Econometrica Supplementary Material

SUPPLEMENT TO "DYNAMIC NOISY RATIONAL EXPECTATIONS EQUILIBRIUM WITH INSIDER INFORMATION"

(*Econometrica*, Vol. 88, No. 6, November 2020, 2697–2737)

JEROME DETEMPLE
Questrom School of Business, Boston University

MARCEL RINDISBACHER
Questrom School of Business, Boston University

SCOTT ROBERTSON
Questrom School of Business, Boston University

This supplementary document contains the proofs of many results in the main paper as well as both statements and proofs of a number of technical lemmas.

S1. LEMMAS REGARDING Θ

WE FIRST PROVE THREE LEMMAS regarding Θ from (4.1),[1] valid under Assumption 2.1. For $M \in \mathbb{S}^d_{++}$ write $0 < \underline{M} < \overline{M}$ as the smallest and largest eigenvalues. Additionally, the bounding constant $\check{K} > 0$ below may change from line to line.

LEMMA S1.1: *There exists $\hat{K} > 0$ such that for $x \in E$, $y \in \mathbb{R}^d$, (i) $\Theta(x, y) \geq -\hat{K}(1 + y'y)$ and (ii) $|\Theta(x, y)| \leq \hat{K}(1 + x'x + y'y)$.*

PROOF: Let $0 < \delta < \underline{M}$. By part (iii) of Assumption 2.1,

$$\Theta(x, y) \geq -K_2\big(1 + |x|^{2-\varepsilon_1}\big) + \frac{1}{2}(\underline{M} - \delta)|x|^2 + \frac{1}{2}\delta x'x + x'\zeta - x'\tilde{M}y.$$

Clearly, $-K_2(1 + |x|^{2-\varepsilon_1}) + (1/2)(\underline{M} - \delta)|x|^2 \geq -\check{K}$. As $\delta x'x - 2x'(\tilde{M}y - \zeta) \geq -(1/\delta) \times (\tilde{M}y - \zeta)'(\tilde{M} - \zeta)$, the Cauchy–Schwarz inequality implies this may further be bounded from below by $-(2\overline{\tilde{M}'\tilde{M}}/\delta) \times y'y - (2/\delta)\zeta'\zeta$. Part (i) now readily follows. Part (ii) is obvious from Assumption 2.1 part (ii) and the Cauchy–Schwarz inequality. *Q.E.D.*

LEMMA S1.2: *The maps $y \to e^{-\gamma\Theta(X_T, y)}$ and $y \to \Psi(X_T)e^{-\gamma\Theta(X_T, y)}$ are analytic from $\mathbb{R}^d$ to $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^d)$, respectively.*

PROOF: Part (i) of Lemma S1.1 implies $e^{-\gamma\Theta(X_T, y)} \in L^1(\mathbb{R})$ for all $y \in \mathbb{R}^d$. Next, using the multidimensional Taylor theorem, for any fixed $g \in \mathbb{R}^d$ we may write

$$e^{-\gamma\theta(x,y)} = e^{-\gamma(\Pi'\Psi(x) + \frac{1}{2}x'Mx + x'\zeta)}e^{\gamma y'\tilde{M}x} = \sum_{\alpha} A_\alpha(x, g)(y - g)^\alpha,$$

Jerome Detemple: detemple@bu.edu
Marcel Rindisbacher: rindisbm@bu.edu
Scott Robertson: scottrob@bu.edu

[1]Note: References to the main paper do not include an "S."

 https://doi.org/10.3982/ECTA17038

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $(y - g)^\alpha = \prod_{i=1}^d (y_i - g_i)^{\alpha_i}$, and

$$A_\alpha(x, g) = e^{-\gamma(\Pi'\Psi(x)+\frac{1}{2}x'Mx+x'\zeta)} \frac{1}{|\alpha|!} D_\alpha\big(e^{\gamma y'\tilde{M}x}\big)\big|_{y=g} = e^{-\gamma\Theta(x,g)} \frac{1}{|\alpha|!} \gamma^{|\alpha|} (\tilde{M}x)^\alpha.$$

Above, $D_\alpha f$ is the $|\alpha|$th partial derivative of f, α_i times with respect to y_i for $i = 1, \dots, d$. Since $|x^\alpha| \le |x|^{|\alpha|}$ and $|\tilde{M}x| \le \overline{\tilde{M}}|x|$, it follows that

$$\big|(\tilde{M}x)^\alpha\big| \le |\tilde{M}x|^{|\alpha|} \le (\overline{\tilde{M}})^{|\alpha|}|x|^{|\alpha|}.$$

Let ε_0 be from Assumption 2.1 and write $M_* = \gamma\overline{\tilde{M}}/(\varepsilon_0)$. If $|\alpha| = n$, then by part (i) of Lemma S1.1 and the bound $|x|^n \le e^{\varepsilon_0|x|}(n/\varepsilon_0)^n e^{-n}$,

$$\mathbb{E}\big[\big|A_\alpha(X_T, g)\big|\big] \le e^{\gamma\hat{K}(1+g'g)} \frac{(nM_*)^n}{n!e^n} \mathbb{E}\big[e^{\varepsilon_0|X_T|}\big] \le \check{K} e^{\gamma\hat{K}(1+g'g)} \frac{(M_*)^n}{\sqrt{n}} \mathbb{E}\big[e^{\varepsilon_0|X_T|}\big], \tag{S1.1}$$

where the last inequality follows from Stirling's formula. Thus, if there is $L > 1$ such that $\max_{i=1,\dots,d} |y_i - g_i| \le 1/(LM_*)$, then for $N = 1, 2, \dots,$

$$\begin{aligned}
\mathbb{E}\Big[\Big|\sum_{n\ge N}\sum_{|\alpha|=n} A_\alpha(X_T, g)(y-g)^\alpha\Big|\Big] &\le \sum_{n\ge N}\sum_{|\alpha|=n} \mathbb{E}\big[\big|A_\alpha(X_T, g)\big|\big] \prod_{i=1}^d |y_i - g_i|^{\alpha_i} \\
&\le \check{K} e^{\gamma\hat{K}(1+g'g)} \mathbb{E}\big[e^{\varepsilon_0|X_T|}\big] \sum_{n\ge N}\sum_{|\alpha|=n} \frac{1}{\sqrt{n}} L^{-n} \\
&= \check{K} e^{\gamma\hat{K}(1+g'g)} \mathbb{E}\big[e^{\varepsilon_0|X_T|}\big] \sum_{n\ge N} \frac{1}{\sqrt{n}} L^{-n} \binom{n+d-1}{n} \\
&\le \check{K} e^{\gamma\hat{K}(1+g'g)} \mathbb{E}\big[e^{\varepsilon_0|X_T|}\big] \sum_{n\ge N} L^{-n} n^{d-\frac{1}{2}}.
\end{aligned}$$

The right hand side of the last inequality goes to 0 as $N \uparrow \infty$, which proves $y \to e^{-\gamma\Theta(X_T,y)}$ is an analytic map from $\mathbb{R}^d$ to $L^1(\mathbb{R})$. We next consider $y \to \Psi(X_T)e^{-\gamma\Theta(X_T,y)}$. As $\Psi(X_T)$ does not depend on y, the proof is very similar and we only show the differences. First, that $\Psi(X_T)e^{-\gamma\Theta(X_T,y)} \in L^1(\mathbb{R}^d)$ follows from Assumption 2.1 and part (i) of Lemma S1.1, since

$$\big|\Psi(X_T)\big|e^{-\gamma\Theta(X_T,y)} \le K_1\big(1 + X_T'X_T\big)e^{\gamma\hat{K}(1+y'y)} \le \check{K} e^{\varepsilon_0|X_T|} e^{\hat{K}(1+y'y)},$$

where the last inequality uses the estimate

$$x^2 \le \frac{2}{k^2}\big(e^{kx} - 1\big) \le \frac{2}{k^2} e^{kx}, \quad x, k > 0. \tag{S1.2}$$

Next, the analytic convergence proof is the same except in (S1.1) the first line (right hand side) should have $|\Psi(X_T)|$ within the expected value. Then, going from the first to the second line, we use, for $\varepsilon_2, \delta > 0$ such that $\delta + \varepsilon_2 < \varepsilon_0$,

$$\big|\Psi(X_T)\big||X_T|^n \leq K_1\big(1 + X_T' X_T\big)e^{\delta|X_T|}\left(\frac{n}{\delta}\right)^n e^{-n} \leq \check{K}e^{\varepsilon_0|X_T|}\left(\frac{n}{\delta}\right)^n e^{-n}.$$

From here, the rest of the proof is the same. *Q.E.D.*

LEMMA S1.3: *There exists a constant $\hat{C}$ so that*

$$\frac{\mathbb{E}\big[X_T' X_T e^{-\gamma\Theta(X_T,y)}\big]}{\mathbb{E}\big[e^{-\gamma\Theta(X_T,y)}\big]} \leq \hat{C}\big(1 + \mathbb{E}\big[e^{\varepsilon_0|X_T|}\big] + y'y\big).$$

PROOF: For $\tau \geq 0$, define $f(\tau) := -(1/\gamma)\log(\mathbb{E}[e^{-\gamma\tau\Theta(X_T,y)}])$. By part (i) of Lemma S1.1, $f(\tau) \geq \hat{K}(1 + y'y)\tau$. By Jensen's inequality, part (ii) of Lemma S1.1, and (S1.2) we deduce $f(\tau) \leq \check{K}(1 + y'y + \mathbb{E}[e^{\varepsilon_0|X_T|}])\tau$, and hence f is linearly bounded. The dominated convergence theorem shows f is smooth (see Dembo and Zeitouni (1998, Lemma 2.5)), and Hölder's inequality shows f is concave. Therefore,

$$\dot{f}(1) \leq \lim_{\varepsilon\downarrow 0}\dot{f}(\varepsilon) = \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}\big[\Theta(X_T,y)e^{-\gamma\varepsilon\Theta(X_T,y)}\big]}{\mathbb{E}\big[e^{-\gamma\varepsilon\Theta(X_T,y)}\big]}.$$

Part (i) of Lemma S1.1 readily implies the uniform integrability of $\{e^{-\gamma\varepsilon\Theta(X_T,y)}\}_{\varepsilon>0}$. Parts (i) and (ii) of Lemma S1.1, along with (S1.2) also imply, for $\varepsilon < 1$,

$$\begin{aligned}\big|\Theta(x,y)\big|e^{-\gamma\varepsilon\Theta(x,y)} &= \Theta(x,y)^+ e^{-\gamma\varepsilon\Theta(x,y)}1_{\Theta(x,y)\geq 0} + \Theta(x,y)^- e^{-\gamma\varepsilon\Theta(x,y)}1_{\Theta(x,y)<0} \\ &\leq \big|\Theta(x,y)\big| + \hat{K}\big(1 + y'y\big)e^{\gamma\hat{K}(1+y'y)} \\ &\leq \check{K}\big(y'y + e^{\varepsilon_0|x|}\big) + \hat{K}\big(1 + y'y\big)e^{\gamma\hat{K}(1+y'y)}.\end{aligned}$$

Thus, by Assumption 2.1 part (i) and dominated convergence we conclude that

$$\dot{f}(1) = \frac{\mathbb{E}\big[\Theta(X_T,y)e^{-\gamma\Theta(X_T,y)}\big]}{\mathbb{E}\big[e^{-\gamma\Theta(X_T,y)}\big]} \leq \mathbb{E}\big[\Theta(X_T,y)\big].$$

Plugging in for Θ gives

$$\begin{aligned}&\frac{\mathbb{E}\big[X_T' M X_T e^{-\gamma\Theta(X_T,y)}\big]}{2\mathbb{E}\big[e^{-\gamma\Theta(X_T,y)}\big]} \\ &\quad\leq \mathbb{E}\big[\big|\Theta(X_T,y)\big|\big] - \frac{\mathbb{E}\big[\big(\Pi'\Psi(X_T) + X_T'\zeta - (X_T)'\tilde{M}y\big)e^{-\gamma\Theta(X_T,y)}\big]}{\mathbb{E}\big[e^{-\gamma\Theta(X_T,y)}\big]}.\end{aligned} \tag{S1.3}$$

We claim Assumption 2.1(iii) implies for $\delta > 0$ there exists a constant $\check{K}$ so that

$$\Pi'\Psi(x) + x'\zeta - x'\tilde{M}y \geq -\check{K}\big(1 + y'y\big) - \frac{\delta}{2}x'Mx. \tag{S1.4}$$

Admitting this, taking $\delta = 1/2$, and using (S1.3), we deduce

$$\frac{1}{4}\frac{\mathbb{E}\big[X_T' M X_T e^{-\gamma\Theta(X_T,y)}\big]}{\mathbb{E}\big[e^{-\gamma\Theta(X_T,y)}\big]} \le \check{K}\big(1+y'y\big) + \mathbb{E}\big[\big|\Theta(X_T,y)\big|\big] \le \check{K}\big(1+y'y+\mathbb{E}\big[e^{\varepsilon_0|X_T|}\big]\big),$$

where the last inequality follows from Lemma S1.1 and (S1.2). The result holds since $x'x \le (1/\underline{M})x'Mx$. It thus remains to prove (S1.4). First, for $\delta > 0$,

$$\begin{aligned}\Pi'\Psi(x) &= \Pi'\Psi(x) \pm \frac{\delta}{2}x'Mx \ge -K_2\big(1+|x|^{2-\varepsilon_1}\big) + \frac{\delta\underline{M}}{2T}x'x - \frac{\delta}{2}x'Mx \\ &\ge -\check{K}(\delta) - \frac{\delta}{2}x'Mx.\end{aligned}$$

A similar calculation gives a commensurate lower bound for $x'\zeta$. Equation (S1.4) follows as

$$-x'\tilde{M}y \ge -x'\tilde{M}y + \frac{\delta\underline{M}}{2}x'x - \frac{\delta}{2}x'Mx \ge -\frac{\overline{\tilde{M}'\tilde{M}}}{2\underline{M}^2\delta^2}y'y - \frac{\delta}{2}x'Mx. \qquad \textit{Q.E.D.}$$

S2. LEMMAS REGARDING THE FULL COMMUNICATION EQUILIBRIUM

Throughout, we enforce Assumptions 2.1 and A.1. Recall u from (4.5) and (B.2), and G_I from (2.5). The following lemma shows that u governs the conditional laws of G_I given $\mathbb{F}^B$, as well as the Brownian motion B^m under $\mathbb{F}^B \vee \sigma(G_I)$.

LEMMA S2.1:

(i) *For each $t \le T$, the law of G_I given $\mathcal{F}_t^B$ has pdf $u(t, X_t, \cdot)$. Therefore, $\mathbb{P}[G_I \in \cdot|\mathcal{F}_t^B] \sim \mathbb{P}[G_I \in \cdot]$ almost surely, with density*

$$\begin{aligned}\tilde{p}_t^g &:= \frac{u(t,X_t,g)}{u(0,X_0,g)} = \mathcal{E}\left(\int_0^{\cdot} \big(\tilde{\mu}_u^g\big)' dB_u\right)_t, \\ \tilde{\mu}_t^g &:= a(X_t)'\nabla_x\big(\log\big(u(t,X_t,g)\big)\big).\end{aligned} \tag{S2.1}$$

(ii) *The filtration $\mathbb{F}^m = \mathbb{F}^B \vee \sigma(G_I)$ is right-continuous, $1/\tilde{p}^{G_I}$ is a $(\mathbb{P},\mathbb{F}^m)$ martingale, and the martingale-preserving measure takes the form*

$$\frac{d\widetilde{\mathbb{P}}^{G_I}}{d\mathbb{P}} = \frac{1}{\tilde{p}_T^{G_I}}; \qquad \tilde{p}_\cdot^{G_I} = \mathcal{E}\left(\int_0^{\cdot} \big(\tilde{\mu}_u^{G_I}\big)' dB_u\right)_\cdot. \tag{S2.2}$$

(iii) *The process B is a $(\widetilde{\mathbb{P}}^{G_I},\mathbb{F}^m)$ Brownian motion with the predictable representation property (PRP), and $B_\cdot^m := B_\cdot - \int_0^\cdot \tilde{\mu}_u^{G_I}\,du$ is a $(\mathbb{P},\mathbb{F}^m)$ Brownian motion on $[0,T]$ with the PRP.*

PROOF: Let $\phi \in C_c^\infty(\mathbb{R}^d)$, $t \le T$, and set $Y^I = 1/\sqrt{T}\sqrt{C_I}W^I$ for a d-dimensional Brownian motion independent of B, and note that Y^I is a Markov process with transition kernel

$$p_{C_I}(\tau,x,y) = \frac{1}{(2\pi)^{d/2}\sqrt{|C_I|}}\sqrt{\frac{T}{\tau}}e^{-\frac{T}{2\tau}(y-x)'C_I^{-1}(y-x)}, \quad \tau > 0, x, y \in \mathbb{R}^d.$$

Similarly to (7.8) set $\hat{p}_{C_I}(y) = p_{C_I}(T, 0, y)$. B,y the tower property,

$$\mathbb{E}\big[\phi(G_N)|\mathcal{F}^B_t\big] = \mathbb{E}\big[\mathbb{E}\big[\phi\big(X_T + Y^I_T\big)|\mathcal{F}^{B,W^I}_t\big]|\mathcal{F}^B_t\big].$$

Using the Markov property yields

$$\mathbb{E}\big[\phi\big(X_T + Y^I_T\big)|\mathcal{F}^{B,W^I}_t\big] = \int \phi(x+y)p(T-t, X_t, x)p_{C_I}\big(T-t, Y^I_t, y\big)\,dx\,dy,$$

where the integration region is $E \times \mathbb{R}^d$. Therefore, by the independence of B and Y^I,

$$\begin{aligned}\mathbb{E}\big[\phi(G_N)|\mathcal{F}^B_t\big] &= \int \phi(x+y)p(T-t, X_t, x)\mathbb{E}\big[p_{C_I}\big(T-t, Y^I_t, y\big)\big]\,dx\,dy \\ &= \int \phi(x+y)p(T-t, X_t, x)\hat{p}_{C_I}(y)\,dx\,dy \\ &= \int \phi(g)\bigg(\int p(T-t, X_t, x)\hat{p}_{C_I}(g-x)\,dx\bigg)\,dg \\ &= \int \phi(g)u(t, X_t, g)\,dg.\end{aligned}$$

Above the second equality holds by the Chapman–Kolmogorov equations. This shows that given $\mathcal{F}^B_t$, G_I has pdf $u(t, X_t, \cdot)$. As $\mathcal{F}^B_0$ is trivial, the Jacod equivalence condition and the first equality in (S2.1) follow. The second equality in (S2.1) follows from (B.3) and Ito's formula, finishing (i). As for (ii), the right-continuity of $\mathbb{F}^m$ and that $1/\tilde{p}^{G_I}$ is a $(\mathbb{P}, \mathbb{F}^m)$ martingale follow from Lemma S4.3, while the second equality in (S2.2) follows from Proposition S4.6. Last, the statement regarding B in part (iii) follows from Fontana (2018, Proposition 2.9) and the statement regarding B^m follows from Fontana (2018, Corollary 2.10). *Q.E.D.*

S3. LEMMAS REGARDING THE PARTIAL COMMUNICATION EQUILIBRIUM

We first prove lemmas in the Markovian noise setting. Assumptions 2.1, 7.1, 7.2, and A.1 are in force. Recall the signal H and market filtration $\mathbb{F}^m$ from Assumption 7.2, and the function ℓ from (4.5) and (7.2). The first lemma collects facts about $\mathbb{F}^m$ and the martingale-reserving measure $\widetilde{\mathbb{P}}^H$ of (7.4).

LEMMA S3.1:

(i) *For each $t \leq T$, the law of H given $\mathcal{F}^B_t$ has pdf $\ell(t, X_t, \cdot)$. In particular, $\mathbb{P}[H \in \cdot|\mathcal{F}^B_t] \sim \mathbb{P}[H \in \cdot]$ almost surely, with density*

$$\begin{aligned}p^h_t &:= \frac{\ell(t, X_t, h)}{\ell(0, X_0, h)} = \mathcal{E}\bigg(\int_0^\cdot \big(\mu^h_u\big)'\,dB_u\bigg)_t, \\ \mu^h_t &:= a(X_t)'\nabla_x\big(\log\big(\ell(t, X_t, h)\big)\big).\end{aligned} \tag{S3.1}$$

(ii) *The filtration $\mathbb{F}^m$ is right-continuous, $1/p^H$ is a $(\mathbb{P}, \mathbb{F}^m)$ martingale, and the martingale-reserving measure takes the form*

$$\frac{d\widetilde{\mathbb{P}}^H}{d\mathbb{P}} = \frac{1}{p^H_T}, \qquad p^H_\cdot = \mathcal{E}\bigg(\int_0^\cdot \big(\mu^H_u\big)'\,dB_u\bigg)_\cdot.$$

(iii) *The process B is a $(\widetilde{\mathbb{P}}^H, \mathbb{F}^m)$ Brownian motion with the PRP, and $B^m_\cdot := B_\cdot - \int_0^\cdot \mu^H_u \, du$ is a $(\mathbb{P}, \mathbb{F}^m)$ Brownian motion with the PRP.*

PROOF: For (i), let $\phi \in C_c^\infty(\mathbb{R}^d)$ and $t \le T$. By the tower property,

$$\mathbb{E}\big[\phi(H)|\mathcal{F}_t^B\big] = \mathbb{E}\big[\mathbb{E}\big[\phi\big(H\big(X_T + Y_T^I, \tau_N\big(X_T + Y_T^I\big) + Y_T^N\big)\big)|\mathcal{F}_t^{B,W^I,W^N}\big]|\mathcal{F}_t^B\big].$$

Using the Markov property yields

$$\begin{aligned}&\mathbb{E}\big[\phi\big(H\big(X_T + Y_T^I, \tau_N\big(X_T + Y_T^I\big) + Y_T^N\big)\big)|\mathcal{F}_t^{B,W^I,W^N}\big] \\ &\quad = \int \phi\big(H\big(x+y, \tau_N(x+y) + \tilde{y}\big)\big) p(T-t, X_t, x) p_I\big(t, Y_t^I, T, y\big) \\ &\qquad \times p_N\big(t, Y_t^N, T, \tilde{y}\big)\, dx\, dy\, d\tilde{y},\end{aligned}$$

where we integrate over $E \times \mathbb{R}^d \times \mathbb{R}^d$. The independence of (Y^I, Y^N, B), along with the Chapman–Kolmogorov equations imply

$$\mathbb{E}\big[p(T-t, X_t, x) p_I\big(t, Y_t^I, T, y\big) p_N\big(t, Y_t^N, T, \tilde{y}\big)|\mathcal{F}_t^B\big] = p(T-t, X_t, x)\hat{p}_I(y)\hat{p}_N(\tilde{y}).$$

Therefore,

$$\mathbb{E}\big[\phi(H)|\mathcal{F}_t^B\big] = \int \phi\big(H\big(x+y, \tau_N(x+y)+\tilde{y}\big)\big) p(T-t, X_t, x)\hat{p}_I(y)\hat{p}_N(\tilde{y})\, dx\, dy\, d\tilde{y}.$$

With x and $\tilde{y}$ fixed, letting $g = x + y$ gives

$$\int \phi\big(H(g, \tau_N g + \tilde{y})\big) p(T-t, X_t, x)\hat{p}_I(g-x)\hat{p}_N(\tilde{y})\, dx\, dg\, d\tilde{y},$$

and we are integrating over $E \times \mathbb{R}^d \times \mathbb{R}^d$. Next, with x and g fixed, let $h = H(g, \tau_N g + \tilde{y})$ so $\tilde{y} = G(g, h) - \tau_N g$, $d\tilde{y} = |J^G|(g, h)\, dh$, and, by Assumption 7.2, h takes values in $\mathcal{R}_H$ for an integration region of $E \times \mathbb{R}^d \times \mathcal{R}_H$. This yields

$$\begin{aligned}&\int \phi(h) p(T-t, X_t, x)\hat{p}_I(g-x)\hat{p}_N\big(G(g,h) - \tau_N g\big)\big|J^G\big|(g,h)\, dx\, dg\, dh \\ &\quad = \int \phi(h)\bigg(\int p(T-t, X_t, x)\hat{p}_I(g-x)\hat{p}_N\big(G(g,h)-\tau_N g\big)\big|J^G\big|(g,h)\, dx\, dg\bigg) dh \\ &\quad = \int \phi(h)\ell(t, X_t, h)\, dh.\end{aligned}$$

Thus, given $\mathcal{F}_t^B$, H has pdf $\ell(t, X_t, \cdot)$. As $\mathcal{F}_0^B$ is trivial, the Jacod equivalence condition and first equality in (S3.1) readily follow. The second equality in (S3.1) holds by Ito's formula, since the PDE for u in (B.3) implies that for a fixed h, ℓ solves $\ell_t + L\ell = 0$ on $(0, T) \times E$. This finishes item (i). The statements in parts (ii) and (iii) follow from the exact same argument used to prove parts (ii) and (iii) in Lemma S2.1. *Q.E.D.*

We next prove similar results for $\mathbb{F}^I = \mathbb{F}^m \vee \mathfrak{s}(G_I)$ with G_I from Assumption 7.1.

LEMMA S3.2:

(i) *For each* $t \le T$, $\mathbb{P}[G_I \in \cdot|\mathcal{F}_t^m] \sim \mathbb{P}[G_I \in \cdot]$ *almost surely with density*

$$p_t^{H,g} := \frac{\hat{p}_N\big(G(g,H) - \tau_N g\big)\big|J^G\big|(g,H)}{\ell(t,X_t,H)} \times \frac{u(t,X_t,g)}{u(0,X_0,g)}, \quad g \in \mathbb{R}^d. \tag{S3.2}$$

In particular, with $\tilde{p}^g$ *and* $\tilde{\mu}^g$ *from* (S2.1), *and* p^H *and* μ^H *from* (S3.1), *we have*

$$\begin{aligned} \mathring{p}_t^{H,g} &:= \frac{p_t^{H,g}}{p_0^{H,g}} = \frac{u(t,X_t,g)}{u(0,X_0,g)} \times \frac{\ell(0,X_0,H)}{\ell(t,X_t,H)} \\ &= \frac{\tilde{p}_t^g}{p_t^H} = \mathcal{E}\bigg(\int_0^\cdot \big(\tilde{\mu}_u^g - \mu_u^H\big)' \, dB_u^m\bigg)_t. \end{aligned} \tag{S3.3}$$

(ii) *The filtration* $\mathbb{F}^I$ *is right-continuous*, $1/p^{H,G_I}$ *is a* $(\mathbb{P},\mathbb{F}^I)$ *martingale, and the* ($\mathbb{F}^m$ *to* $\mathbb{F}^I$) *martingale-preserving measure* $\widetilde{\mathbb{P}}^{H,G_I}$ *for* G_I *is defined by*

$$\begin{aligned} &\frac{d\widetilde{\mathbb{P}}^{H,G_I}}{d\mathbb{P}} := \frac{1}{p_T^{H,G_I}}, \qquad p_\cdot^{H,G_I} = p_0^{H,G_I}\mathcal{E}\bigg(\int_0^\cdot \big(\mu_u^{H,G_I}\big)' \, dB_u^m\bigg)_\cdot, \\ &\mu_\cdot^{H,g} := \tilde{\mu}_\cdot^g - \mu_\cdot^H. \end{aligned} \tag{S3.4}$$

(iii) *The process* B^m *is a* $(\widetilde{\mathbb{P}}^{m,G_I},\mathbb{F}^I)$ *Brownian motion with the PRP, and* $B_\cdot^I := B_\cdot^m - \int_0^\cdot \mu_u^{H,G_I}\, du = B_\cdot - \int_0^\cdot \tilde{\mu}_u^{G_I}\, du$ *is a* $(\mathbb{P},\mathbb{F}^I)$ *Brownian motion with the PRP.*

PROOF: We start with part (i). As these calculations are similar to those in Lemmas S2.1 and S3.1 we will typically omit explanations. Let $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ and $A_t \in \mathbb{F}_t^B$ for $t \le T$. First

$$\begin{aligned} &\mathbb{E}\big[1_{A_t}\phi(G_I)\psi(H)\big] \\ &\quad = \mathbb{E}\big[1_{A_t}\mathbb{E}\big[\phi\big(X_T + Y_T^I\big)\psi\big(H\big(X_T + Y_T^I, \tau_N\big(X_T + Y_T^I\big) + Y_T^N\big)\big)|\mathcal{F}_t^{B,W^I,W^N}\big]\big]. \end{aligned}$$

Next,

$$\begin{aligned} &\mathbb{E}\big[\phi\big(X_T + Y_T^I\big)\psi\big(H\big(X_T + Y_T^I, \tau_N\big(X_T + Y_T^I\big) + Y_T^N\big)\big)|\mathcal{F}_t^{B,W^I,W^N}\big] \\ &\quad = \int \phi(x+y)\psi\big(H\big(x+y, \tau_N(x+y) + \tilde{y}\big)\big)p(T-t,X_t,x)\hat{p}_I(y)\hat{p}_N(\tilde{y})\,dx\,dy\,d\tilde{y} \\ &\quad = \int \phi(g)\psi\big(H(g, \tau_N g + \tilde{y})\big)p(T-t,X_t,x)\hat{p}_I(g-x)\hat{p}_N(\tilde{y})\,dx\,dg\,d\tilde{y} \\ &\quad = \int \phi(g)\psi\big(H(g, \tau_N g + \tilde{y})\big)u(t,X_t,g)\hat{p}_N(\tilde{y})\,dg\,d\tilde{y}. \end{aligned}$$

With g fixed, set $h = H(g, \tau_N g + \tilde{y})$ so that $\tilde{y} = G(g,h) - \tau_N g$, $d\tilde{y} = |J^G|(g,h)\,dh$. Additionally multiplying by 1_{A_t} and taking expectations yields

$$\begin{aligned} &\mathbb{E}\big[1_{A_t}\phi(G_I)\psi(H)\big] \\ &\quad = \mathbb{E}\bigg[1_{A_t}\int \phi(g)\psi(h)\hat{p}_N\big(G(g,h) - \tau_N g\big)\big|J^G\big|(g,h)u(t,X_t,g)\,dg\,dh\bigg]. \end{aligned} \tag{S3.5}$$

By Lemma S3.1 we know that given $\mathcal{F}_t^B$, H has pdf $\ell(t, X_t, \cdot)$. Therefore, for any suitably measurable and integrable function χ,

$$\mathbb{E}\big[1_{A_t}\chi(t, X_t, H)\psi(H)\big] = \mathbb{E}\bigg[1_{A_t}\int \psi(h)\chi(t, X_t, h)\ell(t, X_t, h)\, dh\bigg].$$

Thus, with

$$\chi(t, x, h) = \frac{1}{\ell(t, x, h)} \times \int \phi(g)\hat{p}_N\big(G(g, h) - \tau_N g\big)\big|J^G\big|(g, h)u(t, x, g)\, dg,$$

we see that $\mathbb{E}[1_{A_t}\phi(G_I)\psi(H)] = \mathbb{E}[1_{A_t}\chi(t, X_t, H)\psi(H)]$ for all A_t, ψ, and hence

$$\mathbb{E}\big[\phi(G_I)|\mathcal{F}_t^m\big] = \chi(t, X_t, H) = \int \phi(g)\frac{\hat{p}_N\big(G(g, H) - \tau_N g\big)\big|J^G\big|(g, H)u(t, X_t, g)}{\ell(t, X_t, H)}\, dg,$$

so that given $\mathcal{F}_t^m$, G_I has pdf

$$\frac{\hat{p}_N\big(G(g, H) - \tau_N g\big)\big|J^G\big|(g, H)u(t, X_t, g)}{\ell(t, X_t, H)}.$$

This shows that $p_t^{H,g}$ is the density of $\mathbb{P}[G_I \in \cdot|\mathcal{F}_t^m]$ with respect to $\mathbb{P}[G_I \in \cdot]$ since G_I has unconditional pdf $u(0, X_0, \cdot)$. The statement in (S3.3) follows from (S2.1), (S3.1), and part (iii) of Lemma S3.1. The statements in (ii) follow from Lemma S4.3 and Proposition S4.6. The statements in (iii) follow from Fontana (2018, Proposition 2.9 and Corollary 2.10). *Q.E.D.*

Continuing, we prove results about the filtration $\mathbb{F}^N = \mathbb{F}^m \vee \mathfrak{s}(G_N)$ for G_N from Assumption 7.2. To state the lemma, recall the pdf for G_N given in (C.1).

LEMMA S3.3:

(i) *For each* $t \leq T$, $\mathbb{P}[G_N \in \cdot|\mathcal{F}_t^m] \sim \mathbb{P}[G_N \in \cdot]$ *almost surely with density*

$$p_{N,t}^{H,g} := \frac{u\big(t, X_t, \check{G}(g, H)\big)\big|J^{\check{G}}\big|(g, H)\,\hat{p}_N\big(g - \tau_N\check{G}(g, H)\big)}{\ell(t, X_t, H)u_N(g)}. \tag{S3.6}$$

In particular, with $p^{H,g}$ *from* (S3.2),

$$\frac{p_t^{H,G_I}}{p_{N,t}^{H,G_N}} = \frac{p_0^{G_I,H}}{p_{N,0}^{G_N,H}} = \frac{\big|J^G\big|(G_I, H)u_N(G_N)}{\big|J^{\check{G}}\big|(G_N, H)u(0, X_0, G_I)}. \tag{S3.7}$$

(ii) *We have* $\mathbb{F}^N = \mathbb{F}^I$.

PROOF: We start with (i). These calculations are very similar to those in Lemma S3.2 and as such, we will not include all the steps. Let $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ and $A_t \in \mathbb{F}_t^B$ for $t \leq T$.

First

$$\begin{aligned}
&\mathbb{E}\big[\phi(G_N)\psi(H)|\mathcal{F}_t^{B,W^I,W^N}\big] \\
&\quad = \mathbb{E}\big[\phi\big(\tau_N\big(X_T + Y_T^I\big) + Y_T^N\big)\psi\big(H\big(X_T + Y_T^I, \tau_N\big(X_T + Y_T^I\big) + Y_T^N\big)\big)|\mathcal{F}_t^{B,W^I,W^N}\big] \\
&\quad = \int \phi(\tau_N g + \tilde{y})\psi\big(H(g, \tau_N g + \tilde{y})\big) p(T-t, X_t, x)\hat{p}_I(g-x)\hat{p}_N(\tilde{y})\, dx\, dg\, d\tilde{y} \\
&\quad = \int \phi(\tau_N g + \tilde{y})\psi\big(H(g, \tau_N g + \tilde{y})\big) u(t, X_t, g)\hat{p}_N(\tilde{y})\, dg\, d\tilde{y} \\
&\quad = \int \phi(z)\psi\big(H(g, z)\big) u(t, X_t, g)\hat{p}_N(z - \tau_N g)\, dg\, dz.
\end{aligned}$$

For z fixed, set $h = H(g, z)$ so that $g = \check{G}(z, h)$ and $dg = |J^{\check{G}}|(z, h)\, dh$. This leads to

$$\int \phi(z)\psi(h) u\big(t, X_t, \check{G}(z, h)\big)\big|J^{\check{G}}\big|(z, h)\hat{p}_N\big(z - \tau_N \check{G}(z, h)\big)\, dz\, dh.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\big[1_{A_t}\phi(G_N)\psi(H)\big] \\
&\quad = \mathbb{E}\bigg[1_{A_t}\int \phi(z)\psi(h) u\big(t, X_t, \check{G}(z, h)\big)\big|J^{\check{G}}\big|(z, h)\hat{p}_N\big(z - \tau_N \check{G}(z, h)\big)\, dz\, dh\bigg].
\end{aligned}$$

Repeating the analogous steps as in Lemma S3.2, we deduce that

$$\mathbb{E}\big[\phi(G_N)|\mathcal{F}_t^m\big] = \int \phi(z)\frac{u\big(t, X_t, \check{G}(z, H)\big)\big|J^{\check{G}}\big|(z, H)\hat{p}_N\big(z - \tau_N \check{G}(z, H)\big)}{\ell(t, X_t, H)}\, dz,$$

so that given F_t^m, G_N has pdf (replacing g with z)

$$\frac{u\big(t, X_t, \check{G}(g, H)\big)\big|J^{\check{G}}\big|(g, H)\hat{p}_N\big(g - \tau_N \check{G}(g, H)\big)}{\ell(t, X_t, H)}$$

and hence (S3.6) follows, as u_N is the pdf for G_N. The identity in (S3.7) is immediate. Last, $\mathbb{F}^N = \mathbb{F}^I$ follows because $H(x, y)$ is invertible in both x and y. *Q.E.D.*

With all the preparatory lemmas in place, we prove Theorem 7.3. Note that by Assumption 7.1, $\hat{p}_I$, and hence u, is bounded from above. Similarly, Assumptions 7.1 and 7.2, and Lemma S3.1 imply ℓ is bounded from above, and thus we deduce

$$\begin{aligned}
\int_{\mathbb{R}^d} \big(\log\big(u(0, X_0, g)\big)\big)^+\big(u(0, X_0, g) + u_N(g)\big)\, dg &< \infty, \\
\int_{\mathbb{R}^d\times\mathcal{R}_H} \big(\log\big(u\big(0, X_0, G(g, h)\big)\big)\big)^+ u(0, X_0, g)\ell(0, X_0, h)\, dg\, dh &< \infty, \\
\int_{\mathcal{R}_H} \big(\log\big(\ell(0, X_0, h)\big)\big)^+ \ell(0, X_0, h)\, dh &< \infty.
\end{aligned} \tag{S3.8}$$

PROOF OF THEOREM 7.3: We start by collecting facts regarding $\check{\mathbb{Q}}$. First, $B^{\check{\mathbb{Q}}}_{\cdot} := B_{\cdot} + \int_0^{\cdot} \check{\nu}_u\,du$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ Brownian motion with the PRP. Next, S is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale by construction, with $dS_t = \sigma_t\,dB^{\check{\mathbb{Q}}}_t = \sigma_t(\check{\nu}_t + dB_t)$. Last,

$$Z^{\check{\mathbb{Q}}}_t = \frac{d\check{\mathbb{Q}}}{d\widetilde{\mathbb{P}}}\bigg|_{\mathcal{F}^m_t} \times \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}^m_t} = \frac{\check{Z}_t}{p^H_t} = \frac{\check{Z}_t \ell(0, X_0, H)}{\ell(t, X_t, H)} = \mathcal{E}\left(-\int_0^{\cdot} \nu'_u\,dB^m_u\right)_t \tag{S3.9}$$

for $t \le T$. Indeed, the second equality follows from (7.4) and (C.2), the third equality follows from (S3.1), and the fourth equality follows from (S3.1), (C.2), and $dB^m_t = dB_t - \mu^h_t\,dt$. Now consider the uninformed investor's value function. Clearly, $\check{\mathbb{Q}} \in \tilde{\mathcal{M}}$ and, in fact, we claim $\check{\mathbb{Q}} \in \tilde{\mathcal{M}}^m$. First, $Z^{\check{\mathbb{Q}}}_0 = 1$ so $Z^{\check{\mathbb{Q}}} = \mathring{Z}^{\check{\mathbb{Q}}}$. Second, as $\ell(0, X_0, H)$ is $\mathcal{F}^m_0$ measurable, $\widetilde{\mathbb{E}}[\check{Z}_T | \mathcal{F}^m_0] = 1$, $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\sigma(H)$, and $H \sim \ell(0, X_0, \cdot)$, then

$$\begin{aligned}\mathbb{E}\big[\mathring{Z}^{\check{\mathbb{Q}}}_T \log\big(\mathring{Z}^{\check{\mathbb{Q}}}_T\big)\big] &\le \widetilde{\mathbb{E}}\big[\check{Z}_T \log(\check{Z}_T)\big] + \int_{h \in \mathcal{R}_H} \big(\log\big(\ell(0, X_0, h)\big)\big)^+ \ell(0, X_0, h)\,dh \\ &\quad + \widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\ell(T, X_T, H)\big)^-\big] < \infty,\end{aligned} \tag{S3.10}$$

where the last inequality holds from (S3.8), (C.3), and (C.4)(a). Thus, $H_0(\check{\mathbb{Q}}|\mathbb{P}) < \infty$ almost surely so $\check{\mathbb{Q}} \in \tilde{\mathcal{M}}^m$. Next, we claim that for all $\mathbb{Q} \in \mathcal{M}^m$,

$$Z^{\mathbb{Q}}_{\cdot} = Z^{\mathbb{Q}}_0 \mathring{Z}^{\check{\mathbb{Q}}}_{\cdot}, \tag{S3.11}$$

so $\mathring{Z}^{\mathbb{Q}} = \mathring{Z}^{\check{\mathbb{Q}}}$. Indeed, from Lemma S3.1 part (iii), we can write $Z^{\mathbb{Q}}_T = Z^{\mathbb{Q}}_0 \mathcal{E}(\int_0^{\cdot} \theta'_t\,dB^m_t)_T$ for some $\theta \in \mathcal{P}(\mathbb{F}^m)$. Girsanov's theorem and (7.6) imply S has dynamics

$$dS_t = \sigma_t\big((\nu_t + \theta_t)\,dt + dB^{\mathbb{Q}}_t\big), \qquad B^{\mathbb{Q}}_{\cdot} = B^m_{\cdot} - \int_0^{\cdot} \theta_u\,du.$$

That $\mathbb{Q} \in \mathcal{M}^m$ implies $\int_0^{\cdot} \sigma_u(\nu_u + \theta_u)\,du$ is a continuous $(\mathbb{Q}, \mathbb{F}^m)$ local martingale of finite variation, and hence is identically zero. This gives $\theta = \nu$ $\mathsf{Leb}_{[0,T]} \times \mathbb{P}$ almost surely, which in light of (S3.9) verifies (S3.11).

By duality, for each $\pi \in \mathcal{A}^m$, $\mathbb{Q} \in \tilde{\mathcal{M}}^m$, and $\mathcal{F}^m_0$-measurable $\lambda > 0$,

$$\begin{aligned}\mathbb{E}\left[-\frac{1}{\gamma_U} e^{-\gamma_U \mathcal{W}^\pi_T}\Big|\mathcal{F}^m_0\right] &\le \mathbb{E}\left[\frac{1}{\gamma_U}\big(\lambda Z^{\mathbb{Q}}_T\big)\big(\log\big(\lambda Z^{\mathbb{Q}}_T\big) - 1\big) + \lambda Z^{\mathbb{Q}}_T \mathcal{W}^\pi_T \Big| \mathbb{F}^m_0\right] \\ &\le \frac{1}{\gamma_U}\mathring{\lambda}(\log(\mathring{\lambda}) - 1) + \frac{1}{\gamma_U}\mathring{\lambda}\mathbb{E}\left[\frac{\check{Z}_T}{p^H_T}\log\left(\frac{\check{Z}_T}{p^H_T}\right)\Big|\mathcal{F}^m_0\right],\end{aligned}$$

where we have set $\mathring{\lambda} = \lambda Z^{\mathbb{Q}}_0$ and used (S3.11). The infimum above over $\mathring{\lambda}$ is achieved at $\log(\mathring{\lambda}) = -\mathbb{E}[(\check{Z}_T/p^H_T)\log(\check{Z}_T/p^H_T)|\mathcal{F}^m_0]$, and plugging this in yields

$$\mathbb{E}\left[-\frac{1}{\gamma_U} e^{-\gamma \mathcal{W}^\pi_T}\Big|\mathcal{F}^m_0\right] \le -\frac{1}{\gamma_U} e^{-\mathbb{E}[\frac{\check{Z}_T}{p^H_T}\log(\frac{\check{Z}_T}{p^H_T})|\mathcal{F}^m_0]}.$$

Furthermore, there is equality if and only if

$$\mathcal{W}_T^{\pi} = \frac{1}{\gamma_U}\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H}\log\left(\frac{\check{Z}_T}{p_T^H}\right)\middle|\mathcal{F}_0^m\right] - \frac{1}{\gamma_U}\log\left(\frac{\check{Z}_T}{p_T^H}\right) \tag{S3.12}$$

and $\mathbb{E}[(\check{Z}_T/p_T^H)\mathcal{W}_T^{\pi}|\mathcal{F}_0^m] = 0$, but this latter equality is immediate. By (S3.9), (S3.10), and $|x\log(x)| \le x\log(x) + 2/e$, $x > 0$, we know that

$$M_\cdot^U := -\frac{1}{\gamma_U}\mathbb{E}^{\check{\mathbb{Q}}}\left[\log\left(\frac{\check{Z}_T}{p_T^H}\right)\middle|\mathcal{F}_\cdot^m\right]$$

is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale. Thus, by PRP there is a $\theta^U \in \mathcal{P}(\mathbb{F}^m)$ such that $\int_0^T |\theta_u^U|^2\,du < \infty$ and $M_\cdot^U = M_0^U + \int_0^\cdot (\theta_u^U)'\,dB_u^{\check{\mathbb{Q}}}$. As σ is invertible and $dS_t = \sigma_t\,dB_t^{\check{\mathbb{Q}}}$, if we set $\hat{\pi}_\cdot^U := (\sigma_\cdot')^{-1}\theta_\cdot^U$, then $\mathcal{W}_\cdot^{\hat{\pi}^U} = M_\cdot^U - M_0^U$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale verifying (S3.12). Using (S3.1) and (S3.9), noting that $\ell(0, X_0, H)$ is $\mathcal{F}_0^m$ measurable, and using $\widetilde{\mathbb{E}}[\check{Z}_T|\mathcal{F}_0^m] = 1$, we simplify (S3.12) to deduce the existence of an optimal strategy $\hat{\pi}^U \in \mathcal{A}^m$ such that

$$\begin{aligned} &\mathcal{W}_T^{\hat{\pi}^U} = \frac{1}{\gamma_U}\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_T}{\ell(T, X_T, H)}\right)\middle|\mathcal{F}_0^m\right] - \frac{1}{\gamma_U}\log\left(\frac{\check{Z}_T}{\ell(T, X_T, H)}\right) \\ &\mathcal{W}^{\hat{\pi}^U} \text{ is a } \big(\check{\mathbb{Q}}, \mathbb{F}^m\big) \text{ martingale.} \end{aligned} \tag{S3.13}$$

We next consider the noise trader's value function for a fixed $g \in \mathbb{R}^d$. Since $\mathring{p}^{H,g}$ from (S3.3) is a strictly positive $(\mathbb{P}, \mathbb{F}^m)$ martingale (see (S3.3) and Lemma S4.3), the identity in (S3.3) implies that $\mathbb{P}^g$ from (2.8) is well defined. We have already shown $\check{\mathbb{Q}} \in \mathcal{M}$. Furthermore, from (B.3), (S3.3), and (S3.9) we know that

$$Z_T^{\check{\mathbb{Q}},g} := \frac{d\check{\mathbb{Q}}}{d\mathbb{P}^g}\bigg|_{\mathcal{F}_T^m} = \frac{d\check{\mathbb{Q}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T^m} \times \frac{d\mathbb{P}}{d\mathbb{P}^g}\bigg|_{\mathcal{F}_T^m} = \frac{\check{Z}_T}{p_T^H\mathring{p}_T^{H,g}} = \frac{\check{Z}_T u(0, X_0, g)}{\hat{p}_I(g - X_T)}. \tag{S3.14}$$

As $Z_0^{\check{\mathbb{Q}},g} = 1$,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^g}\big[\mathring{Z}_T^{\check{\mathbb{Q}},g}\log\big(\mathring{Z}_T^{\check{\mathbb{Q}},g}\big)\big] &= \mathbb{E}^{\check{\mathbb{Q}}}\left[\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{H,g}}\right)\right] = \widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_T u(0, X_0, g)}{\hat{p}_I(g - X_T)}\right)\right] \\ &\le \widetilde{\mathbb{E}}\big[\check{Z}_T\log(\check{Z}_T)\big] + \big(\log\big(u(0, X_0, g)\big)\big)^+ + \widetilde{\mathbb{E}}\big[\check{Z}_T\big(\log\big(\hat{p}_I(g - X_T)\big)\big)^-\big] \\ &< \infty. \end{aligned} \tag{S3.15}$$

Above, the finiteness follows from (C.3) and (C.4)(b). Thus $H_0(\check{\mathbb{Q}}|\mathbb{P}^g) < \infty$ and $\tilde{\mathcal{M}}^{m,g} \neq \emptyset$. Now, let $\mathbb{Q} \in \tilde{\mathcal{M}}^{m,g}$. From (S3.11) we know $Z_\cdot^{\mathbb{Q}} = Z_0^{\mathbb{Q}}\check{Z}_\cdot/p_\cdot^H$. Thus, with respect to $\mathbb{P}^g$, $Z_\cdot^{\mathbb{Q},g} = Z_0^{\mathbb{Q}}\check{Z}_\cdot/(p_\cdot^H\mathring{p}_\cdot^{H,g})$. Therefore, repeating the duality argument, a strategy $\pi^{N,g}$ is optimal if and only if

$$\mathcal{W}_T^{\pi^{N,g}} = \frac{1}{\gamma_N}\mathbb{E}^{\mathbb{P}^g}\left[\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{H,g}}\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{H,g}}\right)\middle|\mathcal{F}_0^m\right] - \frac{1}{\gamma_N}\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{H,g}}\right). \tag{S3.16}$$

By (S3.15) and $|x\log(x)| \le x\log(x) + 2/e$, $x > 0$, we know that

$$M^{N,g}_{\cdot} := -\frac{1}{\gamma_N}\mathbb{E}^{\check{\mathbb{Q}}}\bigg[\log\bigg(\frac{\check{Z}_T}{p^H_T\mathring{p}^{H,g}_T}\bigg)\bigg|\mathcal{F}^m_{\cdot}\bigg]$$

is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale. Thus, there is $\theta^{N,g} \in \mathcal{P}(\mathbb{F}^m)$ so that $M^{N,g}_{\cdot} = M^{N,g}_0 + \int_0^{\cdot}(\theta^{N,g}_u)'\,dB^{\check{\mathbb{Q}}}_u$. If we set

$$\hat{\pi}^{N,g}_{\cdot} := (\sigma'_{\cdot})^{-1}\theta^{N,g}_{\cdot}, \tag{S3.17}$$

then $\mathcal{W}^{\hat{\pi}^{N,g}}_{\cdot} = M^{N,g}\cdot - M^{N,g}_0$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale verifying (S3.16). Using (S3.14) and (S3.15), this simplifies to

$$\mathcal{W}^{\hat{\pi}^{N,g}}_T = \frac{1}{\gamma_N}\widetilde{\mathbb{E}}\bigg[\check{Z}_T\log\bigg(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\bigg)\bigg|\mathcal{F}^m_0\bigg] - \frac{1}{\gamma_N}\log\bigg(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\bigg),$$

where the last equality follows as $u(0, X, g)$ is constant and $\widetilde{\mathbb{E}}[\check{Z}_T|\mathcal{F}^m_0] = 1$. Thus, we have shown the existence of an optimal strategy $\hat{\pi}^{N,g} \in \mathcal{A}^{N,g}$ which satisfies

$$\begin{aligned}&\mathcal{W}^{\hat{\pi}^{N,g}}_T = \frac{1}{\gamma_N}\widetilde{\mathbb{E}}\bigg[\check{Z}_T\log\bigg(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\bigg)\bigg|\mathcal{F}^m_0\bigg] - \frac{1}{\gamma_N}\log\bigg(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\bigg),\\ &\mathcal{W}^{\hat{\pi}^{N,g}} \text{ is a } (\check{\mathbb{Q}}, \mathbb{F}^m) \text{ martingale.}\end{aligned} \tag{S3.18}$$

We next turn to the informed investor. Equation (S3.9) implies that $\check{Z}/p^H$ is a $(\mathbb{P}, \mathbb{F}^m)$ martingale starting at 1. Thus, if we define $\check{\mathbb{Q}}^I$ through $d\check{\mathbb{Q}}^I/d\mathbb{P} = \check{Z}_T/(p^H_T p^{H,G_I}_T)$, then Lemma S4.8 part (ii) implies that $\check{\mathbb{Q}}^I \in \mathcal{M}^I$, while calculation shows $\mathring{Z}^{\check{\mathbb{Q}}^I} = \check{Z}/(p^H\mathring{p}^{H,G_I})$. Furthermore, $B^{\check{\mathbb{Q}}}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ Brownian motion with the PRP (Fontana (2018, Proposition 2.9) and Remark S4.4). Now $\check{\mathbb{Q}}^I \in \tilde{\mathcal{M}}^I$ will follow if $\mathbb{E}[\mathring{Z}^{\check{\mathbb{Q}}^I}_T\log(\mathring{Z}^{\check{\mathbb{Q}}^I}_T)|\mathcal{F}^I_0] < \infty$. To show this, we first claim (note the presence of $\mathring{Z}^{\check{\mathbb{Q}}^I}$)

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\log(\mathring{Z}^{\check{\mathbb{Q}}^I}_T)\big|\big] = \mathbb{E}\bigg[\frac{\check{Z}_T}{p^H_T p^{H,G_I}_T}\bigg|\log\bigg(\frac{\check{Z}_T}{p^H_T\mathring{p}^{H,G_I}_T}\bigg)\bigg|\bigg] < \infty. \tag{S3.19}$$

Since $(x/y)|\log(x)| \le (1/y)(x\log(x) + 2/e)$ for $x, y > 0$, we see (for $x = \check{Z}_T/(p^H_T\mathring{p}^{H,G_I}_T)$ and $y = p^{H,G_I}_0$)

$$\mathbb{E}\bigg[\frac{\check{Z}_T}{p^H_T p^{H,G_I}_T}\bigg|\log\bigg(\frac{\check{Z}_T}{p^H_T\mathring{p}^{H,G_I}_T}\bigg)\bigg|\bigg] \le \mathbb{E}\bigg[\frac{\check{Z}_T}{p^H_T p^{H,G_I}_T}\log\bigg(\frac{\check{Z}_T}{p^H_T\mathring{p}^{H,G_I}_T}\bigg) + \frac{2}{ep^{H,G_I}_0}\bigg]. \tag{S3.20}$$

By definition of p^{H,G_I} and since $G_I \sim u(0, X_0, \cdot)$,

$$\mathbb{E}\bigg[\frac{1}{p^{H,G_I}_0}\bigg] = \mathbb{E}\bigg[\int_{g\in\mathbb{R}^d}\frac{1}{p^{H,g}_0}p^{H,g}_0 u(0, X_0, g)\,dg\bigg] = 1. \tag{S3.21}$$

Next,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H p_T^{H,G_I}}\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right)\right]\\
&\quad=\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H}\int_{\mathbb{R}^d}\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right)u(0,X_0,g)\,dg\right]\\
&\quad=\int_{\mathbb{R}^d}\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_T u(0,X_0,g)}{\hat{p}_I(g-X_T)}\right)\right]u(0,X_0,g)\,dg\\
&\quad\le\widetilde{\mathbb{E}}\big[\check{Z}_T\log(\check{Z}_T)\big]+\int_{g\in\mathbb{R}^d}\widetilde{\mathbb{E}}\big[\check{Z}_T\big(\log\big(\hat{p}_I(g-X_T)\big)\big)^-\big]u(0,X_0,g)\,dg\\
&\qquad+\int_{g\in\mathbb{R}^d}\big(\log\big(u(0,X_0,g)\big)\big)^+u(0,X_0,g)\,dg<\infty.
\end{aligned}
\tag{S3.22}
$$

Above, the first equality follows by conditioning on $\mathcal{F}_T^m$ and using part (i) of Lemma S3.2, the second equality using (B.3), (S3.1), and (S3.3), and the second inequality follows from (C.3), (S3.8), and (C.4)(c). Therefore, (S3.19) follows from (S3.20), (S3.21), and (S3.22). But (S3.19) implies $\mathbb{E}[\mathring{Z}_T^{\check{\mathbb{Q}}^I}\log(\mathring{Z}_T^{\check{\mathbb{Q}}^I})|\mathcal{F}_0^I]<\infty$ since

$$
0\le\mathbb{E}\big[\mathring{Z}_T^{\check{\mathbb{Q}}^I}\log\big(\mathring{Z}_T^{\check{\mathbb{Q}}^I}\big)|\mathcal{F}_0^I\big]\le\frac{1}{Z_0^{\check{\mathbb{Q}}^I}}\mathbb{E}\big[Z_T^{\check{\mathbb{Q}}^I}\big|\log\big(\mathring{Z}_T^{\check{\mathbb{Q}}^I}\big)\big||\mathcal{F}_0^I\big].
$$

Next, we claim that any $\mathbb{Q}\in\tilde{\mathcal{M}}^I$ has density process $Z_\cdot^{\mathbb{Q}}=Z_0^{\mathbb{Q}}\mathring{Z}_\cdot^{\check{\mathbb{Q}}^I}$. Indeed, using part (iii) of Lemma S3.2, we deduce the existence of $\theta\in\mathcal{P}(\mathbb{F}^I)$ so that $Z_T^{\mathbb{Q}}=Z_0^{\mathbb{Q}}\mathcal{E}(\int_0^\cdot\theta_t'\,dB_t^I)_T$. Using Girsanov, along with $dB_t^I=dB_t^m-\mu_t^{H,G_I}\,dt$ and (7.6), we see S has dynamics $dS_t=\sigma_t(\nu_t+\mu_t^{H,G_I}+\theta_t)\,dt+\sigma_t\,dB_t^{\mathbb{Q},I}$, where $B_\cdot^{\mathbb{Q},I}=B_\cdot^I-\int_0^\cdot\theta_u\,du$. As S, $B^{\mathbb{Q},I}$ are continuous $(\mathbb{Q},\mathbb{F}^I)$local martingales, $\nu+\mu^{H,G_I}+\theta\equiv0$ from whence

$$
Z_t^{\mathbb{Q}}=Z_0^{\mathbb{Q}}\mathcal{E}\left(-\int_0^\cdot\big(\nu_u+\mu_u^{H,G_I}\big)'\,dB_u^I\right)_t=Z_0^{\mathbb{Q}}\frac{\check{Z}_t}{p_t^H\mathring{p}_t^{H,G_I}}.
$$

To obtain the last equality we use the steps

$$
\begin{aligned}
\frac{\check{Z}_t}{p_t^H\mathring{p}_t^{H,G_I}}&=\frac{\check{Z}_t u(0,X_0,G_I)}{u(t,X_t,G_I)}=\frac{\mathcal{E}\left(-\int_0^\cdot\check{\nu}_u'\,dB_u\right)_t}{\mathcal{E}\left(\int_0^\cdot\big(\tilde{\mu}_u^{G_I}\big)'\,dB_u\right)_t}\\
&=\mathcal{E}\left(-\int_0^\cdot\big(\check{\nu}_u+\tilde{\mu}_u^{G_I}\big)'\big(dB_u-\tilde{\mu}_u^{G_I}\,du\big)\right)_t\\
&=\mathcal{E}\left(-\int_0^\cdot\big(\nu_u+\mu_u^{H,G_I}\big)'\big(dB_u^m-\mu_u^{H,G_I}\,du\big)\right)_t\\
&=\mathcal{E}\left(-\int_0^\cdot\big(\nu_u+\mu_u^{H,G_I}\big)'\,dB_u^I\right)_t.
\end{aligned}
$$

The first equality follows from (S3.3), the second from (S2.2) and part (iii) of Lemma S3.2, the fourth from $\nu = \check{\nu} + \mu^H$, $dB^m_t = dB_t - \mu^H_t\, dt$, and (S3.4), and the fifth also from part (iii) of Lemma S3.2. This proves the assertion.

The duality argument shows π^I is optimal if and only if

$$\mathcal{W}^{\pi^I}_T = \frac{1}{\gamma_I}\mathbb{E}\biggl[\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr)\Big|\mathcal{F}^I_0\biggr] - \frac{1}{\gamma_I}\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr).$$

By (S3.19) we know that

$$M^I_\cdot := -\frac{1}{\gamma_I}\mathbb{E}^{\check{\mathbb{Q}}^I}\biggl[\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr)\Big|\mathcal{F}^I_\cdot\biggr],$$

is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. Thus, there is a $\theta^I \in \mathcal{P}(\mathbb{F}^I)$ such that $M^I_t = M^I_0 + \int_0^t (\theta^I_u)'\, dB^{\check{\mathbb{Q}}}_u$. Thus, if we set $\hat{\pi}^I_\cdot := (\sigma'_\cdot)^{-1}\theta^I_\cdot$, then $\mathcal{W}^{\hat{\pi}^I}_\cdot = M^I_\cdot - M^I_0$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale, and using the conditional Bayes rule,

$$\begin{aligned}
\mathcal{W}^{\hat{\pi}^I}_T &= \frac{1}{\gamma_I}\mathbb{E}^{\check{\mathbb{Q}}^I}\biggl[\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr)\Big|\mathcal{F}^I_0\biggr] - \frac{1}{\gamma_I}\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr) \\
&= \frac{1}{\gamma_I}\mathbb{E}\biggl[\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr)\Big|\mathcal{F}^I_0\biggr] - \frac{1}{\gamma_I}\log\biggl(\frac{\check{Z}_T}{p^H_T \mathring{p}^{H,G_I}_T}\biggr),
\end{aligned}$$

proving optimality of $\hat{\pi}^I$. To simplify this expression note that as above $p^H_T \mathring{p}^{H,G_I}_T = \hat{p}_I(G_I - X_T)/u(0, X_0, G_I)$, and since $u(0, X_0, G_I)$ is $\mathcal{F}^I_0$-measurable, it disappears from the expression for $\mathcal{W}^{\hat{\pi}^I}_T$. Furthermore, Lemma S4.8 implies

$$\begin{aligned}
\mathbb{E}^{\check{\mathbb{Q}}^I}\biggl[\log\biggl(\frac{\check{Z}_T}{\hat{p}_I(G_I - X_T)}\biggr)\Big|\mathcal{F}^I_0\biggr] &= \biggl(\widetilde{\mathbb{E}}\biggl[\check{Z}_T\log\biggl(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\biggr)\Big|\mathcal{F}^m_0\biggr]\biggr)\Big|_{g=G_I} \\
&= \widetilde{\mathbb{E}}\bigl[\check{Z}_T\log(\check{Z}_T)|\mathcal{F}^m_0\bigr] \\
&\quad - \bigl(\widetilde{\mathbb{E}}\bigl[\check{Z}_T\log\bigl(\hat{p}_I(g - X_T)\bigr)|\mathcal{F}^m_0\bigr]\bigr)\big|_{g=G_I}.
\end{aligned}$$

We conclude for the optimal trading strategy $\hat{\pi}^I$ that

$$\begin{aligned}
\mathcal{W}^{\hat{\pi}^I}_T &= \frac{1}{\gamma_I}\bigl(\widetilde{\mathbb{E}}\bigl[\check{Z}_T\log(\check{Z}_T)|\mathcal{F}^m_0\bigr] - \bigl(\widetilde{\mathbb{E}}\bigl[\check{Z}_T\log\bigl(\hat{p}_I(g - X_T)\bigr)|\mathcal{F}^m_0\bigr]\bigr)\big|_{g=G_I}\bigr) \\
&\quad - \frac{1}{\gamma_I}\log\biggl(\frac{\check{Z}_T}{\hat{p}_I(G_I - X_T)}\biggr), \qquad \text{(S3.23)}
\end{aligned}$$

$\mathcal{W}^{\hat{\pi}^I}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale.

Having identified the optimal wealth processes for each investor, we now put them together. We have already shown that $\mathcal{W}^{\hat{\pi}^I}$ is a $(\check{\mathbb{Q}}, \mathbb{F}^I)$ martingale, and from Amendinger (2000, Proposition 3.4), $\mathbb{F}^m \subseteq \mathbb{F}^I$, we know $\hat{\pi}^U$ is both $\mathbb{F}^I$-predictable and S-integrable under $\check{\mathbb{Q}}^I$. Furthermore, the semi-martingales $\mathbb{F}^m - \mathcal{W}^{\hat{\pi}^U}$ and $\mathbb{F}^I - \mathcal{W}^{\hat{\pi}^U}$ have a common version. Thus, as (S3.13) implies $\mathcal{W}^{\hat{\pi}^U}$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale, Amendinger (2000, Theorem 3.2) shows $\mathcal{W}^{\hat{\pi}^U}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. As for the noise trader, Lemma S3.4 below proves

$$\begin{aligned} &\hat{\pi}^{N,G_N} \text{ is well defined, } \mathbb{F}^I\text{-predictable, and } S\text{-integrable under } \big(\check{\mathbb{Q}}^I, \mathbb{F}^I\big), \\ &\mathcal{W}^{\hat{\pi}^{N,G_N}} \text{ is a } \big(\check{\mathbb{Q}}^I, \mathbb{F}^I\big) \text{ martingale, and } \mathcal{W}^{\hat{\pi}^{N,G_N}} = \big(\mathcal{W}^{\hat{\pi}^{N,g}}\big)\big|_{g=G_N}. \end{aligned} \tag{S3.24}$$

Therefore, we use (S3.13), (S3.23), and (S3.18) to obtain

$$\begin{aligned} &\int_0^T \big(\omega_U \hat{\pi}_t^U + \omega_I \hat{\pi}_t^I + \omega_N \hat{\pi}_t^{N,G_N}\big)' \, dS_t \\ &= \omega_U \mathcal{W}_T^{\hat{\pi}^U} + \omega_I \mathcal{W}_T^{\hat{\pi}^I} + \omega_N \mathcal{W}_T^{\hat{\pi}^{N,G_N}} \\ &= \alpha_U \bigg(\widetilde{\mathbb{E}}\big[\check{Z}_T \log(\check{Z}_T) | \mathcal{F}_0^m\big] - \widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\ell(T, X_T, H)\big) | \mathcal{F}_0^m\big] - \log\bigg(\frac{\check{Z}_T}{\ell(T, X_T, H)}\bigg)\bigg) \\ &\quad + \alpha_I \bigg(\widetilde{\mathbb{E}}\big[\check{Z}_T \log(\check{Z}_T) | \mathcal{F}_0^m\big] - \big(\widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\hat{p}_I(g - X_T)\big) | \mathcal{F}_0^m\big]\big)_{g=G_I} \\ &\quad - \log\bigg(\frac{\check{Z}_T}{\hat{p}_I(G_I - X_T)}\bigg)\bigg) \\ &\quad + \alpha_N \bigg(\widetilde{\mathbb{E}}\big[\check{Z}_T \log(\check{Z}_T) | \mathcal{F}_0^m\big] - \big(\widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\hat{p}_I(g - X_T)\big) | \mathcal{F}_0^m\big]\big)_{g=G_N} \\ &\quad - \log\bigg(\frac{\check{Z}_T}{\hat{p}_I(G_N - X_T)}\bigg)\bigg). \end{aligned}$$

Furthermore,

$$\overline{\mathbf{M}} := \int_0^\cdot \big(\omega_U \hat{\pi}_t^U + \omega_I \hat{\pi}_t^I + \omega_N \hat{\pi}_t^{N,G_N}\big)' \, dS_t \text{ is a } \big(\check{\mathbb{Q}}^I, \mathbb{F}^I\big) \text{ martingale.}$$

Recalling the definition of γ in (2.2), the above simplifies to

$$\begin{aligned} \overline{\mathbf{M}}_T = \frac{1}{\gamma}&\big(\widetilde{\mathbb{E}}\big[\check{Z}_T \log(\check{Z}_T) | \mathcal{F}_0^m\big] - \log(\check{Z}_T)\big) \\ &- \alpha_U \big(\widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\ell(T, X_T, H)\big) | \mathcal{F}_0^m\big] - \log\big(\ell(T, X_T, H)\big)\big) \\ &- \alpha_I \big(\big(\widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\hat{p}_{C_I}(g - X_T)\big) | \mathcal{F}_0^m\big]\big)_{g=G_I} - \log\big(\hat{p}_{C_I}(G_I - X_T)\big)\big) \\ &- \alpha_N \big(\big(\widetilde{\mathbb{E}}\big[\check{Z}_T \log\big(\hat{p}_{C_I}(g - X_T)\big) | \mathcal{F}_0^m\big]\big)_{g=G_N} - \log\big(\hat{p}_{C_I}(G_N - X_T)\big)\big). \end{aligned}$$

Now assume a PCE exists. Then $\omega_U \hat{\pi}_t^U + \omega_I \hat{\pi}_t^I + \omega_N \hat{\pi}_t^{N,G_N} = \Pi$ and (7.7) follows. Next assume (7.7). This gives $\overline{\mathbf{M}}_T = \Pi'(\Psi(X_T) - S_0) = \int_0^T \Pi' \, dS_t$. By construction in (7.6), we know S is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale, hences a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. Thus, we see for all $t \leq T$ that $0 = \check{M}_t := \int_0^t (\omega_U \hat{\pi}_u^U + \omega_I \hat{\pi}_u^I + \omega_N \hat{\pi}_u^{N,G_N} - \Pi)' \, dS_u$. Thus, $\check{M}$ is a continuous martingale with quadratic variation

$$0 = \langle \check{M} \rangle_t = \int_0^t \left| \sigma_u' \left(\omega_U \hat{\pi}_u^U + \omega_I \hat{\pi}_u^I + \omega_N \hat{\pi}_u^{N,G_N} - \Pi \right) \right|^2 du.$$

As σ' is nondegenerate $\mathsf{Leb}_{[0,T]} \times \mathbb{P}$ almost surely, we have $\omega_U \hat{\pi}_t^U + \omega_I \hat{\pi}_t^I + \omega_N \hat{\pi}_t^{N,G_N} = \Pi$, and hence a PCE exists, finishing the proof. *Q.E.D.*

LEMMA S3.4: *The statements in* (S3.24) *hold*.

PROOF: Recall (S3.15), which states

$$\mathbb{E}^{\mathbb{P}^g} \left[\mathring{Z}_T^{\check{\mathbb{Q}},g} \log \left(\mathring{Z}_T^{\check{\mathbb{Q}},g} \right) \right] = \mathbb{E}^{\check{\mathbb{Q}}} \left[\log \left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}} \right) \right] < \infty.$$

The inequality $x|\log(x)| \le x\log(x) + 2/e$ implies $\mathbb{E}^{\check{\mathbb{Q}}}[|\log(\check{Z}_T/(p_T^H \mathring{p}_T^{H,g}))|] < \infty$, which in turn identifies the processes $\theta^{N,g}$ and $\hat{\pi}^{N,g}$ from (S3.17). In light of Lemma S3.3 and the above integrability condition, we may apply Proposition S4.6 (with $\mathbb{F} = \mathbb{F}^m$, $\mathbb{P} = \check{\mathbb{Q}}$, and $B = B^{\check{\mathbb{Q}}}$ therein). Part (i) implies θ^{N,G_N}, $\hat{\pi}^{N,G_N}$ are $\mathcal{P}(\mathbb{F}^N)$-measurable, and hence $\mathcal{P}(\mathbb{F}^I)$-measurable, as Lemma S3.3 shows $\mathbb{F}^I = \mathbb{F}^N$. To ease notation, set $\overline{\mathbb{F}}$ as the common filtration. Next define the measure $\check{\mathbb{Q}}^N$ by

$$\frac{d\check{\mathbb{Q}}^N}{d\mathbb{P}} = \frac{\check{Z}_T}{p_T^H p_{N,T}^{H,G_N}}. \tag{S3.25}$$

The filtration $\check{\mathbb{Q}}^N$ is the ($\mathbb{F}^m$ to $\overline{\mathbb{F}}$) martingale-preserving measure for $\check{\mathbb{Q}}$ and G_N. As such, $B^{\check{\mathbb{Q}}}$ is a $(\check{\mathbb{Q}}^N, \overline{\mathbb{F}})$ Brownian motion, and from parts (iii) and (iv) of Proposition S4.6 we know that $\mathbb{Q}^N$ almost surely

$$\int_0^T \left| \theta_t^{N,G_N} \right|^2 dt < \infty, \qquad \sup_{t \le T} \left| \int_0^t \left(\theta_u^{N,G_N} \right)' dB_u^{\check{\mathbb{Q}}} - \left(\int_0^t \left(\theta_u^{N,g} \right)' dB_u^{\check{\mathbb{Q}}} \right) \Big|_{g=G_N} \right| = 0.$$

As $\mathbb{Q}^I$ is equivalent to $\mathbb{Q}^N$ on $\overline{\mathcal{F}}_T$, it follows that $\hat{\pi}^{N,G_N}$ is S-integrable under $\mathbb{Q}^I$, and from the right equality above that $\mathcal{W}^{\hat{\pi}^{N,G_N}} = (\mathcal{W}^{\hat{\pi}^{N,g}})|_{g=G_N}$ under $\mathbb{Q}^I$.

The last thing to show is that $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^I, \overline{\mathbb{F}})$ martingale. To this end, we first show that it is a $(\check{\mathbb{Q}}^N, \overline{\mathbb{F}})$ martingale. Indeed,

$$\mathbb{E}^{\check{\mathbb{Q}}^N} \left[\left| \log \left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_N}} \right) \right| \right] = \int_{\mathbb{R}^d} \mathbb{E}^{\check{\mathbb{Q}}} \left[\left| \log \left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}} \right) \right| \right] u_N(g)\, dg, \tag{S3.26}$$

where $u_N(g)$ from (C.1) is the pdf of G_N. To evaluate this expression set $\chi(g, H) := \mathbb{E}^{\check{\mathbb{Q}}}[|\log(\check{Z}_T/(p_T^H \mathring{p}_T^{H,g}))| | \mathcal{F}_0^m]$. As $\check{Z}_0/(p_0^H \mathring{p}_0^{H,g}) = 1$, $x|\log(x)| \le x\log(x) + 2/e$, and

$\widetilde{\mathbb{E}}[\check{Z}_T|\mathcal{F}_0^m]=1$, calculations similar to (S3.15) show

$$\chi(g,H)\leq\frac{2}{e}+\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_Tu(0,X_0,g)}{\hat{p}_I(g-X_T)}\right)\Big|\mathcal{F}_0^m\right]$$
$$\leq\frac{2}{e}+\widetilde{\mathbb{E}}\big[\check{Z}_T\log(\check{Z}_T)|\mathcal{F}_0^m\big]+\big(\log\big(u(0,X_0,g)\big)\big)^+$$
$$+\widetilde{\mathbb{E}}\big[\check{Z}_T\big(\log\big(\hat{p}_I(g-X_T)\big)\big)^-|\mathcal{F}_0^m\big]. \tag{S3.27}$$

Using this in (S3.26),

$$\mathbb{E}^{\check{\mathbb{Q}}^N}\left[\left|\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{H,G_N}}\right)\right|\right]\leq\frac{2}{e}+\widetilde{\mathbb{E}}\big[\check{Z}_T\log(\check{Z}_T)\big]+\int_{\mathbb{R}^d}\big(\log\big(u(0,X_0,g)\big)\big)^+u_N(g)\,dg$$
$$+\int_{\mathbb{R}^d}\widetilde{\mathbb{E}}\big[\check{Z}_T\big(\log\big(\hat{p}_I(g-X_T)\big)\big)^-\big]u_N(g)\,dg<\infty, \tag{S3.28}$$

where the last inequality follows from (S3.8), (C.3), and (C.4)(c).

From (S3.28), and Lemma S4.8 we know $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^N,\overline{\mathbb{F}})$ martingale. Let us assume for now that

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_t^{\hat{\pi}^{N,G_N}}\big|\big]<\infty,\quad t\leq T. \tag{S3.29}$$

The $(\check{\mathbb{Q}}^I,\overline{\mathbb{F}})$ martingale property follows from that under $(\check{\mathbb{Q}}^N,\overline{\mathbb{F}})$ and Lemma S3.3. Indeed, from part (i) of Lemma S3.3 for $t\leq T$,

$$\frac{d\check{\mathbb{Q}}^N}{d\check{\mathbb{Q}}^I}\bigg|_{\bar{\mathcal{F}}_t}=\frac{d\check{\mathbb{Q}}^N}{d\mathbb{P}}\bigg|_{\bar{\mathcal{F}}_t}\times\frac{d\mathbb{P}}{d\check{\mathbb{Q}}^I}\bigg|_{\bar{\mathcal{F}}_t}=\frac{p_t^{H,G_I}}{p_{N,t}^{H,G_N}}=\frac{p_0^{H,G_I}}{p_{N,0}^{H,G_N}}. \tag{S3.30}$$

As this does not change with t, the martingale property is clear. The last thing to show is (S3.29). Using (S3.30) and that $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^N,\bar{\mathbb{F}})$ martingale, we find $\mathbb{E}^{\check{\mathbb{Q}}^I}[|\mathcal{W}_t^{\hat{\pi}^{N,G_N}}|]\leq\mathbb{E}^{\check{\mathbb{Q}}^I}[|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}|]$ so (S3.29) will hold for $t\leq T$ provided it holds at T. To show this, we use (S3.30), Lemma S4.8, and $G_I=\check{G}(G_N,H)$ to obtain

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\big]=\mathbb{E}^{\check{\mathbb{Q}}^N}\left[\frac{u\big(0,X_0,\check{G}(G_N,H)\big)\big|J^{\check{G}}\big|(G_N,H)}{u_N(G_N)\big|J^G\big|\big(\check{G}(G_N,H),H\big)}\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\right]$$
$$=\mathbb{E}^{\check{\mathbb{Q}}}\left[\int_{\mathbb{R}^d}\frac{u\big(0,X_0,\check{G}(\tilde{g},H)\big)\big|J^{\check{G}}\big|(\tilde{g},H)}{\big|J^G\big|\big(\check{G}(\tilde{g},H),H\big)}\big|\mathcal{W}_T^{\hat{\pi}^{N,\tilde{g}}}\big|\,d\tilde{g}\right].$$

From (S3.16) we deduce

$$\big|\mathcal{W}_T^{\hat{\pi}^{N,\tilde{g}}}\big|\leq\frac{1}{\gamma_N}\left(\mathbb{E}^{\check{\mathbb{Q}}}\left[\left|\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{\tilde{g},H}}\right)\right|\Big|\mathcal{F}_0^m\right]+\left|\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{\tilde{g},H}}\right)\right|\right).$$

Thus, by first conditioning on $\mathcal{F}_0^m$, we obtain, using $\chi(g,H)$ from above (see (S3.27)),

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\big]\leq\frac{2}{\gamma_N}\mathbb{E}^{\check{\mathbb{Q}}}\left[\int_{\mathbb{R}^d}\frac{u\big(0,X_0,\check{G}(\tilde{g},H)\big)\big|J^{\check{G}}\big|(\tilde{g},H)}{\big|J^G\big|\big(\check{G}(\tilde{g},H),H\big)}\chi(\tilde{g},H)\,d\tilde{g}\right],$$

$\check{\mathbb{Q}} = \mathbb{P}$ on $\sigma(H)$ and $H \sim \ell(0, X_0, \cdot)$ under $\mathbb{P}$. Thus,

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\big] \le \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathcal{R}_H} \frac{u\big(0, X_0, \check{G}(\tilde{g}, h)\big)\big|J^{\check{G}}\big|(\tilde{g}, h)}{\big|J^G\big|\big(\check{G}(\tilde{g}, h), h\big)}\chi(\tilde{g}, h)\ell(0, X_0, h)\,d\tilde{g}\,dh.$$

Now, for any appropriately measurable and integrable function f, for $\tilde{g}$ fixed, the substitution $h = H(g, \tilde{g})$ yields

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f\big(g, \tilde{g}, H(g, \tilde{g})\big)\,dg\,d\tilde{g} = \int_{\mathbb{R}^d\times\mathcal{R}_H} f\big(\check{G}(\tilde{g}, h), \tilde{g}, h\big)\big|J^{\check{G}}\big|(\tilde{g}, h)\,d\tilde{g}\,dh.$$

At $f(g, \tilde{g}, h) = u(0, X_0, g)\chi(\tilde{g}, h)\ell(0, X_0, h)/|J^G|(g, h)$, we deduce

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\big] \le \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{u(0, X_0, g)}{\big|J^G\big|\big(g, H(g, \tilde{g})\big)}\chi\big(\tilde{g}, H(g, \tilde{g})\big)\ell\big(0, X_0, H(g, \tilde{g})\big)\,d\tilde{g}\,dg.$$

Keeping g fixed, set $h = H(g, \tilde{g})$ so that $\tilde{g} = G(g, h)$ with $d\tilde{g} = |J^G|(g, h)$. This gives

$$\begin{aligned}\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\big] &\le \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathcal{R}_H} u(0, X_0, g)\chi\big(G(g, h), h\big)\ell(0, X_0, h)\,dg\,dh \\ &= \frac{2}{\gamma_N}\mathbb{E}^{\check{\mathbb{Q}}}\bigg[\int_{\mathbb{R}^d}\chi\big(G(g, H), H\big)u(0, X_0, g)\,dg\bigg].\end{aligned}$$

By first conditioning on $\mathcal{F}_0^m$ and using (S3.27), then using $\widetilde{\mathbb{E}}[\check{Z}_T|\mathcal{F}_0^m] = 1$, $H \overset{\widetilde{\mathbb{P}}}{\sim} \ell(0, X_{,}\cdot)$, and $u(0, X_0, \cdot)$ is a pdf, we see that

$$\begin{aligned}\mathbb{E}^{\check{\mathbb{Q}}^I}\big[\big|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\big|\big] &\le \frac{4}{e\gamma_N} + \frac{2}{\gamma_N}\widetilde{\mathbb{E}}\big[\check{Z}_T\log(\check{Z}_T)\big] \\ &\quad + \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathcal{R}_H}\big(\log\big(u\big(0, X_0, G(g, h)\big)\big)\big)^+u(0, X_0, g)\ell(0, X_0, h)\,dg\,dh \\ &\quad + \frac{2}{\gamma_N}\int_{\mathbb{R}^d}\widetilde{\mathbb{E}}\big[\big(\widetilde{\mathbb{E}}\big[\check{Z}_T\big(\log\big(\hat{p}_I(\tilde{g} - X_T)\big)\big)^-|\mathcal{F}_0^m\big]\big)\big|_{\tilde{g}=G(g,H)}\big]u(0, X_0, g)\,dg \\ &< \infty. \end{aligned} \tag{S3.31}$$

Above, the second inequality follows from (C.3), (S3.8), and (C.4)(d). This verifies (S3.29) and hence $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. Thus, all statements in (S3.24) hold. *Q.E.D.*

S4. ON INITIAL ENLARGEMENTS

In this section, we collect a number of results for parameter-dependent Brownian stochastic integrals in initially enlarged filtrations. Many of these results may be either found in, or deduced from, Stricker and Yor (1978), Amendinger (2000), Gasbarra, Valkeila, and Vostrikova (2006), Esmaeeli and Imkeller (2018), and especially Fontana (2018). We present them for ease of reference.

We take a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ satisfies the usual conditions. There is a d-dimensional $(\mathbb{P}, \mathbb{F})$ Brownian motion B, which has the predictable representation property, but we do not mandate $\mathbb{F} = \mathbb{F}^B$. Next, let $\mathcal{Y} \subseteq \mathbb{R}^m$ be an open set, with Borel sigma-algebra $\mathcal{B}(\mathcal{Y})$. Write $\mathcal{P}(\mathbb{F})$ and $\mathcal{O}(\mathbb{F})$ for the $\mathbb{F}$-predictable and optional sigma algebras. Last, for ease of terminology, we use the following definition.

DEFINITION S4.1: The relationship $\theta : [0, T] \times \Omega \times \mathcal{Y} \to \mathbb{R}^d$ is *$\mathcal{Y}$-predictable* (respectively, *$\mathcal{Y}$-optional*) if θ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{Y})$ (resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{Y})$) measurable.

Let Y be a random variable taking values in $\mathcal{Y}$ and make the following assumption.

ASSUMPTION S4.2: *For $t \leq T$, $\mathbb{P}[Y \in \cdot|\mathcal{F}_t] \sim \mathbb{P}[Y \in \cdot]$ almost surely. Denote by $p_t^y = p(t, \cdot, y)$ the resultant density and by λ the unconditional law of Y.*

Define $\mathbb{G} := \mathbb{F} \vee \mathfrak{s}(Y)$. The first lemma contains three results from Fontana (2018).

LEMMA S4.3: *Let Assumption S4.2 hold. Then (i) p is $\mathcal{Y}$-optional, (ii) $\mathbb{G}$ satisfies the usual conditions of right-continuity and saturation of $\mathbb{P}$-null sets in $\mathcal{G}_0$, and (iii) both p_0^Y/p^Y and $1/p^Y$ are strictly positive $(\mathbb{P}, \mathbb{G})$ martingales with constant expectation 1.*

PROOF: Parts (i), (ii), and (iii) for $1/p^Y$ follow directly from Lemma 2.3, Lemma 4.2, and Proposition 4.4, respectively, in Fontana (2018). As for p_0^Y/p^Y, let $0 \leq s < t \leq T$, $A \in \mathcal{F}_s$, and $H \in \mathcal{B}(\mathcal{Y})$. Recalling that λ is the law of Y,

$$\mathbb{E}\left[1_{A_s} 1_{Y \in H} \frac{p_0^Y}{p_t^Y}\right] = \mathbb{E}\left[1_{A_s} \int_H p_0^y \lambda(dy)\right] = \mathbb{E}\left[1_{A_s} 1_{Y \in H} \frac{p_0^Y}{p_s^Y}\right].$$

The first equality follows by conditioning on t and the (reverse) second follows by conditioning on s. Taking $A_s = \Omega$ and $H = \mathcal{Y}$, and using Fontana (2018, Equation (4.1)) at $f \equiv 1$ shows that $\mathbb{E}[\int_{\mathcal{Y}} p_0^y \lambda(dy)] = 1$, which finishes the result. *Q.E.D.*

REMARK S4.4: Given Lemma S4.3, it follows from Jacod (1985, Section 1) and Gasbarra, Valkeila, and Vostrikova (2006, Lemma 4.2) that $\theta = \theta^y$ is $\mathcal{Y}$-predictable if and only if $\theta^Y \in \mathcal{P}(\mathbb{G})$. Additionally, by part (iii) above, we may define the martingale=preserving measure $\widetilde{\mathbb{P}}^Y$ by either $d\widetilde{\mathbb{P}}_0^Y/d\mathbb{P} = 1/p_T^Y$ or $d\widetilde{\mathbb{P}}^Y/d\mathbb{P} = p_0^Y/p_T^Y$. Note that if $\mathcal{F}_0$ is $\mathbb{P}$-trivial, then $\widetilde{\mathbb{P}}_0^Y = \widetilde{\mathbb{P}}^Y$. Next, Fontana (2018, Proposition 2.9) proves that B is a $(\widetilde{\mathbb{P}}_0^Y, \mathbb{G})$ Brownian motion with the predictable representation property. Similarly, Fontana (2018, Proposition 4.4) implies B is a $(\widetilde{\mathbb{P}}, \mathbb{G})$ Brownian motion. As $d\widetilde{\mathbb{P}}_0^Y/d\widetilde{\mathbb{P}}^Y|_{\mathcal{G}_T} = p_0^Y$, which is $\mathcal{G}_0$ measurable, it follows that B has the predictable representation property under $(\widetilde{\mathbb{P}}, \mathbb{G})$ as well. For technical integrability reasons, it is more convenient for us to work with $\widetilde{\mathbb{P}}^Y$ rather than $\widetilde{\mathbb{P}}_0^Y$.

The first main result concerns martingale representation. To state it, make the following assumption.

ASSUMPTION S4.5: *We have that $\phi = \phi(\omega, y)$ is a $\mathcal{F}_T \otimes \mathcal{B}(\mathcal{Y})$-measurable function such that $\mathbb{E}[|\phi(\cdot, y)|] < \infty$ for each $y \in \mathcal{Y}$.*

Next, denote by $\theta = \theta^y$ the process $\theta^y \in \mathcal{P}(\mathbb{F})$ for each $y \in \mathcal{Y}$ and such that

$$M_\cdot^y := \mathbb{E}\big[\phi(\cdot, y)|\mathcal{F}_t\big] = M_0^y + \int_0^\cdot \left(\theta_u^y\right)' dB_u. \tag{S4.1}$$

We then have the following intuitive result and corollary.

PROPOSITION S4.6: *Let Assumptions S4.2 and S4.5 hold, and let θ be from (S4.1). Then (i) θ is $\mathcal{Y}$-predictable, hence $\mathcal{P}(\mathbb{G})$ measurable, (ii) the stochastic integral $\int_0^\cdot (\theta_u^y)'\,dB_u$ is $\mathcal{Y}$-predictable, (iii) the stochastic integral $\int_0^\cdot (\theta_u^Y)'\,dB_u$ is well defined, and (iv) $\int_0^\cdot (\theta_u^Y)'\,dB_u$ and $(\int_0^\cdot (\theta_u^y)'\,dB_u)|_{y=Y}$ are indistinguishable.*

COROLLARY S4.7: *If additionally θ is strictly positive almost surely, then the same conclusions hold for $\nu = \nu^y$ defined by*

$$\frac{\phi(\cdot, y)}{\mathbb{E}[\phi(\cdot, y)]} = \mathcal{E}\left(\int_0^\cdot (\nu_u^y)'\,dB_u\right)_T.$$

PROOF OF PROPOSITION S4.6: For (i), it follows from Stricker and Yor (1978, Proposition 3) that we can take $M = M^y$ in (S4.1) to be a cadlag and $\mathcal{B}(\mathcal{Y})$-measurable version of the $\mathbb{F}$-optional projection of $\phi(\cdot, y)$ (see also the proof of Fontana (2018, Lemma 4.2)). The result then follows from Fontana (2018, Proposition A.1).

Part (ii) is proved in Stricker and Yor (1978, Proposition 5) when $\mathbb{E}[(\int_0^T |\theta(t, \cdot, y)|^2\,dt)^{1/2}] < \infty$ (and noting the integral sample paths are continuous). For the general case, set $\theta_n = \theta 1_{|\theta| \le n}$ and write M^n as the resultant $\mathcal{Y}$-predictable map. Clearly, we have $\mathbb{P}\text{-}\lim_{n,m\to\infty} \int_0^T |\theta_n(t, \cdot, y) - \theta_m(t, \cdot, y)|^2\,dt = 0$, and hence by Karatzas and Shreve (1991, Proposition 3.2.26) we know $\mathbb{P}\text{-}\lim_{n,m\to\infty} \sup_{t\le T} |M_n(t, \cdot, y) - M_m(t, \cdot, y)| = 0$. The result follows using Stricker and Yor (1978, Proposition 1) with $\mathcal{F}$, $\mathbb{P}$ therein being $\mathcal{P}(\mathbb{F})$ and $\mathbb{P} \times \text{Leb}_{[0,T]}$, respectively. For part (iii), we first note that by part (i), $\theta^Y \in \mathcal{P}(\mathbb{G})$. Next, as B is a $(\widetilde{\mathbb{P}}^Y, \mathbb{G})$ Brownian motion, hence it is $(\mathbb{P}, \mathbb{G})$ semi-martingale. Thus, the result will follow if $\mathbb{P}[\int_0^T |\theta_u^Y|^2\,du < \infty] = 1$. By Fubini, we know that $1_{\int_0^T |\theta_u^Y|^2\,du<\infty} = (1_{\int_0^T |\theta_u^y|^2\,du<\infty})|_{y=Y}$and that $(\omega, g) \to 1_{(\int_0^T |\theta_u^y|^2\,du)(\omega)<\infty}$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathcal{Y})$-measurable. Thus, from Fontana (2018, Equation (4.1)) we conclude

$$\mathbb{P}\left[\int_0^T |\theta_u^Y|^2\,du < \infty\right] = \int_{\mathcal{Y}} \mathbb{E}\left[p_T^y 1_{\int_0^T |\theta_u^y|^2\,du<\infty}\right]\lambda(dy) = \int_{\mathcal{Y}} \mathbb{E}\left[p_T^y\right]\lambda(dy) = 1, \tag{S4.2}$$

where the last equality follows from Fontana (2018, Equation (4.1)) applied to $f \equiv 1$.

That part (iv) holds is stated in the proof of Fontana (2018, Proposition 4.10) as following from (a) an application of the monotone convergence theorem and (b) Stricker and Yor (1978, Proposition 5) combined with the dominated convergence theorem for stochastic integrals (see Protter (2005, Chapter IV, Theorem 32)). Part (iv) is also implicitly used in the proof of Amendinger, Imkeller, and Schweizer (1998, Corollary 2.10). However, for the sake of clarity, we will offer a detailed sketch.

First, assume $\theta(t, \omega, y) = \psi(t, \omega)h(y)$, where $\psi \in \mathcal{P}(\mathbb{F})$ and $h \in \mathcal{B}(\mathcal{Y})$ are bounded. Considering integration with respect to the $(\mathbb{P}, \mathbb{F})$Brownian motion B, it follows that $(\int_0^\cdot (\theta_u^y)'\,dB_u)|_{y=Y} = h(Y)\int_0^\cdot \psi(u, \cdot)'\,dB_u$. Next, considering integration with respect to the $(\widetilde{\mathbb{P}}^Y, \mathbb{G})$ Brownian motion B, we have $\int_0^t (\theta_u^Y)'\,dB_u = h(Y)\int_0^\cdot \psi(u, \cdot)'\,dB_u$. The result follows by path continuity. Next, let bounded $\{\theta_n\}$ converge (boundedly) to a bounded θ. Write the associated integrals as M_n, M. For each $t \le T$.

$$\begin{aligned}
&\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(M(t, \cdot, Y) - \int_0^t \theta(u, \cdot, Y)'\,dB_u\right)^2\right] \\
&\quad \le 2\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(M(t, \cdot, Y) - M_n(t, \cdot, Y)\right)^2\right] + 2\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(\int_0^t \left(\theta_n(u, \cdot, Y) - \theta(u, \cdot, Y)\right)'\,dB_u\right)^2\right].
\end{aligned}$$

First,

$$\begin{aligned}\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\big[\big(M(t,\cdot,Y)-M_n(t,\cdot,Y)\big)^2\big] &= \mathbb{E}\left[\frac{1}{p_t^Y}\big(M(t,\cdot,Y)-M_n(t,\cdot,Y)\big)^2\right] \\ &= \mathbb{E}\left[\int_{\mathcal{Y}}\big(M(t,\cdot,y)-M_n(t,\cdot,y)\big)^2\lambda(dy)\right] \\ &= \mathbb{E}\left[\int_{\mathcal{Y}}\left(\int_0^t\big|\theta(u,\cdot,y)-\theta_n(u,\cdot,y)\big|^2\,du\right)\lambda(dy)\right] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\int_0^t\big|\theta(u,\cdot,Y)-\theta_n(u,\cdot,Y)\big|^2\,du\right].\end{aligned}$$

Above we have used the definition of p^Y and the Ito isometry. Similarly

$$\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(\int_0^t\big(\theta_n(u,\cdot,Y)-\theta(u,\cdot,Y)\big)'\,dB_u\right)^2\right]=\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\int_0^t\big|\theta(u,\cdot,Y)-\theta_n(u,\cdot,Y)\big|^2\,du\right].$$

The bounded convergence theorem implies almost surely for $t\le T$ that $M(t,\cdot,Y)-\int_0^t\theta(u,\cdot,Y)'\,dB_u=0$. As θ is bounded, $\int_0^{\cdot}\theta(u,\cdot,Y)'\,dB_u$ is a $(\widetilde{\mathbb{P}}^Y,\mathbb{G})$ martingale. But this implies $M(t,\cdot,Y)$ is also a $(\widetilde{\mathbb{P}}^Y,\mathbb{G})$ martingale. As martingale representation holds with respect to B, we deduce $M(\cdot,\cdot,Y)$ has continuous paths, and hence $M(\cdot,\cdot,Y)$ and $\int_0^{\cdot}\theta(u,\cdot,Y)'\,dB_u$ are indistinguishable. The monotone class theorem gives the result for bounded θ. We now extend to θ such that $\int_0^T|\theta(u,\cdot,Y)|^2\,du<\infty$. For each t, n,

$$\begin{aligned}M(t,\cdot,Y)-\int_0^t\theta(u,\cdot,Y)'\,dB_u &= \left.\left(\int_0^t\big(\theta(u,\cdot,y)1_{|\theta(u,\cdot,y)|\ge n}\big)'\,dB_u\right)\right|_{y=Y} \\ &\quad -\int_0^t\big(\theta(u,\cdot,Y)1_{|\theta(u,\cdot,Y)|\ge n}\big)'\,dB_u. \qquad \text{(S4.3)}\end{aligned}$$

We first handle the rightmost term above. By construction of p^Y, for each $\varepsilon>0$,

$$\widetilde{\mathbb{P}}^Y\left[\int_0^T\big|\theta(u,\cdot,Y)\big|^2 1_{|\theta(u,\cdot,Y)|\ge n}\,du\ge\varepsilon\right]=\int_{\mathcal{Y}}\mathbb{E}\big[1_{\int_0^T|\theta(u,\cdot,y)|^2 1_{|\theta(u,\cdot,y)|\ge n}\,du\ge\varepsilon}\big]\lambda(dy).$$

Since for each $y\in\mathcal{Y}$, $\int_0^T|\theta(u,\cdot,y)|^2 1_{|\theta(u,\cdot,y)|\ge n}\,du\to 0$ almost surely as $n\uparrow\infty$, two applications of the dominated convergence theorem allow us to conclude that $\lim_{n\uparrow\infty}\int_0^T|\theta(u,\cdot,Y)|^2 1_{|\theta(u,\cdot,Y)|\ge n}\,du=0$ in $\widetilde{\mathbb{P}}^Y$ probability. Therefore, by Karatzas and Shreve (1991, Proposition 3.2.26) we know that in $\widetilde{\mathbb{P}}^Y$ probability,

$$\lim_{n\uparrow\infty}\sup_{t\le T}\left|\int_0^t\big(\theta(u,\cdot,Y)1_{|\theta(u,\cdot,Y)|\ge n}\big)'\,dB_u\right|=0.$$

As for the first term on the right side of (S4.3), set $M_n(t,\cdot,y):=\int_0^t(\theta(u,\cdot,y)\times 1_{|\theta(u,\cdot,y)|\ge n})'\,dB_u$. Since for each $y\in\mathcal{Y}$, $\int_0^T|\theta(u,\cdot,y)|^2 1_{|\theta(u,\cdot,y)|\ge n}\,du$ converges to 0 almost surely, we again deduce from Karatzas and Shreve (1991, Proposition 3.2.26) that in $\mathbb{P}$

probability $\sup_{t\le T}|M_n(t,\cdot,y)|\to 0$. As M_n is $\mathcal{Y}$-optional,

$$\widetilde{\mathbb{P}}^Y\Big[\sup_{t\le T}\big|M_n(t,\cdot,Y)\big|\ge\varepsilon\Big]=\int_Y\mathbb{E}[1_{\sup_{t\le T}|M_n(t,\cdot,y)|\ge\varepsilon}]\lambda(dy),$$

so that $\sup_{t\le T}|M_n(t,\cdot,Y)|\to 0$ in $\widetilde{\mathbb{P}}^Y$ probability. Thus, by taking subsequences where the convergence takes place almost surely $\mathbb{Q}^G$ and hence $\mathbb{P}$, we deduce from (S4.3) that $\sup_{t\le T}|M(t,\cdot,Y)-\int_0^t\theta(u,\cdot,Y)'\,dB_u|=0$ almost surely, finishing the result. *Q.E.D.*

PROOF OF COROLLARY S4.7: It is clear that $\nu=\theta/M$. Thus, by the results on θ above (in particular, the connection to the proof of (iv) which proved indistinguishability for general θ), it suffices to prove that $\int_0^T|\nu_u^Y|^2\,du<\infty$ almost surely. But this will follow provided $\inf_{t\le T}M_t^Y>0$ almost surely. But this latter fact follows using the same calculations as in (S4.2), but now for the random variable $1_{\inf_{t\le T}M_t^y>0}$, which is almost surely 1 for all y since M^y has continuous paths. *Q.E.D.*

Last, we relate $(\mathbb{Q}^{\mathbb{G}},\mathbb{G})$ and $(\mathbb{Q},\mathbb{F})$ conditional expectations, where $\mathbb{Q}$ is a measure on $\mathcal{F}_T$, and $\mathbb{Q}^{\mathbb{G}}$ is built from $\mathbb{Q}$ in a similar manner to how $\widetilde{\mathbb{P}}^Y$ was built from $\mathbb{P}$.

LEMMA S4.8: *Let Z be a strictly positive $(\mathbb{P},\mathbb{F})$ martingale with $\mathbb{E}[Z_0]=1$. Define $\mathbb{Q}$ via $d\mathbb{Q}/d\mathbb{P}:=Z_T$ and $Z_\cdot^{\mathbb{G}}:=Z_\cdot/p_\cdot^Y$. Then $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$ martingale. Next, define $\mathbb{Q}^{\mathbb{G}}$ by $d\mathbb{Q}^{\mathbb{G}}/d\mathbb{P}=Z_T^{\mathbb{G}}$. Let $0\le s<t\le T$, let ϕ be $\mathcal{F}_t$-measurable, taking values in $G\subseteq\mathbb{R}^n$, and let $f:G\times\mathcal{Y}\to\mathbb{R}$ be a measurable function such that either (a) f is nonnegative or (b) $\mathbb{E}^{\mathbb{Q}}[|f(\phi,y)|]<\infty$ for each $y\in\mathcal{Y}$ as well as $\int_{\mathcal{Y}}\mathbb{E}^{\mathbb{Q}}[|f(\phi,y)|]\lambda(dy)<\infty$. Then*

$$\mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\big[f(\phi,Y)|\mathcal{G}_s\big]=\big(\mathbb{E}^{\mathbb{Q}}\big[f(\phi,y)|\mathcal{F}_s\big]\big)\big|_{y=\mathcal{Y}}.$$

PROOF: Let $0\le s<t\le T$, $A_s\in\mathcal{F}_s$, $H\in\mathcal{B}(\mathcal{Y})$ and denote by λ the law of Y. We have

$$\begin{aligned}\mathbb{E}\big[1_{A_s}1_{Y\in H}Z_t^{\mathbb{G}}\big]&=\mathbb{E}\bigg[1_{A_s}1_{Y\in H}Z_t\frac{1}{p_t^Y}\bigg]=\int_H\mathbb{E}[1_{A_s}Z_t1]\lambda(dy)\\&=\int_H\mathbb{E}[1_{A_s}Z_s]\lambda(dy)=\mathbb{E}\big[1_{A_s}1_{Y\in H}Z_s^{\mathbb{G}}\big].\end{aligned}$$

Taking the above for $A_s=\Omega$ and $H=\mathcal{Y}$, we see $\mathbb{E}[Z_t^{\mathbb{G}}]=\int_{\mathcal{Y}}\mathbb{E}[Z_t]\lambda(dy)=\int_{\mathcal{Y}}\mathbb{E}[Z_0]\times\lambda(dy)=1$. Here, we have used the martingale property for Z and $\mathbb{E}[Z_0]=1$. The martingale property readily follows. As for the conditional expectation equality, if f is not nonnegative, the condition $\int_{\mathcal{Y}}\mathbb{E}^{\mathbb{Q}}[|f(\phi,y)|]\lambda(dy)<\infty$ implies $\mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}[|f(\phi,Y)|]<\infty$, so the conditional expectation is well defined. Next, let $A_s\in\mathcal{F}_s$ and $H\in\mathcal{B}(\mathcal{Y})$. Set $\chi_s^t(y):=\mathbb{E}^{\mathbb{Q}}[f(\phi,y)|\mathcal{F}_s^B]$. Note that $(\omega,y)\to\chi_s^t(y)$ is $\mathcal{F}_s\times\mathcal{B}(\mathcal{Y})$-measurable, and hence

$\chi_s^t(Y)$ is $\mathcal{G}_s$-measurable. As $Z^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$ martingale,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\big[1_{A_s}1_{Y\in H}\chi_s^t(Y)\big] &= \mathbb{E}\left[1_{A_s}1_{Y\in H}\chi_s^t(Y)\frac{Z_s}{p_s^Y}\right] = \mathbb{E}\left[1_{A_s}Z_s\mathbb{E}\left[1_{Y\in H}\chi_s^t(Y)\frac{1}{p_s^Y}|\mathcal{F}_s^B\right]\right]\\ &= \mathbb{E}\left[1_{A_s}Z_s\int_H \chi_s^t(y)1\lambda(dy)\right] = \int_H \mathbb{E}\big[1_{A_s}Z_s\chi_s^t(y)\big]\lambda(dy)\\ &= \int_H \mathbb{E}\big[1_{A_s}Z_t f(\phi, y)\big]\lambda(dy) = \mathbb{E}\left[1_{A_s}Z_t\int_H \frac{1}{p_t^y}p_t^y f(\phi, y)\lambda(dy)\right]\\ &= \mathbb{E}\left[1_{A_s}1_{Y\in H}\frac{Z_t}{p_t^Y}f(\phi, Y)\right] = \mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\big[1_{A_s}1_{Y\in H}f(\phi, Y)\big].\end{aligned}$$

Q.E.D.

REFERENCES

AMENDINGER, J. (2000): "Martingale Representation Theorems for Initially Enlarged Filtrations," *Stochastic Process. Appl.*, 89 (1), 101–116. [14,15,18]

AMENDINGER, J., P. IMKELLER, AND M. SCHWEIZER (1998): "Additional Logarithmic Utility of an Insider," *Stochastic Process. Appl.*, 75 (2), 263–286. [20]

DEMBO, A., AND O. ZEITOUNI (1998): *Large Deviations Techniques and Applications* (Second Ed.). Applications of Mathematics (New York), Vol. 38. New York: Springer-Verlag. [3]

ESMAEELI, N., AND P. IMKELLER (2018): "American Options With Asymmetric Information and Reflected BSDE," *Bernoulli*, 24 (2), 1394–1426. [18]

FONTANA, C. (2018): "The Strong Predictable Representation Property in Initially Enlarged Filtrations Under the Density Hypothesis," *Stochastic Process. Appl.*, 128 (3), 1007–1033. [5,8,12,18-20]

GASBARRA, D., E. VALKEILA, AND L. VOSTRIKOVA (2006): "Enlargement of Filtration and Additional Information in Pricing Models: Bayesian Approach," in *From Stochastic Calculus to Mathematical Finance*. Berlin: Springer, 257–285. [18,19]

JACOD, J. (1985): *Grossissement initial, hypothese (H') et theoreme de Girsanovpp*. Berlin, Heidelberg: Springer Berlin Heidelberg, 15–35. [19]

KARATZAS, I., AND S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus* (Second Ed.). Graduate Texts in Mathematics, Vol. 113. New York: Springer-Verlag. [20,21]

PROTTER, P. E. (2005): *Stochastic Integration and Differential Equations* (Second Ed.). Stochastic Modelling and Applied Probability, Vol. 21. Berlin: Springer-Verlag. Version 2.1, Corrected third printing. [20]

STRICKER, C., AND M. YOR (1978): "Calcul stochastique dépendant d'un paramètre," *Z. Wahrsch. Verw. Gebiete*, 45 (2), 109–133. [18,20]

Co-editor Alessandro Lizzeri handled this manuscript.

Manuscript received 5 February, 2019; final version accepted 5 June, 2020; available online 10 June, 2020.