Econometrica Supplementary Material

SUPPLEMENT TO "ROBUST SCREENS FOR NONCOMPETITIVE BIDDING IN PROCUREMENT AUCTIONS"

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This Online Appendix to "Robust Screens for Noncompetitive Bidding in Procurement Auctions" provides extensions, robustness checks, and proofs. We show how to extend our results to allow for multistage bidding in Section OA. Section OB presents theoretical extensions. Section OC presents further empirical results and robustness checks. Section OD collects proofs for Lemmas 1, 2, and 3.

ONLINE APPENDIX OA: MULTISTAGE BIDDING

NATIONAL LEVEL AUCTIONS in our data follow a first-price auction format with a secret reserve price. This means that the auction is a multistage game, with stages $k \in \{1, \dots, \overline{k}\}$. The auctioneer picks a secret reserve price r. At each stage k, bidders submit bids $b_{i,k}$. A winner is declared if and only if $\min_i b_{i,k} \leq r$. In this case, the winner is paid her bid. If instead $\min_i b_{i,k} > r$ the game continues to an additional stage. At the end of each stage without a winner, the lowest bid is revealed. The reserve price is constant across stages. In this Appendix, we extend the revealed preference inequalities of Section 6 to multistage first-price auctions.

In a multistage auction, a bidder's continuation strategy after her first bid is a contingent plan dependent on the information revealed at each stage. We denote by $b_{i,1}$ bidder i's first bid, and by β_i her continuation play, mapping future information to bids.

Given an equilibrium strategy $\sigma_i = (b_{i,1}, \beta_i)$ by player i we consider first-stage-only deviations $\sigma_i' = (b_{i,1}', \beta_i)$ such that player i's initial bid is different, but her continuation contingent plan, as a function of her own private signals, and the play of others, is unchanged.

Let $\mathsf{win}_{i,k}$ denote the event that bidder i wins in round k. Expected profits under σ_i and σ_i' take the form

$$\mathbb{E}_{\sigma_i}[\pi_i] = (b_{i,1} - c_i)\mathsf{prob}_{\sigma_i,\sigma_{-i}}(\mathsf{win}_{i,1}) + \mathbb{E}_{\sigma_i,\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}\right],$$

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$$\mathbb{E}_{\sigma_i'}[\pi_i] = \big(b_{i,1}' - c_i\big)\mathsf{prob}_{\sigma_i',\sigma_{-i}}(\mathsf{win}_{i,1}) + \mathbb{E}_{\sigma_i',\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}\right].$$

We now introduce a classification of histories following upward and downward deviations in the first round as a function of how they affect the continuation play. We say that a deviation is marginal for continuation, if it changes whether the auction continues after period 1. When a deviation is marginal for information, it changes the information available to participants in future periods. If a deviation is nonmarginal, it does not affect continuation play. This corresponds to the following formal definition.

DEFINITION OA.1: Consider an upward deviation $b_{i,1}' > b_{i,1}$. It is *marginal for continuation* (MC) if and only if $b_{i,1} \leq r < \wedge \mathbf{b}_{-i,1}$, and $b_{i,1}' > r$. It is *marginal for information* (MI) if and only if $r < b_{i,1} < \wedge \mathbf{b}_{-i,1}$. It is *nonmarginal* (NM) otherwise.

Consider a downward deviation $b_{i,1}' < b_{i,1}$. It is *marginal for continuation* (MC) if and only if $b_{i,1}' \leq r < \wedge \mathbf{b}_{-i,1}$, and $b_{i,1} > r$. It is *marginal for information* (MI) if and only if $r < b_{i,1}' < \wedge \mathbf{b}_{-i,1}$. It is *nonmarginal* (NM) otherwise.

Note that we can assess the marginality of deviations using data, since it only relies on observed period 1 bids. Note also that bidder i's belief that a given deviation is marginal for continuation or information only depends on the bidder i's beliefs about bids $b_{-i,1}$.

For bids $b_{i,1}$ and $b_{i,1}'$, we have that

$$\begin{aligned}
\mathbb{E}_{\sigma_i}[\pi_i] = {} & (b_{i,1} - c_i)\mathsf{prob}_{\sigma_i,\sigma_{-i}}(\mathsf{win}_{i,1}) + \mathbb{E}_{\sigma_i,\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}} \middle| \mathsf{MC}\right]\mathsf{prob}_{\sigma_{-i}}(\mathsf{MC}) \\
& + \mathbb{E}_{\sigma_i,\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}} \middle| \mathsf{MI}\right]\mathsf{prob}_{\sigma_{-i}}(\mathsf{MI}) \\
& + \mathbb{E}_{\sigma_i,\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}} \middle| \mathsf{NM}\right]\mathsf{prob}_{\sigma_{-i}}(\mathsf{NM}), \\
\mathbb{E}_{\sigma_i'}[\pi_i] = {} & \big(b_{i,1}' - c_i\big)\mathsf{prob}_{\sigma_i',\sigma_{-i}}(\mathsf{win}_{i,1}) + \mathbb{E}_{\sigma_i',\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}} \middle| \mathsf{MC}\right]\mathsf{prob}_{\sigma_{-i}}(\mathsf{MC}) \\
& + \mathbb{E}_{\sigma_i',\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}} \middle| \mathsf{MI}\right]\mathsf{prob}_{\sigma_{-i}}(\mathsf{MI}) \\
& + \mathbb{E}_{\sigma_i',\sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}} \middle| \mathsf{NM}\right]\mathsf{prob}_{\sigma_{-i}}(\mathsf{NM}).
\end{aligned}$$

Equilibrium implies that under player i's beliefs $\mathbb{E}_{\sigma_i}[\pi_i] \geq \mathbb{E}_{\sigma_i'}[\pi_i]$. We now establish implications of this equilibrium condition that can be taken to the data.

For all deviations, the following hold:

- Bids must decrease with the stage of the game: $b_{i,k} > b_{i,k+1}$; indeed, since the reserve price is constant, any bid submitted in period k wins with probability 0 in period $k+1$ if the auction continues.
- Continuation payoffs under σ_i and σ_i' are equal conditional on the deviation being nonmarginal.

If the deviation is *an upward deviation* then,

- Player i's continuation value under σ_i is equal to zero when the deviation is marginal for continuation.
- If continuation strategies β_i, β_{-i} are monotonic in observed bids, then

$$\mathbb{E}_{\sigma_i', \sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}|\mathsf{MI}\right] \geq \mathbb{E}_{\sigma_i, \sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}|\mathsf{MI}\right].$$

It follows from this that $\mathbb{E}_{\sigma_i}[\pi_i] \geq \mathbb{E}_{\sigma_i'}[\pi_i]$ implies

$$(b_{i,1} - c_i)\mathsf{prob}_{\sigma_i,\sigma_{-i}}(\mathsf{win}_{i,1}) \geq \left(b'_{i,1} - c_i\right)\mathsf{prob}_{\sigma_i',\sigma_{-i}}(\mathsf{win}_{i,1}). \tag{O1}$$

This coincides with the IC constraint for upward deviations used in Sections 5 and 6.

If the deviation is *a downward deviation*, then player i's continuation value under σ_i' is equal to zero when the deviation is marginal for continuation. Furthermore, we assume that for some $\alpha \in (0, 1)$,

$$\mathbb{E}_{\sigma_i', \sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}|\mathsf{MI}\right] \geq (1-\alpha)\mathbb{E}_{\sigma_i, \sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}|\mathsf{MI}\right]. \tag{O2}$$

In words, following a downward deviation that is marginal for information (meaning that the bid is in fact above the reserve price, which it would have to beat to win at a later stage) the change in the information provided in the continuation stage does not destroy all the continuation value of the bidder. Note that if at the end of each stage the auctioneer revealed an exogenous signal of the reserve price, rather than the endogenous minimum bid, then condition (O2) would hold with $\alpha = 0$. In our empirical investigation, we use $\alpha = 0.5$.

Finally, we observe that the following bounds hold:

$$\mathbb{E}_{\sigma_i, \sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}|\mathsf{MI}\right] \leq \mathbb{E}\left[(r - c_i)^+\right],$$

$$\mathbb{E}_{\sigma_i, \sigma_{-i}}\left[\sum_{k>1}(b_{i,k} - c_i)\mathbf{1}_{\mathsf{win}_{i,k}}|\mathsf{MC}\right] \leq \mathbb{E}\left[(r - c_i)^+\right].$$

Altogether, with optimality condition $\mathbb{E}_{\sigma_i}[\pi_i] \geq \mathbb{E}_{\sigma_i'}[\pi_i]$ this implies that

$$\begin{aligned}(b_{i,1} - c_i)\mathsf{prob}_{\sigma_i,\sigma_{-i}}(\mathsf{win}_{i,1}) \geq{}& \left(b'_{i,1} - c_i\right)\mathsf{prob}_{\sigma_i',\sigma_{-i}}(\mathsf{win}_{i,1}) \\ &- \left[\mathsf{prob}_{\sigma_{-i}}(\mathsf{MC}) + \alpha\mathsf{prob}_{\sigma_{-i}}(\mathsf{MI})\right]\mathbb{E}\left[(r - c_i)^+\right]. \end{aligned} \tag{O3}$$

Equations (O1) and (O3) replace (IC) in the inference problem defined in Section 6. In addition to disciplining residual demand under μ, expanded consistency requirement $(\widehat{CR})$ must ensure that the probability of events MI and MC under the true historical average distribution of beliefs μ^* must also be close to their sample probability. Denote by mi_h and mc_h the probability that downward deviation ρ is marginal for information or continuation at h. An extension of Proposition 1 implies that for any $\rho < 0$, with large probability as $|H|$ becomes large,

$$\mathbb{E}_{\mu^*}[mi] \equiv \frac{1}{|H|}\sum_{h \in H} mi_h \in \left[\frac{1}{|H|}\sum_{h \in H}\mathbf{1}_{r<(1+\rho)b_{i,1}<\wedge \mathbf{b}_{-i,1}} \pm K\right],$$

$$\mathbb{E}_{\mu^*}[mc] \equiv \frac{1}{|H|}\sum_{h\in H} mc_h \in \left[\frac{1}{|H|}\sum_{h\in H} \mathbf{1}_{(1+\rho)b_{i,1}\le r<\wedge \mathbf{b}_{-i,1}} \pm K\right],$$

where K is an arbitrary fixed tolerance parameter. This implies that we can expand coverage sets $\mathcal{D}_\alpha$ introduced in Section 6.3 to cover not only the true expected vector of demands $\mathbb{E}_{\mu^*}[\mathbf{d}]$, but also the true expected probability that deviations are marginal: $\mathbb{E}_{\mu^*}[mi]$ and $\mathbb{E}_{\mu^*}[mc]$.

ONLINE APPENDIX OB: FURTHER THEORETICAL RESULTS

OB.1. *Connection With Bayes Correlated Equilibrium*

In this section, we further extend the estimator introduced in Section 6 and clarify what would need to be added so that asymptotically, it exploits all implications from equilibrium. This allows us to connect with the work of Bergemann and Morris (2016).

For simplicity, we assume that player identities i, bids b and costs c take a fixed finite number of values $(i, b, c) \in I \times B \times C$ that does not grow with sample size $|H|$. Ties between bids are resolved with uniform probability. Deviations $\rho_n \in (-1, \infty)$ correspond to the ratios of different bids on finite grid B.

In Section 6, we were able to express inference problem (P) as a function of beliefs alone. Because both costs and bids are part of bidders' information set (so that selecting histories on the basis of both beliefs and costs is adapted), to exploit all the information content of equilibrium, we must impose constraints on demand conditional on both bids and costs. For this reason, instead of expressing our inference problem using the distribution of beliefs alone, we express this new inference problem using the historical profile of beliefs, costs, and bids.[1]

For any history $h \in H$, let $\omega_h = ((d_{h,n})_{n\in\mathcal{M}}, c_h)$ be the demand and cost of the firm associated with history h. Let $\omega_H = (\omega_h)_{h\in H}$ denote the profile of demands and costs across all histories in H. Let $\Omega \equiv \{\omega_H : \forall h \in H, (d_{h,n})_{n\in\mathcal{M}} \in \mathcal{F}\}$ be the set of environments ω_H with feasible demands.

For each profile $\omega_H \in \Omega$, define

$$H_{\mathsf{comp}}(\omega_H) \equiv \{h \in H \text{ s.t. } (d_h, c_h) \text{ satisfy (IC) and (MKP)}\},$$

to be the set of histories in H that satisfy markup constraint (MKP) and are rationalizable as competitive under ω_H.

For each set of adapted histories H, each deviation n, and each profile $\omega_H = (\omega_h)_{h\in H}$, let

$$D_n(\omega_H, H) \equiv \frac{1}{|H|}\sum_{h_{i,t}\in H} d_{h_{i,t},n}$$

be the average residual demand when firms' demands and costs are given by ω_H.

We extend problem (P) as follows. For any environment ω_H and $(i, b, c) \in I \times B \times C$, let us define $H_{i,b,c}(\omega_H) \equiv \{h \in H | (i_h, b_h, c_h) = (i, b, c)\}$ to be the histories at which bidder i experiences a cost c and bids b. Note that $H_{i,b,c}$ is adapted to the information of player i. For any tolerance function $K : \mathbb{N} \to \mathbb{R}^+$ such that

$$\lim_{k\to\infty} K(k) = 0 \quad \text{and} \quad \lim_{k\to\infty} \exp(-K(k)^2 k/2N_{\max}) = 0$$

[1] As in Section 6, we could directly consider the joint distribution of beliefs, costs, and bids.

we consider inference problem (P′),

$$\widehat{s}_{\mathsf{comp}} = \max_{\omega_H} \frac{|H_{\mathsf{comp}}(\omega_H)|}{|H|} \tag{P′}$$
$$\text{s.t.} \quad \forall (i,b,c), \forall n, \quad D_n(\omega_H, H_{i,b,c}(\omega_H)) \in \big[\widehat{D}(\rho_n | H_{i,b,c}(\omega_H)) - K(|H_{i,b,c}(\omega_H)|),$$
$$\widehat{D}(\rho_n | H_{i,b,c}(\omega_H)) + K(|H_{i,b,c}(\omega_H)|)\big].$$

Problem (P′) differs from (P) by imposing demand consistency requirements conditional on all triples (i, b, c). Proposition 3 continues to hold with an identical proof: with probability approaching 1 as $|H|$ goes to ∞, $\widehat{s}_{\mathsf{comp}}$ is an upper bound to the share of competitive histories. Imposing consistency requirements conditional on bids and costs lets us establish a converse: if data passes our extended tests, then the joint distribution of bids and costs is an ϵ-Bayes correlated equilibrium in the sense of Hart and Mas-Colell (2000).

Consider an ω_H solving (P′). Let $\widehat{\mu} \in \Delta([B \times C]^I)$ denote the sample distribution over bids and costs implied by (H, ω_H).

PROPOSITION OB.1: *For any $\epsilon > 0$, for $|H|$ large enough, $\widehat{s}_{\mathsf{comp}} = 1$ implies that $\widehat{\mu}$ is an ϵ-Bayes correlated equilibrium of the stage-game first-price auction.*

PROOF: Consider demand and costs $(d_{h,n}, c_h)_{h \in H}$ solving Problem (P′), and $\widehat{\mu}$ the corresponding sample distribution over profiles of bids b and costs c.

In order to deal with ties, we denote by $\wedge \mathbf{b}_{-i} \succ b_i$ the event "$\wedge \mathbf{b}_{-i} > b_i$, or $\wedge \mathbf{b}_{-i} = b_i$ and the tie is broken in favor of bidder i."

For $|H|$ large enough, we have that for all (i, b, c) and all n,

$$\frac{1}{|H|} \left| \sum_{h \in H_{i,b,c}} d_{n,h} - \mathsf{prob}_{\widehat{\mu}}\big(\wedge \mathbf{b}_{-i} \succ (1+\rho_n) b_i | i, b, c\big) \right| \le \epsilon. \tag{O4}$$

In addition, $\widehat{s} = 1$ implies that (IC) holds at all histories: for all h, n,

$$d_{h,n}\big((1+\rho_n)b_h - c_h\big) \le d_{h,0}(b_h - c_h).$$

Summing over histories $h \in H_{i,b,c}$ yields

$$\frac{1}{|H|} \sum_{h \in H_{i,b,c}} d_{h,n}\big((1+\rho_n)b_h - c_h\big) - d_{h,0}(b_h - c_h) \le 0.$$

Hence for $|H|$ large enough, for all (b_i, c_i),

$$\sum_{b_{-i}, c_{-i}} \widehat{\mu}(b_i, c_i, b_{-i}, c_{-i}) \big(\mathbf{1}_{\wedge \mathbf{b}_{-i} \succ (1+\rho_n) b_i}\big((1+\rho_n)b_i - c_i\big) - \mathbf{1}_{\wedge \mathbf{b}_{-i} \succ b_i}(b_i - c_i)\big) \le \epsilon.$$

It follows that $\widehat{\mu}$ is an ϵ-Bayes correlated equilibrium in the sense of Hart and Mas-Colell (2000). *Q.E.D.*

OB.2. *Bounding the Share Competitive Histories in Simple Cases*

This section provides an explicit characterization of bound $\widehat{s}_{\mathsf{comp}}$ when we consider either a single upward deviation or a single downward deviation.

OB.2.1. *Inference From an Upward Deviation*

Consider first the case of a single upward deviation $\rho_1 > 0$. Let $\Lambda = \{(1,0),(0,-1)\}$ and $x_\lambda = x > 0$ for all $\lambda \in \Lambda$. Set $\mathcal{D}_\alpha$ then takes the form:

$$\mathcal{D}_\alpha = \big\{\mathbf{d} = (d_0, d_1) \in \mathcal{F} \text{ s.t. } d_0 \leq \widehat{D}(\rho_0|H) + x \text{ and } d_1 \geq \widehat{D}(\rho_1|H) - x\big\}.$$

Bound $\widehat{s}_{\mathsf{comp}}$ takes the form

$$\widehat{s}_{\mathsf{comp}} = \min\left\{1, 1 - \frac{\big(\widehat{D}(\rho_1|H) - x\big)\left(1 + \rho_1\left(1 + \dfrac{1}{M}\right)\right) - \big(\widehat{D}(\rho_0|H) + x\big)}{\rho_1\left(1 + \dfrac{1}{M}\right)}\right\}.$$

Note that when $M = +\infty$, $\widehat{s}_{\mathsf{comp}} < 1$ is equivalent to the condition that elasticity of demand is larger than -1:

$$\widehat{s}_{\mathsf{comp}} < 1 \quad\iff\quad \frac{\log\big(\widehat{D}(\rho_1|H) - x\big) - \log\big(\widehat{D}(\rho_0|H) + x\big)}{\log(1 + \rho_1)} > -1.$$

When the bidder's demand is unchanged following an upward deviation (i.e., when $\widehat{D}(\rho_1|H) \approx \widehat{D}(\rho_0|H)$), for $x > 0$ small enough we have $\widehat{s}_{\mathsf{comp}} < 1$ even as M goes to $+\infty$.

PROOF: For any competitive history $h \in H$, beliefs $d_{h,1}$ and $d_{h,0}$ must be such that

$$\left(1 + \rho_1 - \frac{1}{1+M}\right)d_{h,1} \leq \left(1 - \frac{1}{1+M}\right)d_{h,0}$$
$$\iff \quad d_{h,0} \geq d_{h,1}\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right), \tag{O5}$$

where the first inequality uses the mark-up constraint $\frac{c_h}{b_h} \geq \frac{1}{1+M}$. Suppose that

$$\widehat{D}(\rho_0|H) + x \geq \big(\widehat{D}(\rho_1|H) - x\big)\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right). \tag{O6}$$

Note that in this case, $\widehat{s}_{\mathsf{comp}} = 1$. Indeed, let $\mu \in \Delta(\mathcal{F})$ be a distribution that puts all its mass at (d_0, d_1), with $d_0 = \widehat{D}(\rho_0|H) + x$, and $d_1 = \widehat{D}(\rho_1|H) - x$. Note that $\mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$, and $\mathbb{E}_\mu[\text{IsComp}(\mathbf{d})] = 1$.

Suppose next that (O6) does not hold. Let $\mu \in \Delta(\mathcal{F})$ be a distribution satisfying the constraint $\mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$, and let $\widehat{s}(\mu) = \mathbb{E}_\mu[\text{IsComp}(\mathbf{d})]$ be the share of competitive histories under μ. Note that

$$\begin{aligned}\widehat{D}(\rho_0|H) + x &\geq \mathbb{E}_\mu[d_0] \\ &= \widehat{s}_{\mathsf{comp}}(\mu)\mathbb{E}_\mu\big[d_0|\text{IsComp}(\mathbf{d}) = 1\big] \\ &\quad + \big(1 - \widehat{s}_{\mathsf{comp}}(\mu)\big)\mathbb{E}_\mu\big[d_0|\text{IsComp}(\mathbf{d}) = 0\big],\end{aligned} \tag{O7}$$

$$\begin{aligned}\widehat{D}(\rho_1|H) - x &\leq \mathbb{E}_\mu[d_1] \\ &= \widehat{s}_{\mathsf{comp}}(\mu)\mathbb{E}_\mu\big[d_1|\text{IsComp}(\mathbf{d}) = 1\big] \\ &\quad + \big(1 - \widehat{s}_{\mathsf{comp}}(\mu)\big)\mathbb{E}_\mu\big[d_1|\text{IsComp}(\mathbf{d}) = 0\big].\end{aligned} \tag{O8}$$

Since equation (O5) holds for $\mathbf{d}$ with $\text{IsComp}(\mathbf{d}) = 1$, we have that

$$\mathbb{E}_\mu\big[d_0|\text{IsComp}(\mathbf{d}) = 1\big] \geq \mathbb{E}_\mu\big[d_1|\text{IsComp}(\mathbf{d}) = 1\big]\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right). \tag{O9}$$

Using (O7)–(O9), we get

$$\begin{aligned}
&\widehat{D}(\rho_0|H) + x - \big(1 - \widehat{s}_{\text{comp}}(\mu)\big)\mathbb{E}_\mu\big[d_0|\text{IsComp}(\mathbf{d}) = 0\big] \\
&\quad\geq \widehat{s}_{\text{comp}}(\mu)\mathbb{E}_\mu\big[d_0|\text{IsComp}(\mathbf{d}) = 1\big] \\
&\quad\geq \widehat{s}_{\text{comp}}(\mu)\mathbb{E}_\mu\big[d_1|\text{IsComp}(\mathbf{d}) = 1\big]\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right) \\
&\quad\geq \big(\widehat{D}(\rho_1|H) - x - \big(1 - \widehat{s}_{\text{comp}}(\mu)\big)\mathbb{E}_\mu\big[d_1|\text{IsComp}(\mathbf{d}) = 0\big]\big)\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right).
\end{aligned} \tag{O10}$$

From equation (O10), we get

$$\begin{aligned}
&\big(\widehat{D}(\rho_1|H) - x\big)\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right) - \big(\widehat{D}(\rho_0|H) + x\big) \\
&\qquad\leq \big(1 - \widehat{s}_{\text{comp}}(\mu)\big) \\
&\qquad\quad\times \left(\mathbb{E}_\mu\big[d_1|\text{IsComp}(\mathbf{d}) = 0\big]\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right) - \mathbb{E}_\mu\big[d_0|\text{IsComp}(\mathbf{d}) = 0\big]\right) \\
&\qquad\leq \big(1 - \widehat{s}_{\text{comp}}(\mu)\big)\rho_1\left(1 + \frac{1}{M}\right) \\
&\implies \quad \widehat{s}_{\text{comp}}(\mu) \leq 1 - \frac{\big(\widehat{D}(\rho_1|H) - x\big)\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right) - \big(\widehat{D}(\rho_0|H) + x\big)}{\rho_1\left(1 + \frac{1}{M}\right)}.
\end{aligned}$$

where the second inequality follows since $1 \geq \mathbb{E}_\mu[d_0|\text{IsComp}(\mathbf{d}) = 0] \geq \mathbb{E}_\mu[d_1|\text{IsComp}(\mathbf{d}) = 0] \geq 0$. Since this inequality holds for all $\mu \in \Delta(\mathcal{F})$ satisfying the constraint $\mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$,

$$\widehat{s}_{\text{comp}} \leq \overline{s}_1 \equiv 1 - \frac{\big(\widehat{D}(\rho_1|H) - x\big)\left(1 + \rho_1\left(1 + \frac{1}{M}\right)\right) - \big(\widehat{D}(\rho_0|H) + x\big)}{\rho_1\left(1 + \frac{1}{M}\right)}.$$

Finally, to see that $\widehat{s}_{\text{comp}} = \overline{s}_1$ when (O6) does not hold, let $\overline{\mu} \in \Delta(\mathcal{F})$ be a distribution that puts weight $1 - \overline{s}_1$ on beliefs $d^{nc} = (d_0^{nc}, d_1^{nc}) = (1, 1)$ and puts weight $\overline{s}_1$ on beliefs $d^c = (d_0^c, d_1^c)$ such that $\overline{s}_1 d_0^c + (1 - \overline{s}_1) = \widehat{D}(\rho_0|H) + x$ and $\overline{s}_1 d_1^c + (1 - \overline{s}_1) = \widehat{D}(\rho_1|H) - x$. One can check that $\mathbb{E}_{\overline{\mu}}[\mathbf{d}] \in \mathcal{D}_\alpha$ and that $\mathbb{E}_{\overline{\mu}}[\text{IsComp}(\mathbf{d})] = \overline{s}_1$. Hence, when (O6) does not hold, $\widehat{s}_{\text{comp}} = \overline{s}_1$. *Q.E.D.*

OB.2.2. *Inference From a Downward Deviation*

Consider next the case of a single downward deviation $\rho_{-1} < 0$. Let $\Lambda = ((-1, 0), (0, 1))$ and $x_\lambda = x > 0$ for all $\lambda \in \Lambda$. Set $\mathcal{D}_\alpha$ then takes the form:

$$\mathcal{D}_\alpha = \left\{\mathbf{d} = (d_{-1}, d_0) \in \mathcal{F} \text{ s.t. } d_{-1} \geq \widehat{D}(\rho_{-1}|H) - x \text{ and } d_0 \leq \widehat{D}(\rho_0|H) + x\right\}.$$

With a single deviation $\rho_{-1} < 0$ such that $m \leq 1/(1+\rho_{-1}) - 1$, all histories can be rationalized as competitive. Indeed, reducing bids by ρ_{-1} would result in negative profits if costs are such that $b_h/c_h = 1 + m$. Instead, if $m > 1/(1+\rho_{-1}) - 1$, the solution to program (P) is

$$\widehat{s}_{\mathsf{comp}} = \min\left\{1, 1 - \frac{(\widehat{D}(\rho_{-1}|H) - x)\left(1 + \rho_{-1}\left(1 + \frac{1}{m}\right)\right) - (\widehat{D}(\rho_0|H) + x)}{1 + \rho_{-1}\left(1 + \frac{1}{m}\right)}\right\}.$$

PROOF: For any competitive history h, beliefs $d_{h,-1}$ and $d_{h,0}$ must be such that

$$\begin{aligned}
&\left(1 + \rho_{-1} - \frac{1}{1+m}\right)d_{h,-1} \leq \left(1 - \frac{1}{1+m}\right)d_{h,0} \\
\iff \quad & d_{h,0} \geq d_{h,-1}\left(1 + \rho_{-1}\left(1 + \frac{1}{m}\right)\right),
\end{aligned} \tag{O11}$$

where the first inequality uses the mark-up constraint $\frac{c_h}{b_h} \leq \frac{1}{1+m}$. Suppose that

$$\widehat{D}(\rho_0|H) + x \geq (\widehat{D}(\rho_{-1}|H) - x)\left(1 + \rho_{-1}\left(1 + \frac{1}{m}\right)\right). \tag{O12}$$

Note that in this case, $\widehat{s}_{\mathsf{comp}} = 1$. Indeed, let $\mu \in \Delta(\mathcal{F})$ be a distribution that puts all its mass at (d_{-1}, d_0), with $d_{-1} = \widehat{D}(\rho_{-1}|H) - x$ and $d_0 = \min\{\widehat{D}(\rho_0|H) + x, \widehat{D}(\rho_{-1}|H) - x\}$. Note that $\mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$, and $\mathbb{E}_\mu[\text{IsComp}(\mathbf{d})] = 1$. Note that (O12) always holds when $m \leq 1/(1+\rho_{-1}) - 1$.

Suppose next that (O12) does not hold. Let $\mu \in \Delta(\mathcal{F})$ be a distribution satisfying the constraint $\mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$, and let $\widehat{s}_{\mathsf{comp}}(\mu) = \mathbb{E}_\mu[\text{IsComp}(\mathbf{d})]$ be the share of competitive histories under μ. Note that equation (O7) must hold. In addition,

$$\begin{aligned}
\widehat{D}(\rho_{-1}|H) - x &\leq \mathbb{E}_\mu[d_{-1}] \\
&= \widehat{s}_{\mathsf{comp}}(\mu)\mathbb{E}_\mu\left[d_{-1}|\text{IsComp}(\mathbf{d}) = 1\right] \\
&\quad + (1 - \widehat{s}_{\mathsf{comp}}(\mu))\mathbb{E}_\mu\left[d_{-1}|\text{IsComp}(\mathbf{d}) = 0\right].
\end{aligned} \tag{O13}$$

Since equation (O11) holds for $\mathbf{d}$ with $\text{IsComp}(\mathbf{d}) = 1$,

$$\mathbb{E}_\mu\left[d_0|\text{IsComp}(\mathbf{d}) = 1\right] \geq \mathbb{E}_\mu\left[d_{-1}|\text{IsComp}(\mathbf{d}) = 1\right]\left(1 + \rho_{-1}\left(1 + \frac{1}{m}\right)\right). \tag{O14}$$

Using (O7), (O13), and (O14), we get

$$\begin{aligned}
\widehat{D}(\rho_0|H) + x &\geq \widehat{D}(\rho_0|H) + x - (1 - \widehat{s}_{\mathsf{comp}}(\mu))\mathbb{E}_\mu\left[d_0|\text{IsComp}(\mathbf{d}) = 0\right] \\
&\geq \widehat{s}_{\mathsf{comp}}(\mu)\mathbb{E}_\mu\left[d_0|\text{IsComp}(\mathbf{d}) = 1\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \widehat{s}_{\mathrm{comp}}(\mu)\mathbb{E}_\mu\big[d_{-1}|\mathrm{IsComp}(\mathbf{d})=1\big]\left(1+\rho_{-1}\left(1+\frac{1}{m}\right)\right)\\
&\geq \big(\widehat{D}(\rho_{-1}|H)-x-\big(1-\widehat{s}_{\mathrm{comp}}(\mu)\big)\\
&\quad\times \mathbb{E}_\mu\big[d_{-1}|\mathrm{IsComp}(\mathbf{d})=0\big]\big)\left(1+\rho_{-1}\left(1+\frac{1}{m}\right)\right)\\
&\geq \big(\widehat{D}(\rho_{-1}|H)-x-\big(1-\widehat{s}_{\mathrm{comp}}(\mu)\big)\big)\left(1+\rho_{-1}\left(1+\frac{1}{m}\right)\right), \qquad \text{(O15)}
\end{aligned}
$$

where the first inequality uses $0 \leq \mathbb{E}_\mu[d_0|\mathrm{IsComp}(\mathbf{d})=0]$ and the last inequality uses $\mathbb{E}_\mu[d_{-1}|\mathrm{IsComp}(\mathbf{d})=0] \leq 1$. It follows from (O15) that

$$
\widehat{s}_{\mathrm{comp}}(\mu) \leq 1-\frac{\big(\widehat{D}(\rho_{-1}|H)-x\big)\left(1+\rho_{-1}\left(1+\frac{1}{m}\right)\right)-\big(\widehat{D}(\rho_0|H)+x\big)}{1+\rho_{-1}\left(1+\frac{1}{m}\right)}.
$$

Since this holds for all $\mu \in \Delta(\mathcal{F})$ satisfying the constraint $\mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$,

$$
\widehat{s}_{\mathrm{comp}} \leq \overline{s}_{-1} \equiv 1-\frac{\big(\widehat{D}(\rho_{-1}|H)-x\big)\left(1+\rho_{-1}\left(1+\frac{1}{m}\right)\right)-\big(\widehat{D}(\rho_0|H)+x\big)}{1+\rho_{-1}\left(1+\frac{1}{m}\right)}.
$$

Finally, to see that $\widehat{s}_{\mathrm{comp}} = \overline{s}_{-1}$ when (O12) does not hold, let $\overline{\mu} \in \Delta(\mathcal{F})$ be a distribution that puts weight $1-\overline{s}_{-1}$ on beliefs $d^{nc} = (d^{nc}_{-1}, d^{nc}_0) = (1, 0)$ and puts weight $\overline{s}_{-1}$ on beliefs $d^c = (d^c_{-1}, d^c_0)$ such that $\overline{s}_{-1}d^c_0 + (1-\overline{s}_{-1})d^{nc}_0 = \overline{s}_{-1}d^c_0 = \widehat{D}(\rho_0|H) + x$ and $\overline{s}_{-1}d^c_{-1} + (1-\overline{s}_{-1})d^{nc}_{-1} = \overline{s}_{-1}d^c_{-1} + (1-\overline{s}_{-1}) = \widehat{D}(\rho_{-1}|H) - x$. One can check that $\mathbb{E}_{\overline{\mu}}[\mathbf{d}] \in \mathcal{D}_\alpha$ and that $\mathbb{E}_{\overline{\mu}}[\mathrm{IsComp}(\mathbf{d})] = \overline{s}_{-1}$. Hence, when (O12) does not hold, $\widehat{s}_{\mathrm{comp}} = \overline{s}_{-1}$. *Q.E.D.*

OB.3. *Complementarities Between Upward and Downward Deviations*

In this Appendix, we clarify complementarities between downward and upward deviations and establish a possibility result in a stylized setting. Even if neither individual deviation implies that a positive share of auctions is noncompetitive, the joint restrictions imposed by upward and downward deviations can imply that a positive share of auctions is noncompetitive. For simplicity, we focus on the case of arbitrarily large data so that we can use limit confidence set $\mathcal{D}_\alpha = \{\widehat{\mathbf{D}}\}$, where $\widehat{\mathbf{D}} = (\widehat{D}(\rho_n|H))_{n\in\mathcal{M}}$.

As we discussed in Section 7, individual upward and downward deviations, respectively, imply strict bounds on the share of competitive histories if and only if

$$
\widehat{D}(0|H)-(1+\rho_1)\widehat{D}(\rho_1|H) < \frac{1}{1+M}\big[\widehat{D}(0|H)-\widehat{D}(\rho_1|H)\big],
$$

$$
(1+\rho_{-1})\widehat{D}(\rho_{-1}|H)-\widehat{D}(0|H) > \frac{1}{1+m}\big[\widehat{D}(\rho_{-1}|H)-\widehat{D}(0|H)\big].
$$
[2]

To clarify the existence of complementarities between upward and downward deviations, we now consider the special case in which

$$\widehat{D}(0|H) - (1+\rho_1)\widehat{D}(\rho_1|H) = \frac{1}{1+M}\big[\widehat{D}(0|H) - \widehat{D}(\rho_1|H)\big], \tag{O16}$$

$$(1+\rho_{-1})\widehat{D}(\rho_{-1}|H) - \widehat{D}(0|H) = \frac{1}{1+m}\big[\widehat{D}(\rho_{-1}|H) - \widehat{D}(0|H)\big]. \tag{O17}$$

Individual upward and downward deviations imply no restrictions on the set of competitive histories. However, different deviations are potentially rationalized by using different costs at the same history. We show this is indeed the case, and that jointly considering upward and downward deviations can yield strict constraints on the share of competitive histories. The following lemma clarifies that markup constraints will play a role in our argument.

LEMMA OB.1: *Under* (O16) *and if* $m = 0$ *and* $M = +\infty$, *then all histories can be rationalized as competitive*.

PROOF: The following demand and costs rationalize the observed bidding behavior while satisfying consistency requirement ($\widehat{CR}$). At every history h such that the bidder wins, we set $d_{h,0} = 1$, $d_{h,-1} = 1$, $d_{h,1} = \widehat{D}(\rho_1|H)/\widehat{D}(0|H)$ and $c_h = 0$. Since $\rho_1 > 0$, $d_{h,1} \le 1$.

At every history h such that the bidder loses, but would win after reducing its bids by ρ_{-1}, we set $d_{h,0} = d_{h,1} = 0$, $d_{h,-1} = 1$, and $c_h = b_h$

At every history such that the bidder loses even after deviation ρ_{-1}, we set $d_{h,-1} = d_{h,0} = d_{h,1} = 0$, and $c_h = b_h$.

It is immediate that these demand and costs are feasible, and satisfy (IC) and ($\widehat{CR}$). *Q.E.D.*

We return now to the case where (O16) and (O17) hold for $m > 0$. We establish lower bounds for the number of histories at which c_h/b_h must be equal to $\frac{1}{1+m}$ and $\frac{1}{1+M}$. Whenever these two lower bounds are mutually incompatible, the share of competitive histories is strictly less than one.

Histories Such That $c_h/b_h = 1/(1+M)$. (IC) for upward deviation ρ_1 implies that for all histories h,

$$d_{h,0} - (1+\rho_1)d_{h,1} \ge (d_{h,0} - d_{h,1})\frac{c_h}{b_h}.$$

Summing over all histories, conditions ($\widehat{CR}$) and (O16) imply that

$$\frac{1}{|H|}\sum_{h\in H}(d_{h,0} - d_{h,1})\frac{c_h}{b_h} \le \frac{1}{|H|}\sum_{h\in H} d_{h,0} - (1+\rho_1)d_{h,1} = \frac{1}{1+M}\big(\widehat{D}(0|H) - \widehat{D}(\rho_1|H)\big)$$

$$= \frac{1}{|H|}\sum_{h\in H}(d_{h,0} - d_{h,1})\frac{1}{1+M}.$$

[2]Checking whether these constrains hold can be performed rapidly, and suggests a rough rationale by which one could pick ρ_{-1} and ρ_1: obtain a smooth estimate of the true demand, and pick ρ_{-1} and ρ_1 so that the conditions above hold with a reliable margin.

Since $d_{h,0} - d_{h,1} \geq 0$ and $c_h/b_h \geq \frac{1}{1+M}$, this implies that whenever $d_{h,0} - d_{h,1} > 0$, $c_h/b_h = 1/(1+M)$.

Note that if $d_{h,0} = d_{h,1} > 0$ then $d_{h,0} - (1+\rho_1)d_{h,1} < 0$ so that (IC) cannot hold. Hence $d_{h,0} - d_{h,1} = 0$ implies $d_{h,0} = d_{h,1} = 0$. This implies that

$$\frac{1}{|H|}\sum_{h\in H}\mathbf{1}_{d_{h,0}-d_{h,1}>0} \geq \frac{1}{|H|}\sum_{h\in H}\mathbf{1}_{d_{h,0}>0} \geq \frac{1}{|H|}\sum_{h\in H} d_{h,0} = \widehat{D}(0|H).$$

Hence the share of histories such that $c_h/b_h = \frac{1}{1+M}$ is at least equal to $\widehat{D}(0|H)$.

Histories Such That $c_h/b_h = 1/(1+m)$. (IC) for downward deviation ρ_{-1} implies that for all histories h,

$$(1+\rho_{-1})d_{h,-1} - d_{h,0} \leq (d_{h,-1} - d_{h,0})\frac{c_h}{b_h}.$$

Summing over all histories in H, conditions ($\widehat{CR}$) and (O17) imply that

$$\begin{aligned}\frac{1}{|H|}\sum_{h\in H}(d_{h,-1} - d_{h,0})\frac{c_h}{b_h} &\geq \frac{1}{|H|}\sum_{h\in H}(1+\rho_{-1})d_{h,-1} - d_{h,0} = \frac{1}{1+m}\big(\widehat{D}(\rho_{-1}|H) - \widehat{D}(0|H)\big)\\ &= \frac{1}{|H|}\sum_{h\in H}(d_{h,-1} - d_{h,0})\frac{1}{1+m}.\end{aligned}$$

Since $d_{h,-1} - d_{h,0} \geq 0$ and $c_h/b_h \leq \frac{1}{1+m}$, this implies that whenever $d_{h,-1} - d_{0,h} > 0$, then $c_h/b_h = 1/(1+m)$. In addition, for all h, we have that

$$(1+\rho_{-1})d_{h,-1} - d_{h,0} = (d_{h,-1} - d_{h,0})\frac{1}{1+m} \Rightarrow d_{h,0} = \frac{1+\rho_{-1} - \dfrac{1}{1+m}}{1 - \dfrac{1}{1+m}} d_{h,-1} = (1-\nu)d_{h,-1}$$

with $\nu \equiv -\rho_{-1}/(1 - \frac{1}{1+m}) > 0$. Hence, we have that

$$\begin{aligned}\frac{1}{|H|}\sum_{h\in H} d_{h,-1} - d_{h,0} &\leq \frac{1}{|H|}\sum_{h\in H}(d_{h,-1} - d_{h,0})\mathbf{1}_{d_{h,-1}-d_{h,0}>0} \leq \frac{1}{|H|}\sum_{h\in H}\nu d_{h,-1}\mathbf{1}_{d_{h,-1}-d_{h,0}>0}\\ &\leq \frac{1}{|H|}\sum_{h\in H}\nu\mathbf{1}_{d_{h,-1}-d_{h,0}>0}.\end{aligned}$$

This implies that the share of histories such that $c_h/b_h = 1/(1+m)$ is greater than $\frac{1}{\nu}(\widehat{D}(\rho_{-1}|H) - \widehat{D}(0|H))$.

Hence, if $\widehat{D}(0|H) + \frac{1}{\nu}(\widehat{D}(\rho_{-1}|H) - \widehat{D}(0|H)) > 1$, then joint upward and downward deviations imply strict constraints on the share of competitive histories. For example, if $m = 3\%$, $\rho_{-1} = -1.5\%$, $\widehat{D}(\rho_{-1}|H) = 65\%$ and $\widehat{D}(0|H) = 25\%$, then $\frac{1}{\nu} \simeq 1.94$, and $\widehat{D}(0|H) + \frac{1}{\nu}(\widehat{D}(\rho_{-1}|H) - \widehat{D}(0|H)) \simeq 1.027$.

OB.4. *Common Values*

We now show how to extend the analysis in Section 6 to allow for common values. Because expected costs conditional on winning now depend on a bidder's bid, costs, and demand associated with history $h \in H$ now take the form $(d_{h,n}, c_{h,n})_{n\in\mathcal{M}}$, where for each $n \in \mathcal{M}$, $c_{h,n} = \mathbb{E}[c|h, \wedge\mathbf{b}_{-i,h} > (1+\rho_n)b_h]$ is the bidder's expected cost at history h conditional on winning at bid $(1+\rho_n)b_h$.

We make the following monotonicity assumption.

ASSUMPTION OB.1: *For all histories h and all bids b, b', b'' with $b < b' < b''$,* $\mathbb{E}[c|h, \wedge\mathbf{b}_{-i,h} \in (b, b')] \le \mathbb{E}[c|h, \wedge\mathbf{b}_{-i,h} \in (b', b'')]$.

In words, bidders' expected costs are increasing in opponents' bids. This implies that expected costs $c_{h,n}$ conditional on winning are weakly increasing in the deviation ρ_n. This condition on costs follows from affiliation when bidders' signals are one-dimensional and bidders use monotone bidding strategies. We now show that, under these conditions, allowing for common values does not relax the constraints in Program (P).

Note first that, for each deviation n, expected costs conditional on winning $(c_{h,n})_{n\in\mathcal{M}}$ satisfy

$$\forall n \in \mathcal{M}, \quad d_{h,n}c_{h,n} = d_{h,0}c_{h,0} + (d_{h,n} - d_{h,0})\hat{c}_{h,n}, \tag{O18}$$

where $\hat{c}_{h,n} = \mathbb{E}[c|h, \wedge\mathbf{b}_{-i,h} \in (b_h, (1+\rho_n)b_h)]$.[3] Our assumptions on costs imply that $\hat{c}_{h,n}$ is weakly increasing in n.

Consider first downward deviations $\rho_n < 0$ (i.e., $n < 0$). For such deviations, incentive compatibility constraints hold if and only if

$$\frac{d_{h,n}(1+\rho_n)b_h - d_{h,0}b_h}{d_{h,n} - d_{h,0}} \le \hat{c}_{h,n}.$$

Consider next upward deviations $\rho_n > 0$ (i.e., $n > 0$). For any such deviation, incentive compatibility constraints become

$$\hat{c}_{h,n} \le \frac{d_{h,0}b_h - d_{h,n}(1+\rho_n)b_h}{d_{h,0} - d_{h,n}}.$$

Since $\hat{c}_{h,\hat{n}}$ is weakly increasing in $\hat{n}$, $\hat{c}_{h,n} \ge \hat{c}_{h,n'}$ for all $n > 0$ and $n' < 0$. Hence, there exist costs $(c_{h,n})_{n\in\mathcal{M}}$ satisfying (IC) if and only if

$$\max_{n<0} \frac{d_{h,n}(1+\rho_n)b_h - d_{h,0}b_h}{d_{h,n} - d_{h,0}} \le \min_{n>0} \frac{d_{h,0}b_h - d_{h,n}(1+\rho_n)b_h}{d_{h,0} - d_{h,n}}. \tag{O19}$$

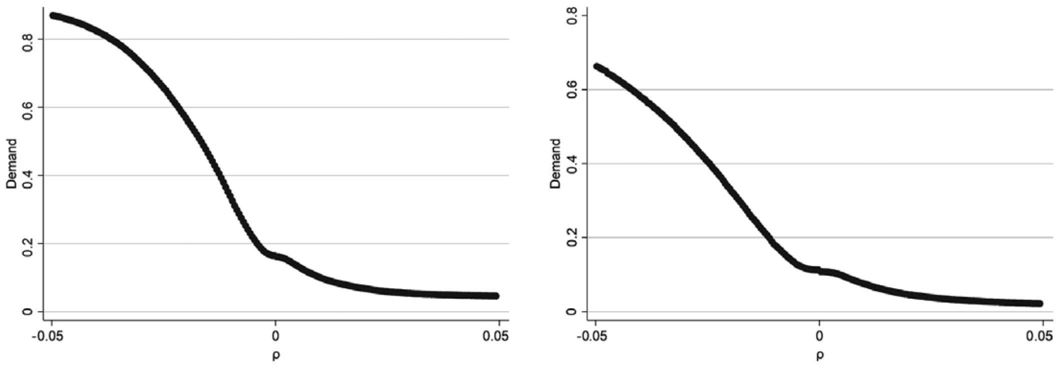
Condition (O19) implies that there also exists a constant profile of costs $c_{h,n} = c_h$ (i.e., a private value cost) that satisfies (IC).

ONLINE APPENDIX OC: FURTHER EMPIRICAL FINDINGS

OC.1. *Additional Figures for Section 2*

Figure OC.1 illustrates the clustering of bids we highlight in Section 2 more directly. The two panels of Figure OC.1 plot the sample demand function, $\widehat{D}(\cdot)$, for city-level and national-level auctions. We define the sample demand as follows: $\widehat{D}(\rho) =$

[3] We replace (b, b') by (b', b) in the event that $b' < b$.



(a) city auctions (b) national auctions

FIGURE OC.1.—Sample demand.

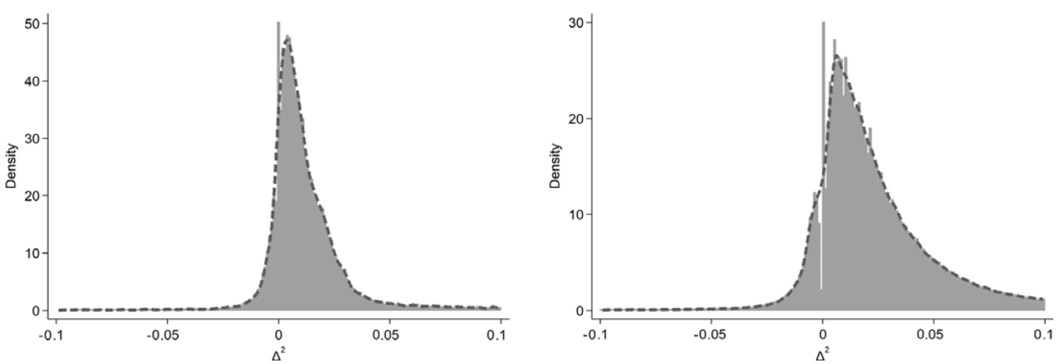
$\frac{1}{|H|}\sum_{i,a}\mathbf{1}_{b_{i,a}(1+\rho)<\wedge\mathbf{b}_{-i,a}}$, where $|H|$ denotes the number of all bids in the data set. In words, $\widehat{D}(\rho)$ is the sample probability with which bidders win an auction if the bids are changed by a factor of $(1+\rho)$. For panel (a), we find that a drop in bids of 2% increases the likelihood of winning by more than 3-fold from 16.3% to 56.9%. For panel (b), we find that a 2% drop in bids also increases the likelihood of winning by about 3-fold from 10.8% to 33.2%.

Figure OC.2 plots the distribution of differences Δ^2 between bids after the lowest bid is excluded. Formally, let $NW(a)$ denote the set of nonwinning bidders in auction a. Then

$$\forall i \in NW(a), \quad \Delta^2_{i,a} = \frac{b_{i,a} - \min\limits_{j \in NW(a)} b_{j,a}}{r}.$$

Although there is some bunching at exactly zero making the distributions of Δ^2 somewhat irregular, the kernel density estimates show that there is no corresponding missing mass.

Figure OC.3 plots the distribution of bid differences Δ for the set of auctions whose prices were not renegotiated up (about 15.3% of the sample). As the figure shows, the



(a) city auctions (b) national auctions

FIGURE OC.2.—Distribution of bid-difference Δ^2 excluding winning bids. The dotted curves correspond to local (6th order) polynomial density estimates with bandwidth set to 0.0075.

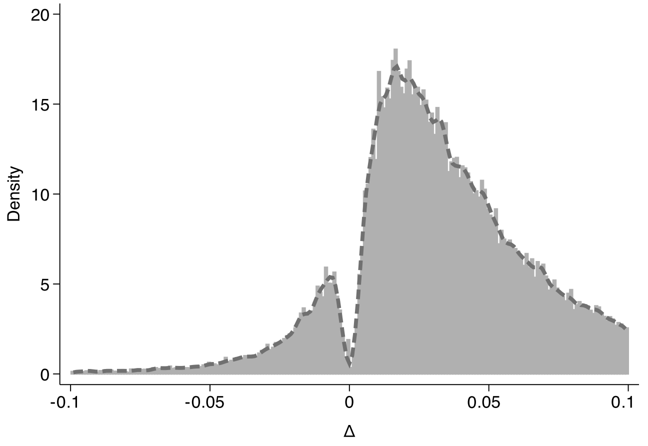


FIGURE OC.3.—Distribution of bid-difference Δ. National data, auctions with no renegotiation. The dotted curves correspond to local polynomial density estimates with bandwidth set to 0.0075.

missing mass at $\Delta = 0$ is just as visible when we focus on this set of auctions. This suggests that renegotiation does not drive the patterns we document.

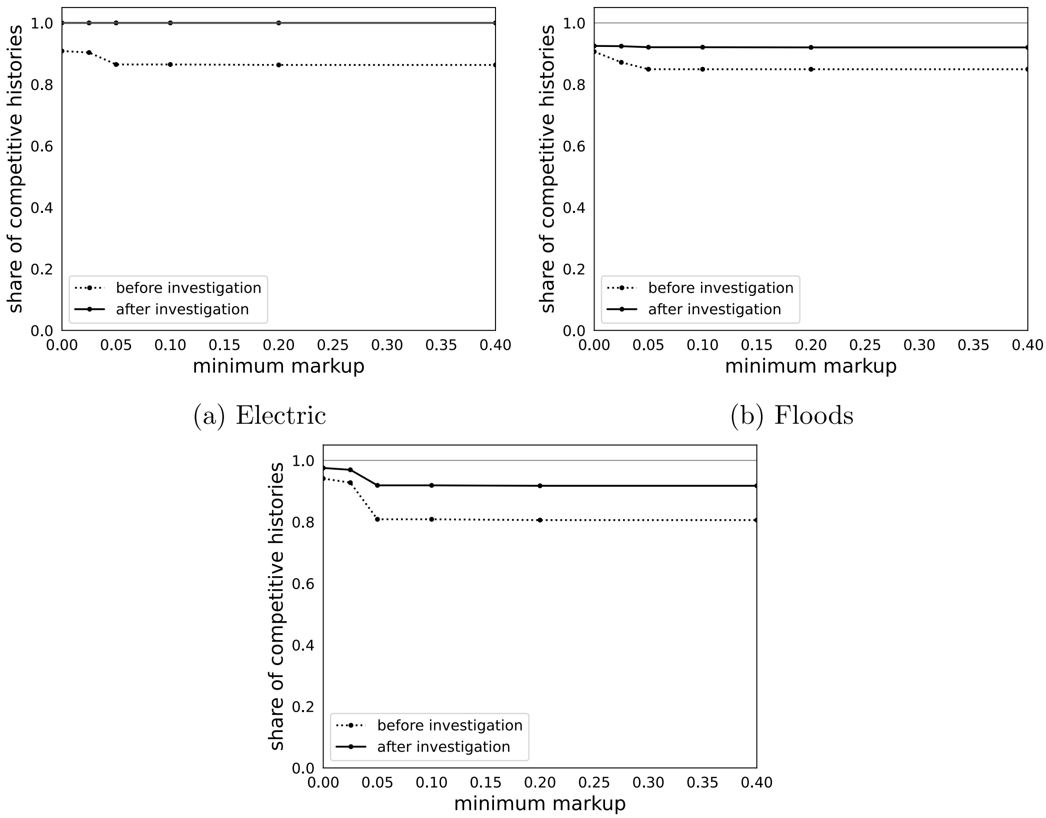
OC.2. *Additional Findings Related to Section 7.4*

This Appendix provides sensitivity analysis of the share of competitive histories for the industries that were investigated for bid rigging by the JFTC to changes in (a) the choice of confidence level, and (b) the set of deviations considered.

Figure OC.4 presents estimates of the 90% confidence bound on the share of competitive histories for the three industries we studied in Section 7.4. Relative to the bounds in Figure 7 in Section 7.4, these bounds are obtained using smaller tolerance parameters $(x_\lambda)_{\lambda\in\Lambda}$ when computing set $\mathcal{D}_\alpha$. Under this less conservative bound, firms labeled Floods and Prestressed Concrete appear to have continued colluding in the period after the investigation.

Figure OC.5 again presents estimates of the 90% confidence bound on the share of competitive histories for the three industries we studied in Section 7.4, but with the set of deviations $\{-0.02, 0, 0.002\}$. Relative to the bounds in Figure OC.4, we now only find evidence of noncompetitive behavior for firms labeled Prestressed Concrete in the after period.

The difference between our estimates in Figure OC.4 and Figure OC.5 can be explained as follows. For Flood auctions occurring after investigation, a 0.1% upward deviation causes no change in demand. As a result, our estimate on the share of competitive histories in the after period are strictly less than 1 when we consider such a small upward deviation. However, we are worried that this insensitivity of demand to a small upward deviation might be a mechanical consequence of the small number of observations we have in the after period for this industry. Indeed, a 0.2% upward deviation causes a small drop in demand, and our estimate on the share of competitive histories in the after period is exactly 1 in Figure OC.5.



(a) Electric

(b) Floods

(c) Prestressed Concrete

FIGURE OC.4.—90% confidence bound on share of competitive histories. Deviations $\{-0.02, 0, 0.001\}$; maximum markup 0.5.

OC.3. *Relation to the Cover Bidding Screen of Imhof, Karagök, and Rutz (2018)*

One screen of collusion that is closely related to ours is the cover bidding screen proposed by Imhof, Karagök, and Rutz (2018). The authors study road construction projects in Switzerland and document a suspicious bidding pattern for a subset of the auctions in which the difference between the two lowest bids is substantially larger than the gap between any two losing bids. Based on this pattern, the authors propose a screen based on a comparison of the gap between the two lowest bids and the standard error of the losing bids. In particular, they consider computing the following statistic for each auction:

$$cover_t = \frac{\mathbf{b}_t^{(2)} - \mathbf{b}_t^{(1)}}{\sigma_t},$$

where $\mathbf{b}_t^{(1)}$ and $\mathbf{b}_t^{(2)}$ are the lowest and second lowest bids in the auction taking place in period t; and σ_t is the standard error of the losing bids in auction t. Imhof, Karagök, and Rutz (2018) identifies a subset of auctions in which $cover_t$ is consistently above 1 and flag them as potentially uncompetitive. Interestingly, the Swiss competition commission

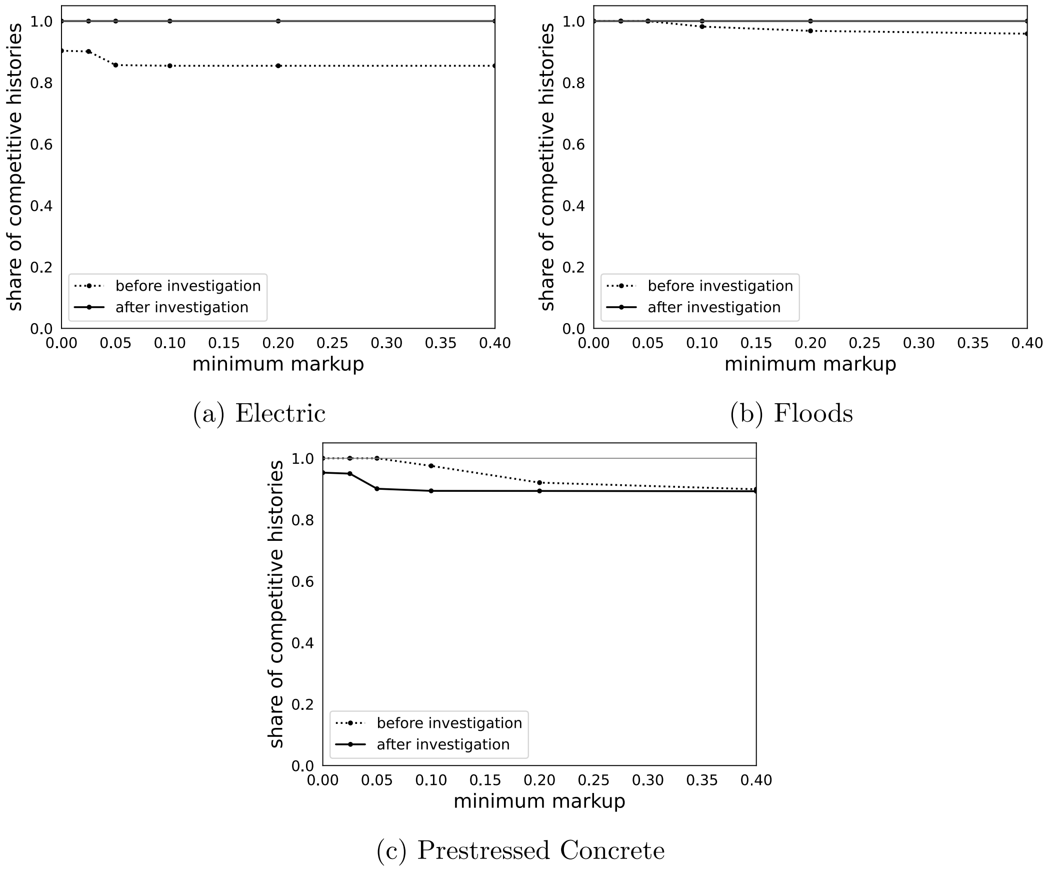


FIGURE OC.5.—90% confidence bound on share of competitive histories. Deviations $\{-0.02, 0, 0.002\}$; maximum markup 0.5.

(COMCO) launched an investigation based partly on the results of this statistical test which led to sanctions against eight firms.

The distribution of the gap between the two lowest bids determine the elasticity of residual demand, and hence, the cover bidding screen of Imhof, Karagök, and Rutz (2018) is related to Proposition 2. Moreover, the distribution of the gap between the two lowest bids determine the upper bound on costs that must hold under competitive bidding. Together with our upper bound on markups, the gap between the lowest bids determine how much power upward deviations have in problem (P) in Section 6. This relates the cover bidding screen to the metric of competitive behavior introduced in Section 6.

Table OC.1 reports the firm-level bounds on the share of competitive auctions and the test statistic of Imhof, Karagök, and Rutz (2018). The left panel of the table corresponds to the top 30 firms in the municipal sample and the right panel corresponds to the top 30 firms in the national sample. Columns (1)–(4) are the same as in Table 2. In column (5), we report for each firm the fraction of auctions it participates in, such that $cover_t$ is above 1. We find that $cover_t$ is above 0.5 for all of the firms except one (firm 4 in the sample of national auction). In fact, except this one firm, we can reject with 95% confidence that $cover_t$ is equal to 0.5. Hence, almost all of the firms in our data set would be considered as somewhat suspicious by the cover bidding screen.

TABLE OC.1

SHARE OF COMPETITIVE HISTORIES, INDIVIDUAL FIRMS.

(1) Rank	(2) Participation	(3) Shr won	(4) Shr comp	(5) Shr cover ≥ 1	(1) Rank	(2) Participation	(3) Shr won	(4) Shr comp	(5) Shr cover ≥ 1
	(a) Municipal Data					(b) National Data			
1	347	0.19	**0.88**	0.69	1	4044	0.17	**0.84**	0.56
2	336	0.21	**0.86**	0.74	2	3854	0.07	**0.91**	0.82
3	299	0.08	**0.98**	0.79	3	3621	0.12	**0.85**	0.58
4	293	0.05	1.00	0.94	4	2998	0.15	1.00	0.49
5	290	0.14	1.00	0.81	5	2919	0.06	**0.92**	0.89
6	287	0.20	1.00	0.70	6	2547	0.08	**0.71**	0.76
7	269	0.14	**0.94**	0.72	7	2338	0.07	**0.74**	0.79
8	268	0.09	**0.97**	0.75	8	2333	0.07	**0.74**	0.77
9	262	0.12	1.00	0.69	9	2328	0.04	**0.95**	0.60
10	259	0.18	**0.90**	0.74	10	2292	0.06	**0.75**	0.77
11	252	0.12	**0.97**	0.85	11	2237	0.08	**0.90**	0.59
12	241	0.12	**0.95**	0.83	12	2211	0.03	**0.96**	0.62
13	239	0.16	**0.93**	0.82	13	2015	0.09	**0.76**	0.78
14	238	0.09	**0.99**	0.76	14	1984	0.08	**0.75**	0.75
15	227	0.11	**0.97**	0.82	15	1727	0.07	1.00	0.81
16	226	0.12	**0.99**	0.84	16	1674	0.05	**0.84**	0.80
17	225	0.08	**0.96**	0.81	17	1661	0.03	**0.94**	0.87
18	223	0.12	**0.98**	0.84	18	1660	0.08	**0.75**	0.78
19	220	0.07	1.00	0.91	19	1589	0.07	**0.79**	0.77
20	218	0.08	1.00	0.93	20	1427	0.10	1.00	0.58
21	211	0.07	1.00	0.77	21	1393	0.06	**0.86**	0.83
22	210	0.14	**0.95**	0.84	22	1392	0.07	1.00	0.76
23	209	0.17	**0.93**	0.78	23	1370	0.04	**0.92**	0.86
24	204	0.15	1.00	0.74	24	1368	0.14	1.00	0.54
25	203	0.11	**0.98**	0.72	25	1353	0.05	**0.80**	0.80
26	199	0.06	1.00	0.75	26	1342	0.09	1.00	0.72
27	190	0.12	1.00	0.82	27	1337	0.04	**0.87**	0.85
28	189	0.06	1.00	0.75	28	1326	0.08	**0.92**	0.75
29	188	0.16	**0.94**	0.82	29	1291	0.06	**0.86**	0.83
30	187	0.08	1.00	0.85	30	1260	0.06	**0.93**	0.74

Note: 95% confidence bound on the share of competitive auctions for top thirty most active firms. The first column corresponds to the ranking of the firms and the second column corresponds to the number of auctions in which each firm participates. Column 3 shows the fraction of auctions that each of these firms wins. Column 4 present our 95% confidence bound on the share of competitive histories for each firm based on Proposition 3. For our estimates of column 5, we use deviations $\{-0.02, 0, 0.001\}$, minimum markup $m = 0.025$ and maximum markup $M = 0.5$.

The overall correlation between the share of competitive histories that we report in column (4) and column (5) is essentially zero across both municipal and national data. If we instead take the correlation between a dummy of whether or not the estimated share of competitive histories is less than 1 and the statistic in column (5), the two statistics become more negatively correlated, with a correlation coefficient of -0.18.

A possible reason why the association between the two measures is somewhat weak is because our measure exploits information from downward deviations as well as from upward deviations. The metric proposed in Imhof, Karagök, and Rutz (2018) only captures information from upward deviations. This may explain why the correlation between the two is not very strong.[4]

OC.4. *Bounds on Other Moments*

This Appendix shows how to adapt the approach of Section 6 to obtain robust bounds on other moments of interest: (i) the share of competitive auctions, and (ii) the total deviation temptation.

Maximum Share of Competitive Auctions. The bound on the share of competitive histories provided by Proposition 3 allows some histories in the same auctions to have different competitive versus noncompetitive status. This may underestimate the prevalence of noncompetition in a given data set. In particular, if one player is noncompetitive, she must expect other players to be noncompetitive in the future. Otherwise, if all of her opponents played competitively, her stage-game best reply would be a profitable dynamic deviation.

For this reason, one might be interested in providing an upper bound on the share of competitive *auctions*, where an auction is considered to be competitive if and only if every player is competitive at their respective histories.

Take as given an adapted set of histories H, corresponding to a set A of auctions. Recall from Appendix OB that $\omega_H = (\omega_h)_{h\in H}$ denotes an environment, with $\omega_h = ((d_{h,n})_{n\in\mathcal{M}}, c_h)$. Recall further that $\Omega = \{\omega_H : \forall h \in H, (d_{h,n})_{n\in\mathcal{M}} \in \mathcal{F}\}$ is the set of environments ω_H with feasible demands.

For every environment $\omega_H \in \Omega$, let

$$A_{\mathsf{comp}}(\omega_H) \equiv \{A' \subset A \text{ s.t. } \forall a \in A', \forall h \in a, (d_h, c_h) \text{ satisfy (IC) and (MKP)}\}$$

be the set of competitive auctions under ω_H. Consider the following program:

$$\widehat{s}_{\mathsf{auc}} = \max_{\omega_H} \frac{|A_{\mathsf{comp}}(\omega_H)|}{|A|}$$
$$\text{s.t.} \quad \forall n, D_n(\omega_H, H) \in [\widehat{D}(\rho_n|H) - K, \widehat{D}(\rho_n|H) + K],$$

where, for each n and each ω_H, $D_n(\omega_H, H) = \frac{1}{|H|}\sum_{h\in H} d_{h,n}$. $\widehat{s}_{\mathsf{auc}}$ provides an upper bound to the fraction of competitive auctions.

Total Deviation Temptation. Regulators may want to investigate an industry only if firms fail to optimize in a significant way. Our methods can be used to derive a lower bound on the bidders' deviation temptation.

Given demand and costs ω_H, define

$$U(\omega_H) \equiv \frac{1}{|H|}\sum_{h\in H}\Big[(b_h - c_h)d_{h,0} - \max_{n\in\{-\underline{n},\dots,\overline{n}\}}\big[(1+\rho_n)b_h - c_h\big]d_{h,n}\Big].$$

Our inference problem now becomes

$$\widehat{DT} = \max_{\omega_H} U(\omega_H)$$
$$\text{s.t.} \quad \forall n, D_n(\omega_H, H) \in [\widehat{D}(\rho_n|H) - K, \widehat{D}(\rho_n|H) + K].$$

[4]Another potential reason is that, for the municipal data, the presence of the public reserve price compresses the distribution of the losing bids.

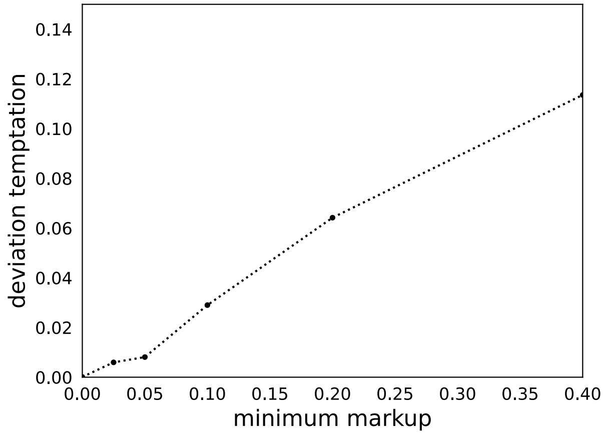


FIGURE OC.6.—Total deviation temptation as a fraction of profits, Tsuchiura. Deviations $\{-.02, 0, .001\}$. Maximum markup 0.5.

In this case, with probability approaching 1 as $|H|$ gets large, $-\widehat{DT}$ is a lower bound for the average total deviation-temptation per auction. This lets a regulator assess the extent of firms' failure to optimize before launching a costly audit. In addition, since the sum of deviation temptations must be compensated by a share of the cartel's future excess profits (along the lines of Levin (2003)), $\widehat{DT}$ provides an indirect measure of the excess profits generated by the cartel.

Figure OC.6 reports estimates for firms in the city of Tsuchiura, as a function of minimum markup m.

OC.5. *Sensitivity to Economic Plausibility Constraints*

Figure OC.7 shows that, for our city-level data, our estimates on the share of competitive histories are insensitive to changes in maximum markup M. Figure OC.8 illustrates

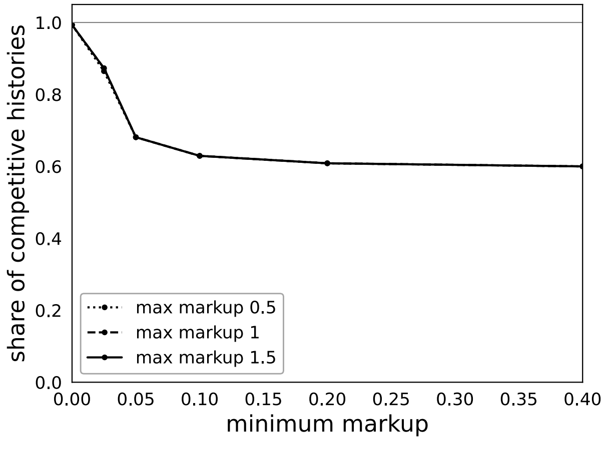


FIGURE OC.7.—Share of competitive histories for different maximum markups, city data, deviations $\{-0.02, 0, 0.001\}$.

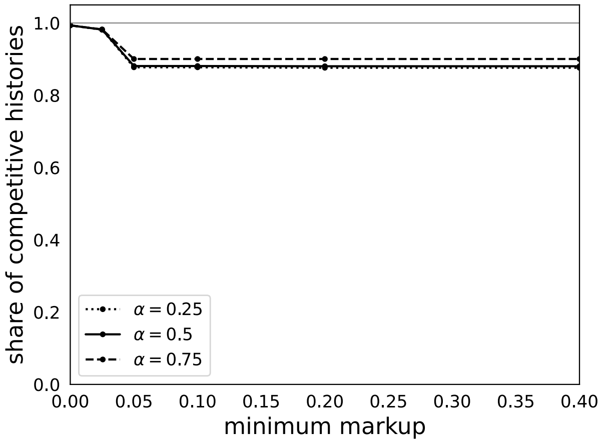


FIGURE OC.8.—Share of competitive histories, national-level data. Deviations $\{-0.02, 0, 0.001\}$, $M = 0.5$.

the sensitivity of our estimates to parameter $\alpha \in [0, 1]$ in downward deviation IC constraint (O3) for auctions with rebidding. Recall that parameter α measures the extent to which a deviation by a firm in round 1 affects her continuation profits in the following rounds when the deviation changes the information bidders have in the following rounds.

ONLINE APPENDIX OD: PROOFS FOR LEMMAS 1, 2, AND 3

PROOF OF LEMMA 1: Let us first establish that problem (P) does admit a solution. Since $\mathcal{F}$ is a compact subset of $\mathbb{R}^{\mathcal{M}}$, Prokhorov's theorem implies the set $\Delta(\mathcal{F})$ of distributions over $\mathcal{F}$ is compact under the weak topology. Since $\mathcal{D}_\alpha$ is compact and IsComp is upper semicontinuous, it follows that

$$\sup_{\mu \in \Delta(\mathcal{F})} \mathbb{E}_\mu\big[\text{IsComp}(\mathbf{d})\big]$$

$$\text{s.t. } \mathbb{E}_\mu[\mathbf{d}] \in \mathcal{D}_\alpha$$

does admit a solution, and the supremum is in fact a maximum. Let us denote by $\widehat{\mu}$ this solution.

Let us denote by C the set of competitive belief profiles $\mathbf{d}$ satisfying (IC-MKP), and by C^{o} the interior of set C, that is, the set of beliefs $\mathbf{d}$ satisfying (IC-MKP) with strict inequalities.

Since $\mathcal{F}_0^n$ becomes dense in $\mathcal{F}$ as n grows, it follows that $\mathcal{F}_0^n \cap C^{\mathrm{o}}$ becomes dense in C^{o}. Since C is equal to the closure of C^{o}, it follows that $\mathcal{F}_0^n \cap C^{\mathrm{o}}$ becomes dense in C.

Pick $\epsilon > 0$. Since C is compact, it is covered with finitely many balls of radius ϵ, and since $\mathcal{F}_0^n \cap C^{\mathrm{o}}$ becomes dense in C, for n large enough and for all $\mathbf{d} \in C$, there exists $\mathbf{d}' \in \mathcal{F}_0^n \cap C^{\mathrm{o}}$ such that $\|\mathbf{d} - \mathbf{d}'\| \le \epsilon$, where $\| \cdot \|$ is the Euclidean distance on $\mathbb{R}^{\mathcal{M}}$.

Hence, for n large enough and for every $\mathbf{d} \in \operatorname{supp} \widehat{\mu} \cap C$ we can associate $f(\mathbf{d}) = \mathbf{d}' \in \mathcal{F}_0^n \cap C^{\mathrm{o}}$ such that $\|\mathbf{d} - f(\mathbf{d})\| \le \epsilon$.

Let $\mathbf{d}_{\neg C} \equiv \mathbb{E}_{\widehat{\mu}}[\mathbf{d} \,|\, \mathbf{d} \notin C]$ denote the weighted average of $\mathbf{d}$ under $\widehat{\mu}$ conditional on $\mathbf{d} \notin C$ (i.e., $\mathbf{d}$ not competitive). For n large enough, there exists $f(\mathbf{d}_{\neg C}) \in \mathcal{F}_0^n$ such that $\|f(\mathbf{d}_{\neg C}) - \mathbf{d}_{\neg C}\| \le \epsilon$.

Similarly, consider the sample demands $\widehat{\mathbf{D}}$. For n large enough, we can find $f(\widehat{\mathbf{D}}) \in \mathcal{F}_0^n$ such that $\|f(\widehat{\mathbf{D}}) - \widehat{\mathbf{D}}\| \le \epsilon$. We assume for simplicity that $\widehat{\mathbf{D}} \neq \mathbf{d}_{\neg C}$.

Finally consider distributions $\mu_n \in \Delta(\mathcal{F}_0^n)$ such that

$$\mu_n \circ f(\widehat{\mathbf{D}}) = \nu,$$
$$\mu_n \circ f(\mathbf{d}_{\neg C}) = (1-\nu) \times \widehat{\mu}(\mathcal{F} - C),$$
$$\forall \mathbf{d} \in C, \mu_n \circ f(\mathbf{d}) = (1-\nu) \times \widehat{\mu}(\mathbf{d}),$$

where $\nu > 0$.[5]

For each $\nu > 0$, we can find $\epsilon > 0$ small and $\overline{N}$ large such that, for all $n \geq \overline{N}$, $\mathbb{E}_{\mu_n}[\mathbf{d}] \in \mathcal{D}_\alpha$. In addition, by construction $\mu_n \in \Delta(\mathcal{F}_0^n)$ and $\mathbb{E}_{\mu_n}[\text{IsComp}(\mathbf{d})] \geq (1-\nu) \times \mathbb{E}_{\widehat{\mu}}[\text{IsComp}(\mathbf{d})] = (1-\nu)\widehat{s}_{\text{comp}}$. Since $\widehat{s}^n_{\text{comp}} \leq \widehat{s}_{\text{comp}}$ by construction, this implies that $\lim_{n\to\infty} \widehat{s}^n_{\text{comp}} = \widehat{s}_{\text{comp}}$. *Q.E.D.*

PROOF OF LEMMA 2: Along the lines of the proof of Proposition 1, we define

$$\varepsilon_t \equiv \sum_{h_{i,t}} \langle \lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}} \rangle \quad \text{and} \quad S_T \equiv \sum_{t=0}^{T} \varepsilon_t,$$

where for each history $h_{i,t}$, $\widehat{\mathbf{d}}_{h_{i,t}} = (\mathbf{1}_{\wedge \mathbf{b}_{-i,t} > (1+\rho_n) b_{i,t}})_{n\in\mathcal{M}}$. S_T is a sum of martingale increments ε_t whose absolute value is bounded by $\|\lambda\|_1 \overline{N}_t$, where $\|\lambda\|_1 \equiv \sum_{n\in\mathcal{M}} |\lambda_n|$ and $\overline{N}_t$ denotes the number of bidders participating at time t with histories in H.

The Azuma–Hoeffding inequality implies that

$$\text{prob}\big(S_T > x_\lambda |H|\big) \leq \exp\left(-\frac{x_\lambda^2 |H|^2}{2\|\lambda\|_1^2 \sum_{t=0}^{T} \overline{N}_t^2}\right).$$

Observing that $\sum_{t=0}^{T} \overline{N}_t^2 \leq \sum_{t=0}^{T} \overline{N}_t N_{\max} = N_{\max}|H|$, this implies that

$$\text{prob}\big(S_T > x_\lambda |H|\big) \leq \exp\left(-\frac{x_\lambda^2 |H|}{2\|\lambda\|_1^2 N_{\max}}\right).$$

This concludes the proof. *Q.E.D.*

PROOF OF LEMMA 3: As in the proof of Lemma 2, we define

$$\varepsilon_t \equiv \sum_{h_{i,t}} \langle \lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}} \rangle \quad \text{and} \quad S_T \equiv \sum_{t=0}^{T} \varepsilon_t = |H| \big\langle \lambda, \mathbb{E}_{\mu^*}[\mathbf{d}] - \widehat{\mathbf{D}} \big\rangle.$$

S_T is a sum of martingale increments ε_t whose absolute value is bounded by $\|\lambda\|_1 N_{\max}$. This implies that the central limit theorem for sums of martingale increments holds (Billingsley (1995), Theorem 35.11):

$$\lim_{T\to\infty} \text{prob}\left(\frac{1}{\sigma_\lambda \sqrt{T+1}} S_T \geq x\right) = 1 - \Phi(x) = \Phi(-x)$$

[5]If $\widehat{\mathbf{D}} = \mathbf{d}_{\neg C}$, we can define μ_n to be such that: $\mu_n \circ f(\mathbf{d}_{\neg C}) = \nu + (1-\nu) \times \widehat{\mu}(\mathcal{F} - C)$, and such that $\forall \mathbf{d} \in C$, $\mu_n \circ f(\mathbf{d}) = (1-\nu) \times \widehat{\mu}(\mathbf{d})$.

with

$$\sigma_\lambda \equiv \sqrt{\frac{1}{T+1}\sum_{t=0}^{T}\mathsf{var}\left(\sum_{h_{i,t}\in H}\langle\lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}}\rangle | h_t^0\right)}.$$

We cannot directly exploit this result to get explicit bounds on the distribution of $\langle\lambda, \mathbb{E}_{\mu^*}[\mathbf{d}] - \widehat{\mathbf{D}}\rangle$ because $\mathbf{d}_{h_{i,t}}$ is not directly observed, so that we cannot form a consistent estimate of σ_λ. Instead, we show that

$$\widehat{\sigma}_\lambda = \sqrt{\frac{1}{T+1}\sum_{t=0}^{T}\left|\{h_{i,t}\in H\}\right|\sum_{h_{i,t}\in H}\langle\lambda, \widehat{\mathbf{D}}_t - \widehat{\mathbf{d}}_{h_{i,t}}\rangle^2}.$$

can be used as an asymptotic upper bound to σ_λ.

Indeed, for any period t, Jensen's inequality implies that

$$\begin{aligned}\mathsf{var}\left(\sum_{h_{i,t}\in H}\langle\lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}}\rangle | h_t^0\right) &= \left|\{h_{i,t}\in H\}\right|^2\mathsf{var}\left(\frac{1}{\left|\{h_{i,t}\in H\}\right|}\sum_{h_{i,t}\in H}\langle\lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}}\rangle | h_t^0\right)\\ &\le \left|\{h_{i,t}\in H\}\right|\sum_{h_{i,t}\in H}\mathsf{var}\big(\langle\lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}}\rangle | h_t^0\big).\end{aligned}$$

Furthermore, since $\mathbf{d}_{h_{i,t}} = \mathbb{E}[\widehat{\mathbf{d}}_{h_{i,t}} | h_{i,t}]$, and since $h_{i,t}$ includes all the information provided in history h_t^0, it follows that for any h_t^0-measurable random variable $Z_t \in \mathbb{R}^{\mathcal{M}}$,

$$\mathsf{var}\big(\langle\lambda, \mathbf{d}_{h_{i,t}} - \widehat{\mathbf{d}}_{h_{i,t}}\rangle | h_t^0\big) \le \mathbb{E}\big[\langle\lambda, Z_t - \widehat{\mathbf{d}}_{h_{i,t}}\rangle^2 | h_t^0\big].$$

The law of large number for martingale increments implies that almost surely,

$$\lim_{T\to\infty}\frac{1}{T+1}\sum_{t=0}^{T}\sum_{h_{i,t}\in H}\mathbb{E}\big[\langle\lambda, Z_t - \widehat{\mathbf{d}}_{h_{i,t}}\rangle^2 | h_t^0\big] - \langle\lambda, Z_t - \widehat{\mathbf{d}}_{h_{i,t}}\rangle^2 = 0.$$

Hence, setting $Z_t = \widehat{\mathbf{D}}_t$, it follows that for any $\epsilon > 0$, almost surely as T becomes large, $(1+\epsilon)\widehat{\sigma}_\lambda \ge \sigma_\lambda$. Since $x > 0$, this implies that

$$\begin{aligned}\limsup \mathsf{prob}\left(\frac{1}{\widehat{\sigma}_\lambda\sqrt{T+1}}S_T \ge x\right) &\le \limsup \mathsf{prob}\left(\frac{1}{\sigma_\lambda\sqrt{T+1}}S_T \ge x(1+\epsilon)\right)\\ &= \Phi\big(-x(1+\epsilon)\big).\end{aligned}$$

Observing that $\frac{1}{\widehat{\sigma}_\lambda\sqrt{T+1}}S_T = \frac{|H|}{\widehat{\sigma}_\lambda\sqrt{T+1}}\langle\lambda, \mathbb{E}_{\mu^*}[\mathbf{d}] - \widehat{\mathbf{D}}\rangle$, Lemma 3 follows from the fact that the result holds for any $\epsilon > 0$, and Φ is continuous. *Q.E.D.*

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