V(\nu) + h(V(\mu))H(\nu)$. Equations (S.18) and (S.19) completely characterize the solution of the HJB.

Suppose $\sigma(\mu) \neq 0$, it suggests that using Gaussian signal is optimal; hence,

$$\rho V(\mu) = \sup_{\sigma} \frac{1}{2}\sigma^T \operatorname{Hess} V(\mu)\sigma - C\Big(-\frac{1}{2}\sigma^T \operatorname{Hess} H(\mu)\sigma\Big).$$

Same analysis as in the previous case (omitted) implies the optimality condition:

$$h(V(\mu)) = -\frac{\sigma(\mu)^T \operatorname{Hess} V(\mu)\sigma(\mu)}{\sigma(\mu)^T \operatorname{Hess} H(\mu)\sigma(\mu)} = \sup_\sigma -\frac{\sigma^T \operatorname{Hess} V(\mu)\sigma}{\sigma^T \operatorname{Hess} H(\mu)\sigma}. \tag{S.20}$$

Step 2. Prove $V(\nu(\mu)) > V(\mu)$ when $\sigma(\mu) = 0$. Suppose for the purpose of contradiction that $V(\nu(\mu)) \le V(\mu)$. Consider $V(\mu_\alpha) = V(\alpha\nu(\mu) + (1-\alpha)\mu)$ for $\alpha \in [0,1]$. Since $\Delta(X)$ is convex, $\mu_\alpha \in \Delta(X)$. By equation (S.19), $G(\mu_\alpha) \le G(\mu) + \nabla G(\mu)(\mu_\alpha - \mu)$. For α sufficiently small, $\mu_\alpha \in E$. Now define $G_1 = V + \lambda_1 H$ for $\lambda_1 < h(V(\mu))$. Then, since H is strictly concave, G_1 is strictly more convex than G; hence,

$$\begin{aligned}
&\begin{cases} G_1(\mu_\alpha) - G_1(\mu) - \nabla G_1(\mu)(\mu_\alpha - \mu) < 0; \\ G_1(\mu_\alpha) - G_1(\nu(\mu)) - \nabla G_1(\nu(\mu))(\mu_\alpha - \nu(\mu)) < 0 \end{cases} \\
&\quad \Longrightarrow \quad G_1(\mu_\alpha) + \nabla G_1(\mu_\alpha)(\mu - \mu_\alpha) < G_1(\mu) \\
&\text{or} \quad G_1(\mu_\alpha) + \nabla G_1(\mu_\alpha)(\nu(\mu) - \mu_\alpha) < G_1(\nu(\mu)) \\
&\qquad \Longrightarrow \quad \lambda_1 < \sup_\nu \frac{V(\nu) - V(\mu_\alpha) - \nabla V(\mu_\alpha)(\nu - \mu_\alpha)}{H(\mu_\alpha) - H(\nu) + \nabla H(\mu_\alpha)(\nu - \mu_\alpha)}.
\end{aligned}$$

The second set of inequalities are from the fact that $(\nabla G_1(\mu) - \nabla G_1(\mu_\alpha))(\mu_\alpha - \mu)$ and $(\nabla G_1(\mu) - \nabla G_1(\mu_\alpha))(\nu(\mu) - \mu_\alpha)$ are of opposite signs. This suggests that $\lambda_1 < h(V(\mu_\alpha))$. Since $\lambda_1 < h(V(\mu))$ can be chosen arbitrarily, $V(\mu_\alpha) \ge V(\mu)$, and this implies

$$\begin{aligned}
&\left.\frac{\mathrm{d}V(\mu_\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} \ge 0 \\
&\quad \Longleftrightarrow \quad \nabla V(\mu)(\nu(\mu) - \mu) \ge 0 \\
&\quad \Longrightarrow \quad V(\nu(\mu)) - V(\mu) - \nabla V(\mu)(\nu(\mu) - \mu) \le 0.
\end{aligned}$$

This implies $\rho V(\mu) \le 0$, contradiction. Therefore, we conclude that $V(\nu(\mu)) > V(\mu)$.

Step 3. Prove $V(\nu(\mu)) = F(\nu(\mu))$ when $\sigma(\mu) = 0$. Suppose $V(\nu(\mu)) > F(\nu(\mu))$. Define $G = V + h(V(\mu))H$ and $G_1 = V + h(V(\nu(\mu)))H$. Then $\forall \nu \in \Delta(X)$:

$$\begin{aligned}
&G(\nu) \le G(\nu(\mu)) + \nabla G(\nu(\mu))(\nu - \nu(\mu)) \\
&\quad \Longrightarrow \quad G_1(\nu) = G(\nu) + \big(h(V(\nu(\mu))) - h(V(\mu))\big)H(\nu) \\
&\qquad\qquad \le G(\nu(\mu)) + \nabla G(\nu(\mu))(\nu - \nu(\mu)) + \big(h(V(\nu(\mu))) - h(V(\mu))\big)H(\nu) \\
&\qquad\qquad < G(\nu(\mu)) + \nabla G(\nu(\mu))(\nu - \nu(\mu)) \\
&\qquad\qquad\quad + \big(h(V(\nu(\mu))) - h(V(\mu))\big)\big(H(\nu(\mu)) + \nabla H(\nu(\mu))(\nu - \nu(\mu))\big) \\
&\qquad\qquad = G_1(\nu(\mu)) + \nabla G_1(\nu(\mu))(\nu - \nu(\mu)).
\end{aligned}$$

The second inequality is by replacing $G(\nu)$ using the first inequality. The third inequality is from the strict concavity of H. The strict inequality $G_1(\nu) < G_1(\nu(\mu)) + \nabla G_1(\nu(\mu))(\nu - \nu(\mu))$ $\forall \nu$ implies that equation (S.19) has no solution. On the other hand, $G(\nu) \le G(\nu(\mu)) + \nabla G(\nu(\mu))(\nu - \nu(\mu))$ $\forall \nu$ implies that $\operatorname{Hess} G(\nu(\mu))$ is negative semidefinite, and hence $\operatorname{Hess} G_1(\nu(\mu))$ is negative definite $\Longrightarrow$ $h(V(\nu(\mu))) > \sup_\sigma -\frac{\sigma^T \operatorname{Hess} V(\nu(\mu))\sigma}{\sigma^T \operatorname{Hess} H(\nu(\mu))\sigma}$.

Hence, equation (S.20) has no solution either. This suggests that $V(\nu(\mu))$ could not possibly satisfy equation (3), contradiction. Therefore, $V(\nu(\mu)) = F(\nu(\mu))$.

Step 4. Let $K = \{\mu \in E | h(V(\mu)) = \sup_\sigma -\frac{\sigma^T \operatorname{Hess} V(\mu)\sigma}{\sigma^T \operatorname{Hess} H(\mu)\sigma}\}$. Prove that K is nowhere dense. Suppose for the purpose of contradiction that $\exists$ nondegenerate close ball O in the interior of K. Let μ^* be the center of O. Since V is quasiconvex, the sublevel set $\{\mu | V(\mu) \le V(\mu^*)\}$ is convex. By the supporting hyperplane theorem, there exists a linear function $A\mu + b$ such that

$$\begin{cases} A\mu^* + b = 0; \\ A\mu + b < 0 \quad \Longrightarrow \quad V(\mu) > V(\mu^*). \end{cases}$$

Define set $Q = \{\mu | \mu \in O \& A\mu + b \le 0\}$, $\underline{Q} = \{\mu | \mu \in O \& A\mu + b = 0\}$, and $\partial\bar{Q} = \{\mu | \partial Q \setminus \underline{Q}\}$. Now we are ready to draw a contradiction. Let $L(\mu) = V(\mu) + h(V(\mu^*))H(\mu)$, then $L(\mu)$ satisfies (i) $L(\mu^*) = 0$, (ii) $L(\mu) \le 0, \forall \mu \in \Delta(X)$, and (iii) $\nabla L(\mu^*) = \mathbf{0}$. This implies that $\forall \mu \in Q \setminus \mu^*$, $L(\mu) < 0$.[S.4] Since L is continuous, $\delta \triangleq \sup_{\mu \in \partial\bar{Q}} L(\mu) < 0$. There exists $\varepsilon > 0$ sufficiently small that: $\varepsilon(A\mu + b) \ge \delta$ when $\mu \in Q$. Therefore,

$$\varepsilon(A\mu + b) \ge L(\mu) \quad \forall \mu \in \underline{Q} \cup \partial\bar{Q}. \tag{S.21}$$

Now consider $\mu_\alpha = \mu^* - \alpha A$. $A\mu_\alpha + b = -\alpha|A| < 0$ so μ_α is in the interior of Q for α sufficiently small:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\big(L(\mu_\alpha) - \varepsilon(A\mu_\alpha + b)\big)\bigg|_{\alpha=0} = \nabla L(\mu^*) \cdot A + \varepsilon|A| > 0.$$

Therefore, for α sufficiently small, $L(\mu_\alpha) > \varepsilon(A\mu_\alpha + b)$. By continuity of L, there exists $b' > b$ such that $L(\mu) \le \varepsilon(A\mu + b')$, and equality holds at some μ'. Since $b' > b$, equation (S.21) implies that μ' is in the interior of Q. Therefore, $\operatorname{Hess} L(\mu')$ is negative semidefinite. Since by construction $V(\mu') > V(\mu^*)$, $\operatorname{Hess} V(\mu') + h(V(\mu')) \operatorname{Hess} H(\mu')$ is negative definite. This contradicts the fact that $\mu' \in K$. Therefore, property (i) is proved.

Step 5. Prove the policy function $(\nu(\mu), I(\mu))$ exists and $\nu(\mu) \in E^C$ (property (v)). $\forall \mu \in E \setminus K$, the proof is done according to step 3. Now consider $\mu \in K$. Since K is nowhere dense, there exists $\mu_n \in E \setminus K$ such that $\mu_n \to \mu$. By step 3, $\nu(\mu_n) \in E^C$ so $\nu(\mu_n)$ are bounded away from μ. Since E^C is bounded and closed, there exists converging subsequence of $\nu(\mu_n)$. Without loss of generality, we assume $\nu(\mu_n) \to \nu \in E^C$. Since V is continuous, $V, C \in C^2(E)$ and $h \in C^1(\mathbb{R}^+)$:

$$\begin{aligned} h\big(V(\mu)\big) &= \lim_{n\to\infty} h\big(V(\mu_n)\big) = \lim_{n\to\infty} \frac{V\big(\nu(\mu_n) - V(\mu_n) - \nabla V(\mu_n)\big(\nu(\mu_n) - \mu_n\big)\big)}{H(\mu_n) - H\big(\nu(\mu_n)\big) + \nabla H(\mu_n)\big(\nu(\mu_n) - \mu_n\big)} \\ &= \frac{V(\nu) - V(\mu) - \nabla V(\mu)(\nu - \mu)}{H(\mu) - H(\nu) + \nabla H(\mu)(\nu - \mu)}. \end{aligned}$$

Let $\nu(\mu) = \nu$ and $I(\mu) = f^{-1}(V(\mu))$, then $(\nu(\mu), I(\mu))$ solves equation (3). Since f^{-1} is C^1, $I(\mu) \in C^1(E)$. Moreover, since f^{-1} is an increasing function, $I(\mu)$ increases in $V(\mu)$ (property (iv)).

[S.4] If $L(\mu) = 0$, then it is a local maximizer of L; hence, $\operatorname{Hess} L(\mu)$ is negative semidefnitie. If $V(\mu) > V(\mu^*)$, then $\operatorname{Hess} V(\mu) + h(V(\mu)) \operatorname{Hess} H(\mu)$ is negative definite. If $V(\mu) = V(\mu^*)$, then $\mu \in \underline{Q}$ and by quasiconvexity of V, $V(\alpha\mu + (1-\alpha)\mu^*)$ is constant for $\alpha \in [0, 1]$. This implies $L(\alpha\mu + (1-\alpha)\mu^*)$ being strictly concave in α, and hence is strictly negative at μ. Both lead to contradiction.

Step 6. Prove properties (ii) and (iii). On $E \setminus K$, by the envelope theorem,

$$D_{\nu(\mu)-\mu} h\big(V(\mu)\big) = \big(\nu(\mu)-\mu\big)^T \cdot \frac{\partial}{\partial\mu} \frac{F\big(\nu(\mu)\big)-V(\mu)-\nabla V(\mu)\big(\nu(\mu)-\mu\big)}{H(\mu)-H\big(\nu(\mu)\big)+\nabla H(\mu)\big(\nu(\mu)-\mu\big)}$$

$$\Longrightarrow \quad D_{\nu(\mu)-\mu} V(\mu) = \frac{1}{h'\big(V(\mu)\big)} \cdot \big(\nu(\mu)-\mu\big)^T \times \left(-\frac{\operatorname{Hess} V(\mu) + h\big(V(\mu)\big) \operatorname{Hess} H(\mu)}{H(\mu)-H\big(\nu(\mu)\big)+\nabla H(\mu)\big(\nu(\mu)-\mu\big)}\big(\nu(\mu)-\mu\big)\right) > 0.$$

This proves property (ii). On $E \setminus K$, equation (S.19) implies

$$\begin{aligned}
&\nabla F\big(\nu(\mu)\big) - \nabla V(\mu) + h\big(V(\mu)\big)\big(\nabla H\big(\nu(\mu)\big) - \nabla H(\mu)\big) = 0 \\
&\quad\Longrightarrow\quad -\operatorname{Hess} V(\mu)\cdot\alpha + h'\big(V(\mu)\big)\big(\nabla V(\mu)\cdot\alpha\big)\big(\nabla H\big(\nu(\mu)\big) - \nabla H(\mu)\big) \\
&\qquad\qquad + h(V(\mu)\big(\operatorname{Hess} H\big(\nu(\mu)\big)\cdot D_\alpha \nu(\mu) - \operatorname{Hess} H(\mu)\cdot\alpha\big) = 0 \\
&\quad\Longrightarrow\quad \operatorname{Hess} H\big(\nu(\mu)\big)\cdot D_\alpha\nu(\mu) \\
&\qquad\qquad = \operatorname{Hess} H(\mu)\cdot\alpha + \frac{\operatorname{Hess} V(\mu)\cdot\alpha}{h\big(V(\mu)\big)} \\
&\qquad\qquad\quad + \frac{h'\big(V(\mu)\big)}{h\big(V(\mu)\big)}\nabla V(\mu)\cdot\alpha\big(\nabla H(\mu) - \nabla H\big(\nu(\mu)\big)\big).
\end{aligned}$$

Let $\alpha = \nu(\mu) - \mu$ and replace $D_{\nu(\mu)-\mu}V(\mu)$:

$$\begin{aligned}
&\big(\nu(\mu)-\mu\big)^T \cdot \operatorname{Hess} H\big(\nu(\mu)\big)\cdot D_{\nu(\mu)-\mu}\nu(\mu) \\
&\quad = -\frac{h'\big(V(\mu)\big)}{h\big(V(\mu)\big)}\big(H(\mu) - H\big(\nu(\mu)\big) + \nabla H(\mu)\big(\nu(\mu)-\mu\big)\big) D_{\nu(\mu)-\mu}V(\mu) \\
&\qquad + \frac{h'\big(V(\mu)\big)}{h\big(V(\mu)\big)} D_{\nu(\mu)-\mu}V(\mu)\big(\nabla H(\mu) - \nabla H\big(\nu(\mu)\big)\big)\cdot\big(\nu(\mu)-\mu\big) \\
&\quad = \frac{h'\big(V(\mu)\big)}{h\big(V(\mu)\big)} D_{\nu(\mu)-\mu}\big(H\big(\nu(\mu)\big) - H(\mu) + \nabla H\big(\nu(\mu)\big)\big(\mu - \nu(\mu)\big)\big) > 0.
\end{aligned}$$

This proves property (iii). *Q.E.D.*

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