Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL MONITORING DESIGN"
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PROOF OF LEMMA 1: Fix an arbitrary stopping rule τ, and assume that the wage scheme $W(\boldsymbol{\omega}_\tau)$ solves (1) subject to (3) and (5). In what follows, we define a new wage scheme, $\widehat{W}$, which only depends on the score. According to this scheme, after a realized path $\boldsymbol{\omega}_\tau$, the Agent's wage is the average wage according to W conditional on the score being $B_\tau(\boldsymbol{\omega}_\tau)$. We will argue that $\widehat{W}$ is feasible (i.e., it satisfies (3) and (5)) and has the same expectation as W. Finally, we show that if W does not only depend on the score with positive probability, then the relaxed incentive constraint, (5), is slack at $\widehat{W}$ and hence, this wage scheme can be further modified to strictly reduce the Principal's expected cost.

Formally, we define the new wage scheme by

$$\widehat{W}(s) = \mathbb{E}_{a^*}[W | B_\tau = s].$$

By construction, this wage scheme bears the same expected cost to the Principal. In addition, since $W \geq \underline{w}$, this new scheme also satisfies (3). Next, we show that $\widehat{W}$ also satisfies (5). Notice that

$$\begin{aligned} \mathbb{E}_{a^*}\big[u\big(\widehat{W}(s_\tau)\big)s_\tau\big] &= \mathbb{E}_{a^*,s_\tau}\big[u\big(\mathbb{E}_{a^*}[W | B_\tau = s_\tau]\big)s_\tau\big] \\ &\geq \mathbb{E}_{a^*,s_\tau}\big[\mathbb{E}_{a^*}\big[u(W) | B_\tau = s_\tau\big]s_\tau\big] \\ &= \mathbb{E}_{a^*}\big[u(W)B_\tau\big] \geq c'\big(a^*\big), \end{aligned}$$

where the first equality follows the definition of $\widehat{W}$, the first inequality is implied by Jensen's inequality, the second equality follows from $s_\tau = B_\tau$, and the last inequality follows from the assumption that W satisfies (5). This inequality chain implies that $\widehat{W}$ also solves (5). Furthermore, if the probability of s for which $W(s) \neq \mathbb{E}_{a^*}[W | B_\tau) = s]$ is positive, the first inequality is strict and hence, the incentive constraint at $\widehat{W}$ is slack. Therefore, $\widehat{W}$ can be modified by reducing it at those values at which $\widehat{W}(s) \neq \underline{w}$ so that this modified wage scheme still satisfies (3) and (5). This wage scheme then would be strictly less costly for the Principal than W. This would contradict the hypothesis that W solves (1) subject to (3) and (5). *Q.E.D.*

PROOF OF LEMMA 2: Let τ be a stopping time with finite expectation. For each $n \in \mathbb{N}$, define $\tau_n := \min\{\tau, n\}$, and note that τ_n is bounded and converges to τ pointwise as $n \to \infty$. Recall that $s_t = B_t$ if $a = a^*$. Since $\{s_t\}_{t\geq 0}$ and $\{s_t^2 - t\}_{t\geq 0}$ are martingales and τ_n is bounded, it follows from Doob's optional sampling theorem that for each $n \in \mathbb{N}$,

$$\mathbb{E}_{a^*}[s_{\tau_n}] = \mathbb{E}_{a^*}[s_0] = 0 \quad \text{and} \quad \mathbb{E}_{a^*}\big[s_{\tau_n}^2 - \tau_n\big] = \mathbb{E}_{a^*}\big[s_0^2 - 0\big] = 0. \tag{S1}$$

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The second inequality chain and $\mathbb{E}_{a^*}[\tau_n] \le n$ imply that $\mathbb{E}_{a^*}[s^2_{\tau_n}] < \infty$. It remains to show that these properties are preserved in the limit.

Observe that for any $m < n$,

$$\mathbb{E}_{a^*}\big[(s_{\tau_n} - s_{\tau_m})^2\big] = \mathbb{E}_{a^*}\big[s^2_{\tau_n} - s^2_{\tau_m}\big] = \mathbb{E}_{a^*}[\tau_n - \tau_m],$$

where the first equality follows from $\mathbb{E}_{a^*}[s_{\tau_n} s_{\tau_m}] = \mathbb{E}_{a^*}[s_{\tau_m}\mathbb{E}_{a^*}[s_{\tau_n}|s_{\tau_m}]] = \mathbb{E}_{a^*}[s^2_{\tau_m}]$ and the second equality follows from (S1). Since τ_n, τ_m converges to τ and $\mathbb{E}_{a^*}[\tau] < \infty$, the right-hand side vanishes as n, m go to infinity. Therefore, $\{s_{\tau_n}\}_{n\in\mathbb{N}}$ is an L^2-Cauchy sequence, and s_{τ_n} converges to s_τ as n goes to infinity in L^2. Hence, s_{τ_n} also converges to s_τ in L^1, and so $\lim_{n\to\infty} \mathbb{E}_{a^*}[s_{\tau_n}] = \mathbb{E}_{a^*}[s_\tau]$. Since $\mathbb{E}_{a^*}[s_{\tau_n}] = 0$ by the first equality chain in (S1), $\mathbb{E}_{a^*}[s_\tau] = 0$ also follows.

Next, note that

$$\mathbb{E}_{a^*}\big[s^2_\tau\big] = \mathbb{E}_{a^*}\Big[\liminf_{n\to\infty} s^2_{\tau_n}\Big] \le \liminf_{n\to\infty} \mathbb{E}_{a^*}\big[s^2_{\tau_n}\big] = \lim_{n\to\infty} \mathbb{E}_{a^*}[\tau_n] = \mathbb{E}_{a^*}[\tau] < \infty,$$

where the first equality follows from $\lim_{n\to\infty} s^2_{\tau_n} = s^2_\tau$ almost surely, the first inequality follows from Fatou's lemma, the second equality is implied by the second equality chain in (S1), the third equality follows from Lebesgue's dominated convergence theorem because $\tau_n \le \tau$ for every n, and the last inequality follows by assumption.

Thus, letting F_τ denote the distribution of s_τ when the agent chooses $a = a^*$, we have shown that for any stopping time such that $\mathbb{E}_{a^*}[\tau] < \infty$, we have $\mathbb{E}_{F_\tau}[s] = 0$ and $\mathbb{E}_{F_\tau}[s^2] < \infty$ as desired. *Q.E.D.*

PROOF OF LEMMA 4: To prove part (i), note that

$$L(\lambda, F) = \int \big[w(\lambda, s) + \gamma s^2\big] + \lambda\big[c'\big(a^*\big) - su\big(w(\lambda, s)\big)\big]\, dF(s)$$

because the wage scheme $w(\lambda, \cdot)$ (defined by (8)) minimizes the integrand in (7) pointwise. Hence, the dual problem is

$$\sup_{\lambda\ge 0} \int \big[w(\lambda, s) + s^2\big] + \lambda\big[c'\big(a^*\big) - su\big(w(\lambda, s)\big)\big]\, dF(s). \tag{S2}$$

It is easy to show that the objective function is concave in λ, so the first-order condition is necessary and sufficient for an optimal solution. The Envelope Condition implies that

$$L_1(\lambda, F) = c'\big(a^*\big) - \int su\big(w(\lambda, s)\big)\, dF(s). \tag{S3}$$

Note that $L_1(0, F) = c'(a^*) > 0$, so there are two cases to be considered.

Case 1: There exists a $\widehat{\lambda} > 0$ such that $L_1(\widehat{\lambda}, F) = 0$. Then, by (S2) and (S3),

$$L(\widehat{\lambda}, F) = \int \big[w(\widehat{\lambda}, s) + s^2\big]\, dF(s). \tag{S4}$$

Observe that $w(\widehat{\lambda}, \cdot)$ is a feasible wage scheme because it satisfies the limited liability constraint, (LL), by construction and it also satisfies the relaxed incentive constraint, (IC), by (S3) and $L_1(\widehat{\lambda}, F) = 0$. Therefore, (S4) implies that $\Pi(F) \le \int [w(\widehat{\lambda}, s) + s^2]\, dF(s)$. On the other hand, weak duality implies that $\Pi(F) \geq L(\widehat{\lambda}, F)$, and thus we have $L(\widehat{\lambda}, F) = \Pi(F)$.

Case 2: $L_1(\lambda, F) > 0$ for all $\lambda \geq 0$. Then, by (S2) and (S3),

$$\sup_{\lambda \geq 0} L(\lambda, F) \geq \sup_{\lambda \geq 0} \int \left[w(\lambda, s) + s^2\right] dF(s). \tag{S5}$$

Since $w(\lambda, s)$ converges to infinity if $s > 0$ and to $\underline{w}$ if $s \leq 0$ as λ goes to infinity, the right-hand side of (S5) is infinity unless F is the degenerate distribution, $F = \mathbb{I}_{\{s \geq 0\}}$. Hence, $\sup_{\lambda \geq 0} L(\lambda, F) = \infty$. If $F = \mathbb{I}_{\{s \geq 0\}}$, then, by $w(\lambda, 0) = \underline{w}$ and (S2), $L(\lambda, F) \geq \underline{w} + \lambda c'(a^*)$, so $\sup_{\lambda \geq 0} L(\lambda, F) = \infty$. Again, weak duality implies that $\Pi(F) = \infty$. Finally, notice that this equality implies that, in this case, the problem in (6) does not have a solution.

To prove part (ii), first observe from the proof of part (i) it follows that if there exists a $\widehat{\lambda} > 0$ such that $L_1(\widehat{\lambda}, F) = 0$, then the problem in (6) has a solution (see Case 1). In particular, (S4) implies that the wage scheme $w(\widehat{\lambda}, \cdot)$ solves (6). Moreover, by (S3) and $L_1(\widehat{\lambda}, F) = 0$, the incentive constraint, (IC), indeed binds at $w(\widehat{\lambda}, \cdot)$. Furthermore, since $w(\widehat{\lambda}, s)$ is strictly increasing in λ if $s > s_*(\lambda)$ and $s_*(\lambda)$ is strictly decreasing, the right-hand side of (S4) is strictly increasing in λ. This implies the uniqueness of $\widehat{\lambda}$. Also notice that if $L_1(\lambda, F) > 0$ for all $\lambda \geq 0$, then $\Pi(F) = \infty$ (see Case 2), and hence, the problem in (6) does not have a solution.

It remains to show that the wage scheme $w(\widehat{\lambda}, s)$ uniquely solves (6) subject to (IC) and (LL). Towards a contradiction, suppose that there exists a wage scheme $\tilde{w}(\cdot)$ which differs from $w(\widehat{\lambda}, \cdot)$ on a set of positive measure, it satisfies the constraints (IC) and (LL), and bears a weakly lower expected cost to the Principal than the scheme $w(\widehat{\lambda}, \cdot)$, that is, $\mathbb{E}_F(w(\widehat{\lambda}, s)) \geq \mathbb{E}_F(\tilde{w}(s))$. For each $\varepsilon \in [0, 1]$, define the wage scheme, w^ϵ, by

$$u\big(w^\epsilon(s)\big) = (1 - \epsilon)u\big(w(\widehat{\lambda}, s)\big) + \epsilon u\big(\tilde{w}(s)\big)$$

for all s. This is the certainty equivalent of a $(1-\epsilon, \epsilon)$ lottery between $w(\widehat{\lambda}, s)$ and $\tilde{w}(s)$. To obtain a contradiction, we show that $\mathbb{E}_F(w(\widehat{\lambda}, s)) \geq \mathbb{E}_F(\tilde{w}(s))$ implies that $\partial \mathbb{E}_F(w^\epsilon)/\partial \epsilon < 0$ at $\epsilon = 0$. On the other hand, we argue that $\partial \mathbb{E}_F(w^\epsilon)/\partial \epsilon \geq 0$ at $\epsilon = 0$ follows from $w(\widehat{\lambda}, \cdot)$ satisfying the incentive constraint, (IC), with equality.

To this end, note that

$$\begin{aligned}
\frac{\partial w^\epsilon(s)}{\partial \epsilon} &= \frac{1}{u'\big(w^\epsilon(s)\big)}\big[u\big(\tilde{w}(s)\big) - u\big(w(\widehat{\lambda}, s)\big)\big] \quad \text{and} \\
\frac{\partial^2 w^\epsilon(s)}{\partial \epsilon^2} &= -\frac{u''\big(w^\epsilon(s)\big)}{\big[u'\big(w^\epsilon(s)\big)\big]^3}\big[u\big(\tilde{w}(s)\big) - u\big(w(\widehat{\lambda}, s)\big)\big]^2 \geq 0,
\end{aligned} \tag{S6}$$

where the inequality is strict if $w(\widehat{\lambda}, s) \neq \tilde{w}(s)$. Since $\tilde{w}(\cdot)$ and $w(\widehat{\lambda}, \cdot)$ differ on a set of positive measure, the Principal's expected cost associated with the wage scheme w^ϵ, $\mathbb{E}_F(w^\epsilon)$, is strictly convex in ϵ. Therefore, since $w^0(s) = w(\widehat{\lambda}, s)$, $w^1(s) = \tilde{w}(s)$, and $\mathbb{E}_F(w(\widehat{\lambda}, s)) \geq \mathbb{E}_F(\tilde{w}(s))$, it must be that

$$\left.\frac{\partial \mathbb{E}_F\big(w^\epsilon\big)}{\partial \epsilon}\right|_{\epsilon=0} < 0. \tag{S7}$$

Next, we show that

$$\frac{1}{u'\big(w(\widehat{\lambda},s)\big)}\big[u\big(\tilde{w}(s)\big)-u\big(w(\widehat{\lambda},s)\big)\big]\geq\widehat{\lambda}s\big[u\big(\tilde{w}(s)\big)-u\big(w(\widehat{\lambda},s)\big)\big] \tag{S8}$$

for all s. This inequality holds with equality for all $s>s_*(\widehat{\lambda})$ since $1/u'(w(\widehat{\lambda},s))=\widehat{\lambda}s$ for such s (see (8)). If $s\leq s_*(\widehat{\lambda})$, then $w(\widehat{\lambda},s)=\underline{w}$, and the desired inequality follows from the facts that $u(\tilde{w}(s))-u(w(\widehat{\lambda},s))\geq 0$ (as $\tilde{w}(\cdot)$ satisfies (LL)) and

$$\frac{1}{u'\big(w(\widehat{\lambda},s)\big)}=\frac{1}{u'(\underline{w})}\geq\widehat{\lambda}s.$$

Therefore,

$$\begin{aligned}\frac{\partial\mathbb{E}_F\big(w^\epsilon\big)}{\partial\epsilon}\bigg|_{\epsilon=0}&=\int\frac{1}{u'\big(w^\epsilon(s)\big)}\big[u\big(\tilde{w}(s)\big)-u\big(w(\widehat{\lambda},s)\big)\big]\,dF(s)\\&\geq\widehat{\lambda}\int s\big[u\big(\tilde{w}(s)\big)-u\big(w(\widehat{\lambda},s)\big)\big]\,dF(s)\\&\geq\widehat{\lambda}\bigg[\int su\big(\tilde{w}(s)\big)\,dF(s)-c'\big(a^*\big)\bigg]\geq 0,\end{aligned}$$

where the equality follows from (S6), the first inequality follows from (S8), the second inequality holds because $w(\widehat{\lambda},\cdot)$ satisfies (IC) with equality, and the last inequality follows because $\tilde{w}$ satisfies (IC). Notice that this inequality chain contradicts (S7), so we conclude that $w(\widehat{\lambda},\cdot)$ is uniquely optimal. *Q.E.D.*

PROOF OF LEMMA 5: If $\{\lambda^*,F^*\}$ is an equilibrium in the zero-sum game defined above, then

$$\inf_{F\in\mathcal{F}}\sup_{\lambda\geq 0}L(\lambda,F)\leq\sup_{\lambda\geq 0}L\big(\lambda,F^*\big)=L\big(\lambda^*,F^*\big)=\inf_{F\in\mathcal{F}}L\big(\lambda^*,F\big)\leq\sup_{\lambda\geq 0}\inf_{F\in\mathcal{F}}L(\lambda,F),$$

where the two equalities hold because λ^* and F^* are best responses to each others. Since $\inf_{F\in\mathcal{F}}\sup_{\lambda\geq 0}L(\lambda,F)\geq\sup_{\lambda\geq 0}\inf_{F\in\mathcal{F}}L(\lambda,F)$ always holds, the two inequalities are equalities in the previous chain. This proves the equation in the statement of the lemma.

Finally, by part (ii) of Lemma 4, $\sup_{\lambda\geq 0}L(\lambda,F^*)=L(\lambda^*,F^*)$ implies that the wage scheme $w(\lambda^*,\cdot)$ solves the problem in (6) with $F=F^*$. Therefore, since $L(\lambda^*,F^*)=\inf_{F\in\mathcal{F}}\sup_{\lambda\geq 0}L(\lambda,F)$, it follows that $w(\lambda^*,\cdot)$ and F^* solve the problem in (Obj). *Q.E.D.*

PROOF OF LEMMA 9: Note that $\underline{s}(\lambda)$ and $\overline{s}(\lambda)$ are defined by the following equations:

$$\begin{aligned}Z_2(\lambda,\underline{s})-Z_2(\lambda,\overline{s})&=0,\\Z(\lambda,\underline{s})+(\overline{s}-\underline{s})Z_2(\lambda,\underline{s})-Z(\lambda,\overline{s})&=0.\end{aligned}$$

The first equation requires that the derivatives of $Z(\lambda,s)$ with respect to s are the same at $s=\underline{s}$ and at $s=\overline{s}$. The second equation requires that the point $(\overline{s},Z(\lambda,\overline{s}))$ lies on the line crossing $(\underline{s},Z(\lambda,\underline{s}))$ with slope $Z_2(\lambda,\underline{s})$. The Jacobian matrix corresponding to this mapping is

$$\begin{vmatrix}Z_{22}(\lambda,\underline{s}) & -Z_{22}(\lambda,\overline{s})\\-\underline{s}Z_{22}(\lambda,\underline{s}) & 0\end{vmatrix}.$$

Since $Z_{22}(\lambda, \overline{s}) > 0$, the determinant of this matrix is nonzero. Then, by the implicit function theorem, part (i) of the lemma follows.

To prove part (ii), first, noting that $s_*(\lambda_n)$ converges to $s_*(\lambda^c)$ as $n \to \infty$ and $\underline{s}(\lambda_n) < s_*(\lambda_n) < \overline{s}(\lambda_n)$, it is enough to show that $\overline{s}(\lambda_n) - \underline{s}(\lambda_n)$ tends to zero as $n \to \infty$. Suppose, by contradiction, that there is a subsequence $(\lambda_{n_k})_{n_k} \subset (\lambda_n)_n$ and an $\varepsilon > 0$ such that

$$\overline{s}(\lambda_{n_k}) - \underline{s}(\lambda_{n_k}) > \varepsilon.$$

Therefore, since $s_*(\lambda_{n_k}) \to s_*(\lambda^c)$ and $\underline{s}(\lambda_{n_k}) < s_*(\lambda_n) < \overline{s}(\lambda_{n_k})$, there must exist $s_1, s_2 \in (s_*(\lambda^c) - \varepsilon, s_*(\lambda^c) + \varepsilon)$ and a subsequence $(\lambda_{n_l})_{n_l} \subset (\lambda_{n_k})_{n_k}$ such that $s_2 - s_1 > \varepsilon/2$ and

$$\underline{s}(\lambda_{n_l}) \le s_1 \quad \text{and} \quad s_2 \le \overline{s}(\lambda_{n_l}).$$

Then

$$\begin{aligned}
&\limsup_{n_l \to \infty} Z_2\big(\lambda_{n_l}, \underline{s}(\lambda_{n_l})\big) \\
&\quad \le \limsup_{n_l \to \infty} Z_2(\lambda_{n_l}, s_1) = Z_2\big(\lambda^c, s_1\big) \\
&\quad < Z_2\big(\lambda^c, s_2\big) = \liminf_{n_l \to \infty} Z_2(\lambda_{n_l}, s_2) \le \liminf_{n_l \to \infty} Z_2\big(\lambda_{n_l}, \overline{s}(\lambda_{n_l})\big),
\end{aligned}$$

where the first and last inequalities follow from $Z(\lambda_{n_l}, s)$ being convex in s on $(-\infty, \underline{s}(\lambda_{n_l})] \cup [\overline{s}(\lambda_{n_l}), \infty)$, the two equalities follow from continuity, and the strict inequality follows from $Z(\lambda^c, s)$ being strictly convex (see Lemma 7). Note, however, that $Z_2(\lambda_{n_l}, \underline{s}(\lambda_{n_l})) = Z_2(\lambda_{n_l}, \overline{s}(\lambda_{n_l}))$ and hence

$$\limsup_{n_l \to \infty} Z_2\big(\lambda_{n_l}, \underline{s}(\lambda_{n_l})\big) \ge \liminf_{n_l \to \infty} Z_2\big(\lambda_{n_l}, \overline{s}(\lambda_{n_l})\big),$$

which contradicts the previous displayed inequality chain. *Q.E.D.*

PROOF OF LEMMA 10: First, note that if the IC is *not* slack for a given λ, then the Principal's payoff is bounded from below by $\underline{w}$. Indeed, even if the Principal does not acquire any information, she has to pay at least $\underline{w}$ to the Agent. Therefore, in order to prove the lemma, it is enough to show that if λ is large enough, then the Principal's payoff is smaller than $\underline{w}$.

Let $\overline{u}$ denote $\lim_{w \to \infty} u(w)$ and fix an $\widetilde{s} > 0$ such that

$$\widetilde{s} > \frac{2c'\big(a^*\big)}{\overline{u} - u(\underline{w})}. \tag{S9}$$

(If $\overline{u} = \infty$, then this inequality imposes no restriction on $\widetilde{s}$ in addition to requiring it to be positive.) Consider the binary distribution, $\widetilde{F}$, which specifies probability half on $\widetilde{s}$ and $-\widetilde{s}$.[1] Recall that

$$\frac{\partial \mathbb{E}_{\widetilde{F}}\big[Z(\lambda, s)\big]}{\partial \lambda} = -\int su\big(w(\lambda, s)\big)\, d\widetilde{F}(s) + c'\big(a^*\big)$$

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[1] $\widetilde{F}(s) = \begin{cases} 0 & \text{if } s < -\widetilde{s}, \\ \frac{1}{2} & \text{if } s \in [-\widetilde{s}, \widetilde{s}), \\ 1 & \text{if } s \ge \widetilde{s}. \end{cases}$

$$= \frac{1}{2}\widetilde{s}u(\underline{w}) - \widetilde{s}\frac{1}{2}u\big(w(\lambda, \widetilde{s})\big) + c'\big(a^*\big).$$

Since $\lim_{\lambda\to\infty} w(\lambda, \widetilde{s}) = \infty$, $\lim_{\lambda\to\infty} u(w(\lambda, \widetilde{s})) = \overline{u}$. Therefore,

$$\lim_{\lambda\to\infty} \frac{\partial \mathbb{E}_{\widetilde{F}}\big[Z(\lambda, s)\big]}{\partial\lambda} = -\frac{\widetilde{s}\big[\overline{u} - u(\underline{w})\big]}{2} + c'\big(a^*\big) < 0,$$

where the inequality follows from (S9). (If $\overline{u} = \infty$, then this limit is minus infinity.) Since $w(\lambda, \widetilde{s})$ is strictly increasing in λ, it follows that there exists a $\overline{\lambda}$ such that for all $\lambda > \overline{\lambda}$,

$$\frac{\partial \mathbb{E}_{\widetilde{F}}\big[Z(\lambda, s)\big]}{\partial\lambda} < 0.$$

Since $\partial \mathbb{E}_{\widetilde{F}}[Z(\lambda, s)]/\partial\lambda$ is strictly decreasing in λ (because $u(w(\lambda, \widetilde{s}))$ is strictly increasing in λ), it follows that there exists a Λ such that the Principal's payoff is smaller than $\underline{w}$ whenever $\lambda > \Lambda$ and the Principal chooses $\widetilde{F}$. Of course, the Principal's payoff is even smaller if she best-responds to λ. *Q.E.D.*

PROOF OF LEMMA 11: Suppose, by contradiction, that (IC') is slack at λ^*. It follows from Lemma 7 that $\lambda^* > \lambda_c$, for otherwise the Principal would choose the degenerate distribution and (IC') would be violated. By part (i) of Lemma 9 and continuity, there exists $\lambda < \lambda^*$ such that (IC') is also slack at λ, that is,

$$\underline{p}(\lambda)\underline{s}(\lambda)u(\underline{w}) + \overline{p}(\lambda)\overline{s}(\lambda)u\big(w\big(\lambda, \overline{s}(\lambda)\big)\big) > c'\big(a^*\big).$$

This contradicts the definition of λ^* given in (24).

Suppose now that (IC') is violated. First, we argue that $\lambda^* > \lambda_c$, and hence, the function $Z(\lambda^*, s)$ is non-convex. To see this, first observe that by continuity and the definition of λ^*, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} > \lambda^*$ such that $\lim_{n\to\infty} \lambda_n = \lambda^*$, and for all $n \in \mathbb{N}$,

$$\underline{p}(\lambda_n)\underline{s}(\lambda_n)u(\underline{w}) + \overline{p}(\lambda_n)\overline{s}(\lambda_n)u\big(w\big(\lambda_n, \overline{s}(\lambda_n)\big)\big) > c'\big(a^*\big). \tag{S10}$$

It must be the case that

$$\underline{s}(\lambda_n) < 0 < \overline{s}(\lambda_n), \tag{S11}$$

for otherwise, $Z^c(\lambda_n, 0) = Z(\lambda_n, 0)$, so the Principal would choose the degenerate distribution and (IC') would be violated. Suppose, by contradiction, that $Z(\lambda^*, s)$ is convex in s. By Lemma 7, this implies that $\lambda^* = \lambda_c$, and the convexity of $Z(\lambda^*, s)$ in s together with the fact that $s_*(\lambda) > 0$ for all λ imply that

$$Z_2\big(\lambda^*, 0\big) < Z_2\big(\lambda^*, s_*\big(\lambda^*\big)\big).$$

By continuity and part (ii) of Lemma 9,

$$\lim_{n\to\infty} Z_2(\lambda_n, 0) = Z_2\big(\lambda^*, 0\big) \quad \text{and} \quad \lim_{n\to\infty} Z_2\big(\lambda_n, \underline{s}(\lambda_n)\big) = Z_2\big(\lambda^*, s_*\big(\lambda^*\big)\big).$$

The convexity of $Z(\lambda^*, s)$ in s and the previous two displayed equations imply that $0 < \underline{s}(\lambda_n)$ for sufficiently large n. This contradicts (S11), and we conclude that $\lambda^* > \lambda_c$ and $Z(\lambda^*, s)$ is non-convex in s.

If (IC') is violated at $\lambda^* (> \lambda^c)$, then by continuity and Lemma 9(i), there exists an $\varepsilon > 0$ such that (IC') is violated for all $\lambda \in [\lambda^*, \lambda^* + \varepsilon]$. This, again, contradicts the definition of λ^* in (24). *Q.E.D.*

PROOF OF THEOREM 2: We first describe a pair of equations that define an equilibrium. Note that the proof of Lemma 6 does not rely on Assumption 1 and consequently the pair (λ^*, F_1^*) satisfy equation (16). Observe that part (i) of Lemma 8 (characterizing the Principal's best response) is still valid but the support of the distribution is not necessarily binary. However, the value of the function $Z(\lambda^*, \cdot)$ must still coincide with the line defining the convexification of $Z(\lambda^*, \cdot)$ around zero, denoted by L, at each point in the support of F_1^*. Therefore, the equilibrium (λ^*, F_1^*) satisfies the following two conditions:

$$\int su\big(w\big(\lambda^*, s\big)\big)\, dF_1^*(s) = c'\big(a^*\big) \quad \text{and} \qquad \text{(S12)}$$
$$Z\big(\lambda^*, s\big) = L(s) \quad \text{for all } s \in \operatorname{supp} F_1^*,$$

where the first equation implies that λ^* is a best response against F_1^* and the second equation says that F_1^* is a best response against λ^*. It is enough to construct an $F_2^* \in \mathcal{F}$ such that the first line of (S12) is satisfied, $\operatorname{supp} F_2^* \subset \operatorname{supp} F_1^*$, and $|\operatorname{supp} F_2^*| \in \{2, 3\}$. Indeed, such an F_2^* would satisfy both lines of (S12), so the pair (λ^*, F_2^*) would be an equilibrium.

By the proof of Lemma 7, $Z_{22}(\lambda^*, s) > 0$ whenever $s < s_*(\lambda^*)$. Therefore, the set of negative elements of the support of F_1^* is a singleton, that is, $\operatorname{supp} F_1^* \cap \mathbb{R}_- = \{\underline{s}\}$. Let $\overline{S}$ denote $\operatorname{supp} F_1^* \cap \mathbb{R}_+$. Using these notations, we can rewrite equation (16) as follows:

$$\begin{aligned} &F_1^*(\underline{s})\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \big(1 - F_1^*(\underline{s})\big) \int su\big(w\big(\lambda^*, \overline{s}\big)\big)\, dF_1^*(\overline{s}|\overline{s} > 0) \\ &\quad = \int \left[\frac{F_1^*(\underline{s})}{\big(1 - F_1^*(\underline{s})\big)} \underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \big(1 - F_1^*(\underline{s})\big)\overline{s}u\big(w\big(\lambda^*, \overline{s}\big)\big) \right] dF_1^*(\overline{s}|\overline{s} > 0) \\ &\quad = c'\big(a^*\big). \end{aligned}$$

Let us define $\rho(\overline{s}) = \overline{s}/(\overline{s} - \underline{s})$ and note that

$$\begin{aligned} \int_0^\infty \frac{\rho(\overline{s})}{1 - \rho(\overline{s})} \underline{s}\, dF(\overline{s}) + \int_0^\infty \overline{s}\, dF(\overline{s}) &= \int_0^\infty \left[\frac{\rho(\overline{s})}{1 - \rho(\overline{s})} \underline{s} + \overline{s} \right] dF(\overline{s}) \\ &= 0 = F(\underline{s})\underline{s} + \int_0^\infty \overline{s}\, dF(\overline{s}), \end{aligned}$$

where the second equality follows from $\rho(\overline{s})\underline{s} + (1 - \rho(\overline{s}))\overline{s} = 0$ and the third one from $F \in \mathcal{F}$, that is, $\mathbb{E}_{F_1^*}[s] = 0$. This equality chain implies that

$$F(\underline{s}) = \int_0^\infty \frac{\rho(\overline{s})}{1 - \rho(\overline{s})}\, dF(\overline{s}). \qquad \text{(S13)}$$

Therefore,

$$\begin{aligned} &F_1^*(\underline{s})\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \int_0^\infty \overline{s}u\big(w\big(\lambda^*, \overline{s}\big)\big)\, dF_1^*(\overline{s}) \\ &\quad = \int_0^\infty \frac{\rho(\overline{s})}{1 - \rho(\overline{s})} \underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \overline{s}u\big(w\big(\lambda^*, \overline{s}\big)\big)\, dF_1^*(\overline{s}) \\ &\quad = \int_0^\infty \frac{1}{1 - \rho(\overline{s})} \big[\rho(\overline{s})\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \big(1 - \rho(\overline{s})\big)\overline{s}u\big(w\big(\lambda^*, \overline{s}\big)\big)\big]\, dF_1^*(\overline{s}) \end{aligned}$$

$$= c'\big(a^*\big), \tag{S14}$$

where the first equality follows from (S13).

We now explain that $dG = [1/(1-\rho)]\,dF_1^*$ is a probability measure on $\mathbb{R}_+$. To see this, note that

$$\int_0^\infty \frac{1}{1-\rho(\bar{s})}\,dF_1^*(\bar{s}) = 1 - F_1^*(\underline{s}) + \int_0^\infty \left(\frac{1}{1-\rho(\bar{s})} - 1\right) dF_1^*(\bar{s})$$
$$= F_1^*(\underline{s}) + \int_0^\infty \frac{\rho(\bar{s})}{1-\rho(\bar{s})}\,dF_1^*(\bar{s}) = 1,$$

where the first equality follows from $F_1^*(\underline{s}) = 1 - \int_0^\infty 1\,dF_1^*(\bar{s})$ and the third one follows from equation (S13). Consequently, the last line of (S14) can be rewritten as

$$\int_0^\infty \big[\rho(\bar{s})\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \big(1-\rho(\bar{s})\big)\bar{s}u\big(w\big(\lambda^*, \bar{s}\big)\big)\big]\,dG(\bar{s}) = c'\big(a^*\big).$$

Suppose first that there exists an $\bar{s} \in \overline{S}$ such that $\rho(\bar{s})\underline{s}u(w(\lambda^*, \underline{s})) + (1-\rho(\bar{s}))\bar{s}u(w(\lambda^*, \bar{s})) = c'(a^*)$. Then the binary distribution placing probability masses $\rho(\bar{s})$ and $1-\rho(\bar{s})$ on the scores $\underline{s}$ and $\bar{s}$, respectively, satisfy the second line of (S12) and the proof is complete. Otherwise, the previous displayed equation implies that there must exist $\bar{s}_1, \bar{s}_2 \in \overline{S}$ such that

$$\rho(\bar{s}_1)\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \big(1-\rho(\bar{s}_1)\big)\bar{s}_1 u\big(w\big(\lambda^*, \bar{s}_1\big)\big) > c'\big(a^*\big),$$
$$\rho(\bar{s}_2)\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) + \big(1-\rho(\bar{s}_2)\big)\bar{s}_1 u\big(w\big(\lambda^*, \bar{s}_2\big)\big) < c'\big(a^*\big),$$

and hence, there must exist $\kappa \in (0, 1)$ such that

$$\begin{aligned}
&\big[\kappa\rho(\bar{s}_1) + (1-\kappa)\rho(\bar{s}_2)\big]\underline{s}u\big(w\big(\lambda^*, \underline{s}\big)\big) \\
&\quad + \kappa\big(1-\rho(\bar{s}_1)\big)\bar{s}_1 u\big(w\big(\lambda^*, \bar{s}_1\big)\big) + (1-\kappa)\big(1-\rho(\bar{s}_2)\big)\bar{s}_1 u\big(w\big(\lambda^*, \bar{s}_2\big)\big) \\
&= c'\big(a^*\big).
\end{aligned} \tag{S15}$$

Note that the pair (ρ, κ) defines a probability distribution over the points $(\underline{s}, \bar{s}_1, \bar{s}_2)$ such that

$$\Pr(\underline{s}) = \kappa\rho(\bar{s}_1) + (1-\kappa)\rho(\bar{s}_2), \qquad \Pr(\underline{s}_1) = \kappa\big(1-\rho(\bar{s}_1)\big), \qquad \Pr(\underline{s}_2) = (1-\kappa)\big(1-\rho(\bar{s}_2)\big),$$

and let $F_2^* \in \mathcal{F}$ denote the corresponding CDF. By (S15), the CDF F_2^* indeed satisfies the second line of (S12). *Q.E.D.*

PROOF OF THEOREM 3: To establish the result, we construct a sequence of binary distributions and corresponding wages that satisfy (IC) and (LL) so that the Principal's expected cost converges to $\underline{w}$. To this end, for each $n \in \mathbb{N}$, let us define $F_n \in \mathcal{F}$ as follows:

$$F_n(s) = \begin{cases} 0 & \text{if } s < -n^{-\zeta}, \\ \dfrac{n}{n + n^{-\zeta}} & \text{if } s \in [-n^{-\zeta}, n), \\ 1 & \text{if } s \geq n. \end{cases} \tag{S16}$$

Note that the support of F_n is $\{\underline{s}_n, \overline{s}_n\} = \{-n^{-\zeta}, n\}$. Furthermore, $\Pr(\underline{s}_n) = \overline{s}_n/(\overline{s}_n - \underline{s}_n)$ and $\Pr(\overline{s}_n) = -\underline{s}_n/(\overline{s}_n - \underline{s}_n)$. Next, we define a wage scheme for each n, so that the Agent's incentive constraint, (IC), binds. That is, $w(\underline{s}) = \underline{w}$ and $w(\overline{s}_n)$ satisfies

$$\left(1 + \frac{\underline{s}_n}{\overline{s}_n - \underline{s}_n}\right)\underline{s}_n u(\underline{w}) - \frac{\underline{s}_n}{\overline{s}_n - \underline{s}_n}\overline{s}_n u\big(w(\overline{s}_n)\big) = c'\big(a^*\big),$$

or equivalently,

$$w(\overline{s}_n) = u^{-1}\left(u(\underline{w}) - \frac{\overline{s}_n - \underline{s}_n}{\overline{s}_n \underline{s}_n} c'\big(a^*\big)\right). \tag{S17}$$

Since $w(\overline{s}_n) > \underline{w}$, the Agent's limited liability constraint, (LL), is satisfied.

The Principal's expected cost is

$$\frac{\overline{s}_n}{\overline{s}_n - \underline{s}_n}\big[\underline{w} + \underline{s}_n^2\big] - \frac{\underline{s}_n}{\overline{s}_n - \underline{s}_n}\big[w(\overline{s}_n) + \overline{s}_n^2\big] = \frac{\overline{s}_n}{\overline{s}_n - \underline{s}_n}\underline{w} - \frac{\underline{s}_n}{\overline{s}_n - \underline{s}_n} w(\overline{s}_n) + \frac{\overline{s}_n \underline{s}_n^2 - \underline{s}_n \overline{s}_n^2}{\overline{s}_n - \underline{s}_n}. \tag{S18}$$

Next, we show that this cost converges to $\underline{w}$ as n goes to infinity. First, note that the last term, corresponding to the Principal's cost of information acquisition, tends to zero because

$$\lim_{n\to\infty} \frac{\overline{s}_n \underline{s}_n^2 - \underline{s}_n \overline{s}_n^2}{\overline{s}_n - \underline{s}_n} = \lim_{n\to\infty} \frac{n^{1-2\zeta} + n^{2-\zeta}}{n + n^{-\zeta}} \le \lim_{n\to\infty} \frac{n^{1-2\zeta} + n^{2-\zeta}}{n} = \lim_{n\to\infty}\big(n^{-2\zeta} + n^{1-\zeta}\big) = 0, \tag{S19}$$

where the last equality follows from $\zeta > 1$.

It remains to show that the expected wage cost of the Principal converges to $\underline{w}$. First, we show that the first term on the right-hand side of (S18) goes to $\underline{w}$. Note that

$$\lim_{n\to\infty} \frac{\overline{s}_n}{\overline{s}_n - \underline{s}_n}\underline{w} = \lim_{n\to\infty} \frac{n}{n + n^{-\zeta}}\underline{w} = \underline{w}.$$

In what follows, we show that the second term of the right-hand side of (S18) converges to zero. We do this by sandwiching this term between two sequences and showing that both of these sequences go to zero. To this end, note that

$$\begin{aligned} -\frac{\underline{s}_n}{\overline{s}_n - \underline{s}_n}\underline{w} \le -\frac{\underline{s}_n}{\overline{s}_n - \underline{s}_n} w(\overline{s}_n) &= \frac{n^{-\zeta}}{n + n^{-\zeta}} u^{-1}\left(u(\underline{w}) + \frac{n + n^{-\zeta}}{n^{1-\zeta}} c'\big(a^*\big)\right) \\ &\le \frac{n^{-\zeta}}{n + n^{-\zeta}} u^{-1}\Big(u(\underline{w}) + \big(1 + n^{\zeta}\big)c'\big(a^*\big)\Big), \end{aligned}$$

where the first inequality follows from $\underline{w} \le w(\overline{s}_n)$, the equality follows from (S17), and the second inequality from $1/n \le 1$. Since, $\lim_{n\to\infty}[-\underline{s}_n/(\overline{s}_n - \underline{s}_n)] = \lim_{n\to\infty}[n^{-\zeta}/(n + n^{\zeta})] = 0$, it is enough to show that the right-hand side also converges to zero. That is, by letting v denote $u(\underline{w}) + c'(a^*)$, we have to show that

$$\lim_{n\to\infty} \frac{u^{-1}\big(v + n^{\zeta} c'\big(a^*\big)\big)}{n^{\zeta+1} + 1} = 0.$$

Observe that the denominator goes to infinity, hence, if u is bounded and the numerator does not go to infinity, this result follows. If u is unbounded, then the numerator also goes to infinity and, applying L'Hospital's rule, we have that

$$\lim_{n\to\infty}\frac{u^{-1}\big(v+n^{\zeta}c'(a^*)\big)}{n^{\zeta+1}+1}=\lim_{n\to\infty}\frac{\dfrac{\zeta n^{\zeta-1}c'(a^*)}{u'\big(u^{-1}\big(v+n^{\zeta}c'(a^*)\big)\big)}}{(\zeta+1)n^{\zeta}}\le c'(a)\lim_{n\to\infty}\frac{\dfrac{1}{u'\big(u^{-1}\big(v+n^{\zeta}c'(a^*)\big)\big)}}{n},$$

where the inequality follows from $\zeta/[n(\zeta+1)]<1$. Since $\lim_{n\to\infty}u^{-1}(v+n^{\zeta}c'(a^*))=\infty$ by supposition and $\lim_{w\to\infty}u'(w)=0$ by assumption, both the numerator and the denominator of the right-hand-side term above go to infinity. Applying L'Hospital's rule again, we have that

$$c'(a)\lim_{n\to\infty}\frac{\dfrac{1}{u'\big(u^{-1}\big(v+n^{\zeta}c'(a^*)\big)\big)}}{n}=\zeta\big[c'(a^*)\big]^2\lim_{n\to\infty}\frac{-u''\big(u^{-1}\big(v+n^{\zeta}c'(a^*)\big)\big)}{\big[u'\big(u^{-1}\big(v+n^{\zeta}c'(a^*)\big)\big)\big]^3}n^{\zeta-1}.$$

Letting $w=u^{-1}(v+n^{\zeta}c'(a^*))$, the last expression can be rewritten as

$$\zeta\big[c'(a^*)\big]^2\lim_{w\to\infty}\frac{-u''(w)}{\big[u'(w)\big]^3}\left[\frac{u(w)-v}{c'(a^*)}\right]^{\frac{\zeta-1}{\zeta}}\le\zeta\big[c'(a^*)\big]^{\frac{\zeta+1}{\zeta}}\lim_{w\to\infty}\frac{-u''(w)}{\big[u'(w)\big]^3}\big[u(w)\big]^{\frac{\zeta-1}{\zeta}}=0,$$

where the inequality follows because $u(w)>v$ and $\zeta>1$ and the equality follows from (26). *Q.E.D.*

Proofs Related to Proposition 1

PROOF OF LEMMA 12: Since $c''_{\kappa}(a)/c'_{\kappa}(a)$ is increasing in κ, its derivative in κ is positive, that is,

$$\frac{c'_{\kappa}(a)\dfrac{\partial c''_{\kappa}(a)}{\partial\kappa}-c''_{\kappa}(a)\dfrac{\partial c'_{\kappa}(a)}{\partial\kappa}}{\big[c'_{\kappa}(a)\big]^2}\ge 0.$$

Note that the left-hand side of the previous inequality is also the derivative of $[\partial c'_{\kappa}(a)/\partial\kappa]/c'_{\kappa}(a)$ in a and hence, this function is increasing in a. That is, for all $a>\widetilde{a}$,

$$\frac{\big[\partial c'_{\kappa}(a)/\partial\kappa\big]}{c'_{\kappa}(a)}\ge\frac{\big[\partial c'_{\kappa}(\widetilde{a})/\partial\kappa\big]}{c'_{\kappa}(\widetilde{a})},$$

which is equivalent to

$$\frac{\big[\partial c'_{\kappa}(a)/\partial\kappa\big]c'_{\kappa}(\widetilde{a})-\big[\partial c'_{\kappa}(\widetilde{a})/\partial\kappa\big]c'_{\kappa}(a)}{\big[c'_{\kappa}(a)\big]^2}\ge 0.$$

Note that the left-hand side is $\partial[\frac{c'_{\kappa}(a)}{c'_{\kappa}(\widetilde{a})}]/\partial\kappa$ and hence, the previous inequality implies that $c'_{\kappa}(a)/c'_{\kappa}(\widetilde{a})$ is increasing in κ.

Finally, observe that

$$\lim_{\kappa\to\infty}\frac{c'_{\kappa}(a)}{c'_{\kappa}(\widetilde{a})}=\lim_{\kappa\to\infty}\frac{\displaystyle\int_{\widetilde{a}}^{a}c''_{\kappa}(x)\,dx}{c'_{\kappa}(\widetilde{a})}\ge\lim_{\kappa\to\infty}\int_{\widetilde{a}}^{a}\frac{c''_{\kappa}(x)}{c'_{\kappa}(x)}\,dx=\infty,$$

where the first equality follows from the fundamental theorem of calculus, the inequality follows from c'_κ being increasing in a, and the last equality is implied by Condition 1 and Lebesgue's monotone convergence theorem. *Q.E.D.*

We now establish two additional lemmas.

LEMMA S1: *Suppose that the Agent's utility function, u, satisfies Assumption* 1. *Let $\{\lambda^*, F^*\}$ denote the unique equilibrium characterized in Theorem* 1, *and let $\delta^* := c'(a^*)$. Then λ^*, and the* (*two*) *scores in the support of F^*, $\underline{s}$ and $\overline{s}$, are continuously differentiable in δ^*, and*

$$\frac{d\lambda^*}{d\delta^*} > 0, \qquad \frac{d\underline{s}'}{d\delta^*} < 0 \quad \textit{and} \quad \frac{d\overline{s}'}{d\delta^*} > 0.$$

Moreover, both the Agent's bonus, $w(\lambda^*, \overline{s}) - \underline{w}$, and the Principal's expected information-acquisition cost, $\mathbb{E}_{F^*}[s^2]$, are strictly increasing in δ^*.

PROOF OF LEMMA S1: It follows from Sections 5.1.1 and 5.1.2 that λ^*, $\overline{s}$, and $\underline{s}$ satisfy the following equations:

$$\begin{aligned} Z_2(\lambda^*, \overline{s}) &= Z_2(\lambda^*, \underline{s}), \\ Z(\lambda^*, \overline{s}) &= Z(\lambda^*, \underline{s}) + (\overline{s} - \underline{s}) Z_2(\lambda^*, \underline{s}), \\ \mathbb{E}_{F^*}\big[su\big(w(\lambda^*, s)\big)\big] &= \delta^*. \end{aligned}$$

The first two equations specify that the points $(\underline{s}, Z(\lambda^*, \underline{s}))$ and $(\overline{s}, Z(\lambda^*, \overline{s}))$ lie on a line that is tangent to $Z(\lambda^*, \cdot)$ at $s \in \{\underline{s}, \overline{s}\}$, as mandated by (21) and Lemma 8. The third equation stipulates that the Agent's incentive constraint must be satisfied with equality per Lemma 6. Moreover, we must have $Z_{22}(\lambda^*, \overline{s}) = (\lambda^*)^2[(u')^3/u''] + 2 > 0$, where u' and u'' are evaluated at $w(\lambda^*, \overline{s})$. Theorem 1 guarantees the uniqueness of λ^*, $\overline{s}$, and $\underline{s}$ when Assumption 1 is satisfied.

Using the definition of $Z(\lambda^*, \cdot)$ and that $\mathbb{E}_{F^*}[s] = 0$, these equations can be rewritten as

$$\begin{aligned} \lambda^*\big[u\big(w(\lambda^*, \overline{s})\big) - u(\underline{w})\big] &= 2(\overline{s} - \underline{s}), \\ w(\lambda^*, \overline{s}) - \underline{w} + (\overline{s} - \underline{s})^2 &= \lambda^* \overline{s}\big[u\big(w(\lambda^*, \overline{s})\big) - u(\underline{w})\big], \\ -\frac{\overline{s}\underline{s}}{\overline{s} - \underline{s}}\big[u\big(w(\lambda^*, \overline{s})\big) - u(\underline{w})\big] &= \delta^*, \end{aligned} \tag{S20}$$

where $\lambda^* \overline{s} u'(w(\lambda^*, \overline{s}))) = 1$. Using (S20), the second and third equations can be rewritten as

$$\lambda^* \delta^* = -2\overline{s}\underline{s} \quad \text{and} \tag{S21}$$

$$w(\lambda^*, \overline{s}) - \underline{w} = \overline{s}^2 - \underline{s}^2, \tag{S22}$$

respectively. Therefore, an equilibrium to the zero-sum game is fully characterized by a three-tuple $\{\lambda^*, \underline{s}, \overline{s}\}$ that satisfies (S20)–(S22). Define

$$f_1(\overline{s}, \underline{s}, \lambda) := \lambda\big[u\big(w(\lambda, \overline{s})\big) - u(\underline{w})\big] - 2(\overline{s} - \underline{s}),$$

$$f_2(\overline{s}, \underline{s}, \lambda) := \lambda\delta^* + 2\overline{s}\underline{s},$$
$$f_3(\overline{s}, \underline{s}, \lambda) := w(\lambda, \overline{s}) - \underline{w} - (\overline{s}^2 - \underline{s}^2).$$

One can show using straightforward algebra that the determinant of the Jacobian matrix

$$J := \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \overline{s}} & \frac{\partial f_1}{\partial \underline{s}} & \frac{\partial f_1}{\partial \lambda} \\ \frac{\partial f_2}{\partial \overline{s}} & \frac{\partial f_2}{\partial \underline{s}} & \frac{\partial f_2}{\partial \lambda} \\ \frac{\partial f_3}{\partial \overline{s}} & \frac{\partial f_3}{\partial \underline{s}} & \frac{\partial f_3}{\partial \lambda} \end{bmatrix}$$

evaluated at a three-tuple $\{\lambda^*, \underline{s}, \overline{s}\}$ that satisfies (S20)–(S22) is strictly positive, which implies that J is invertible, and so we can apply the implicit function theorem. Then its inverse can be written as

$$J^{-1} = \frac{1}{\det(J)} \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix},$$

where $j_{kl}/\det(J)$ denotes the entry of row k and column l of the inverse of matrix J. By the implicit function theorem, we have that

$$\begin{bmatrix} \frac{d\overline{s}}{d\delta^*} \\ \frac{d\underline{s}}{d\delta^*} \\ \frac{d\lambda^*}{d\delta^*} \end{bmatrix} = -J^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial \delta^*} \\ \frac{\partial f_2}{\partial \delta^*} \\ \frac{\partial f_3}{\partial \delta^*} \end{bmatrix} = -\frac{1}{\det(J)} \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} \begin{bmatrix} 0 \\ \lambda^* \\ 0 \end{bmatrix} = \underbrace{-\frac{\lambda^*}{\det(J)}}_{<0} \begin{bmatrix} j_{12} \\ j_{22} \\ j_{32} \end{bmatrix}.$$

Therefore, to establish (i), it suffices to show that $j_{12} < 0$, $j_{22} > 0$, and $j_{32} < 0$, which is easy to show using the facts that $u'' < 0$, $\underline{s} < 0$, and $Z_{22}(\lambda^*, \overline{s}) > 0$.

Finally, (ii) follows immediately by observing that

$$\frac{dw(\lambda^*, \overline{s})}{d\delta^*} = \frac{dw(\lambda^*, \overline{s})}{\partial \lambda^*}\frac{d\lambda^*}{d\delta^*} + \frac{dw(\lambda^*, \overline{s})}{\partial \overline{s}}\frac{d\overline{s}}{d\delta^*} = -\frac{u'}{\lambda^* u''}\frac{d\lambda^*}{d\delta^*} - \frac{u'}{\overline{s}u''}\frac{d\overline{s}}{d\delta^*} > 0 \quad \text{and}$$
$$\frac{d}{d\delta^*}\mathbb{E}_{F^*}[s^2] = \frac{d(-\overline{s}\underline{s})}{d\delta^*} = -\overline{s}\frac{d\underline{s}}{d\delta^*} - \underline{s}\frac{d\overline{s}}{d\delta^*} > 0.$$

Q.E.D.

Notice that the fraction $p_1(a, \delta)/p_1(a^*, \delta)$ is not defined for $\delta = 0$. The following lemma shows that this function can be continuously extended to the compact interval $[0, \overline{d}]$ by showing that the limit of this fraction exists and it is not zero as δ converges to zero.

LEMMA S2: *For all* $a, a^* \in \mathbb{R}_+$, $\lim_{d\to 0} p_1(a, d)/p_1(a^*, d) > 0$.

PROOF OF LEMMA S2: Using L'Hospital's rule, it is easy to show that

$$\frac{p_1(a, \delta)}{p_1(a^*, \delta)} = 2\left(-\frac{\overline{s} - \underline{s}}{\underline{s}\overline{s}}\right)\frac{\overline{s}e^{-2(a-a^*)\overline{s}} - \underline{s}e^{-2(a-a^*)\underline{s}} - (\overline{s} - \underline{s})e^{-2(a-a^*)(\overline{s}+\underline{s})}}{\left[e^{-2(a-a^*)\underline{s}} - e^{-2(a-a^*)\overline{s}}\right]^2}, \tag{S23}$$

where we drop the dependence of $\overline{s}$ and $\underline{s}$ for notational convenience. By Lemma S1, $\overline{s}$ increases in δ, while $\underline{s}$ decreases in δ. Let $s_H := \inf_{\delta\in(0,1]}\{\overline{s}(\delta)\}$ and $s_L := \sup_{\delta\in(0,1]}\{\underline{s}(\delta)\}$. It follows from the fact that $\overline{s} > 0 > \underline{s}$ and the monotone convergence theorem that $\lim_{\delta\to 0}\overline{s} = s_H$ and $\lim_{\delta\to 0}\underline{s} = s_L$. If $s_H > 0 > s_L$, then because both $p_1(a,\delta) > 0$ and $p_1(a^*,\delta) > 0$, the desired conclusion holds. We consider three cases.

First, suppose that $s_H > 0$ and $s_L = 0$. Applying L'Hospital's rule once, we have

$$\lim_{\underline{s}\to 0}\frac{p_1(a,\delta)}{p_1(a^*,\delta)} = 2\frac{1-e^{-2(a-a^*)\overline{s}} - 2(a-a^*)\overline{s}e^{-2(a-a^*)\overline{s}}}{\left[1-e^{-2(a-a^*)\overline{s}}\right]^2} > 0.$$

Next, suppose that $s_H = 0$ and $s_L < 0$. Applying L'Hospital's rule once, we have

$$\lim_{\overline{s}\to 0}\frac{p_1(a,\delta)}{p_1(a^*,\delta)} = 2\frac{1-e^{-2(a-a^*)\underline{s}} - 2(a-a^*)\underline{s}e^{-2(a-a^*)\underline{s}}}{\left[1-e^{-2(a-a^*)\overline{s}}\right]^2} > 0.$$

Finally, we rule out the possibility that $s_H = s_L = 0$. Note that it is enough to show that $\overline{s}+\underline{s}$ cannot converge to zero as δ goes to zero. Recall from the proof of Lemma S1 that for every $\delta > 0$, there exists an equilibrium $\{\lambda^*, \overline{s}, \underline{s}\}$, which is characterized by a solution to the following system of equations:

$$\lambda^*\left[u\left(w\left(\lambda^*,\overline{s}\right)\right) - u(\underline{w})\right] = 2(\overline{s}-\underline{s}), \tag{S24}$$

$$\lambda^* d = -2\overline{s}\underline{s}, \tag{S25}$$

$$w\left(\lambda^*,\overline{s}\right) - \underline{w} = \overline{s}^2 - \underline{s}^2, \tag{S26}$$

where $\lambda^*\overline{s}u'(w(\lambda^*,\overline{s}))) = 1$ and $\lambda^* > 0$. Observe that

$$\lim_{\delta\to 0}\frac{2}{\overline{s}+\underline{s}} = \lim_{\delta\to 0}\lambda^*\frac{u\left(w\left(\lambda^*,\overline{s}\right)\right) - u(\underline{w})}{w\left(\lambda^*,\overline{s}\right) - \underline{w}} \le \lim_{\delta\to 0}\lambda^* u'(\underline{w}),$$

where the equality follows from dividing both sides of (S24) by the corresponding sides of (S26) and the inequality follows from the concavity of u. Note that since, by Lemma S1, λ^* is increasing in δ and $u'(\underline{w}) < \infty$, the right-hand side of the previous inequality chain is bounded from above. Consequently, the left-hand side must also be bounded from above and hence, $\lim_{\delta\to\infty}(\overline{s}+\underline{s}) \neq 0$. *Q.E.D.*

Finally, we are in a position to prove Proposition 1.

PROOF OF PROPOSITION 1: First, we show that if $a^* \le \overline{a}$, the Agent's cost function is c_κ, and the Principal's contract is defined by the scores $\{\underline{s}(\delta^*_\kappa), \overline{s}(\delta^*_\kappa)\}$ and wages $\{\underline{w}, \widetilde{W}(\overline{s}(\delta^*_\kappa))\}$, where $\delta^*_\kappa = c'_\kappa(a^*)$, then there exists $\widehat{a}(>\overline{a})$ such that the Agent is better off exerting effort zero than any $a > \widehat{a}$. Importantly, $\widehat{a}$ depends neither on κ nor on a^*. Observe that a consequence of the existence of such an $\widehat{a}$ is that if the incentive constraint is satisfied at zero, then it is also satisfied at any effort level above $\widehat{a}$. To this end, let $\widehat{a}$ be defined as follows:

$$\widehat{a} = \overline{a} + \frac{\left[u\left(\widetilde{W}\left(\overline{s}(\overline{d})\right)\right) - u(\underline{w})\right]}{\underline{d}}.$$

Observe that for all $a > \widehat{a}$,

$$c_\kappa(a) \geq c_\kappa(\overline{a}) + c_\kappa'(\overline{a})(a-\overline{a}) \geq c_\kappa(\overline{a}) + \underline{d}(a-\overline{a}) > \underline{d}(a-\overline{a}), \tag{S27}$$

where the first inequality follows from the convexity of c_κ, the second one from Condition 2, and the third one from $c_\kappa(\overline{a}) > 0$. Note that the Agent's payoff gain from exerting effort $a(>\overline{a})$ instead of zero is

$$u\big(\widetilde{W}\big(\overline{s}\big(\delta_\kappa^*\big)\big)\big) - u(\underline{w}) - c_\kappa(a),$$

because the increase in the probability of getting wage $\widetilde{W}(\overline{s}(\delta_\kappa^*))$ instead of $\underline{w}$ is bounded from above by 1. It is enough to show that this payoff gain is negative. Observe that for all $a > \widehat{a}$,

$$\begin{aligned}
&u\big(\widetilde{W}\big(\overline{s}\big(\delta_\kappa^*\big)\big)\big) - u(\underline{w}) - c_\kappa(a) \\
&\quad \leq \sup_{\delta^* \in (0,\overline{d}]} u\big(\widetilde{W}\big(\overline{s}\big(\delta^*\big)\big)\big) - u(\underline{w}) - c_\kappa(a) \\
&\quad = u\big(\widetilde{W}\big(\overline{s}(\overline{d})\big)\big) - u(\underline{w}) - c_\kappa(a) < u\big(\widetilde{W}\big(\overline{s}(\overline{d})\big)\big) - u(\underline{w}) - \underline{d}(a-\overline{a}) < 0,
\end{aligned}$$

where the first equality follows Lemma S1, the second inequality follows from equation (S27), and the last inequality follows from the definition of $\widehat{a}$.

Next, let us define $p_1(a,0)/p_1(a^*,0)$ to be $\lim_{\delta\to 0} p_1(a,\delta)/p_1(a^*,\delta)$ for each $a^* \in (0,\overline{a}]$ and $a \in \mathbb{R}_+$. We show that for each $(a^*, a) \in [\underline{a},\overline{a}] \times [0,\infty)$, there exists K such that whenever $\kappa > K$,

$$\begin{aligned}
&\inf_{\delta\in[0,\overline{d}]} \frac{p_1(a,\delta)}{p_1\big(a^*,\delta\big)} > \frac{c_\kappa'(a)}{c_\kappa'\big(a^*\big)} \quad \text{if } a < a^* \quad \text{and} \\
&\sup_{\delta\in[0,\overline{d}]} \frac{p_1(a,\delta)}{p_1\big(a^*,\delta\big)} < \frac{c_\kappa'(a)}{c_\kappa'\big(a^*\big)} \quad \text{if } a^* < a \leq \widehat{a}.
\end{aligned} \tag{S28}$$

Observe that $p'(a,\delta)/p'(a^*,\delta)$ is strictly positive for all $a, a^* \in \mathbb{R}_+$ and $\delta \in (0,\overline{d}]$. Therefore, by Lemma S2 and the theorem of the maximum, the two terms on the left-hand sides of the inequalities in (S28) are continuous in a and strictly positive. In what follows, we show that for each pair $(a^*, a) \in [\underline{a},\overline{a}] \times [0,\widehat{a}]$, there exists a $K_{a^*}^a$ and an open neighborhood of (a^*, a), $N_{a^*}^a$, such that (S28) is satisfied for any $\kappa \geq K_{a^*}^a$ and for each pair in $N_{a^*}^a$.

First, fix an effort level $a < a^*$. Since $\inf_{\delta\in[0,\overline{d}]}[p_1(a,\delta)/p_1(a^*,\delta)] > 0$, Lemma 12 implies that there exists $K_{a^*}^a \in \mathbb{R}_+$ such that

$$\inf_{\delta\in[0,\overline{d}]} \frac{p_1(a,\delta)}{p_1\big(a^*,\delta\big)} > \frac{c_{K_{a^*}^a}'(a)}{c_{K_{a^*}^a}'\big(a^*\big)}.$$

Furthermore, by the continuity of $c_{K_a^a}'$ and $\inf_{\delta\in[0,\overline{d}]}[p_1(a,\delta)/p_1(a^*,\delta)]$, there exists a neighborhood around (a^*, a), $N_{a^*}^a$, such that the previous inequality is satisfied for all $(\widetilde{a}^*, \widetilde{a}) \in N_{a^*}^a$. Since $c_\kappa'(\widetilde{a})/c_\kappa'(a^*)$ is decreasing in k by Lemma 12, it follows that for all $(\widetilde{a}^*, \widetilde{a}) \in N_{a^*}^a$ and $\kappa \geq K_{a^*}^a$,

$$\inf_{\delta\in[0,\overline{d}]} \frac{p_1(\tilde{a},\delta)}{p_1\big(\widetilde{a}^*,\delta\big)} > \frac{c_\kappa'(\tilde{a})}{c_\kappa'\big(\widetilde{a}^*\big)}. \tag{S29}$$

Second, fix an effort level $a > a^*$. Again, Lemma 12 implies the existence of $K^a_{a^*} \in \mathbb{R}_+$ such that

$$\sup_{\delta\in[0,\overline{d}]} \frac{p_1(a,\delta)}{p_1(a^*,\delta)} < \frac{c'_{K^a_{a^*}}(a)}{c'_{K^a_{a^*}}(a^*)}.$$

Furthermore, by the continuity of $c'_{K^a_{a^*}}$ and $\sup_{\delta\in(0,\overline{d}]}[p_1(a,\delta)/p_1(a^*,\delta)]$, there exists an open neighborhood of (a^*,a), $N^a_{a^*}$, such that the previous inequality is satisfied for all $(\widetilde{a}^*,\widetilde{a}) \in N^a_{a^*}$. Since $c'_\kappa(\widetilde{a})/c'_\kappa(a^*)$ is increasing in κ by Lemma 12, it follows that for all $(\widetilde{a}^*,\widetilde{a}) \in N^a_{a^*}$ and $\kappa \geq K^a_{a^*}$,

$$\sup_{\delta\in[0,\overline{d}]} \frac{p_1(\tilde{a},\delta)}{p_1(\widetilde{a}^*,\delta)} < \frac{c'_\kappa(\tilde{a})}{c'_\kappa(\widetilde{a}^*)}. \tag{S30}$$

Now, consider $a = a^*$. We first show that there exists an open neighborhood of (a^*,a^*), $N^{a^*}_{a^*}$, and $K^{a^*}_{a^*} \in \mathbb{R}_+$, such that if $(\widetilde{a}^*,\widetilde{a}) \in N^{a^*}_{a^*}$ then

$$\begin{aligned} \inf_{\delta\in[0,\overline{d}]} \frac{p_1(\tilde{a},\delta)}{p_1(\widetilde{a}^*,\delta)} &> \frac{c'_{K^{a^*}_{a^*}}(\widetilde{a})}{c'_{K^{a^*}_{a^*}}(\widetilde{a}^*)} \quad \text{if } \widetilde{a} < \widetilde{a}^*, \\ \sup_{\delta\in[0,\overline{d}]} \frac{p_1(\tilde{a},\delta)}{p_1(\widetilde{a}^*,\delta)} &< \frac{c'_{K^{a^*}_{a^*}}(\widetilde{a})}{c'_{K^{a^*}_{a^*}}(\widetilde{a}^*)} \quad \text{if } \widetilde{a} > \widetilde{a}^*. \end{aligned} \tag{S31}$$

We note that $p_1(a,\delta)/p_1(a^*,\delta)$ is continuously differentiable in a and the derivative is continuous in δ and a^*. Therefore, there exists an open ball around (a^*,a^*), N_1, such that $p_{11}(\widetilde{a},\delta)/p_1(\widetilde{a}^*,\delta) < B$ for all $(\widetilde{a}^*,\widetilde{a}) \in N_1$ and $\delta \in [0,\overline{d}]$. This implies that for all $(\widetilde{a}^*,\widetilde{a}) \in N_1$ and $\delta \in [0,\overline{d}]$,

$$\begin{aligned} \frac{p_1(\widetilde{a},\delta)}{p_1(\widetilde{a}^*,\delta)} &> 1 - B(\widetilde{a}^* - \widetilde{a}) \quad \text{if } \widetilde{a} < \widetilde{a}^* \quad \text{and} \\ \frac{p_1(\widetilde{a},\delta)}{p_1(\widetilde{a}^*,\delta)} &< 1 + B(\widetilde{a} - \widetilde{a}^*) \quad \text{if } \widetilde{a} > \widetilde{a}^*. \end{aligned} \tag{S32}$$

By Condition 1, there exist $K^{a^*}_{a^*} \in \mathbb{R}_+$ and an $\varepsilon > 0$ such that $c''_{K_{a^*}}(\widetilde{a}^*)/c'_{K_{a^*}}(\widetilde{a}^*) > 2B$ for all $\widetilde{a}^* \in (a^* - \varepsilon, a^* + \varepsilon)$. Hence, by the continuity of $c''_{K^{a^*}_{a^*}}$, there is an open neighborhood of (a^*,a^*), N_2, such that for all $(\widetilde{a}^*,\widetilde{a}) \in N_2$,

$$\begin{aligned} \frac{c'_{K^{a^*}_{a^*}}(\widetilde{a})}{c'_{K^{a^*}_{a^*}}(\widetilde{a}^*)} &< 1 - B(\widetilde{a}^* - \widetilde{a}) \quad \text{if } \widetilde{a} < \widetilde{a}^* \quad \text{and} \\ \frac{c'_{K^{a^*}_{a^*}}(\widetilde{a})}{c'_{K^{a^*}_{a^*}}(\widetilde{a}^*)} &> 1 + B(\widetilde{a} - \widetilde{a}^*) \quad \text{if } \widetilde{a} > \widetilde{a}^*. \end{aligned} \tag{S33}$$

Observe that equations (S32) and (S33) imply equation (S31) for all $(\widetilde{a}^*, \widetilde{a}) \in N_1 \cap N_2 =: N_{a^*}^{a^*}$. Finally, by Lemma 12, it follows that for all $(\widetilde{a}^*, \widetilde{a}) \in N_{a^*}^{a^*}$ and $\kappa \geq K_{a^*}^{a^*}$,

$$\begin{aligned} &\inf_{\delta \in [0,\overline{d}]} \frac{p_1(\tilde{a}, \delta)}{p_1(\widetilde{a}^*, \delta)} > \frac{c'_\kappa(\widetilde{a})}{c'_\kappa(\widetilde{a}^*)} \quad \text{if } \widetilde{a} < a^*, \\ &\sup_{\delta \in [0,\overline{d}]} \frac{p_1(\tilde{a}, \delta)}{p_1(\widetilde{a}^*, \delta)} < \frac{c'_\kappa(\widetilde{a})}{c'_\kappa(\widetilde{a}^*)} \quad \text{if } \widetilde{a} > a^*. \end{aligned} \tag{S34}$$

Since the set $[\underline{a}, \overline{a}] \times [0, \widehat{a}]$ is compact and $N_{a^*}^{a}$ is open for all $(a^*, a) \in [\underline{a}, \overline{a}] \times [0, \widehat{a}]$, there exist finitely many points, $\{(a_j^*, a_j)\}_1^m \subset [\underline{a}, \overline{a}] \times [0, \widehat{a}]$, such that

$$[\underline{a}, \overline{a}] \times [0, \widehat{a}] = \bigcup_{j \in \{1, \ldots, m\}} N_{a_j^*}^{a_j}. \tag{S35}$$

Now, let us define

$$K = \max\{K_{a_1^*}^{a_1}, \ldots, K_{a_m^*}^{a_m}\}, \tag{S36}$$

and let us consider c_κ such that $\kappa > K$. We show that c_k satisfies (30) whenever $a \leq \widehat{a}$. By (S35), for each $(a^*, a) \in [\underline{a}, \overline{a}] \times [0, \widehat{a}]$, there is $j \in \{1, \ldots, m\}$ such that $(a^*, a) \in N_{a_j^*}^{a_j}$. If $a_j < a_j^*$, then (a^*, a) satisfies (S29) because $\kappa > K \geq K_{a_j^*}^{a_j}$ by (S36). If $a_j = a_j^*$, then (a^*, a) satisfies (S34) because $\kappa > K \geq K_{a_j^*}^{a_j}$. If $a_j > a^*$, then (a^*, a) satisfies (S30) because $\kappa > K \geq K_{a_j^*}^{a_j}$ by (S36). *Q.E.D.*

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