Econometrica Supplementary Material

SUPPLEMENT TO "REPUTATION EFFECTS UNDER INTERDEPENDENT VALUES": ADDITIONAL APPENDIX

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APPENDIX SA: GENERALIZATION AND PROOF OF THEOREM 2

I STATE A GENERALIZED VERSION of Theorem 2 by allowing for arbitrary correlations between the two dimensions of player 1's private information in μ. For every $\theta \in \Theta$, let $\mu(\theta)$ be the probability of commitment type θ. For every $a_1^* \in A_1^*$, let $\mu(a_1^*)$ be the probability of commitment type a_1^*. I say that μ is optimistic if

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta_g \cup \Theta_p} \mu(\theta)\mathcal{D}(\theta, \overline{a}_1) > 0 \tag{SA.1}$$

and is pessimistic otherwise, which generalizes the optimistic and pessimistic belief conditions in the main text.

THEOREM 2′: *If the game satisfies Assumptions* 1, 2, *and* 3, *and* μ *has full support and satisfies Assumption* 4 *and* (SA.1), *then*

$$\liminf_{\delta \to 1} \underline{v}_\theta(\delta, \mu, u_1, u_2) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) \quad \textit{for every } \theta \in \Theta^*.$$

SA.1. *Defining Useful Constants*

Recall the definitions of Θ_g, Θ_p, and Θ_n in Appendix D of the main text. Let θ_g, θ_p, and θ_n be typical elements of these sets and recall that Lemma D.1 shows that $\theta_g \succ \theta_p \succ \theta_n$.

My proof starts by defining several useful constants which only depend on μ, u_1, and u_2, but are independent of σ and δ. Let $M \equiv \max_{\theta, a_1, a_2} |u_1(\theta, a_1, a_2)|$ and

$$K \equiv \frac{\max_{\theta \in \Theta}\{u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \underline{a}_2)\}}{\min_{\theta \in \Theta}\{u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \underline{a}_2)\}}.$$

Since $\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) > 0$, the optimistic belief condition (SSA.1) implies the existence of $\kappa \in (0, 1)$ such that

$$\kappa\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} \mu(\theta)\mathcal{D}(\theta, \overline{a}_1) > 0.$$

For any $\kappa \in (0, 1)$, let

$$\rho_0(\kappa) \equiv \frac{(1-\kappa)\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1)}{2 \max_{(\theta, a_1) \in \Theta \times A_1} |\mathcal{D}(\theta, a_1)|} > 0 \tag{SA.2}$$

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and

$$\overline{T}_0(\kappa) \equiv \lceil 1/\rho_0(\kappa) \rceil. \tag{SA.3}$$

Let

$$\rho_1(\kappa) \equiv \frac{\kappa\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1)}{\max_{(\theta,a_1)}|\mathcal{D}(\theta, a_1)|} \tag{SA.4}$$

and

$$\overline{T}_1(\kappa) \equiv \lceil 1/\rho_1(\kappa) \rceil. \tag{SA.5}$$

Let $\overline{\delta} \in (0, 1)$ be close enough to 1 such that for every $\delta \in [\overline{\delta}, 1)$ and $\theta_p \in \Theta_p$,

$$\begin{aligned} &(1 - \delta^{\overline{T}_0(0)})u_1(\theta_p, \overline{a}_1, \underline{a}_2) + \delta^{\overline{T}_0(0)} u_1(\theta_p, \overline{a}_1, \overline{a}_2) \\ &\quad > \frac{1}{2}\big(u_1(\theta_p, \overline{a}_1, \overline{a}_2) + u_1(\theta_p, \underline{a}_1, \underline{a}_2)\big). \end{aligned} \tag{SA.6}$$

SA.2. *Random History and Random Path*

Let $\Omega \equiv A_1^* \cup \Theta$ be the set of types with ω a typical element of Ω. Abusing notation, I write μ as a full support distribution on Ω. Let $h^t \equiv (a^t, r^t)$, with $a^t \equiv (a_{1,s})_{s\le t-1}$ and $r^t \equiv (a_{2,s}, \xi_s)_{s\le t-1}$. Let $a_*^t \equiv (\overline{a}_1, \dots, \overline{a}_1)$. I call h^t a *public history*, r^t a *random history*, and r^∞ a *random path*. Let $\mathcal{H}$ and $\mathcal{R}$ be the set of public histories and random histories, respectively, with $\succ$, $\succsim$, $\prec$, and $\precsim$ naturally defined. Recall that a strategy profile σ consists of $(\sigma_\theta)_{\theta\in\Theta}$ with $\sigma_\theta : \mathcal{H} \to \Delta(A_1)$ and $\sigma_2 : \mathcal{H} \to \Delta(A_2)$. Let $\mathcal{P}^\sigma(\theta)$ be the probability measure over public histories induced by $(\sigma_\theta, \sigma_2)$. Let $\mathcal{P}^\sigma \equiv \sum_{\omega\in\Omega} \mu(\omega)\mathcal{P}^\sigma(\omega)$ be the probability measure induced by σ, taking into account the possibilities of commitment types. Let $v^\sigma(h^t) \equiv \{v_\theta^\sigma(h^t)\}_{\theta\in\Theta} \in \mathbb{R}^{|\Theta|}$ be the continuation payoff vector for strategic types at h^t under strategy profile σ.

Let $\mathcal{H}^\sigma \subset \mathcal{H}$ be the set of histories h^t such that $\mathcal{P}^\sigma(h^t) > 0$, and let $\mathcal{H}^\sigma(\omega) \subset \mathcal{H}$ be the set of histories h^t such that $\mathcal{P}^\sigma(\omega)(h^t) > 0$. Let

$$\mathcal{R}_*^\sigma \equiv \{r^\infty | (a_*^t, r^t) \in \mathcal{H}^\sigma \text{ for all } t \text{ and } r^t \prec r^\infty\}$$

be the set of random paths consistent with player 1 playing $\overline{a}_1$ in every period. For every $h^t = (a^t, r^t)$, let $\overline{\sigma}_1[h^t] : \mathcal{H} \to A_1$ be a strategy in the continuation game at h^t that satisfies $\overline{\sigma}_1[h^t](h^s) = \overline{a}_1$ for all $h^s \succsim h^t$ with $h^s = (a^t, \overline{a}_1, \dots, \overline{a}_1, r^s) \in \mathcal{H}^\sigma$. Let $\underline{\sigma}_1[h^t] : \mathcal{H} \to A_1$ be a strategy in the continuation game at h^t that satisfies $\underline{\sigma}_1[h^t](h^s) = \underline{a}_1$ for all $h^s \succsim h^t$ with $h^s = (a^t, \underline{a}_1, \dots, \underline{a}_1, r^s) \in \mathcal{H}^\sigma$. For every $\theta \in \Theta$, let

$$\overline{\mathcal{R}}^\sigma(\theta) \equiv \{r^t | \overline{\sigma}_1[a_*^t, r^t] \text{ is type } \theta\text{'s best reply to } \sigma_2\}$$

and

$$\underline{\mathcal{R}}^\sigma(\theta) \equiv \{r^t | \underline{\sigma}_1[a_*^t, r^t] \text{ is type } \theta\text{'s best reply to } \sigma_2\}.$$

SA.3. *Beliefs and Best Response Sets*

Let $\mu(a^t, r^t) \in \Delta(\Omega)$ be player 2's posterior belief at (a^t, r^t) and, specifically, let $\mu^*(r^t) \equiv \mu(a^t_*, r^t)$. Let

$$\mathcal{B}_\kappa \equiv \left\{ \tilde{\mu} \in \Delta(\Omega) \middle| \kappa \tilde{\mu}(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} \tilde{\mu}(\theta)\mathcal{D}(\theta, \overline{a}_1) \geq 0 \right\}. \tag{SA.7}$$

By definition, $\mathcal{B}_{\kappa'} \subsetneq \mathcal{B}_\kappa$ for every $\kappa, \kappa' \in [0, 1]$ with $\kappa' < \kappa$.

For every $r^t \in \mathcal{R}^t$ and $\omega \in \Omega$, let $q^*(r^t)(\omega)$ be the ex ante probability that (a) player 1 is type ω and (b) player 1 has played $\overline{a}_1$ from period 0 to $t-1$, *conditional on* the realization of random history being r^t. Let $q^*(r^t) \equiv \{q^*(r^t)(\omega)\}_{\omega \in \Omega}$. For every $\delta \in (0, 1)$ and strategy profile $\sigma \in \mathrm{NE}(\delta, \mu)$, the following statements hold:

(i) For every a^t and $r^t, \hat{r}^t \succ r^{t-1}$ satisfying $(a^t, r^t), (a^t, \hat{r}^t) \in \mathcal{H}^\sigma$, we have $\mu(a^t, r^t) = \mu(a^t, \hat{r}^t)$.

(ii) For every $r^t, \hat{r}^t \succ r^{t-1}$ with $(a^t_*, r^t), (a^t_*, \hat{r}^t) \in \mathcal{H}^\sigma$, we have $q^*(r^t) = q^*(\hat{r}^t)$.

This is because player 1's action in period $t-1$ depends on r^t only through r^{t-1}, so is player 2's belief at every on-path history. Since the commitment type plays $\overline{a}_1$ in every period, we have $q^*(r^t)(\overline{a}_1) = \mu_0(\overline{a}_1)$.

For future reference, I introduce two sets of random histories based on player 2's posterior beliefs. Let

$$\mathcal{R}^\sigma_g \equiv \{ r^t | (a^t_*, r^t) \in \mathcal{H}^\sigma \text{ and } \mu^*(r^t)(\Theta_p \cup \Theta_n) = 0 \} \tag{SA.8}$$

and let

$$\widehat{\mathcal{R}}^\sigma_g \equiv \{ r^t | \text{there exists } r^T \succsim r^t \text{ such that } r^T \in \mathcal{R}^\sigma_g \}. \tag{SA.9}$$

Intuitively, $\widehat{\mathcal{R}}^\sigma_g$ is the set of on-path random histories under which all the strategic types in $\Theta_p \cup \Theta_n$ will be separated from commitment type $\overline{a}_1$ at some random histories in the future.

SA.4. *Four Useful Lemmas*

Recall that $\sigma_\theta : \mathcal{H} \to \Delta(A_1)$ is type θ's strategy. The first lemma outlines the implications of monotone-supermodularity on different types of player 1's equilibrium strategies.

LEMMA SA.1: *Suppose σ is an equilibrium under (δ, μ), $\theta \succ \tilde{\theta}$ and $h^t_* = (a^t_*, r^t) \in \mathcal{H}^\sigma(\theta) \cap \mathcal{H}^\sigma(\tilde{\theta})$:*

(i) *If $r^t \in \overline{\mathcal{R}}^\sigma(\tilde{\theta})$, then $\sigma_\theta(a^s_*, r^s)(\overline{a}_1) = 1$ for every $(a^s_*, r^s) \in \mathcal{H}^{(\overline{\sigma}_1(h^t_*), \sigma_2)}(\theta)$ with $r^s \succsim r^t$.*

(ii) *If $r^t \in \underline{\mathcal{R}}^\sigma(\theta)$, then $\sigma_{\tilde{\theta}}(a^s, r^s)(\underline{a}_1) = 1$ for every $(a^s, r^s) \in \mathcal{H}^{(\underline{\sigma}_1(h^t_*), \sigma_2)}(\tilde{\theta})$ with $(a^s, r^s) \succsim (a^t_*, r^t)$.*

PROOF: I only need to show the first part, as the second part is symmetric after switching signs. Without loss of generality, I focus on history h^0. For notational simplicity, let $\overline{\sigma}_1[h^0] = \overline{\sigma}_1$. For every σ_ω and σ_2, let $P^{(\sigma_\omega, \sigma_2)} : A_1 \times A_2 \to [0, 1]$ be defined as

$$P^{(\sigma_\omega, \sigma_2)}(a_1, a_2) \equiv \sum_{t=0}^{+\infty} (1-\delta)\delta^t p_t^{(\sigma_\omega, \sigma_2)}(a_1, a_2),$$

where $p_t^{(\sigma_\omega,\sigma_2)}(a_1,a_2)$ is the probability of (a_1,a_2) occurring in period t under (σ_ω,σ_2). Let $P_i^{(\sigma_1,\sigma_2)} \in \Delta(A_2)$ be $P^{(\sigma_1,\sigma_2)}$'s marginal distribution on A_i, for $i \in \{1,2\}$.

Suppose toward a contradiction that $\overline{\sigma}_1$ is type $\tilde{\theta}$'s best reply and there exists σ_θ with $P_1^{(\sigma_\theta,\sigma_2)}(\overline{a}_1) < 1$ such that σ_θ is type θ's best reply. Then type $\tilde{\theta}$'s and type θ's incentive constraints require that

$$\sum_{a_2\in A_2}\big(P_2^{(\overline{\sigma}_1,\sigma_2)}(a_2)-P_2^{(\sigma_\theta,\sigma_2)}(a_2)\big)u_1(\tilde{\theta},\overline{a}_1,a_2) \geq \sum_{a_2\in A_2,a_1\neq\overline{a}_1} P^{(\sigma_\theta,\sigma_2)}(a_1,a_2)\big(u_1(\tilde{\theta},a_1,a_2)-u_1(\tilde{\theta},\overline{a}_1,a_2)\big)$$

and

$$\sum_{a_2\in A_2}\big(P_2^{(\overline{\sigma}_1,\sigma_2)}(a_2)-P_2^{(\sigma_\theta,\sigma_2)}(a_2)\big)u_1(\theta,\overline{a}_1,a_2) \leq \sum_{a_2\in A_2,a_1\neq\overline{a}_1} P^{(\sigma_\theta,\sigma_2)}(a_1,a_2)\big(u_1(\theta,a_1,a_2)-u_1(\theta,\overline{a}_1,a_2)\big).$$

Since $P_1^{(\sigma_\theta,\sigma_2)}(\overline{a}_1) < 1$ and u_1 has strictly increasing differences in θ and a_1, we have

$$\sum_{a_2\in A_2,a_1\neq\overline{a}_1} P^{(\sigma_\theta,\sigma_2)}(a_1,a_2)\big(u_1(\tilde{\theta},a_1,a_2)-u_1(\tilde{\theta},\overline{a}_1,a_2)\big) > \sum_{a_2\in A_2,a_1\neq\overline{a}_1} P^{(\sigma_\theta,\sigma_2)}(a_1,a_2)\big(u_1(\theta,a_1,a_2)-u_1(\theta,\overline{a}_1,a_2)\big),$$

which implies that

$$\sum_{a_2\in A_2}\big(P_2^{(\sigma_\theta,\sigma_2)}(a_2)-P_2^{(\overline{\sigma}_1,\sigma_2)}(a_2)\big)\big(u_1(\theta,\overline{a}_1,a_2)-u_1(\tilde{\theta},\overline{a}_1,a_2)\big) > 0. \tag{SA.10}$$

On the other hand, since u_1 is strictly decreasing in a_1, we have

$$\sum_{a_2\in A_2,a_1\neq\overline{a}_1} P^{(\sigma_\theta,\sigma_2)}(a_1,a_2)\big(u_1(\tilde{\theta},a_1,a_2)-u_1(\tilde{\theta},\overline{a}_1,a_2)\big) > 0.$$

Strategic type $\tilde{\theta}$'s incentive constraint implies that

$$\sum_{a_2\in A_2}\big(P_2^{(\overline{\sigma}_1,\sigma_2)}(a_2)-P_2^{(\sigma_\theta,\sigma_2)}(a_2)\big)u_1(\tilde{\theta},\overline{a}_1,a_2) > 0. \tag{SA.11}$$

Since both $P_2^{(\sigma_\theta,\sigma_2)}$ and $P_2^{(\overline{\sigma}_1,\sigma_2)}$ are probability distributions, we have

$$\sum_{a_2\in A_2}\big(P_2^{(\sigma_\theta,\sigma_2)}(a_2)-P_2^{(\overline{\sigma}_1,\sigma_2)}(a_2)\big) = 0.$$

Since $u_1(\theta,\overline{a}_1,a_2)-u_1(\tilde{\theta},\overline{a}_1,a_2)$ is weakly increasing in a_2, inequality (SA.10) implies that $P_2^{(\sigma_\theta,\sigma_2)}(\overline{a}_2)-P_2^{(\overline{\sigma}_1,\sigma_2)}(\overline{a}_2) > 0$. Since $u_1(\tilde{\theta},\overline{a}_1,a_2)$ is strictly increasing in a_2, (SA.11) implies that $P_2^{(\sigma_\theta,\sigma_2)}(\overline{a}_2)-P_2^{(\overline{\sigma}_1,\sigma_2)}(\overline{a}_2) < 0$, leading to a contradiction. *Q.E.D.*

The next lemma establishes a uniform upper bound on the number of periods in which $\overline{a}_2$ is not player 2's strict best reply although $\overline{a}_1$ has been played in all previous periods and $\mu^*(r^t) \in \mathcal{B}_\kappa$.

LEMMA SA.2: *If $\mu^*(r^t) \in \mathcal{B}_\kappa$ and $\overline{a}_2$ is not a strict best reply at (a_*^t, r^t), then for every $r^{t+1} \succ r^t$ with $(a_*^{t+1}, r^{t+1}) \in \mathcal{H}^\sigma$, we have*

$$\sum_{\theta\in\Theta}\big(q^*\big(r^t\big)(\theta) - q^*\big(r^{t+1}\big)(\theta)\big) \geq \rho_0(\kappa). \tag{SA.12}$$

PROOF: If $\mu^*(r^t) \in \mathcal{B}_\kappa$, then

$$\kappa\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta\in\Theta} q^*\big(r^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) \geq 0.$$

Suppose $\overline{a}_2$ is not a strict best reply at (a_*^t, r^t). Then

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta\in\Theta} q^*\big(r^{t+1}\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) + \sum_{\theta\in\Theta}\big(q^*\big(r^t\big)(\theta) - q^*\big(r^{t+1}\big)(\theta)\big)\mathcal{D}(\theta, \underline{a}_1) \leq 0$$

for every $r^{t+1} \succ r^t$ with $(a_*^{t+1}, r^{t+1}) \in \mathcal{H}^\sigma$ or, equivalently,

$$\underbrace{\kappa\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta\in\Theta} q^*\big(r^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1)}_{\geq 0} + \underbrace{(1-\kappa)\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1)}_{>0}$$
$$+ \sum_{\theta\in\Theta}\big(q^*\big(r^{t+1}\big)(\theta) - q^*\big(r^t\big)(\theta)\big)\mathcal{D}(\theta, \overline{a}_1) + \sum_{\theta\in\Theta}\big(q^*\big(r^t\big)(\theta) - q^*\big(r^{t+1}\big)(\theta)\big)\mathcal{D}(\theta, \underline{a}_1) \leq 0.$$

According to (SA.2), we have

$$\sum_{\theta\in\Theta}\big(q^*\big(r^t\big)(\theta) - q^*\big(r^{t+1}\big)(\theta)\big) \geq \frac{(1-\kappa)\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1)}{2\max_{(\theta,a_1)\in\Theta\times A_1}\big|\mathcal{D}(\theta, a_1)\big|} = \rho_0(\kappa).$$

Q.E.D.

Lemma SA.2 implies that for every $\sigma \in \text{NE}(\delta, \mu)$ and along every $r^\infty \in \mathcal{R}_*^\sigma$, the number of r^t such that $\mu^*(r^t) \in \mathcal{B}_\kappa$ but $\overline{a}_2$ is not a strict best reply is at most $\overline{T}_0(\kappa)$. The next lemma obtains an upper bound for player 1's continuation payoff after separating from commitment type $\overline{a}_1$ at a history with a *pessimistic posterior belief*.

LEMMA SA.3: *For every $\sigma \in \text{NE}(\delta, \mu)$ and $h^t \in \mathcal{H}^\sigma$ with*

$$\mu\big(h^t\big)(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta\in\Theta} \mu\big(h^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) < 0, \tag{SA.13}$$

we have $v_{\underline{\theta}}(h^t) = u_1(\underline{\theta}, \underline{a}_1, \underline{a}_2)$ with $\underline{\theta} \equiv \min\{\text{supp}(\mu(h^t))\}$.

PROOF: Let

$$\Theta' \equiv \big\{\tilde{\theta} \in \Theta_p \cup \Theta_n | \mu\big(h^t\big)(\tilde{\theta}) > 0\big\}.$$

Since $\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) > 0$, (SA.13) implies that $\Theta' \neq \{\emptyset\}$. The rest of the proof is done via induction on $|\Theta'|$. When $|\Theta'| = 1$, there exists a pure strategy $\sigma^*_{\underline{\theta}} : \mathcal{H} \to A_1$ in the support of $\sigma_{\underline{\theta}}$ such that (SA.13) holds for all h^s satisfying $h^s \in \mathcal{H}^{(\sigma^*_{\underline{\theta}}, \sigma_2)}$ and $h^s \succsim h^t$. At every such h^s, $\underline{a}_2$ is player 2's strict best reply. When playing $\sigma^*_{\underline{\theta}}$, type $\underline{\theta}$'s stage game payoff is no more than $u_1(\underline{\theta}, \underline{a}_1, \underline{a}_2)$ in every period.

Suppose toward a contradiction that the conclusion holds when $|\Theta'| \leq k - 1$ but fails when $|\Theta'| = k$. Then there exists $h^s \in \mathcal{H}^\sigma(\underline{\theta})$ with $h^s \succsim h^t$ such that

(i) $\mu(h^\tau) \notin \mathcal{B}_\kappa$ for all $h^s \succsim h^\tau \succsim h^t$,

(ii) $v_{\underline{\theta}}(h^s) > u_1(\underline{\theta}, \underline{a}_1, \underline{a}_2)$,

(iii) for all a_1 such that $\mu(h^s, a_1) \notin \mathcal{B}_\kappa$, $\sigma_{\underline{\theta}}(h^s)(a_1) = 0$.

Since belief is a martingale, there exists a_1 such that $(h^s, a_1) \in \mathcal{H}^\sigma$ and $\mu(h^s, a_1)$ satisfies (SA.13). Since $\mu(h^s, a_1)(\underline{\theta}) = 0$, there exists $\tilde{\theta} \in \Theta^* \setminus \{\underline{\theta}\}$ such that $(h^s, a_1) \in \mathcal{H}^\sigma(\tilde{\theta})$. My induction hypothesis suggests that $v_{\tilde{\theta}}(h^s) = u_1(\tilde{\theta}, \underline{a}_1, \underline{a}_2)$. The incentive constraints of type $\underline{\theta}$ and type $\tilde{\theta}$ at h^s require the existence of $(\alpha_{1,\tau}, \alpha_{2,\tau})_{\tau=0}^{\infty}$ with $\alpha_{i,\tau} \in \Delta(A_i)$ such that

$$\mathbb{E}\left[\sum_{\tau=0}^{\infty}(1-\delta)\delta^\tau\big(u_1(\underline{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\underline{\theta}, \underline{a}_1, \underline{a}_2)\big)\right]$$

$$> 0 \geq \mathbb{E}\left[\sum_{\tau=0}^{\infty}(1-\delta)\delta^\tau\big(u_1(\tilde{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\tilde{\theta}, \underline{a}_1, \underline{a}_2)\big)\right],$$

where $\mathbb{E}[\cdot]$ is taken over probability measure $\mathcal{P}^\sigma$. However, the supermodularity condition implies that

$$u_1(\underline{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\underline{\theta}, \underline{a}_1, \underline{a}_2) \leq u_1(\tilde{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\tilde{\theta}, \underline{a}_1, \underline{a}_2).$$

This leads to a contradiction. *Q.E.D.*

The next lemma outlines an important implication of $r^t \notin \widehat{\mathcal{R}}^\sigma_g$.

LEMMA SA.4: *If $r^t \notin \widehat{\mathcal{R}}^\sigma_g$ and $(a^t_*, r^t) \in \mathcal{H}^\sigma$, then there exists $\theta \in (\Theta_p \cup \Theta_n) \cap \mathrm{supp}(\mu^*(r^t))$ such that $r^t \in \overline{R}^\sigma(\theta)$.*

PROOF: Suppose toward a contradiction that $r^t \notin \widehat{\mathcal{R}}^\sigma_g$ but no such θ exists. Let

$$\theta_1 \equiv \max\Big\{(\Theta_p \cup \Theta_n) \cap \mathrm{supp}\big(\mu^*\big(r^t\big)\big)\Big\}.$$

The set on the RHS is nonempty according to the definition of $\widehat{\mathcal{R}}^\sigma_g$ and $\mathcal{R}^\sigma_g$

Let $(a^{t_1}_*, r^{t_1}) \succsim (a^t_*, r^t)$ be the history at which type θ_1 has a strict incentive not to play $\overline{a}_1$ with $(a^{t_1}_*, r^{t_1}) \in \mathcal{H}^\sigma$. For any $(a^{t_1+1}_*, r^{t_1+1}) \succ (a^{t_1}_*, r^{t_1})$ with $(a^{t_1+1}_*, r^{t_1+1}) \in \mathcal{H}^\sigma$, on one hand, we have $\mu^*(r^{t_1+1})(\theta_1) = 0$. On the other hand, the fact that $r^t \notin \widehat{\mathcal{R}}^\sigma_g$ implies that $\mu^*(r^{t_1+1})(\Theta_n \cup \Theta_p) > 0$. Let

$$\theta_2 \equiv \max\Big\{(\Theta_p \cup \Theta_n) \cap \mathrm{supp}\big(\mu^*\big(r^{t_1+1}\big)\big)\Big\},$$

and let us examine type θ_1 and θ_2 incentive constraints at $(a^{t_1}_*, r^{t_1})$. According to Lemma SA.1, there exists $r^{t_2} \succ r^{t_1}$ such that type θ_2 has a strict incentive not to play $\overline{a}_1$ at

$(a_*^{t_2}, r^{t_2}) \in \mathcal{H}^\sigma$. One can iterate the above process and construct $r^{t_3} \succ r^{t_4} \dots$. Since

$$\left|\operatorname{supp}\left(\mu^*\left(r^{t_{k+1}}\right)\right)\right| \leq \left|\operatorname{supp}\left(\mu^*\left(r^{t_k}\right)\right)\right| - 1$$

for any $k \in \mathbb{N}$, there exists $m \leq |\Theta_p \cup \Theta_n|$ such that $(a_*^{t_m}, r^{t_m}) \in \mathcal{H}^\sigma$, $r^{t_m} \succsim r^t$, and $\mu^*(r^{t_m})(\Theta_n \cup \Theta_p) = 0$, which contradicts $r^t \notin \widehat{\mathcal{R}}_g^\sigma$. *Q.E.D.*

SA.5. *Proof of Theorem 2: $\Theta_n = \{\emptyset\}$*

PROPOSITION SA.1: *If $\Theta_n = \{\emptyset\}$ and $\mu \in \mathcal{B}_\kappa$, then for every $\theta \in \Theta$, we have*

$$v_\theta\left(a_*^0, r^0\right) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - 2M\left(1 - \delta^{\overline{T}_0(\kappa)}\right).$$

Despite Proposition SA.1 being stated in terms of player 1's guaranteed payoff at h^0, the conclusion applies to all r^t and $\theta \in \Theta_g \cup \Theta_p$ as long as $\mu^*(r^t) \in \mathcal{B}_\kappa$ and $(a_*^t, r^t) \in \mathcal{H}^\sigma(\theta)$ but $(a_*^t, r^t) \notin \bigcup_{\theta_n \in \Theta_n} \mathcal{H}^\sigma(\theta_n)$. I show Lemma SA.5 and Lemma SA.6, which together imply Proposition SA.1.

LEMMA SA.5: *For every $\sigma \in \mathrm{NE}(\delta, \mu)$, if $\mu^*(r^t) \in \mathcal{B}_\kappa$ for all $r^t \in \widehat{\mathcal{R}}_g^\sigma$, then for every $r^\infty \in \mathcal{R}_*^\sigma$,*

$$\left|\left\{t \in \mathbb{N} | r^\infty \succ r^t \text{ and } \overline{a}_2 \text{ is not a strict best reply at } \left(a_*^t, r^t\right)\right\}\right| \leq \overline{T}_0(\kappa). \tag{SA.14}$$

PROOF: Pick any $r^\infty \in \mathcal{R}_*^\sigma$. If $r^0 \notin \widehat{\mathcal{R}}_g^\sigma$, then let $t^* = -1$; otherwise, let

$$t^* \equiv \max\left\{t \in \mathbb{N} \cup \{+\infty\} | r^t \in \widehat{\mathcal{R}}_g^\sigma \text{ and } r^\infty \succ r^t\right\}.$$

According to Lemma SA.2, for every $t \leq t^*$, if $\overline{a}_2$ is not a strict best reply at (a_*^t, r^t), then we have inequality (SA.12).

Next, I show that $\mu^*(r^{t^*+1}) \in \mathcal{B}_\kappa$. If $t^* = -1$, this is a direct implication of (SA.1). If $t^* \geq 0$, then there exists $\hat{r}^{t^*+1} \succ r^{t^*}$ such that $\hat{r}^{t^*+1} \in \widehat{\mathcal{R}}_g^\sigma$. Letting $r^{t^*+1} \prec r^\infty$, we have $q^*(r^{t^*+1}) = q^*(\hat{r}^{t^*+1})$. Moreover, since $\mu^*(r^t) \in \mathcal{B}_\kappa$ for every $r^t \in \widehat{\mathcal{R}}_g^\sigma$, we have $\mu^*(r^{t^*+1}) = \mu^*(\hat{r}^{t^*+1}) \in \mathcal{B}_\kappa$.

Since $r^{t^*+1} \notin \hat{\mathcal{R}}_g^\sigma$, Lemma SA.4 implies the existence of

$$\theta \in (\Theta_p \cup \Theta_n) \cap \operatorname{supp}\left(\mu^*\left(r^{t^*+1}\right)\right)$$

such that $r^{t^*+1} \in \overline{R}^\sigma(\theta)$. Since $\theta_g \succ \theta$ for all $\theta_g \in \Theta_g$, Lemma SA.1 implies that for every θ_g and $r^\infty \succ r^t \succsim r^{t^*+1}$, we have $\sigma_{\theta_g}(a_*^t, r^t) = 1$, and, therefore, $q^*(r^t)(\theta_g) = q^*(r^{t+1})(\theta_g)$. This implies that $\mu^*(r^t) \in \mathcal{B}_\kappa$ for every $r^\infty \succ r^t \succsim r^{t^*+1}$. If $\overline{a}_2$ is not a strict best reply at (a_*^t, r^t) for any $t > t^*$, inequality (SA.12) again applies.

To sum up, for every $t \in \mathbb{N}$, if $\overline{a}_2$ is not a strict best reply at (a_*^t, r^t), then

$$\sum_{\theta \in \Theta}\left(q^*\left(r^t\right)(\theta) - q^*\left(r^{t+1}\right)(\theta)\right) \geq \rho_0(\kappa),$$

from which we obtain (SA.14). *Q.E.D.*

Next, I show that the condition required in Lemma SA.5 holds in every equilibrium when δ is large enough.

LEMMA SA.6: *For every* $\sigma \in \mathrm{NE}(\delta, \mu)$ *with* $\delta > \overline{\delta}$, $\mu^*(r^t) \in \mathcal{B}_0$ *for every* $r^t \in \widehat{\mathcal{R}}_g^\sigma$ *with* $\mu^*(r^t)(\Theta_n) = 0$.

PROOF: For any given $\delta > \overline{\delta}$, according to (SA.6), there exists $\kappa^* \in (0, 1)$ such that

$$\big(1 - \delta^{\overline{T}_0(\kappa^*)}\big)u_1(\theta_p, \overline{a}_1, \underline{a}_2) + \delta^{\overline{T}_0(\kappa^*)}u_1(\theta_p, \overline{a}_1, \overline{a}_2) > \frac{1}{2}\big(u_1(\theta_p, \overline{a}_1, \overline{a}_2) + u_1(\theta_p, \underline{a}_1, \underline{a}_2)\big). \tag{SA.15}$$

Suppose toward a contradiction that there exist r^{t_1} and r^{T_1} such that $r^{T_1} \succ r^{t_1}$, $r^{T_1} \in \mathcal{R}_g^\sigma$, and $\mu^*(r^{t_1}) \notin \mathcal{B}_0$. Since $\mu^*(r^{T_1}) \in \mathcal{B}_0$, let t_1^* be the largest $t \in \mathbb{N}$ such that $\mu^*(r^t) \notin \mathcal{B}_0$ for $r^{T_1} \succ r^t \succsim r^{t_1}$. Then there exists $a_1 \neq \overline{a}_1$ and $r^{t_1^*+1} \succ r^{t_1^*}$ such that $\mu((a_*^{t_1^*}, a_1), r^{t_1^*+1}) \notin \mathcal{B}_0$ and $((a_*^{t_1^*}, a_1), r^{t_1^*+1}) \in \mathcal{H}^\sigma$. This also implies the existence of $\theta_p \in \Theta_p \cap \mathrm{supp}(\mu((a_*^{t_1^*}, a_1), r^{t_1^*+1}))$.

According to Lemma SA.3, type θ_p's continuation payoff at $(a_*^{t_1^*}, r^{t_1^*})$ by playing a_1 is at most

$$(1 - \delta)u_1(\theta_p, \underline{a}_1, \overline{a}_2) + \delta u_1(\theta_p, \underline{a}_1, \underline{a}_2). \tag{SA.16}$$

His incentive constraint at history $(a_*^{t_1^*}, r^{t_1^*})$ requires that his expected payoff from $\overline{\sigma}_1$ is weakly lower than (SA.16), that is, there exists $r^{t_1^*+1} \succ r^{t_1^*}$ satisfying $(a_*^{t_1^*+1}, r^{t_1^*+1}) \in \mathcal{H}^\sigma$ and type θ_p's continuation payoff at $(a_*^{t_1^*+1}, r^{t_1^*+1})$ is no more than

$$\frac{1}{2}\big(u_1(\theta_p, \overline{a}_1, \overline{a}_2) + u_1(\theta_p, \underline{a}_1, \underline{a}_2)\big). \tag{SA.17}$$

If $\mu^*(r^t) \in \mathcal{B}_{\kappa^*}$ for every $r^t \in \widehat{\mathcal{R}}_g^\sigma \cap \{r^t \succsim r^{t_1^*}\}$, then according to Lemma SA.5, his continuation payoff at $(a_*^{t_1^*}, r^{t_1^*})$ by playing $\overline{\sigma}_1$ is at least

$$\big(1 - \delta^{\overline{T}_0(\kappa^*)}\big)u_1(\theta_p, \overline{a}_1, \underline{a}_2) + \delta^{\overline{T}_0(\kappa^*)}u_1(\theta_p, \overline{a}_1, \overline{a}_2),$$

which is strictly larger than (SA.17) by the definition of κ^* in (SA.15), leading to a contradiction.

Suppose, on the other hand, there exists $r^{t_2} \succ r^{t_1^*}$ such that $r^{t_2} \in \hat{\mathcal{R}}_g^\sigma$ while $\mu^*(r^{t_2}) \notin \mathcal{B}_{\kappa^*}$. There exists $r^{T_2} \succ r^{t_2}$ such that $r^{T_2} \in \mathcal{R}_g^\sigma$ and $r^{T_2} \succ r^{t_2}$. Again, we can find $r^{t_2^*}$ such that t_2^* is the largest $t \in \{t_2, t_2 + 1, \dots, T_2\}$ such that $\mu^*(r^t) \notin \mathcal{B}_0$ for $r^{T_2} \succ r^t \succsim r^{t_2}$. Then there exists $a_1 \neq \overline{a}_1$ and $r^{t_2^*+1} \succ r^{t_2^*}$ such that $\mu((a_*^{t_2^*}, a_1), r^{t_2^*+1}) \notin \mathcal{B}_0$ and $((a_*^{t_2^*}, a_1), r^{t_2^*+1}) \in \mathcal{H}^\sigma$.

Iterating the above process and repeatedly applying the aforementioned argument, we know that for every $k \geq 1$, in order to satisfy player 1's incentive constraint to play $a_1 \neq \overline{a}_1$ at $(a_*^{t_k^*}, r^{t_k^*})$, we can find a triple $(r^{t_{k+1}}, r^{t_{k+1}^*}, r^{T_{k+1}})$. It implies that this process cannot stop after a finite number of rounds. Since $\mu^*(r^{t_k}) \notin \mathcal{B}_{\kappa^*}$ but $\mu^*(r^{t_k^*+1}) \in \mathcal{B}_0$ as well as $r^{t_{k+1}} \succ r^{t_k^*+1}$, we have

$$\sum_{\theta \in \Theta} q^*\big(r^{t_k}\big)(\theta) - q^*\big(r^{t_{k+1}}\big)(\theta) \geq \sum_{\theta \in \Theta} q^*\big(r^{t_k}\big)(\theta) - q^*\big(r^{t_k^*+1}\big)(\theta) \geq \rho_1\big(\kappa^*\big) \tag{SA.18}$$

for every $k \geq 2$. Equations (SA.18) and (SA.5) together suggest that this iteration process cannot last for more than $\overline{T}_1(\kappa^*)$ rounds, which is an integer independent of δ, leading to a contradiction. *Q.E.D.*

LEMMA SA.7: *For every* $\delta \geq \overline{\delta}$ *and* $\sigma \in \mathrm{NE}(\delta, \mu)$, *if* r^t *satisfies* $(a_*^t, r^t) \in \mathcal{H}^\sigma$, $\mu^*(r^t)(\Theta_n) = 0$, $r^t \notin \widehat{\mathcal{R}}_g^\sigma$, *and*

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta\in\Theta} q^*(r^t)(\theta)\mathcal{D}(\theta, \overline{a}_1) > 0, \tag{SA.19}$$

then $\overline{a}_2$ *is player 2's strict best reply at every* $(a_*^s, r^s) \succsim (a_*^t, r^t)$ *with* $(a_*^s, r^s) \in \mathcal{H}^\sigma$.

PROOF: Since $\mu^*(r^t)(\Theta_n) = 0$ and $r^t \notin \widehat{\mathcal{R}}_g^\sigma$, Lemma SA.4 implies the existence of $\theta_p \in \Theta_p \cap \mathrm{supp}(\mu^*(r^t))$ such that $r^t \in \overline{R}^\sigma(\theta_p)$. According to Lemma SA.1, $\sigma_\theta(a_*^s, r^s)(\overline{a}_1) = 1$ for every $(a_*^s, r^s) \in \mathcal{H}^\sigma(\theta)$ with $r^s \succsim r^t$. From (SA.19), we know that $\overline{a}_2$ is not a strict best reply only if there exists type $\theta_p \in \Theta_p$ who plays $a_1 \neq \overline{a}_1$ with positive probability. In particular, (SA.19) implies the existence of $\overline{\kappa} \in (0, 1)$ such that[1]

$$\overline{\kappa}\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta\in\Theta} q^*(r^t)(\theta)\mathcal{D}(\theta, \overline{a}_1) > 0.$$

According to (SA.12), we have

$$\sum_{\theta\in\Theta_p} \big(q^*(r^s)(\theta) - q^*(r^{s+1})(\theta)\big) \geq \rho_0(\overline{\kappa})$$

whenever $\overline{a}_2$ is not a strict best reply at $(a_*^s, r^s) \succsim (a_*^t, r^t)$. Therefore, there can be at most $\overline{T}_0(\overline{\kappa})$ such periods. Hence, there exists r^N with $(a_*^N, r^N) \in \mathcal{H}^\sigma$ such that

(i) $\overline{a}_2$ is not a strict best reply at (a_*^N, r^N),

(ii) $\overline{a}_2$ is a strict best reply for all $(a_*^s, r^s) \succ (a_*^N, r^N)$ with $(a_*^s, r^s) \in \mathcal{H}^\sigma$.

Then there exists $\theta_p \in \Theta_p$ who plays $a_1 \neq \overline{a}_1$ in equilibrium at (a_*^N, r^N): his continuation payoff by playing $\overline{a}_1$ in every subsequent period is at least $(1-\delta)u_1(\theta_p, \overline{a}_1, \underline{a}_2) + \delta u_1(\theta_p, \overline{a}_1, \overline{a}_2)$ while his equilibrium continuation payoff from playing a_1 is at most $(1-\delta)u_1(\theta_p, \underline{a}_1, \overline{a}_2) + \delta u_1(\theta_p, \underline{a}_1, \underline{a}_2)$ according to Lemma SA.3. The latter is strictly less than the former when $\delta > \overline{\delta}$. This leads to a contradiction. *Q.E.D.*

SA.6. *Proof of Theorem 2: Incorporating Types in* Θ_n

Next, we extend the proof in Appendix SA.5 by allowing for types in Θ_n. Lemmas SA.5 and SA.6 imply the following result in this general environment.

PROPOSITION SA.2: *For every* $\delta > \overline{\delta}$ *and* $\sigma \in \mathrm{NE}(\delta, \mu)$, *there exists no* $\theta_p \in \Theta_p$, *random histories* r^{t+1} *and* r^t *with* $r^{t+1} \succ r^t$ *and* $a_1 \neq \overline{a}_1$ *that simultaneously satisfy the three requirements*

(i) $r^{t+1} \in \widehat{\mathcal{R}}_g^\sigma$,

(ii) $((a_*^t, a_1), r^{t+1}) \in \mathcal{H}^\sigma(\theta_p)$,

(iii) $v_{\theta_p}(((a_*^t, a_1), \hat{r}^{t+1})) = u_1(\theta_p, \underline{a}_1, \underline{a}_2)$ *for all* $\hat{r}^{t+1} \succ r^t$.

[1]There are two reasons why one cannot directly apply the conclusion in Lemma SA.2. First, a stronger conclusion is required for Lemma SA.7. Second, $\overline{\kappa}$ can be arbitrarily close to 1, while κ is uniformly bounded below 1 for any given μ.

PROOF: Suppose toward a contradiction that such $\theta_p \in \Theta_p$, r^t, r^{t+1}, and a_1 exist. From requirement (iii), we know that $r^t \in \underline{\mathcal{R}}^\sigma(\theta_p)$. According to Lemma D.1 in the main text, $\theta_n \prec \theta_p$ for all $\theta_n \in \Theta_n$. The second part of Lemma SA.1 then implies that $\mu^*(\hat{r}^{t+1})(\Theta_n) = 0$ for all $\hat{r}^{t+1} \succ r^t$ with $(a_*^{t+1}, \hat{r}^{t+1}) \in \mathcal{H}^\sigma$.

If $\mu^*(r^{t+1}) \in \mathcal{B}_\kappa$, then requirement (ii) and Proposition SA.1 result in a contradiction when examining type θ_p's incentive at (a_*^t, r^t) to play a_1 as opposed to $\overline{a}_1$. If $\mu^*(r^{t+1}) \notin \mathcal{B}_\kappa$, since $\delta > \overline{\delta}$ and $r^{t+1} \in \widehat{\mathcal{R}}_g^\sigma$, we obtain a contradiction from Lemma SA.6. *Q.E.D.*

The rest of the proof considers a given $\sigma \in \mathrm{NE}(\delta, \mu)$ when δ is large enough. First,

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} q^*\big(r^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) \geq 0 \tag{SA.20}$$

for all $t \geq 1$ and r^t satisfying $(a_*^t, r^t) \in \mathcal{H}^\sigma$. This is because otherwise, according to Lemma SA.3, there exists $\theta \in \mathrm{supp}(\mu^*(r^t))$ such that $v_\theta(a_*^t, r^t) = u_1(\theta, \underline{a}_1, \underline{a}_2)$. But then, at (a_*^{t-1}, r^{t-1}) with $r^{t-1} \prec r^t$, he could obtain strictly higher payoff by playing $\underline{a}_1$ instead of $\overline{a}_1$, leading to a contradiction.

LEMMA SA.8: *If μ is optimistic, then $v_\theta(a_*^t, r^t) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - 2M(K+1)(1-\delta)$ for every θ and $r^t \notin \widehat{\mathcal{R}}_g^\sigma$ satisfying the following two requirements*:

(i) *We have $(a_*^t, r^t) \in \mathcal{H}^\sigma$.*

(ii) *Either $t = 0$ or $t \geq 1$, but there exists $\hat{r}^t$ such that $r^t, \hat{r}^t \succ r^{t-1}$, $(a_*^t, \hat{r}^t) \in \mathcal{H}^\sigma$, and $\hat{r}^t \in \widehat{\mathcal{R}}_g^\sigma$.*

PROOF: If $\mu^*(r^t) \in \mathcal{B}_\kappa$ and $r^t \notin \widehat{\mathcal{R}}_g^\sigma$, then Lemmas SA.1 and SA.4 suggest that $\mu^*(r^s) \in \mathcal{B}_\kappa$ for all $r^s \succsim r^t$ and the conclusion is straightforward from Lemma SA.2.

Therefore, for the rest of the proof, I consider the adverse circumstance in which $\mu^*(r^t) \notin \mathcal{B}_\kappa$. I consider two cases. First, when $\mu^*(r^t)(\Theta_n) > 0$, then according to (SA.20),

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta_p \cup \Theta_g} q^*\big(r^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) > 0.$$

Since $r^t \notin \widehat{\mathcal{R}}_g^\sigma$, according to Lemma SA.4, there exists $\theta \in \Theta_p \cup \Theta_n$ with $(a_*^t, r^t) \in \mathcal{H}^\sigma(\theta)$ such that $r^t \in \overline{\mathcal{R}}^\sigma(\theta)$. According to Lemma SA.1, for all $\theta_g \in \Theta_g$ with $(a_*^t, r^t) \in \mathcal{H}^\sigma(\theta_g)$ and every $(a_*^s, r^s) \in \mathcal{H}^\sigma(\theta)$ with $r^s \succsim r^t$, we have $\sigma_{\theta_g}(a_*^s, r^s)(\overline{a}_1) = 1$. This implies that for every $h^s = (a^s, r^s) \succ (a_*^t, r^t)$ with $a^s \neq a_*^s$ and $h^s \in \mathcal{H}^\sigma$, we have $\mu(h^s)(\Theta_g) = 0$. Therefore,

$$v_\theta\big(h^s\big) = u_1(\theta, \underline{a}_1, \underline{a}_2) \quad \text{for every } \theta \in \Theta. \tag{SA.21}$$

Let $\tau : \mathcal{R}_*^\sigma \to \mathbb{N} \cup \{+\infty\}$ be such that for $r^\tau \prec r^{\tau+1} \prec r^\infty$, we have $\mu^*(r^\tau)(\Theta_n) > 0$ while $\mu^*(r^{\tau+1})(\Theta_n) = 0$. Let

$$\overline{\theta}_n \equiv \max\Big\{\mathrm{supp}\big(\mu^*\big(r^t\big)\big) \cap \Theta_n\Big\}.$$

The second part of Lemma SA.1 and (SA.21) together imply that $\mu^*(r^\tau)(\overline{\theta}_n) > 0$. Let us examine type $\overline{\theta}_n$'s incentive at (a_*^t, r^t) to play his equilibrium strategy as opposed to playing $\underline{a}_1$ in every period. This requires that

$$\mathbb{E}\left[\sum_{s=t}^{\tau-1}(1-\delta)\delta^{s-t}u_1(\overline{\theta}_n,\overline{a}_1,\alpha_{2,s})+\left(\delta^{\tau-t}-\delta^{\tau+1-t}\right)u_1(\overline{\theta}_n,a_{1,\tau},\alpha_{2,\tau})+\delta^{\tau+1-t}u_1(\overline{\theta}_n,\underline{a}_1,\underline{a}_2)\right]$$
$$\geq u_1(\overline{\theta}_n,\underline{a}_1,\underline{a}_2),$$

where $\mathbb{E}[\cdot]$ is taken over $\mathcal{P}^\sigma$ and $\alpha_{2,s}\in\Delta(A_2)$ is player 2's action in period s.

Using the fact that $u_1(\overline{\theta}_n,\underline{a}_1,\underline{a}_2)\geq u_1(\overline{\theta}_n,\overline{a}_1,\overline{a}_2)$, the above inequality implies that

$$\mathbb{E}\Bigg[\sum_{s=t}^{\tau-1}(1-\delta)\delta^{s-t}\big(u_1(\overline{\theta}_n,\overline{a}_1,\alpha_{2,s})-u_1(\overline{\theta}_n,\overline{a}_1,\overline{a}_2)\big)$$
$$+\left(\delta^{\tau-t}-\delta^{\tau+1-t}\right)\big(u_1(\overline{\theta}_n,\underline{a}_1,\alpha_{2,\tau})-u_1(\overline{\theta}_n,\underline{a}_1,\underline{a}_2)\big)\Bigg]\leq 0.$$

According to the definitions of K and M, we know that for all θ,

$$\mathbb{E}\left[\sum_{s=t}^{\tau}(1-\delta)\delta^{s-t}\big(u_1(\theta_n,\overline{a}_1,\alpha_{2,s})-u_1(\theta_n,\overline{a}_1,\overline{a}_2)\big)\right]\leq 2M(K+1)(1-\delta). \tag{SA.22}$$

This bounds the loss (relative to the payoff from the highest action profile) from above in periods before all types in Θ_n separate from the commitment type. For every $r^\infty\in\mathcal{R}^\sigma_*$, since $r^t\notin\widehat{\mathcal{R}}^\sigma_g$, we have

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1},\overline{a}_1)+\sum_{\theta\in\Theta}q^*\big(r^{\tau(r^\infty)+1}\big)(\theta)\mathcal{D}(\theta,\overline{a}_1)$$
$$\geq\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1},\overline{a}_1)+\sum_{\theta\in\Theta_p\cup\Theta_g}q^*\big(r^t\big)(\theta)\mathcal{D}(\theta,\overline{a}_1)$$
$$>\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1},\overline{a}_1)+\sum_{\theta\in\Theta}q^*\big(r^t\big)(\theta)\mathcal{D}(\theta,\overline{a}_1)\geq 0.$$

According to Lemma SA.7, we know that $v_\theta(a_*^{\tau(r^\infty)+1},r^{\tau(r^\infty)+1})=u_1(\theta,\overline{a}_1,\overline{a}_2)$ for all $\theta\in\Theta_g\cup\Theta_p$ and $r^\infty\in\mathcal{R}^\sigma_*$. This together with (SA.22) gives the conclusion.

Second, when $\mu^*(r^t)(\Theta_n)=0$, if $t=0$, the conclusion directly follows from Proposition SA.1. If $t\geq 1$ and there exists $\hat{r}^t$ such that $r^t,\hat{r}^t\succ r^{t-1}$, $(a_*^t,\hat{r}^t)\in\mathcal{H}^\sigma$, and $\hat{r}^t\in\widehat{\mathcal{R}}^\sigma_g$, then since

$$\mu^*\big(r^t\big)=\mu^*\big(\hat{r}^t\big),$$

we have $\mu^*(\hat{r}^t)(\Theta_n)=0$. Since $\hat{r}^t\in\widehat{\mathcal{R}}^\sigma_g$, according to Lemma SA.6, $\mu^*(\hat{r}^t)=\mu^*(r^t)\in\mathcal{B}_\kappa$. The conclusion then follows from Lemma SA.7. *Q.E.D.*

The next lemma puts an upper bound on type $\theta_n\in\Theta_n$'s continuation payoff at (a_*^t,r^t) with $r^t\notin\widehat{\mathcal{R}}^\sigma_g$.

LEMMA SA.9: *For every $\theta_n \in \Theta_n$ such that $\overline{a}_2 \notin BR_2(\theta_n, \overline{a}_1|u_2)$, and $r^t \notin \widehat{\mathcal{R}}_g^\sigma$ with $(a_*^t, r^t) \in \mathcal{H}_{\theta_n}^\sigma$ and $\mu^*(r^t) \notin \mathcal{B}_\kappa$, we have*

$$v_{\theta_n}\big(a_*^t, r^t\big) \leq u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M. \tag{SA.23}$$

This is implied by Lemma SA.8(i). Let

$$A(\delta) \equiv 2M(K+1)(1-\delta), \qquad B(\delta) \equiv 2M\big(1-\delta^{\overline{T}_0(\kappa)}\big)$$

and

$$C(\delta) \equiv 2MK|\Theta_n|(1-\delta).$$

Notice that when $\delta \to 1$, all three functions converge to 0. The next lemma establishes a uniform upper bound on player 1's payoff when $r^t \in \widehat{\mathcal{R}}_g^\sigma$.

LEMMA SA.10: *When $\delta > \overline{\delta}$ and $\sigma \in \mathrm{NE}(\delta, \mu)$, for every $r^t \in \widehat{\mathcal{R}}_g^\sigma$,*

$$v_\theta\big(a_*^t, r^t\big) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - \big(A(\delta)+B(\delta)\big) - 2\overline{T}_1(\kappa)\big(A(\delta)+B(\delta)+C(\delta)\big) \tag{SA.24}$$

for all θ such that $(a_*^t, r^t) \in \mathcal{H}^\sigma(\theta)$.

PROOF: The nontrivial part of the proof deals with situations where $\mu^*(r^t) \notin \mathcal{B}_\kappa$. Since $r^t \in \widehat{\mathcal{R}}_g^\sigma$, Lemma SA.6 implies that $\mu^*(r^t)(\Theta_n) \neq 0$. Without loss of generality, assume $\Theta_n \subset \mathrm{supp}(\mu^*(r^t))$. Let me introduce $|\Theta_n|+1$ integer-valued random variables on the space $\mathcal{R}_*^\sigma$.

- Let $\tau : \mathcal{R}_*^\sigma \to \mathbb{N} \cup \{+\infty\}$ be the first period $s \in \mathbb{N}$ along random path r^∞ such that either one of the following two conditions is met.
  (i) We have $\mu^*(r^{s+1}) \in \mathcal{B}_{\kappa/2}$ for $r^{s+1} \succ r^s$ with $(a_*^{s+1}, r^{s+1}) \in \mathcal{H}^\sigma$.
  (ii) We have $r^s \notin \widehat{\mathcal{R}}_g^\sigma$.

  In the first case, there exists $a_1 \neq \overline{a}_1$ and $r^{\tau+1} \succ r^\tau$ such that $((a_*^\tau, a_1), r^{\tau+1}) \in \mathcal{H}^\sigma(\tilde{\theta})$ for some $\tilde{\theta} \in \Theta_p \cup \Theta_n$ and, moreover, $\mu((a_*^\tau, a_1), r^{\tau+1}) \notin \mathcal{B}_0$.

  Lemma SA.3 implies the existence of $\theta \in \Theta_p \cup \Theta_n$ with $((a_*^\tau, a_1), r^{\tau+1}) \in \mathcal{H}^\sigma(\theta)$ such that $v_\theta((a_*^\tau, a_1), r^{\tau+1}) = u_1(\theta, \underline{a}_1, \underline{a}_2)$.

  Suppose toward a contradiction that $\theta \in \Theta_p$. Then Lemma SA.1 implies that $\mu^*(r^{\tau+1})(\Theta_n) = 0$. Since $\mu^*(r^{\tau+1}) \in \mathcal{B}_{\kappa/2}$, Proposition SA.1 implies that type θ's continuation payoff by playing $\overline{a}_1$ in all subsequent periods is at least

  $$\big(1-\delta^{\overline{T}_0(\kappa/2)}\big)u_1(\theta, \overline{a}_1, \underline{a}_2) + \delta^{\overline{T}_0(\kappa/2)}u_1(\theta, \overline{a}_1, \overline{a}_2),$$

  which is strictly larger than his payoff from playing a_1, which is at most $2M(1-\delta) + u_1(\theta, \underline{a}_1, \underline{a}_2)$. This leads to a contradiction. Hence, there exists $\theta_n \in \Theta_n$ such that $v_{\theta_n}((a_*^\tau, a_1), r^{\tau+1}) = u_1(\theta_n, \underline{a}_1, \underline{a}_2)$, which implies that $v_{\theta_n}(a_*^\tau, r^\tau) \leq u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M$. In the second case, Lemma SA.9 implies that $v_{\theta_n}(a_*^\tau, r^\tau) \leq u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M$ for all $\theta_n \in \Theta_n$ with $r^\tau \in \mathcal{H}^\sigma(\theta_n)$.
- For every $\theta_n \in \Theta_n$, let $\tau_{\theta_n} : \mathcal{R}_*^\sigma \to \mathbb{N} \cup \{+\infty\}$ be the first period s along random path r^∞ such that any one of the following three conditions is met.
  (i) We have $\mu^*(r^{s+1}) \in \mathcal{B}_{\kappa/2}$ for $r^{s+1} \succ r^s$ with $(a_*^{s+1}, r^{s+1}) \in \mathcal{H}^\sigma$.
  (ii) We have $r^s \notin \widehat{\mathcal{R}}_g^\sigma$.
  (iii) We have $\mu^*(r^{s+1})(\theta_n) = 0$ for $r^{s+1} \succ r^s$ with $(a_*^{s+1}, r^{s+1}) \in \mathcal{H}^\sigma$.

By definition, $\tau \geq \tau_{\theta_n}$, so $\tau \geq \max_{\theta_n \in \Theta_n}\{\tau_{\theta_n}\}$. Next, I show that

$$\tau = \max_{\theta_n \in \Theta_n}\{\tau_{\theta_n}\}. \tag{SA.25}$$

Suppose toward a contradiction that $\tau > \max_{\theta_n \in \Theta_n}\{\tau_{\theta_n}\}$ for some $r^\infty \in \mathcal{R}_*^\sigma$. Then there exists $(a_*^s, r^s) \succsim (a_*^t, r^t)$ such that $r^s \in \widehat{\mathcal{R}}_g^\sigma$, $\mu^*(r^s) \notin \mathcal{B}_\kappa$, and $\mu^*(r^s)(\Theta_n) = 0$. This contradicts Lemma SA.6 when $\delta > \overline{\delta}$.

Next, I show by induction on the number of states in Θ_n that

$$\mathbb{E}\left[\sum_{s=t}^{\tau}(1-\delta)\delta^{\tau-t}\big(u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \hat{\alpha}_{2,s})\big)\right] \leq 2MK|\Theta_n|(1-\delta) \tag{SA.26}$$

for all $\theta \in \Theta$ and

$$v_{\tilde{\theta}_n}\big(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}}\big) \leq u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M \tag{SA.27}$$

for

$$\tilde{\theta} \equiv \min\Big\{\Theta_n \cap \operatorname{supp}\big(\mu^*\big(r^{\tau_{\theta_n}+1}\big)\big)\Big\}$$

with $\theta_n, \tilde{\theta}_n \in \Theta_n$, where $\mathbb{E}[\cdot]$ is taken over $\mathcal{P}^\sigma$ and $\hat{\alpha}_{2,s} \in \Delta(A_2)$ is player 2's (mixed) action at (a_*^s, r^s).

When $|\Theta_n| = 1$, let θ_n be its unique element. Consider player 1's pure strategy of playing $\overline{a}_1$ until r^τ and then playing $\underline{a}_1$ forever. This is one of type θ_n's best responses according to (SA.25), which results in payoff at most

$$\mathbb{E}\left[\sum_{s=t}^{\tau-1}(1-\delta)\delta^{s-t}u_1(\theta_n, \overline{a}_1, \hat{\alpha}_{2,s}) + \delta^{\tau-t}\big(u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M\big)\right].$$

The above expression cannot be smaller than $u_1(\theta_n, \underline{a}_1, \underline{a}_2)$, which is the payoff he can guarantee by playing $\underline{a}_1$ in every period. Since $u_1(\theta_n, \underline{a}_1, \underline{a}_2) \geq u_1(\theta_n, \overline{a}_1, \overline{a}_2)$, and using the definition of K, we get for all θ,

$$\mathbb{E}\left[\sum_{s=t}^{\tau-1}(1-\delta)\delta^{s-t}\big(u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \hat{\alpha}_{2,s})\big)\right] \leq 2MK(1-\delta).$$

We can then obtain (SA.27) for free since $\tau = \tau_{\theta_n}$ and type θ_n's continuation value at (a_*^τ, r^τ) is at most $u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M$ by Lemma SA.3.

Suppose the conclusion holds for all $|\Theta_n| \leq k-1$, consider when $|\Theta_n| = k$, and let $\theta_n \equiv \min \Theta_n$. If $(a_*^\tau, r^\tau) \notin \mathcal{H}^\sigma(\theta_n)$, then there exists $(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}}) \prec (a_*^\tau, r^\tau)$ with $(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}}) \in \mathcal{H}^\sigma(\theta_n)$ at which type θ_n plays $\overline{a}_1$ with probability 0. I put an upper bound on type θ_n's continuation payoff at $(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}})$ by examining type $\tilde{\theta}_n \in \Theta_n \setminus \{\theta_n\}$'s incentive to play $\overline{a}_1$ at $(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}})$, where

$$\tilde{\theta} \equiv \min\Big\{\Theta_n \cap \operatorname{supp}\big(\mu^*\big(r^{\tau_{\theta_n}+1}\big)\big)\Big\}.$$

This requires that

$$\mathbb{E}\left[\sum_{s=0}^{\infty}(1-\delta)\delta^s u_1(\tilde{\theta}_n, \alpha_{1,s}, \alpha_{2,s})\right] \leq \underbrace{u_1(\tilde{\theta}_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M}_{\text{by induction hypothesis}},$$

where $\{(\alpha_{1,s}, \alpha_{2,s})\}_{s\in\mathbb{N}}$ is the equilibrium continuation play following $(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}})$. By definition, $\tilde{\theta}_n \succ \theta_n$, the supermodularity condition implies that

$$u_1(\theta_n, \underline{a}_1, \underline{a}_2) - u_1(\tilde{\theta}_n, \underline{a}_1, \underline{a}_2) \geq u_1(\theta_n, \alpha_{1,s}, \alpha_{2,s}) - u_1(\tilde{\theta}_n, \alpha_{1,s}, \alpha_{2,s}).$$

Therefore, we have

$$\begin{aligned} v_{\theta_n}\big(a_*^{\tau_{\theta_n}}, r^{\tau_{\theta_n}}\big) &= \mathbb{E}\left[\sum_{s=0}^{\infty}(1-\delta)\delta^s u_1(\theta_n, \alpha_{1,s}, \alpha_{2,s})\right] \\ &\leq \mathbb{E}\left[\sum_{s=0}^{\infty}(1-\delta)\delta^s\big(u_1(\tilde{\theta}_n, \alpha_{1,s}, \alpha_{2,s}) + u_1(\theta_n, \underline{a}_1, \underline{a}_2) - u_1(\tilde{\theta}_n, \underline{a}_1, \underline{a}_2)\big)\right] \\ &\leq u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M. \end{aligned}$$

Since it is optimal for type θ_n to play $\overline{a}_1$ until $r^{\tau_{\theta_n}}$ and then play $\underline{a}_1$ forever, doing so must give him a higher payoff than playing $\underline{a}_1$ forever starting from r^t, which gives

$$\mathbb{E}\left[\sum_{s=t}^{\tau_{\theta_n}-1}(1-\delta)\delta^{s-t}u_1(\theta_n, \overline{a}_1, \hat{\alpha}_{2,s}) + \delta^{\tau_{\theta_n}}\big(u_1(\theta_n, \underline{a}_1, \underline{a}_2) + 2(1-\delta)M\big)\right] \geq u_1(\theta_n, \underline{a}_1, \underline{a}_2).$$

This implies that

$$\mathbb{E}\left[\sum_{s=t}^{\tau_{\theta_n}-1}(1-\delta)\delta^{s-t}\big(u_1(\theta_n, \overline{a}_1, \overline{a}_2) - u_1(\theta_n, \overline{a}_1, \hat{\alpha}_{2,s})\big)\right] \leq 2M(1-\delta),$$

which also implies that for every $\theta \in \Theta$,

$$\mathbb{E}\left[\sum_{s=t}^{\tau_{\theta_n}-1}(1-\delta)\delta^{s-t}\big(u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \hat{\alpha}_{2,s})\big)\right] \leq 2MK(1-\delta). \tag{SA.28}$$

When $\tau > \tau_{\theta_n}$, the induction hypothesis implies that

$$\mathbb{E}\left[\sum_{s=\tau_{\theta_n}}^{\tau_\theta-1}(1-\delta)\delta^{s-\tau_{\theta_n}}\big(u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \alpha_{2,s})\big)\right] \leq 2MK(k-1)(1-\delta). \tag{SA.29}$$

According to (SA.28) and (SA.29),

$$\mathbb{E}\left[\sum_{s=t}^{\tau}(1-\delta)\delta^{\tau-t}\big(u_1(\theta, \overline{a}_1, \overline{a}_2) - u_1(\theta, \overline{a}_1, \hat{\alpha}_{2,s})\big)\right] \leq 2MKk(1-\delta),$$

which establishes (SA.26) when $|\Theta_n| = k$. Equation (SA.27) can be obtained directly from the induction hypothesis.

I examine player 1's continuation payoff at on-path histories after $(a_*^{\tau+1}, r^{\tau+1}) \in \mathcal{H}^\sigma$ in three cases.

Case 1. If $r^{\tau+1} \notin \widehat{\mathcal{R}}_g^\sigma$, by Lemma SA.8, then for every θ,

$$v_\theta\left(a_*^{\tau+1}, r^{\tau+1}\right) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - A(\delta).$$

Case 2. If $r^{\tau+1} \in \widehat{\mathcal{R}}_g^\sigma$ and $\mu^*(r^s) \in \mathcal{B}_\kappa$ for all r^s satisfying $r^s \succsim r^{\tau+1}$ and $r^s \in \widehat{\mathcal{R}}_g^\sigma$, then for every θ,

$$v_\theta\left(a_*^{\tau+1}, r^{\tau+1}\right) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - B(\delta).$$

Case 3. If there exists r^s such that $\mu^*(r^s) \notin \mathcal{B}_\kappa$ with $r^s \succsim r^{\tau+1}$ and $r^s \in \widehat{\mathcal{R}}_g^\sigma$, then repeat the procedure in the beginning of this proof by defining random variables

- $\tau' : \mathcal{R}_*^\sigma \to \{n \in \mathbb{N} \cup \{+\infty\} | n \geq s\}$,
- $\tau'_{\theta_n} : \mathcal{R}_*^\sigma \to \{n \in \mathbb{N} \cup \{+\infty\} | n \geq s\}$

similarly as we have defined τ and τ_{θ_n}, and then examine continuation payoffs at $r^{\tau'+1}$....

Since $\mu^*(r^{\tau+1}) \in \mathcal{B}_{\kappa/2}$ but $\mu^*(r^s) \notin \mathcal{B}_\kappa$, then

$$\sum_{\theta \in \Theta}\left(q^*\left(r^{\tau+1}\right)(\theta) - q^*\left(r^s\right)(\theta)\right) \geq \frac{\rho_1(\kappa)}{2}. \tag{SA.30}$$

Therefore, such iterations can last for at most $2\overline{T}_1(\kappa)$ rounds.

Next, I establish the payoff lower bound in Case 3. For future reference, I introduce the notion of *trees*. Let

$$\mathcal{R}_b^\sigma \equiv \left\{r^t | \mu^*\left(r^t\right) \notin \mathcal{B}_\kappa \text{ and } r^t \in \hat{\mathcal{R}}_g^\sigma\right\}.$$

For $k \in \mathbb{N}$, I define set $\mathcal{R}^\sigma(k) \subset \mathcal{R}$ recursively as follows. Let

$$\mathcal{R}^\sigma(1) \equiv \left\{r^t | r^t \in \mathcal{R}_b^\sigma \text{ and there exists no } r^s \prec r^t \text{ such that } r^s \in \mathcal{R}_b^\sigma\right\}.$$

For every $r^t \in \mathcal{R}^\sigma(1)$, let $\tau[r^t] : \mathcal{R}_*^\sigma \to \mathbb{N} \cup \{+\infty\}$ be the first period $s > t$ (starting from r^t) such that either one of the following two conditions is met:

(i) We have $\mu^*(r^{s+1}) \in \mathcal{B}_{\kappa/2}$ for $r^{s+1} \succ r^s$ with $(a_*^{s+1}, r^{s+1}) \in \mathcal{H}^\sigma$,

(ii) We have $r^s \notin \widehat{\mathcal{R}}_g^\sigma$.

Then

$$\mathcal{T}\left(r^t\right) \equiv \left\{r^s | r^{\tau[r^{t_1}]} \succsim r^s \succsim r^t\right\}$$

is called a *tree* with root r^t. For any $k \geq 2$, let

$$\mathcal{R}^\sigma(k) \equiv \Big\{r^t | r^t \in \mathcal{R}_b^\sigma,\ r^t \succ r^{\tau[r^s]} \text{ for some } r^s \in \mathcal{R}^\sigma(k-1) \text{ and there exists no } r^s \prec r^t \text{ that satisfy these two conditions}\Big\}.$$

Let T be the largest integer such that $\mathcal{R}^\sigma(T) \neq \{\emptyset\}$. According to (SA.30), we know that $T \leq 2\overline{T}_1(\kappa)$. Similarly, one can define trees with roots in $\mathcal{R}(k)$ for every $k \leq T$.

In what follows, I show that for every θ and every $r^t \in \mathcal{R}^\sigma(k)$,

$$v_\theta\left(a_*^t, r^t\right) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - (T + 1 - k)\left(A(\delta) + B(\delta) + C(\delta)\right). \tag{SA.31}$$

The proof is done by induction on k from T to 0. When $k = T$, player 1's continuation value at $(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1})$ is at least $u_1(\theta, \overline{a}_1, \overline{a}_2) - A(\delta) - B(\delta)$ according to Lemma SA.2

and Lemma SA.8. His continuation value at r^t is at least

$$u_1(\theta, \overline{a}_1, \overline{a}_2) - A(\delta) - B(\delta) - C(\delta).$$

Suppose the conclusion holds for all $k \geq n+1$. Then when $k = n$, type θ's continuation payoff at (a_*^t, r^t) is at least

$$\mathbb{E}\big[\big(1-\delta^{\tau[r^t]-t}\big)u_1(\theta, \overline{a}_1, \overline{a}_2) + \delta^{\tau[r^t]-t}V_\theta\big(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1}\big)\big] - C(\delta).$$

Pick any $(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1})$ and consider the set of random paths r^∞ that it is consistent with. Denote this set by

$$\mathcal{R}^\infty\big(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1}\big).$$

Partition it into the following two subsets:

(i) Subset $\mathcal{R}_+^\infty(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1})$ consists of r^∞ such that for all $s \geq \tau[r^t]+1$ and $r^s \prec r^\infty$, we have $r^s \notin \mathcal{R}_b^\sigma$.

(ii) Subset $\mathcal{R}_-^\infty(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1})$ consists of r^∞ such that there exists $s \geq \tau[r^t]+1$ and $r^s \prec r^\infty$ at which $r^s \in \mathcal{R}^\sigma(n+1)$.

Conditional on $r^\infty \in \mathcal{R}_+^\infty(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1})$, we have

$$v_\theta\big(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1}\big) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - A(\delta) - B(\delta).$$

Conditional on $r^\infty \in \mathcal{R}_-^\infty(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1})$, type θ's continuation payoff is no less than

$$v_\theta\big(a_*^s, r^s\big) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - (T-n)\big(A(\delta) + B(\delta) + C(\delta)\big)$$

after reaching $r^s \in \mathcal{R}^\sigma(n)$ according to the induction hypothesis. Moreover, since his payoff loss is at most $A(\delta) + B(\delta)$ before reaching r^s (according to Lemmas SA.2 and SA.8), we have

$$v_\theta\big(a_*^{\tau[r^t]+1}, r^{\tau[r^t]+1}\big) \geq u_1(\theta, \overline{a}_1, \overline{a}_2) - (T+1-n)\big(A(\delta) + B(\delta) + C(\delta)\big),$$

which obtains (SA.31). Equation (SA.24) is implied by (SA.31) since player 1's loss is bounded above by $A(\delta) + B(\delta)$ from r^0 to every $r^t \in \mathcal{R}^\sigma(0)$. *Q.E.D.*

Theorem 2′ is implied by Lemmas SA.8, SA.9, and SA.10.

APPENDIX SB: PROOF OF THEOREM 3

SB.1. *Proof of Theorem 3: Equilibrium Payoff*

First, I show that for every $\theta \in \Theta$, strategic type θ secures payoff $w_\theta(\phi)$ in all equilibria. Let $\kappa \in (0,1)$. Given $\delta > \overline{\delta}$ and $\sigma \in \mathrm{NE}(\delta, \mu)$, let us examine r^1 such that $(a_*^1, r^1) \in \mathcal{H}^\sigma$. If $\mu^*(r^1) \in \mathcal{B}_\kappa$, then for every $\hat{r}^1$ with $(a_*^1, \hat{r}^1) \in \mathcal{H}^\sigma$, we have $\mu^*(\hat{r}^1) \in \mathcal{B}_\kappa$. The conclusion is then implied by Theorem 2. If $\mu^*(r^1) \notin \mathcal{B}_\kappa$, then we still have

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} q^*\big(r^1\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) \geq 0. \tag{SB.1}$$

This is because otherwise there exists $\theta \in \operatorname{supp} \mu^*(r^1)$ such that $v_\theta(a_*^1, r^1) = u_1(\theta, \underline{a}_1, \underline{a}_2)$ according to Lemma SA.3, contradicting type θ's incentive to play $\overline{a}_1$ in period 0. I consider two cases separately.

Case 1. If $\Theta_n \cap \operatorname{supp} \mu^*(r^1) = \{\emptyset\}$, then Lemma SA.6 implies that $r^1 \notin \widehat{\mathcal{R}}^\sigma_g$. According to Lemma SA.4, there exists $\theta \in (\Theta_p \cup \Theta_n) \cap \operatorname{supp} \mu^*(r^1)$ such that $r^1 \in \overline{\mathcal{R}}^\theta$. According to Lemma SA.1, for every $\theta_g \in \Theta_g$, type θ_g will play $\overline{a}_1$ at every $(a^t_*, r^t) \succsim (a^1_*, r^1)$ with $(a^t_*, r^t) \in \mathcal{H}^\sigma(\theta_g)$.

According to the definition of $w_\theta(\phi)$, and given that the two dimensions of player 1's private information are independently distributed, we know that type θ can secure payoff $w_\theta(\phi)$ at r^1 for every $\theta \in \Theta$. Since $\mu^*(r^1) \notin \mathcal{B}_\kappa$, $\mu^*(\hat{r}^1) \notin \mathcal{B}_\kappa$ for every $\hat{r}^1$ with $(a^1_*, \hat{r}^1) \in \mathcal{H}^\sigma$. The argument in the previous paragraph applies symmetrically, which implies that type θ's discounted average payoff at h^0 is at least

$$(1-\delta)u_1(\theta, \overline{a}_1, \underline{a}_2) + \delta w_\theta(\phi).$$

Case 2. If $\Theta_n \cap \operatorname{supp} \mu^*(r^1) \neq \{\emptyset\}$, then according to Lemma SA.10, type θ can guarantee payoff at least the RHS of (SA.24), which leads to the same conclusion.

Next, I uniquely pin down every type's equilibrium payoff when the total probability of commitment types is arbitrarily small. The key is to show that for every Nash equilibrium σ, we have

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} q^*\big(r^1\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) = 0$$

for every r^1 such that $(a^1_*, r^1) \in \mathcal{H}^\sigma$. This is because when the total probability of commitment types is small enough and ϕ is pessimistic,

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta^*} q_0(\theta)\mathcal{D}(\theta, \overline{a}_1) < 0.$$

Suppose toward a contradiction that

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta} q^*\big(r^1\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) > 0.$$

On one hand, Theorem 2 suggests that every type $\theta \in \Theta^*$ receives continuation payoff at least $u_1(\theta, \overline{a}_1, \overline{a}_2)$ after playing $\overline{a}_1$ in period 0. On the other hand, it also implies that there exists type $\theta \in \Theta^*$ that plays actions other than $\overline{a}_1$ with positive probability, and according to Lemma C.3, this type's continuation payoff in period 1 is $u_1(\theta, \underline{a}_1, \underline{a}_2)$. As a result, this type has a strict incentive to deviate by playing $\overline{a}_1$ in period 0, which leads to a contradiction. Similarly, one can show by induction that for every $t \geq 1$ and $(a^t_*, r^t) \in \mathcal{H}^\sigma$,

$$\mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta^*} q^*\big(r^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) = 0.$$

The rest of proof follows the same steps as Appendix D in the main text.

SB.2. *Proof of Theorem 3: On-Path Behavior*

Step 1. Let

$$X\big(h^t\big) \equiv \mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1) + \sum_{\theta \in \Theta_g \cup \Theta_p} q\big(h^t\big)(\theta)\mathcal{D}(\theta, \overline{a}_1) \tag{SB.2}$$

and

$$Y(h^t) \equiv \mu(\mathcal{A}_1^*)\mathcal{D}(\overline{\theta}, \overline{a}_1) + \sum_{\theta \in \Theta_g \cup \Theta_p} q(h^t)(\theta)\mathcal{D}(\theta, \overline{a}_1). \tag{SB.3}$$

When belief is pessimistic, $X(h^0) < 0$ and $Y(h^0) < 0$. Moreover, at every $h^t \in \mathcal{H}^\sigma$ with $Y(h^t) < 0$, player 2 has a strict incentive to play $\underline{a}_2$. According to Lemma SA.3, there exists $\theta_p \in \Theta_p$ with $h^t \in \mathcal{H}(\theta_p)$ such that type θ_p's continuation value at h^t is $u_1(\theta_p, \underline{a}_1, \underline{a}_2)$, which further implies that playing $\underline{a}_1$ in every period is one of his best replies. According to Lemma SA.1 and using the implication that $Y(h^0) < 0$, every $\theta_n \in \Theta_n$ plays $\underline{a}_1$ with probability 1 at every $h^t \in \mathcal{H}(\theta_n)$.

Step 2. Let us examine the equilibrium behaviors of the types in $\Theta_p \cup \Theta_g$. I claim that for every $h^1 = (\overline{a}_1, r^1) \in \mathcal{H}^\sigma$, we have

$$\sum_{\theta \in \Theta_g \cup \Theta_p} q(h^1)(\theta)\mathcal{D}(\theta, \overline{a}_1) < 0. \tag{SB.4}$$

Suppose toward a contradiction that $\sum_{\theta \in \Theta_g \cup \Theta_p} q(h^1)(\theta)\mathcal{D}(\theta, \overline{a}_1) \geq 0$. Then $X(h^1) \geq \mu(\overline{a}_1)\mathcal{D}(\phi_{\overline{a}_1}, \overline{a}_1)$. According to Proposition SA.1, there exists $K \in \mathbb{R}_+$ independent of δ such that type θ's continuation payoff is at least $u_1(\theta, \overline{a}_1, \overline{a}_2) - (1-\delta)K$ at every $h_*^1 \in \mathcal{H}^\sigma$. When δ is large enough, this contradicts the conclusion in the previous step that there exists $\theta_p \in \Theta_p$ such that type θ_p's continuation value at h^0 is $u_1(\theta_p, \underline{a}_1, \underline{a}_2)$, as he can profitably deviate by playing $\overline{a}_1$ in period 0.

Step 3. According to (SB.4), we have $\mu^*(r^1) \notin \mathcal{B}_0$. Step 1 also implies that $\mu^*(r^1)(\Theta_n) = 0$. According to Lemma SA.6, we have $r^1 \notin \widehat{\mathcal{R}}_g^\sigma$. According to Lemma SA.1, type θ_g plays $\overline{a}_1$ at every $h^t \in \mathcal{H}(\theta_g)$ with $t \geq 1$ for every $\theta_g \in \Theta_g$. Next, I show that $r^0 \notin \widehat{\mathcal{R}}_g^\sigma$. Suppose toward a contradiction that $r^0 \in \widehat{\mathcal{R}}_g^\sigma$. Then there exists $h^T = (a_*^T, r^T) \in \mathcal{H}^\sigma$ such that $\mu(h^T)(\Theta_p \cup \Theta_n) = 0$. If $T \geq 2$, it contradicts our previous conclusion that $r^1 \notin \widehat{\mathcal{R}}_g^\sigma$. If $T = 1$, then it contradicts (SB.4). Therefore, we have $r^0 \notin \widehat{\mathcal{R}}_g^\sigma$. This implies that type θ_g plays $\overline{a}_1$ at every $h^t \in \mathcal{H}(\theta_g)$ with $t \geq 0$ for every $\theta_g \in \Theta_g$.

Step 4. In the last step, I pin down the strategies of type θ_p by showing that $X(h^t) = 0$ for every $h^t = (a_*^t, r^t) \in \mathcal{H}^\sigma$ with $t \geq 1$. First, I show that $X(h^1) = 0$. The argument at other histories follows similarly. Suppose first that $X(h^1) > 0$. Then according to Lemma SA.7, type θ_p's continuation payoff at (a_*^{t+1}, r^{t+1}) is $u_1(\theta_p, \overline{a}_1, \overline{a}_2)$ by playing $\overline{a}_1$ in every period, while his continuation payoff at (a_*^t, a_1, r^{t+1}) is $u_1(\theta_p, \underline{a}_1, \underline{a}_2)$, leading to a contradiction. Suppose next that $X(h^1) < 0$. Similar to the previous argument, there exists type $\theta_p \in \Theta_p$ with $h^1 \in \mathcal{H}(\theta_p)$ such that his incentive constraint is violated. Similarly, one can show that $X(h^t) = 0$ for every $t \geq 1$, $h^t = (a_*^t, r^t) \in \mathcal{H}^\sigma$. This establishes the uniqueness of player 1's equilibrium behavior.

APPENDIX SC: HIGHEST GUARANTEED PAYOFF IN BINARY-ACTION MS GAMES

I show that the payoff lower bound in Theorem 2 is tight in the sense that when the total probability of commitment types is sufficiently small and the set Θ_p is nonempty, no type of strategic player 1 can guarantee payoff strictly higher than $\max\{u_1(\theta, \overline{a}_1, \overline{a}_2), u_1(\theta, \underline{a}_1, \underline{a}_2)\}$.

ASSUMPTION SC.1: *There exists $\theta \in \Theta^*$ such that $BR_2(\theta, \overline{a}_1) = \{\underline{a}_2\}$.*

Intuitively, Assumption SC.1 implies that there exists a state θ under which (a) playing $\overline{a}_1$ is individually rational and (b) player 2 does not have an incentive to play the desirable action when she knows that player 1 is strategic type θ. The result is stated as Proposition SC.1.

PROPOSITION SC.1: *Suppose the game satisfies Assumptions* 2 *and* SC.1. *For every* $\phi \in \Delta(\Theta)$, *there exist* $\overline{\varepsilon} \in (0,1)$ *and* $\underline{\delta} \in (0,1)$, *such that for every* $\delta > \underline{\delta}$, *and every* μ *that attaches probability less than* $\overline{\varepsilon}$ *to all commitment types, and the marginal state distribution is* ϕ, *there exists an equilibrium such that for all* $\theta \in \Theta$, *strategic type* θ*'s payoff is no more than* $\max\{u_1(\theta,\overline{a}_1,\overline{a}_2), u_1(\theta,\underline{a}_1,\underline{a}_2)\}$.

This proposition applies *regardless* of the set of commitment actions A_1^* as well as the distributions of the states conditional on each commitment type $\{\phi_{a_1^*}\}_{a_1^* \in A_1^*}$. This contrasts to the private-value benchmark, in which a patient player can guarantee his commitment payoff from $a_1 \in A_1$ when a_1 is one of the commitment actions.

PROOF OF PROPOSITION SC.1: Since $w_\theta(\phi) \leq \max\{u_1(\theta,\overline{a}_1,\overline{a}_2), u_1(\theta,\underline{a}_1,\underline{a}_2)\}$ for every $\theta \in \Theta$, the case in which ϕ is pessimistic is implied by the payoff uniqueness result of Theorem 3. When ϕ is optimistic, let

$$\underline{\theta} \equiv \min \Theta^* \quad \text{and} \quad \overline{\theta} \equiv \max \Theta^*.$$

Assumption SC.1 and Assumption 2 in the main text together imply that $\text{BR}_2(\underline{\theta},\overline{a}_1) = \{\underline{a}_2\}$. The assumption that ϕ is optimistic implies that $\text{BR}_2(\overline{\theta},\overline{a}_1) = \{\overline{a}_2\}$. For every full support $\phi \in \Delta(\Theta)$, let $\overline{\varepsilon}$ be bounded from above by

$$\overline{\varepsilon} < \min\left\{ \frac{|\phi(\underline{\theta})\mathcal{D}(\underline{\theta},\overline{a}_1)|}{\mathcal{D}(\overline{\theta},\overline{a}_1)}, \frac{\phi(\overline{\theta})\mathcal{D}(\overline{\theta},\overline{a}_1)}{|\mathcal{D}(\min\Theta,\overline{a}_1)|} \right\}. \tag{SC.1}$$

Recall that A_1^* is the set of commitment actions. For every $a_1^* \in A_1^*$, let $\phi_{a_1^*} \in \Delta(\Theta)$ be the distribution of θ conditional on player 1 being commitment type a_1^*. Let A_1^g be the subset of A_1^* such that

$$\text{BR}_2(\phi_{a_1^*}, a_1^*) = \{\overline{a}_2\}.$$

When A_1^g is nonempty, consider the following equilibrium:

- Strategic types outside Θ^* play $\underline{a}_1$ in every period on the equilibrium path.
- Strategic types in $\Theta^* \setminus \{\underline{\theta}\}$ play $\overline{a}_1$ in every period on the equilibrium path.
- Strategic type $\underline{\theta}$ mixes between actions in $\{\overline{a}_1\} \cup A_1^g$ in period 0 and on the equilibrium path, repeats the same action that he has played in period 0 in all subsequent periods. The probability with which he plays a_1^* in period 0 is denoted by $p(a_1^*)$, given by

$$p(a_1^*) \equiv \begin{cases} \dfrac{\mu(a_1^*)\mathcal{D}(\phi_{a_1^*,a_1^*})}{|(1-\varepsilon)\phi(\underline{\theta})\mathcal{D}(\underline{\theta},a_1^*)|} & \text{if } a_1^* \in A_1^g \setminus \{\underline{a}_1,\overline{a}_1\}, \\ 1 - \displaystyle\sum_{\widehat{a}_1 \in A_1^g \setminus \{\underline{a}_1,\overline{a}_1\}} p(\widehat{a}_1) & \text{if } a_1^* = \overline{a}_1, \end{cases} \tag{SC.2}$$

  where $\mu(a_1^*)$ denotes the probability that player 2's prior belief attaches to commitment type a_1^*, and ε denotes the probability it attaches to all the commitment types. Intuitively, after player 2 observes $a_1^* \in A_1^g \setminus \{\underline{a}_1,\overline{a}_1\}$ in period 0, her posterior belief makes her indifferent between $\overline{a}_2$ and $\underline{a}_2$ against a_1^*.

- Starting from period 1, player 2 plays $\overline{a}_2$ in every period if player 1 has played $\overline{a}_1$ in all previous period; she mixes between $\overline{a}_2$ and $\underline{a}_2$ if player 1 has played $a_1^* \in A_1^g \setminus \{\underline{a}_1, \overline{a}_1\}$ in all previous period and the probability of playing $\overline{a}_2$ is such that type $\underline{\theta}$ is indifferent between playing $\overline{a}_1$ in every period and playing a_1^* in every period at period 0. At all other histories, she plays $\underline{a}_2$ with probability 1.

In the above equilibrium, all types in Θ^* receives payoff approximately $u_1(\theta, \overline{a}_1, \overline{a}_2)$, and all types outside Θ^* receives payoff approximately $u_1(\theta, \underline{a}_1, \underline{a}_2)$. This establishes Proposition SC.1. *Q.E.D.*

APPENDIX SD: COUNTEREXAMPLES

SD.1. *Conflicts Between Reputation Building and Signaling Under MS Stage-Game Payoff*

I show that when Assumptions 1–4 are satisfied and the prior belief about θ is optimistic, there exist equilibria such that player 1's highest action signals the low state at some on-path history. Players' stage-game payoffs are

$\theta = \theta_h$	T	N
G	$1, 1$	$-1, 0$
B	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
G	$1 - \eta, -1$	$-1 - \eta, 0$
B	$2, -2$	$0, 0$

There is only one commitment plan, given by

$$\gamma(\theta) \equiv \begin{cases} G & \text{if } \theta = \theta_h, \\ B & \text{if } \theta = \theta_l. \end{cases}$$

The equilibrium construction focus on settings in which $\eta \in (0, 1)$ and the prior probability of state θ_h, denoted by ϕ_h, is strictly between 10/11 and 1.

Consider the following strategy profile. In period 0, player 2 plays N, strategic type θ_h plays G with probability

$$\frac{2\phi_h \varepsilon}{3\phi_h(1 - \varepsilon)},$$

and strategic type θ_l plays G with probability

$$\frac{\phi_h \varepsilon}{6(1 - \phi_h)(1 - \varepsilon)}.$$

According to Bayes rule, the probability of state θ after observing G in period 0 is 10/11, which is strictly less than ϕ_h. Namely, observing player 1 playing his highest action G leads to negative inferences about θ. In period 1, the following situations exist:

- If the history is (B, N), then future play is dictated by the realization of the public randomization device. With probability $(1 - \delta)/\delta$, players play (B, N) in every subsequent period on the equilibrium path; with complementary probability, players play (G, T) in every subsequent period on the equilibrium path. Off-path deviations are deterred by grim-trigger strategies, namely, whenever player 2 observes player 1 playing B in periods in which he is supposed to play G, player 2 plays N in all subsequent periods.

- If the history is (G, N), then both strategic types play B with probability 1 and player 2 plays T. This is incentive compatible for player 2 since at history (G, N), the probability of commitment type G is $6/11$, the probability of strategic type θ_h is $4/11$, and the probability of strategic type θ_l is $1/11$.

In period 2, players' actions at histories (B, N, B, N), (B, N, G, T), and (B, N, B, T) have been specified. At history (G, N, G, T), players play (G, T) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim-trigger strategies. At history (G, N, B, T), the following situations exist:

- With probability $(1-\delta)/\delta$, players play (B, N) in every subsequent period on the equilibrium path.
- With probability $1-\frac{1-\delta}{\delta^2}-\frac{1-\delta}{\delta}$, players play (G, T) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim-trigger strategies.
- With probability $(1-\delta)/\delta^2$, type θ_l plays B for sure, and type θ_h plays B with probability $1/4$ and plays G with probability $3/4$. Player 2 plays T.

In period 3, the following situations exist:

- At history (G, N, B, T, G, T), play (G, T) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim-trigger strategies.
- At history (G, N, B, T, B, T), future play is determined by the realization of public randomization. With probability $(1-\delta)/\delta$, play (B, N) in every subsequent period on the equilibrium path. With complementary probability, play (G, T) in every subsequent period on the equilibrium path, with off-path deviations deterred via grim-trigger strategies.

The above strategy profiles an equilibrium when δ is large enough. Despite that the game satisfies Assumptions 1–4 and the prior belief about state is optimistic, playing G in period 0 signals state θ_l.

SD.2. *Reputation Failure in Common Interest Games*

I present an example of a *common interest game* with nontrivial interdependent values, under which there exists equilibrium such that all strategic types' equilibrium payoffs are *arbitrarily low* compared to their commitment payoffs. Consider the game

$\theta=\theta_1$	h	l
H	$1, 1$	$0, 0$
L	$0, 0$	ϵ, ϵ

$\theta=\theta_2$	h	l
H	$0, 0$	ϵ, ϵ
L	$1, 1$	$0, 0$

with $\epsilon \in (0, 1)$ being a parameter. Suppose $\Gamma \equiv \{\gamma\}$ in which the committed player 1 plays his Stackelberg action in every state:

$$\gamma(\theta) \equiv \begin{cases} H & \text{if } \theta=\theta_1, \\ L & \text{if } \theta=\theta_2. \end{cases} \tag{SD.1}$$

PROPOSITION SD.1: *For every full support $\phi \in \Delta\{\theta_1, \theta_2\}$ and $\epsilon \in (0, 1)$, there exists $\overline{\varepsilon} > 0$, such that when player 1 is committed with probability less than $\overline{\varepsilon}$, there exists an equilibrium in which strategic player 1's payoff is ϵ in every state.*

PROOF: Let

$$\overline{\varepsilon} \equiv \min\{\phi(\theta_1), \phi(\theta_2)\}\frac{\epsilon}{1+\epsilon}. \tag{SD.2}$$

I verify that the following strategy profile is an equilibrium for every $\delta \in (0, 1)$:

- Player 2 plays l at every history.
- Strategic type θ_1 plays L at every history. Strategic type θ_2 plays H at every history.

First, given player 2's strategy, player 1's strategy maximizes his payoff at each state and at each history. Second, given player 1's strategy, I show that player 2 has a strict incentive to play l for all histories.

This is because if player 1 plays L, then he is either strategic type θ_1 or commitment type L. The likelihood ratio between these two types is strictly greater than $\frac{\phi(\theta_1)-\overline{\varepsilon}}{\overline{\varepsilon}}$, which according to (SD.2) is at least $1/\epsilon$. This implies that player 2 strictly prefers l to h in the event that player 1 plays L. Similarly, in the event that player 1 plays H, player 2 strictly prefers l to h. *Q.E.D.*

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