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# SUPPLEMENT TO "VALUE OF PERSISTENT INFORMATION"

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IN THIS SUPPLEMENT, we provide detailed accounts of the comparative statics result for non-zero-sum games mentioned in the Introduction and of the extensions discussed in Section 5 of the main text. We also provide some counterexamples and discuss the relationship of Theorem 1 to the results in Renault and Venel (2017).

Section S.1 contains the application to non-zero-sum games. The sections that follow consider imperfect public monitoring of actions (Section S.2), games with public information about the state (Section S.3), and comparison of operators for a fixed discount factor (Section S.4). In Section S.5, we show that the polynomial characterization implied by Corollary 2 does not extend to operators with complex eigenvalues. And in Section S.6, we show that $P \preceq Q$ does not necessarily imply a ranking of the minimizer's information under the corresponding optimal information structures. We conclude in Section S.7 by discussing the connection to Renault and Venel (2017).

We use the same notation as in the main text. All numbered objects such as Sections, Theorems, and equations are numbered S.1, S.2, etc. in this supplement. Numbers without the prefix S refer to those in the main text.

## S.1. APPLICATION TO REPEATED NON-ZERO-SUM GAMES

The partial order $\preceq$ can be used to characterize how the limit equilibrium payoff set depends on the persistence of types in a class of repeated non-zero-sum games.

Specifically, let $u : C \times \Theta \to \mathbb{R}^2$ be a two-player game of incomplete information, where $C = \times_{i=1}^{2} C_i$ is the finite set of action profiles and $\Theta = \times_{i=1}^{2} \Theta_i$ is the finite set of type profiles. We assume that $u$ has private values, that is, that $u_i(c, \theta) = u_i(c, \theta_i)$ for all $(c, \theta) \in C \times \Theta$, $i = 1, 2$.[1]

A repeated Bayesian game with Markovian types consists of the infinite repetition of a private-value game $u$, with the players' types following independent Markov chains governed by ergodic stochastic operators $P_i : \Delta\Theta_i \to \Delta\Theta_i$, $i = 1, 2$. (That is, we have "independent private values.") In each period, each player $i$ first privately observes his current type $\theta_i \in \Theta_i$ and then chooses an action $c_i \in C_i$ simultaneously with the other player. Both players observe the realized action profile $c \in C$. Payoffs are given by the discounted average stage-game payoffs $(1 - \delta) \sum_{t=0}^{\infty} \delta^t u(c_t, \theta_t)$. Following the literature, we allow unmediated communication by augmenting the game with a round of simultaneous cheap

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[1]The restriction to two players ensures that the minmax value defined by (S.1) is indeed the value of a zero-sum game. With more than two players, the actions of players other than $i$ should be chosen independently. Handling this requires additional analysis, which we leave for future investigations.

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talk in each period after the players have learned their types, but before they choose actions.

Let $P = (P_1, P_2)$, and let $\pi_P = (\pi_{P_1}, \pi_{P_2}) \in \Delta\Theta_1 \times \Delta\Theta_2$ denote the stationary distribution of types. Consider an auxiliary static mechanism design problem with quasilinear transfers where feasible allocations are given by $C$, allocation utilities are given by $u$, and the type distribution is $\pi_P$. Let $W(u, P) \in \mathbb{R}^2$ denote the set of incentive-feasible payoffs, defined as the set of expected allocation utility vectors (i.e., expected payoff profiles net of transfer payments) that can be implemented in a Bayesian Nash equilibrium with some quasilinear transfers.[2] Note that, by definition, $W(u, P)$ depends on $P$ only through $\pi_P$.

Define player $i$'s limit minmax value as

$$\underline{w}_i = \lim_{\delta\to 1} \min_{\sigma_{-i}} \max_{\sigma_i} \mathcal{E}_\sigma \left[ (1-\delta) \sum_{t=0}^{\infty} \delta^t u_i(c_t, \theta_{i,t}) \right], \tag{S.1}$$

where the expectation is with respect to the probability measure over histories induced by the strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ and the type process. Let

$$W^*(u, P) = \{ w \in W(u, P) : w_i \geq \underline{w}_i \text{ for all } i \}.$$

Hörner, Takahashi, and Vieille (2015), Theorem 4 showed that if the interior of $W^*(u, P)$ is nonempty and an additional non-degeneracy assumption (their Assumption A1 on page 1815) is satisfied, then the sets of Bayes-Nash and sequential equilibrium payoffs converge to $W^*(u, P)$ as $\delta \to 1$. Given operators $P = (P_1, P_2)$, let $U(P)$ denote the set of finite stage games $u$ with private values such that the nonempty-interior and non-degeneracy assumptions are satisfied. Then $W^*(u, P)$ is the limit equilibrium payoff set given any $P$ and $u \in U(P)$, and we have the following comparative statics:

PROPOSITION S.1: *For any pairs of ergodic operators $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ on $\Delta\Theta_1 \times \Delta\Theta_2$, we have $W^*(u, Q) \subseteq W^*(u, P)$ for every stage game $u \in U(P) \cap U(Q)$ if and only if $P_i \preceq Q_i$ for $i = 1, 2$.*

That is, the limit equilibrium payoff set increases (in the sense of set inclusion) as types are made more persistent, or smaller, according to the partial order $\preceq$. This follows because a player can be punished more effectively when his type is more persistent and thus more predictable.

More precisely, suppose that $P_i \preceq Q_i$ for $i = 1, 2$. By Corollary 1, we then have $\pi_P = (\pi_{P_1}, \pi_{P_2}) = (\pi_{Q_1}, \pi_{Q_2}) = \pi_Q$. Because the set $W(u, P)$ of incentive-feasible payoffs depends only on the stationary distribution, this implies that the difference between $W^*(u, P)$ and $W^*(u, Q)$ is only due to differences in the minmax values defined by (S.1). But, by inspection, $\underline{w}_i$ is the (limit) value of a stochastic zero-sum game with incomplete information, where $S = \Theta_i$ and $(g, A, B) = (u_i, C_i, C_{-i})$. Therefore, each $\underline{w}_i$ is lower under $P_i$ than under $Q_i$ by definition of $\preceq$, which implies $W^*(u, Q) \subseteq W^*(u, P)$.

The fact that each stage game $u$ is required to satisfy the nonempty-interior and non-degeneracy assumptions of Hörner, Takahashi, and Vieille (2015) generates some additional work in the proof of the converse. The idea is to show first that if $W^*(u, Q) \subseteq$

[2] Hörner, Takahashi, and Vieille (2015) showed that $W(u, P)$ can be characterized as follows: Let $E = \{e \in \mathbb{R}^2 : \|e\| = 1\}$. For each $e \in E$, let $N(e) = \{i : e_i > 0\}$, and define $k(e) = \max \int e \cdot u(\rho(\theta), \theta)\, d\pi_P(\theta)$, where the maximum is over allocation rules $\rho : \times_{i\in N(e)} \Theta_i \to C$. Then $W(u, P) = \bigcap_{e\in E} \{w \in \mathbb{R}^2 : e \cdot w \leq k(e)\}$.

$W^*(u, P)$ for all stage games $u \in U(P) \cap U(Q)$, then $\pi_P = \pi_Q$. This follows by noting that if $\pi_P \neq \pi_Q$, then it is possible to construct a simple game $u$ for which $W^*(u, Q) \nsubseteq W^*(u, P)$. But given that the stationary distribution under $P$ is the same as under $Q$, the set inclusion $W^*(u, Q) \subseteq W^*(u, P)$ is equivalent to the minmax values being lower under $P$ than under $Q$. The conclusion that $P_i \preceq Q_i$ for $i = 1, 2$ then follows from the definition of $\preceq$ by showing that the restriction $u \in U(P) \cap U(Q)$ still allows for a sufficiently rich set of "test games." See the proof below for details.

REMARK S.1: The result of Hörner, Takahashi, and Vieille (2015) applies also to imperfect public monitoring games where monitoring has product structure and certain identifiability conditions are satisfied. Using results from Section S.2 of this supplement, Proposition S.1 can be shown to extend, as stated, to this larger class of games.

### S.1.1. *Proof of Proposition S.1*

Given the above discussion, it remains to show that $W^*(u, Q) \subseteq W^*(u, P)$ for all $u \in U(P) \cap U(Q)$ implies $P_i \preceq Q_i$, $i = 1, 2$.

*Preliminaries.* We make two preliminary observations. First, we note that given any ergodic operators $P = (P_1, P_2)$ and any fixed number of actions for the players, the nondegeneracy assumption of Hörner, Takahashi, and Vieille (2015) holds for an open and dense set of games. For $k, l \in \mathbb{N}$, let $U_{k,l} = \mathbb{R}^{kl(|\Theta_1|+|\Theta_2|)}$ be the set of games $u : C \times \Theta \to \mathbb{R}^2$ with private values such that $|C_1| = k$ and $|C_2| = l$. Let $U^0_{k,l}(P) \subseteq U_{k,l}$ be the nonempty set of games $u$ that satisfy Assumption A1 of Hörner, Takahashi, and Vieille (2015), p. 1815. Take any $u \in U^0_{k,l}(P)$. Assumption A1 is a requirement that $u$ does not satisfy certain linear equalities (since the relative values are linear in $u$). Hence, it continues to hold for any $u'$ sufficiently close to $u$, showing that $U^0_{k,l}(P)$ is open in $U_{k,l}$. To show denseness, let $u \in U_{k,l}$ and $u' \in U^0_{k,l}(P)$. Define $u^\lambda = \lambda u' + (1 - \lambda)u \in U_{k,l}$ for $\lambda \in (0, 1)$. Then we have $u^\lambda \in U^0_{k,l}(P)$ for all $\lambda$ small enough, because the linear equalities in Assumption A1 for $u^\lambda$ are a convex combination of the equalities for $u$ and $u'$, and they do not hold for the latter. This implies that $U^0_{k,l}(P)$ is dense in $U_{k,l}$.

Second, we note that the set $W^*(u, P)$ varies continuously with $u$, implying that the set of games $u$ for which $W^*(u, P)$ has a nonempty interior is open in $U_{k,l}$. Indeed, the set $W(u, P)$ of incentive-feasible payoffs varies clearly continuously with $u$. Furthermore, the limit minmax value $\underline{w}_i$ can be viewed as the value of a stochastic zero-sum game with incomplete information, which is continuous in payoffs by the characterization in Theorem 1.

*Equivalence of stationary distributions.* We then show that $W^*(u, Q) \subseteq W^*(u, P)$ for all $u \in U(P) \cap U(Q)$ implies $\pi_P = \pi_Q$. Suppose to the contrary that $\pi_P \neq \pi_Q$. Without loss of generality, let $\pi_{P_1} \neq \pi_{Q_1}$. Fix $\hat{\theta}_1 \in \Theta_1$ such that $\pi_{Q_1}(\hat{\theta}_1) > \pi_{P_1}(\hat{\theta}_1)$.

Let $\hat{u} \in U_{2,2}$ be the game depicted in Figure 1. (Note that only player 1's payoff depends on his type.) By inspection, the minmax value is zero for both players. The expected payoff

| | $L$ | $R$ |
|---|---|---|
| $T$ | $\mathbf{1}\{\theta_1 = \hat{\theta}_1\}, 1$ | $0, 1$ |
| $B$ | $\mathbf{1}\{\theta_1 = \hat{\theta}_1\}, 0$ | $0, 0$ |

FIGURE 1.—Game $\hat{u}$.

under $\pi_Q$ from playing $(T, L)$ regardless of the type profile is $(\pi_{Q_1}(\hat{\theta}_1), 1)$, and hence $(\pi_{Q_1}(\hat{\theta}_1), 1) \in W^*(\hat{u}, Q)$. On the other hand, the highest feasible payoff to player 1 under $\pi_P$ is similarly obtained by playing $(T, L)$ and it equals $\pi_{P_1}(\hat{\theta}_1) < \pi_{Q_1}(\hat{\theta}_1)$. We conclude that $W^*(\hat{u}, Q) \nsubseteq W^*(\hat{u}, P)$. However, we are not quite done yet, since $\hat{u}$ is not necessarily in $U(P) \cap U(Q)$.

Observe that $W^*(\hat{u}, P)$ and $W^*(\hat{u}, Q)$ have nonempty interiors. (Indeed, we have $W^*(\hat{u}, P) = \text{con}\{(0,0), (0,1), (\pi_{P_1}(\hat{\theta}_1), 0), (\pi_{P_1}(\hat{\theta}_1), 1)\}$, and similarly for $W^*(\hat{u}, Q)$.) Moreover, $W^*$ varies continuously with the game on $U_{2,2}$. We can thus choose from the open and dense set $U^0_{2,2}(P) \cap U^0_{2,2}(Q) \subseteq U_{2,2}$ a game $u$ close enough to $\hat{u}$ so that the interiors of $W^*(u, P)$ and $W^*(u, Q)$ are nonempty and $W^*(u, Q) \nsubseteq W^*(u, P)$. But then $u \in U(P) \cap U(Q)$, a contradiction. Therefore, $\pi_P = \pi_Q$.

*Ranking of operators.* Since $\pi_P = \pi_Q$ and $W(u, P)$ depends on $P$ only through the stationary distribution, we have, for each $u$, $W^*(u, Q) \subseteq W^*(u, P)$ if and only if the minmax values satisfy $\underline{w}_i(u, P) \leq \underline{w}_i(u, Q)$ for $i = 1, 2$. We complete the proof by showing that if $P_i \npreceq Q_i$ for some player $i$, then there exists a game $u \in U(P) \cap U(Q)$ such that $\underline{w}_i(u, P) > \underline{w}_i(u, Q)$, contradicting $W^*(u, Q) \subseteq W^*(u, P)$.

To this end, suppose without loss of generality that $P_1 \npreceq Q_1$. By definition of $\preceq$, there exist $\varepsilon > 0$ and a zero-sum game $g : A \times B \times S \to \mathbb{R}$, with $S = \Theta_1$, such that $v(g, P_1) > v(g, Q_1) + 2\varepsilon$. Let $\bar{g} = \max_{a,b,s} |g(a, b, s)|$. Construct a stage game as follows. Let $C_1 = A \cup \{c'_1\}$ and let $C_2 = B \cup \{c'_2\}$. Define $u : C \times \Theta \to \mathbb{R}^2$ by setting

$$u(c, \theta) = \big(u_1(c_1, c_2, \theta_1), u_2(c_1, c_2, \theta_2)\big) = \begin{cases} \big(g(c_1, c_2, \theta_1), 0\big) & \text{if } (c_1, c_2) \in A \times B, \\ (-\bar{g} - 1, 0) & \text{if } c_1 = c'_1, \\ (\bar{g} + 1, 1) & \text{if } c_2 = c'_2 \text{ and } c_1 \neq c'_1. \end{cases}$$

We observe first that, for any ergodic operators $R = (R_1, R_2)$, player 1's minmax value satisfies $\underline{w}_1(u, R) = v(u_1, R_1) = v(g, R_1) \in [-\bar{g}, \bar{g}]$. Indeed, player 1's action $c'_1$ is dominated by each $a \in A$, and player 2's action $c'_2$ is dominated by each $b \in B$ for the purposes of minimizing player 1's payoff. Such inferior actions do not affect the value by inspection of formula (3.2) in Theorem 1. We note also that $\underline{w}_2 = 0$.

We claim that $\text{int}\, W^*(u, R) \neq \emptyset$ for any ergodic $R$. Fix some $(a, b) \in A \times B$. Then

$$\int u(a, b, \theta)\, d\pi_R(\theta) = \left(\int g(a, b, \theta_1)\, d\pi_{R_1}(\theta_1), 0\right) = (w, 0),$$

where the last equality defines $w \in [-\bar{g}, \bar{g}]$. Thus,

$$\text{con}\big\{(-\bar{g} - 1, 0), (w, 0), (\bar{g} + 1, 1)\big\} \subseteq W(u, R),$$

where the set on the left has a nonempty interior, and hence so does $W(u, R)$. Moreover, the minmax payoffs $(v(g, R), 0)$ are strictly worse for both players than the highest feasible payoffs $(\bar{g} + 1, 1)$, implying that the interior of $W^*(u, R)$ is nonempty as well. Therefore, the interiors of $W^*(u, P)$ and $W^*(u, Q)$ are nonempty.

To conclude the proof, we recall that $U^0_{|A|+1,|B|+1}(P) \cap U^0_{|A|+1,|B|+1}(Q)$ is dense in $U_{|A|+1,|B|+1}$. Hence, there exists $u' \in U^0_{|A|+1,|B|+1}(P) \cap U^0_{|A|+1,|B|+1}(Q)$ close enough to $u$ such that (i) $\text{int}\, W^*(u', R) \neq \emptyset$ for $R \in \{P, Q\}$, and (ii) we have

$$\underline{w}_1\big(u', P\big) = v\big(u'_1, P_1\big) > v(u_1, P_1) - \varepsilon = v(g, P_1) - \varepsilon,$$

and

$$\underline{w}_1(u', Q) = v(u'_1, Q_1) < v(u_1, Q_1) + \varepsilon = v(g, Q_1) + \varepsilon,$$

where (i) follows by our preliminary observation about the continuity of $W^*$ in payoffs, and (ii) follows because the value is continuous in payoffs by the characterization in Theorem 1. But (ii) and the choice of $g$ imply

$$\underline{w}_1(u', P) > v(g, P_1) - \varepsilon > v(g, Q_1) + \varepsilon > \underline{w}_1(u', Q),$$

whereas (i) and the choice of $u'$ imply $u' \in U(P) \cap U(Q)$, as desired. Proposition S.1 now follows.

## S.2. IMPERFECT PUBLIC MONITORING

In this section, we extend the model to incorporate imperfect public monitoring of actions. To do so, we assume that the minimizer only observes a signal $z$ that is drawn from a distribution $F_a \in \Delta Z$ over some finite set $Z$. The distribution $F_a$ depends on the maximizer's action $a$, but not on the minimizer's own action. The signal is public, that is, it is also observed by the maximizer. The observability of the minimizer's actions plays no role in the analysis, and hence we remain agnostic about what the maximizer observes about them.

A zero-sum game $g$ and monitoring structure $F = (\{F_a\}_{a \in A}, Z)$ define an imperfect monitoring game $(g, F)$. A stochastic imperfect monitoring game $(\delta, \pi, g, F, P)$ is defined as the infinite repetition of the game $(g, F)$ analogously to the case of perfect monitoring considered in the main text. Let $v(\pi; g, F, P)$ be the limit value of the stochastic imperfect monitoring game as $\delta \to 1$.

The definition of the auxiliary game $\hat{g}$ in (3.1) has to be adjusted to account for the noisy monitoring. To this end, fix a game $(g, F)$. For each stage-game strategy $\alpha : S \to \Delta A$ and each prior $p$, let $\nu_F^{\alpha,p} \in \Delta^2 S$ be the distribution over posteriors induced by the strategy $\alpha$ and the signal $F$. (More precisely, for each signal $z$, define $q_F^{\alpha,p}(z) \in \Delta S$ as the posterior after signal $z$. That is, for each $s$,

$$q_F^{\alpha,p}(s; z) = \frac{\displaystyle\sum_a F_a(z) p(s) \alpha(a|s)}{\displaystyle\sum_{a,s'} F_a(z) p(s') \alpha(a|s')}.$$

We then define $\nu_F^{\alpha,p}(q) = \sum_{a,s,z} \mathbf{1}\{q_F^{\alpha,p}(z) = q\} F_a(z) \alpha(a|s) p(s)$ for each $q$.)

Given $\nu \in \Delta^2 S$, let $p = \mathcal{E}\nu$ and define

$$\hat{g}_F(\nu) = \min_\beta \max_{\alpha : \nu_F^{\alpha,p} \leq_B \nu} \sum_s p(s) g(\alpha(s), \beta, s). \tag{S.2}$$

(We show below that the minimum and the maximum exist.) The function $\hat{g}_F$ plays the same role as $\hat{g}$ in the main text; the two coincide if monitoring is perfect. Given a prior $p$ and a mean-preserving spread $m$, we can still interpret $\hat{g}_F(m(p))$ as the value of an auxiliary one-shot game, with prior $p$, where the minimizer chooses his action and then observes an exogenous signal about the state leading to the distribution of his posteriors being $m(p)$, and the maximizer is restricted to playing a strategy that reveals no more information (in the Blackwell sense) than the signal.

We then have the following generalization of Theorem 1.

THEOREM S.1: *For every initial distribution* $\pi$, *every imperfect monitoring game* $(g, F)$, *and every operator* $P$,

$$v(\pi; g, F, P) = \max_{(\mu,m)\in\Delta^2 S\times\mathcal{M}\,:\,\psi_{\pi,P}\leq^B\mu \text{ and } P(\mu*m)\leq^B\mu} \int \hat{g}_F\big(m(p)\big)\, d\mu(p). \tag{S.3}$$

The proof can be found below.

We can now extend the comparison of operators to imperfect monitoring games.

DEFINITION S.1: For any initial distribution $\pi$ and any operators $P$ and $Q$, we say that *$Q$ is better for the informed player than $P$ given $\pi$*, or $P \preceq^*_\pi Q$, if for every imperfect monitoring game $(g, F)$, $v(\pi; g, F, P) \leq v(\pi; g, F, Q)$. If $P$ and $Q$ are ergodic, then the relation $\preceq^*_\pi$ is independent of $\pi$ and we denote it by $\preceq^*$.

Note that the above definition requires the comparison of values to hold for each game $g$ *and* each monitoring structure $F$.[3] In particular, because perfect monitoring is a special case of $F$, we have the following relationship.

FACT S.1: *$P \preceq^*_\pi Q$ implies $P \preceq_\pi Q$.*

Given Definition S.1, Theorem 2′ holds as stated in the main text if we replace $\preceq_\pi$ with $\preceq^*_\pi$ in condition (a); we include the ergodic case here as (d) for completeness.

THEOREM S.2: *For any $\pi \in \Delta S$ and operators $P$ and $Q$, the following are equivalent*:

(a) *$P \preceq^*_\pi Q$.*

(b) *$\psi_{\pi,Q} \leq^B \psi_{\pi,P}$ and for every pair $(\mu, m) \in \Delta^2 S \times \mathcal{M}$ such that $\psi_{\pi,P} \leq^B \mu$ and $P(\mu * m) \leq^B \mu$, we have $Q(\mu * m) \leq^B \mu$.*

(c) *$\psi_{\pi,Q} \leq^B \psi_{\pi,P}$ and for every $\nu \in \Delta^2 S$ such that $\psi_{\pi,P} \leq^B \nu$ and $P\nu \leq^B \nu$, we have $Q\nu \leq^B P\nu$.*

*If $P$ and $Q$ are ergodic, then each of the above conditions is equivalent to*

(d) *For every $\nu \in \Delta^2 S$ such that $P\nu \leq^B \nu$, we have $Q\nu \leq^B P\nu$.*

The equivalence of (b) and (c) (and of (d), if $P$ and $Q$ are ergodic) is unaffected since these conditions only involve the initial distribution $\pi$ and the operators $P$ and $Q$. That (b) implies (a) follows since (b) amounts to saying that the feasible set in (S.3) under $P$ is a subset of that under $Q$. The other direction follows because, by Fact S.1, $P \preceq^*_\pi Q$ implies $P \preceq_\pi Q$, which in turn implies (b) by Theorem 2′.

The rest of this section presents the proof of Theorem S.1. The main difficulty in extending the proof of Theorem 1 from the main text is the fact that we do not know whether the function $\hat{g}_F : \Delta^2 S \to \mathbb{R}$ defined by (S.2) is continuous. However, the proof of Theorem 1 relies on continuity of $\hat{g}$ in only two places. First, it is used to find a measurable optimal revelation strategy. Second, it is used in the proof of Lemma 4 to establish the existence of a maximizer to the value problem. We show below that, with some more care, both of these issues can be addressed.

[3]The alternative would be to somehow fix a monitoring structure $F$ and only vary the game $(g, A, B)$. But we find Definition S.1 to be a more natural way to extend Definition 1 from the main text to imperfect monitoring games. First, it is not immediately clear how to fix the monitoring structure independently of the game, because in order to specify the monitoring structure $F$, one needs to know the action space $A$, which is part of the definition of the game $g$. Second, this definition treats the monitoring structure and the payoff function symmetrically.

### S.2.1. *Preliminaries*

Fix an initial distribution $\pi$, an imperfect monitoring game $(g, F)$, and an operator $P$. We start with preliminary definitions and results, which culminate in showing that $\hat{g}_F$ is upper semi-continuous.

Given a stage-game strategy $\alpha \in (\Delta A)^S$ and prior $p \in \Delta S$, let $\phi_F^{\alpha,p} \in \Delta(S \times A \times Z)$ be the induced distribution over states, actions, and signals defined by

$$\phi_F^{\alpha,p}(s, a, z) = p(s)\alpha(a|s)F_a(z).$$

We record the following observation for future reference; the proof is immediate from the definition.

LEMMA S.1: *The map $(\alpha, p) \mapsto \phi_F^{\alpha,p}$ is continuous in the weak topology.*

Given $\phi \in \Delta(A \times S \times Z)$, let $\phi_S = \text{marg}_S\phi$. That is, $\phi_S(\cdot) = \sum_{a,z} \phi(\cdot, a, z)$. The marginals $\phi_Z$ and $\phi_{S\times Z}$ are defined analogously. Let $\phi(\cdot|z) \in \Delta(S \times A)$ denote the conditional distribution given $z \in Z$. The marginal $\phi_S(\cdot|z)$ on $S$ is defined as above.

For each $\phi \in \Delta(S \times A \times Z)$, let $\psi(\phi) \in \Delta^2 S$ be the induced distribution of $\phi_S(\cdot|z)$. (That is, if $q = \phi_S(\cdot|z)$ for some $z$, then $q$ has probability $\phi_Z(z)$ under $\psi(\phi)$; otherwise $q$ is assigned zero probability.) Notice that $\nu_F^{\alpha,p} = \psi(\phi_F^{\alpha,p})$.

LEMMA S.2: *The map $\phi \mapsto \psi(\phi)$ is continuous in the weak topology.*

PROOF: For any continuous $f : \Delta S \to \mathbb{R}$ and any $\phi \in \Delta(S \times A \times Z)$, we have

$$\int f(q)\, d\psi(q|\phi) = \sum_{s,z:\phi_Z(z)>0} f\left(\frac{\phi_{S\times Z}(s, z)}{\phi_Z(z)}\right)\phi_Z(z),$$

where the right-hand side is continuous in $\phi$. *Q.E.D.*

Define the correspondence $\mathcal{A} : \Delta^2 S \rightrightarrows (\Delta A)^S$ by

$$\mathcal{A}(\nu) = \{\alpha \in (\Delta A)^S : \nu_F^{\alpha,\mathcal{E}\nu} \leq^B \nu\}.$$

$\mathcal{A}(\nu)$ is the set of strategies that reveal no more information than $\nu$ given prior $\mathcal{E}\nu$.

LEMMA S.3: *$\mathcal{A}$ is upper hemi-continuous and nonempty, compact, and convex-valued.*

PROOF: $\mathcal{A}$ is nonempty-valued, since any strategy $\alpha$ that always plays the same action independently of the state satisfies $\nu_F^{\alpha,\mathcal{E}\nu} = e_{\mathcal{E}\nu} \leq^B \nu$ for every $\nu$.

Take a sequence $\nu_n$ and $\alpha_n \in \mathcal{A}(\nu_n)$ such that $\nu_n \to \nu$ and $\alpha_n \to \alpha$ for some $\alpha$. Let $p_n = \mathcal{E}\nu_n$. Then, $p_n \to p = \mathcal{E}\nu$, and for any concave $f : \Delta S \to \mathbb{R}$,

$$\int f(q)\, d\psi(q|\phi_F^{\alpha_n,p_n}) = \int f(q)\, d\nu_F^{\alpha_n,p_n}(q) \leq \int f(q)\, d\nu_n(q).$$

The right-hand side converges to $\int f(q)\, d\nu(q)$ as $n \to \infty$. The left-hand side converges to $\int f(q)\, d\psi(q|\phi_F^{\alpha,p})$ by Lemmas S.1 and S.2, showing that $\mathcal{A}$ is u.h.c. Taking $\nu_n = \nu$ for all $n$ shows that $\mathcal{A}$ is compact-valued, since $(\Delta A)^S$ is a bounded set in $\mathbb{R}^S$.

To see convexity, let $\alpha_0, \alpha_1 \in \mathcal{A}(\nu)$ and $\gamma \in (0,1)$. Define $\alpha = \gamma\alpha_0 + (1-\gamma)\alpha_1$. Letting $p = \mathcal{E}\nu$, we then have $\nu_F^{\alpha_i,p} \leq^B \nu$ for $i = 1, 2$. Because the Blackwell ordering is preserved under convex combinations, this implies $\gamma\nu_F^{\alpha_0,p} + (1-\gamma)\nu_F^{\alpha_0,p} \leq^B \nu$. Note that $\gamma\nu_F^{\alpha_0,p} + (1-\gamma)\nu_F^{\alpha_1,p}$ is the distribution of posteriors if the minimizer first observes $i = 0$ with probability $\gamma$ and $i = 1$ with probability $(1-\gamma)$, and then observes a signal generated by the correspoding strategy $\alpha_i$. Because this provides more information than simply observing the signal generated by strategy $\alpha = \gamma\alpha_0 + (1-\gamma)\alpha_1$, we have $\nu_F^{\alpha,p} \leq^B \gamma\nu_F^{\alpha_0,p} + (1-\gamma)\nu_F^{\alpha_0,p}$. It follows that $\alpha \in \mathcal{A}(\nu)$. *Q.E.D.*

Define the function $g^\Delta : \Delta(S \times A \times Z) \to \mathbb{R}$ by

$$g^\Delta(\phi) = \min_\beta \sum_{a,s,z} g(a, \beta, s)\phi(a, s, z).$$

LEMMA S.4: *$g^\Delta$ is concave and continuous.*

PROOF: Concavity follows from $g^\Delta(\phi)$ being the minimum over terms linear in $\phi$. Continuity (also at the boundary of the domain) follows by the maximum theorem. *Q.E.D.*

LEMMA S.5: *For every $\nu \in \Delta^2 S$ and $p = \mathcal{E}\nu$,*

$$\max_{\alpha\in\mathcal{A}(\nu)} g^\Delta\big(\phi_F^{\alpha,p}\big) = \min_{\beta\in\Delta B} \max_{\alpha\in\mathcal{A}(\nu)} \sum_s p(s) g\big(\alpha(s), \beta, s\big) = \hat{g}_F(\nu).$$

*In particular, the maxima and minima in the above statement are well-defined.*

PROOF: The first maximum is well-defined because $\phi_F^{\alpha,p}$ is continuous in $\alpha$ by Lemma S.1, $g^\Delta$ is continuous by Lemma S.4, and $\mathcal{A}(\nu)$ is compact by Lemma S.3. The first equality follows by definition of $g^\Delta$ and the minmax theorem as $\mathcal{A}(\nu)$ is convex by Lemma S.3. The second equality is by definitions of $\hat{g}_F$ and $\mathcal{A}(\nu)$. *Q.E.D.*

LEMMA S.6: *$\hat{g}_F$ is upper semi-continuous on $\Delta^2 S$.*

PROOF: By Lemma S.5, for each $\nu \in \Delta^2 S$ and $p = \mathcal{E}\nu$,

$$\hat{g}_F(\nu) = \max_{\alpha\in\mathcal{A}(\nu)} g^\Delta\big(\phi_F^{\alpha,p}\big).$$

Take any sequence $\nu_n \to \nu$. For each $n$, let $p_n = \mathcal{E}\nu_n$ and let $\alpha_n \in \mathcal{A}(\nu_n)$ be a solution to the maximization problem on the right-hand side of the above equation for $(\nu_n, p_n)$. By Lemma S.3, there exists a subsequence $(\nu_{n_k}, \alpha_{n_k})$ and $\bar{\alpha} \in \mathcal{A}(\nu)$ such that $\alpha_{n_k} \to \bar{\alpha}$. By continuity (Lemmas S.1 and S.4), we have $g^\Delta(\phi_F^{\alpha_{n_k},p_{n_k}}) \to g^\Delta(\phi_F^{\bar{\alpha},p})$. Therefore, $\hat{g}_F(\nu) = \max_{\alpha\in\mathcal{A}(\nu)} g^\Delta(\phi_F^{\alpha,p}) \geq g^\Delta(\phi_F^{\bar{\alpha},p})$. *Q.E.D.*

### S.2.2. *Proof of Theorem S.1*

We now show how the proof of Theorem 1 from the main text can be adapted to the present setting.

The first step in the proof of Theorem 1 is Lemma 2, which gives a recursion for the discounted value. We replace it with the following analogous result.

LEMMA S.7: *For every $\delta < 1$, there exists a (measurable) mean-preserving spread $m^\delta \in \mathcal{M}$ such that for every $p \in \Delta S$,*

$$v^\delta(p; g, P) = (1-\delta)\hat{g}_F\big(m(p)\big) + \delta \int v^\delta(Pq; g, P)\, dm(q|p).$$

PROOF: The standard recursive formula e.g., Mertens, Sorin, and Zamir (2015) implies that

$$v^\delta(p; g, P) = \max_{\alpha \in (\Delta A)^S} \min_{\beta \in \Delta B} (1-\delta) \sum_{a,s,z} \phi_F^{\alpha,p}(a, s, z) g(a, \beta, s) + \delta \int v^\delta(Pq; g, P)\, d\nu_F^{\alpha,p}(q).$$

The right-hand side is further equal to

$$\begin{aligned}
&\max_{\nu \in \Delta^2 S : \mathcal{E}\nu = p} \max_{\alpha \in \mathcal{A}(\nu)} \min_{\beta \in \Delta B} (1-\delta) \sum_{a,s,z} \phi_F^{\alpha,p}(a, s, z) g(a, \beta, s) + \delta \int v^\delta(Pq; g, P)\, d\nu_F^{\alpha,p}(q) \\
&\quad = \max_{\nu \in \Delta^2 S : \mathcal{E}\nu = p} (1-\delta) \Big[ \max_{\alpha \in \mathcal{A}(\nu)} \min_{\beta \in \Delta B} \sum_{a,s,z} \phi_F^{\alpha,p}(a, s, z) g(a, \beta, s) \Big] + \delta \int v^\delta(Pq; g, P)\, d\nu(q) \\
&\quad = \max_{\nu \in \Delta^2 S : \mathcal{E}\nu = p} (1-\delta)\hat{g}_F(\nu) + \delta \int v^\delta(Pq; g, P)\, d\nu(q).
\end{aligned}$$

The second line follows because $v^\delta(p; g, P)$ is concave in $p$ and $\nu_F^{\alpha,p} \leq^B \nu$ for every $\alpha \in \mathcal{A}(\nu)$ by definition. The third line follows by Lemma S.5. To see that the maximizer on the last line can be selected to be a measurable function of $p$, note that in the first display in the proof, the maximum theorem implies that the optimal $\alpha$ can be selected to be a measurable function of $p$. The maximizer on the last line of the second display can be obtained as a continuous function of this optimal $\alpha$. *Q.E.D.*

The second step in the proof of Theorem 1 is Lemma 3. Because Lemma S.7 gives us the existence of a measurable optimal revelation policy for each $\delta$, this step goes through without changes if $\hat{g}$ is replaced with $\hat{g}_F$. (The proof is otherwise verbatim.)

The third step, which establishes the existence of an optimal $(\mu, m)$, requires one adjustment. Continuity of $\hat{g}$ is invoked in the proof of Lemma 4 in display (6.6). But because $\hat{g}_F$ is upper semi-continuous by Lemma S.6, display (6.6) can be replaced by

$$\limsup_{n\to\infty} \int \hat{g}_F\big(m^n(p)\big)\, d\mu^n(p) \leq \int \hat{g}_F(\nu)\, d\omega(\nu) \leq \int \hat{g}\big(m^*(p)\big)\, d\mu^*(p).$$

The rest of the argument then goes through verbatim.

The fourth step, which finishes the proof, is completed exactly as in the main text.

## S.3. PUBLIC SIGNALS

In this section, we extend the model to incorporate exogenous public information about the state. For simplicity, we assume that actions are perfectly monitored (i.e., observed without noise) and we restrict attention to ergodic operators.

We model public information by augmenting the game from the main text with an exogenous public signal observed at the beginning of the period before actions are chosen.

More precisely, we assume that in each period, the two players observe a signal $z \in Z$ that is drawn from a distribution $F_s \in \Delta Z$ that depends on the current state $s$. The signal provides the minimizer information about the state beyond what he can infer from the maximizer's past actions.

In what follows, we use the following terminology for the minimizer's beliefs:

- the belief in the beginning of the period before observing the public signal is called the *prior*;
- the belief after observing the public signal, but before actions are chosen, is called the *interim belief*;
- the belief after the maximizer's action is observed is called the *posterior*.

Let $v(g, P, F)$ denote the (limit) value given game $g$, ergodic operator $P$, and public signal $F = (\{F_s\}_{s\in S}, Z)$.[4] For each prior $p$ and signal $z$, let $q^p(z) \in \Delta Z$ be the interim belief after observing signal $z$ (i.e., $q^p(s|z) = \frac{F_s(z)p(s)}{\sum_{s'} F_{s'}(z)p(s')}$). Let $n^F(p) \in \Delta^2 Z$ be the distribution of these interim beliefs (i.e., $n^F(q|p) = \sum_{s,z} \mathbf{1}\{q^p(z) = q\} p(s) F_s(z)$). Then the map $p \mapsto n^F(p)$ is a mean-preserving spread.

The following result generalizes the ergodic case of Theorem 1 from the main text.

THEOREM S.3: *For every game $g$, every ergodic operator $P$, and every public signal $F$,*

$$v(g, P, F) = \max_{(\mu,m)\in\Delta^2 S\times\mathcal{M} : P(\mu*m)*n^F \leq^B \mu} \int \hat{g}\big(m(p)\big)\, d\mu(p). \tag{S.4}$$

To interpret the constraint in (S.4), let $\mu$ be the distribution of interim beliefs so that, given revelation policy $m$ (which now maps interim beliefs to posteriors), the distribution of posteriors is $\mu * m$. The distribution of the next period's priors is then $P(\mu * m)$, from which the distribution of the next period's interim beliefs, $P(\mu * m) * n^F$, is obtained by applying the exogenous mean-preserving spread $n^F$ induced by the public signal. For $\mu$ to be an invariant distribution of interim beliefs under this process, we thus need $P(\mu * m) * n^F = \mu$. The constraint simply relaxes this to an inequality in the sense of the Blackwell ordering analogously to Theorem 1 in the main text.

We can now describe how the value of the game to the informed player depends on the operator as well as the public signal.

DEFINITION S.2: For any ergodic operators $P$ and $Q$ and any public signals $F$ and $G$, we say that $(Q, G)$ *is better for the informed player than* $(P, F)$, or $(P, F) \preceq_{\text{Pub}} (Q, G)$, if for every game $g$, $v(g, P, F) \leq v(g, Q, G)$.

Note that the case $F = G$ yields a comparison of operators for a fixed public signal. Similarly, letting $P = Q$ yields a comparison of public signals for a fixed operator.

THEOREM S.4: *For any ergodic operators $P$ and $Q$, and any public signals $F$ and $G$, the following are equivalent*:

(a) $(P, F) \preceq_{\text{Pub}} (Q, G)$.
(b) *For every* $(\mu, m) \in \Delta^2 S \times \mathcal{M}$, $P(\mu * m) * n^F \leq^B \mu$ *implies* $Q(\mu * m) * n^G \leq^B \mu$.
(c) *For every* $\nu \in \Delta^2 S$ *such that* $P\nu * n^F \leq^B \nu$, *we have* $Q\nu * n^G \leq^B P\nu * n^F$.

[4]We are not aware of a suitable reference for the existence of $v(g, P, F) = \lim_{\delta\to 1} v^\delta(\pi; g, P, F)$, but since the operator $P$ is ergodic, this can be established along the lines of the simple argument given for the ergodic case (without a public signal) in Renault (2006).

Condition (c) shows that the pair $(Q, G)$ is better for the informed player than the pair $(P, F)$ if and only if the combined effect of the state transitioning from one period to another and the public information released by the signal result in less information being preserved under $(Q, G)$ than under $(P, F)$ given any distribution of posteriors $\nu$ stabilizable under $(P, F)$ (in the sense that $P\nu * n^F \leq^B \nu$; such stabilizable $\nu$ are exactly the ones that can arise as invariant distributions of posteriors).

EXAMPLE S.1: If $P = Q = D_\pi$ for some i.i.d. operator $D_\pi$, then, for every $\nu$, we have $P\nu = Q\nu = D_\pi \nu = e_\pi$. In this case, condition (c) simplifies to the requirement that $e_\pi * n^G \leq^B e_\pi * n^F$. Thus, having $(D_\pi, F) \preceq_{\text{Pub}} (D_\pi, G)$ for all $\pi \in \Delta S$ is equivalent to the public signal $G$ being better than the public signal $F$ in the usual Blackwell sense of comparing signals for decision problems.

EXAMPLE S.2: Let $F = G = F^*$, where $F^*$ is the perfectly informative signal that reveals the current state (i.e., $Z = S$ and $F_s(s) = 1$ for all $s$). Then $n^{F^*}$ maps each $p \in \Delta S$ to a distribution over Dirac measures $e_s$, $s \in S$, such that $n^{F^*}(e_s | p) = p(s)$. Hence, if $P\nu * n^F \leq^B \nu$, then $\nu$ can put positive probability only to Dirac measures. On the other hand, we have $P\mathcal{E}\nu = \mathcal{E}\nu$, which implies $\mathcal{E}\nu = \pi_P$. Thus $\nu(e_s) = \pi_P(s)$. In other words, there is a unique stabilizable $\nu$, which corresponds to the minimizer always knowing the state. Moreover, this stabilizable distribution is the same for all operators with the same invariant distribution and we see from (c) that such operators paired with $F^*$ form an equivalence class. In particular, $\preceq_{\text{Pub}}$ is not a partial order.

The rest of this section is devoted to the proofs of Theorems S.3 and S.4. The former is a simple adaptation of the proof of Theorem 1 specialized to the ergodic setting. The latter is more challenging than the proof of Theorem 2′ since the public signal makes it harder to construct a game that shows the necessity of (b) for (a).

### S.3.1. *Proof of Theorem S.3*

Fix a game $g$, ergodic operator $P$, and public signal $F$. The proof mostly follows that of Theorem 1. The main difference is in the recursive formula where now, if $p$ is the current interim belief and $m$ is the maximizer's information revelation policy (mapping interim beliefs to posteriors), the distribution of the next period's interim beliefs is $Pm(p) * n^F$, whereas the relevant next-period distribution (that of priors) was $Pm(p)$ without the public signal. More precisely, we have the following formula.

LEMMA S.8: *For every $\delta < 1$ and every $p \in \Delta S$,*

$$v^\delta(p; g, P) = \max_{\nu \in \Delta^2 S : \mathcal{E}\nu = p} (1 - \delta)\hat{g}(\nu) + \delta \int v^\delta(q; g, P)\, d\big(P\nu * n^F\big)(q). \tag{S.5}$$

The proof is the same as that of Lemma 2 in the main text and hence omitted.

The recursive formula for the discounted value implies the following stationary characterization for any discount factor. (This substitutes for the "almost stationary" formula in Lemma 3 in the proof of Theorem 1, which does not rely on ergodicity.)

LEMMA S.9: *For every $\delta < 1$, there exist a distribution $\mu \in \Delta^2 S$ and a mean-preserving spread $m \in \mathcal{M}$ such that $P(\mu * m) * n^F = \mu$ and*

$$\int v^\delta(p; g, P)\, d\mu(p) = \int \hat{g}\big(m(p)\big)\, d\mu(p).$$

PROOF: Let $\Sigma(p)$ be the set of measures $\nu \in \Delta^2 S$ that achieve the maximum in (S.5). Notice that the objective function is continuous and concave in $\nu$, since $\hat{g}$ is concave. Furthermore, the feasible set $\{\nu : \mathcal{E}\nu = p\}$ is convex and compact, and continuous as a correspondence of $p$. It follows that the solution correspondence $p \mapsto \Sigma(p)$ is upper hemi-continuous with nonempty, compact, and convex values.

By Theorem 19.31 from Aliprantis and Border (2007), there exist a measurable selection $m : \Delta S \to \Delta^2 S$, $m(p) \in \Sigma(p)$, and a probability distribution $\mu \in \Delta^2 S$ such that $P(\mu * m) * n^F = \mu$. By (S.5), we then have

$$\int v^\delta(p; g, P)\, d\mu(p) = \int \left( (1-\delta)\hat{g}\big(m(p)\big) + \delta \int v^\delta(q; g, P)\, d\big(Pm(p) * n^F\big)(q) \right) d\mu(p),$$

so the result follows from the invariance $P(\mu * m) * n^F = \mu$. *Q.E.D.*

Let $V \subseteq \Delta^2 S \times \mathcal{M}$ be the set of $(\mu, m)$ such that $P(\mu * m) * n^F \leq^B \mu$. The next result is a simplified version of Lemma 4 from the main text. It shows that $\int \hat{g}(m(p))\, d\mu(p)$ attains its supremum on $V$.

LEMMA S.10: *There exists $(\mu^*, m^*) \in V$ such that*

$$\int \hat{g}\big(m^*(p)\big)\, d\mu^*(p) = \sup_{(\mu,m)\in V} \int \hat{g}\big(m(p)\big)\, d\mu(p).$$

PROOF: Take a sequence $(\mu^n, m^n) \in V$ such that

$$\lim_{n\to\infty} \int \hat{g}\big(m^n(p)\big)\, d\mu^n(p) = \sup_{(\mu,m)\in V} \int \hat{g}\big(m(p)\big)\, d\mu(p).$$

As in the proof of Lemma 4 in the main text, it is useful to view $\Delta^2 S \times \mathcal{M}$ as a subset of $\Delta^3 S$ by embedding it with the mapping $\iota : \Delta^2 S \times \mathcal{M} \to \Delta^3 S$ such that for each continuous function $\phi : \Delta^2 S \to \mathbb{R}$,

$$\int \phi\big(m(p)\big)\, d\mu(p) = \int \phi(\nu)\, d\iota(\mu, m)(\nu).$$

Because $\Delta^3 S$ is compact, by moving to a subsequence if necessary, we can assume that the sequence $(\mu^n, m^n)$ converges. That is, there exists a distribution $\omega \in \Delta^3 S$ such that $\lim_n \iota(\mu^n, m^n) = \omega$. Notice that for each continuous $\phi : \Delta^2 S \to \mathbb{R}$,

$$\lim_{n\to\infty} \int \phi\big(m^n(p)\big)\, d\mu^n(p) = \lim_{n\to\infty} \int \phi(\nu)\, d\iota\big(\mu^n, m^n\big)(\nu) = \int \phi(\nu)\, d\omega(\nu).$$

Constructing $(\mu^*, m^*)$ exactly as in the proof of Lemma 4, we then have

$$\begin{aligned} \int \hat{g}\big(m^n(p)\big)\, d\mu^n(p) &\to \int \hat{g}(\nu)\, d\omega(\nu) \\ &= \int \left( \int \hat{g}(\nu)\, d\omega(\nu | \mathcal{E}\nu = p) \right) d\mu^*(p) \leq \int \hat{g}\big(m^*(p)\big)\, d\mu^*(p). \end{aligned}$$

It remains to show that $(\mu^*, m^*) \in V$. One shows exactly as in the proof of Lemma 4 that $\mu^n \to \mu^*$ and $\mu^n * m^n \to \mu^* * m^*$, which implies that $P(\mu^n * m^n) * n^F \to P(\mu^* * m^*) * n^F$. Therefore, if $f$ is concave, then

$$\int f\, dP\big(\mu^* * m^*\big) * n^F = \lim_{n\to\infty} \int f\, dP\big(\mu^n * m^n\big) * n^F \geq \lim_{n\to\infty} \int f\, d\mu^n = \int f\, d\mu^*,$$

where the inequality follows since $(\mu^n, m^n) \in V$ for all $n$. Thus, $(\mu^*, m^*) \in V$. *Q.E.D.*

To complete the proof, we show first that $v(g, P, F)$ is not larger than the right-hand side of (S.3). By Lemma S.9, there are pairs $(\mu^\delta, m^\delta) \in V$, $\delta < 1$, such that

$$\int v^\delta(p; g, P, F)\, d\mu^\delta(p) = \int \hat{g}\big(m^\delta(p)\big)\, d\mu^\delta(p) \leq \max_{(\mu,m)\in V} \int \hat{g}\big(m(p)\big)\, d\mu(p).$$

(The maximum exists by Lemma S.10.) The claim now follows since the left-hand side tends (possibly along a subsequence) to $v(g, P, F)$ as $\delta \to 1$ by compactness of $\Delta^2 S$.

Next, we show that $v(g, P, F)$ is not smaller than the right-hand side of (S.3). By Lemma S.10, the maximum on the right-hand side is attained by some $(\mu^*, m^*) \in V$. By the recursive formula (S.5), for each $\delta < 1$ and $p \in \Delta S$, we have

$$v^\delta(p; g, P, F) \geq (1-\delta)\hat{g}\big(m^*(p)\big) + \delta \int v^\delta(q; g, P, F)\, d\big(Pm^*(p) * n^F\big)(q).$$

Taking expectations with respect to $\mu^*$ gives

$$\begin{aligned} &\int v^\delta(p; g, P, F)\, d\mu^*(p) \\ &\quad \geq (1-\delta) \int \hat{g}\big(m^*(p)\big)\, d\mu^*(p) + \delta \int v^\delta(q; g, P, F)\, d\big(P\big(\mu^* * m^*\big) * n^F\big)(q) \\ &\quad \geq (1-\delta) \int \hat{g}\big(m^*(p)\big)\, d\mu^*(p) + \delta \int v^\delta(p; g, P, F)\, d\mu^*(p), \end{aligned}$$

where the second line follows by concavity of $v^\delta(p; g, P, F)$ in $p$ as $(\mu^*, m^*) \in V$. Thus,

$$\int v^\delta(p; g, P, F)\, d\mu^*(p) \geq \int \hat{g}\big(m^*(p)\big)\, d\mu^*(p).$$

The claim follows by taking the limit $\delta \to 1$. This establishes Theorem S.3.

### S.3.2. *Proof of Theorem S.4*

The proof follows similar ideas as the proof of Theorem 2′ in the main text. The main difference lies in the part "(a) implies (b)" which is here considerably more complicated due to the public signal.

We start with the equivalence of (b) and (c). To see that (b) implies (c), suppose $P\nu * n^F \leq^B \nu$. Then $P\nu * n^F * m = \nu$ for some $m \in \mathcal{M}$, and $P(P\nu * n^F * m) * n^F = P\nu * n^F$. Thus, taking $\mu = P\nu * n^F$, (b) implies $Q(P\nu * n^F * m) * n^G \leq^B P\nu * n^F$, or $Q\nu * n^G \leq^B P\nu * n^F$. In the other direction, suppose $P(\mu * m) * n^F \leq^B \mu$. Then $P(\mu * m) * n^F \leq^B \mu * m$, so, taking $\nu = \mu * m$, (c) implies $Q(\mu * m) * n^G \leq^B P(\mu * m) * n^F \leq^B \mu$.

We then turn to the equivalence of (a) and (b). That (b) implies (a) follows immediately from Theorem S.3 since by (b), any $(\mu, m)$ that is feasible in the maximization problem in (S.4) given $(P, F)$ is also feasible given $(Q, G)$.

We show that (a) implies (b) by establishing the contrapositive. In negation of (b), let $(\mu_0, m_0) \in \Delta^2 S \times \mathcal{M}$ be such that $P(\mu_0 * m_0) * n^F \leq^B \mu_0$ and $Q(\mu_0 * m_0) * n^G \nleq^B \mu_0$. Then, there exists a concave function $f : \Delta S \to \mathbb{R}$ such that

$$\begin{aligned} &\int f\, d\mu_0 - \int f\, d\big(Q(\mu_0 * m_0) * n^G\big) > 0 \\ &\quad \geq \sup_{(\mu,m):Q(\mu*m)*n^G \leq^B \mu} \int f\, d\mu - \int f\, d\big(Q(\mu * m) * n^G\big). \end{aligned} \tag{S.6}$$

We will construct a sequence of zero-sum games $g_n$ such that

$$\begin{aligned} \lim_{n\to\infty} v(g_n, P, F) &\geq \int f\, d\mu_0 - \int f\, d\big(Q(\mu_0 * m_0) * n^G\big) \\ &> \sup_{(\mu,m):Q(\mu*m)*n^G \leq^B \mu} \int f\, d\mu - \int f\, d\big(Q(\mu * m) * n^G\big) = \lim_{n\to\infty} v(g_n, Q, G). \end{aligned}$$

Then for $n$ large enough, $v(g_n, P, F) > v(g_n, Q, G)$, contradicting (a) as desired.

Define the function $f^* : \Delta S \to \mathbb{R}$ by

$$f^*(p) = \int f(q)\, dn^G(q|p) = \sum_{z,s} f\big(q^p(z)\big) F_s(z) p(s).$$

We show that $f^*$ is concave. Indeed, take any $\lambda \in (0, 1)$ and $p_0, p_1 \in \Delta S$ and let $p = \lambda p_0 + (1 - \lambda) p_1$. Notice that

$$\begin{aligned} f^*(p) &= \sum_z f\big(q^p(z)\big) \sum_s F_s(z) p(s) \\ &= \sum_z f\left(\frac{\lambda p_0(z)}{p(z)} q^{p_0}(z) + \frac{(1-\lambda) p_1(z)}{p(z)} q^{p_1}(z)\right) p(z) \\ &\geq \sum_z \left(\frac{\lambda p_0(z)}{p(z)} f\big(q^{p_0}(z)\big) + \frac{(1-\lambda) p_1(z)}{p(z)} f\big(q^{p_1}(z)\big)\right) p(z) \\ &= \sum_z \big(\lambda p_0(z) f\big(q^{p_0}(z)\big) + (1-\lambda) p_1(z) f\big(q^{p_1}(z)\big)\big) \\ &= \lambda f^*(p_0) + (1-\lambda) f^*(p_1), \end{aligned}$$

where we write $p(z) = \sum_s F_s(z) p(s)$ (with analogous notation for priors $p_0$ and $p_1$) and use the observation that $q^p(z) = \frac{\lambda p_0(z)}{p(z)} q^{p_0}(z) + \frac{(1-\lambda) p_1(z)}{p(z)} q^{p_1}(z)$.

Because $f^*$ and $f$ are concave and $\Delta S$ is compact, there exist sequences of finite sets $A_n, B_n$ of affine functions $l : \Delta S \to \mathbb{R}$ such that $f^*(p) = \lim \min_{a\in A_n} a(p)$ and $f(p) = \lim \min_{b\in B_n} b(p)$, where convergence is uniform in $p$. Let $f_n^*(p) = \min_{a\in A_n} a(p)$ and $f_n(p) = \min_{b\in B_n} b(p)$.

We now construct the sequence of games $(A_n, B_n, g_n)$. Recall that for each affine function $l$, there are constants $g^l_s$, $s \in S$, such that $l(p) = \sum_s g^l_s p_s$. (Since $\sum_s p_s = 1$, this allows for an additive constant term in $l(p)$.) Define $g_n : A_n \times B_n \times S \to \mathbb{R}$ by

$$g_n(a, b, s) = g^b_s - a(Qe_s).$$

(Each $g_n$ depends on $n$ only through $A_n, B_n$.) Then for each $\beta \in \Delta B$ and $q \in \Delta S$,

$$\max_{a\in A_n} g_n(a, \beta, q) = \sum_{b,s} \beta(b)q(s)g^b_s - \min_a a(Qq) = \sum_{b,s} \beta(b)q(s)g^b_s - f^*_n(Qq).$$

Thus, for each $\mu \in \Delta^2 S$ and $m \in \mathcal{M}$, we have

$$\begin{aligned}\int \hat{g}_n\big(m(p)\big)\, d\mu(p) &= \int \left(\min_{\beta\in\Delta B_n} \int \sum_{b,s} \beta(b)q(s)g^b_s\, dm(q|p) - \int f^*_n(Qq)\, dm(q|p)\right) d\mu(p)\\ &= \int \min_{b\in B_n} \sum_s g^b_s p(s)\, d\mu(p) - \int f^*_n(Qp)\, d(\mu * m)(p)\\ &= \int f_n(p)\, d\mu(p) - \int f^*_n(p)\, dQ(\mu * m)(p).\end{aligned}$$

Letting $n \to \infty$ gives

$$\lim_{n\to\infty} \int \hat{g}_n\big(m(p)\big)\, d\mu(p) = \int f(p)\, d\mu(p) - \int f(p)\, d\big(Q(\mu * m) * n^G\big)(p),$$

where we have used the definition of $f^*$. Because $P(\mu_0 * m_0) * n^F \leq^B \mu_0$, Theorem S.3 and the first inequality in (S.6) imply that there exists $\varepsilon > 0$ such that for any $n$ large enough,

$$v(g_n, P, F) \geq \int \hat{g}_n\big(m_0(p)\big)\, d\mu_0(p) > \varepsilon > 0.$$

On the other hand, by the uniform convergence of $f_n$ to $f$ and $f^*_n$ to $f^*$, we have

$$\begin{aligned}0 &\geq \sup_{(\mu,m):Q(\mu*m)*n^G\leq^B\mu} \int f\, d\mu - \int f\, d\big(Q(\mu * m) * n^G\big)\\ &= \sup_{(\mu,m):Q(\mu*m)*n^G\leq^B\mu} \int f\, d\mu - \int f^*\, dQ(\mu * m)\\ &= \lim_{n\to\infty} \sup_{(\mu,m):Q(\mu*m)*n^G\leq^B\mu} \int f_n\, d\mu - \int f^*_n\, d\big(Q(\mu * m) * n^G\big) = \lim_{n\to\infty} v(g_n, Q, G).\end{aligned}$$

Thus, for $n$ large enough, $v(g_n, Q, G) < \varepsilon < v(g_n, P, F)$ as desired.

## S.4. DISCOUNTED GAMES

In this section, we consider comparison of operators in discounted games. This is difficult in general because the discounted value $v^\delta(\pi; g, P)$ is not easy to characterize. Nevertheless, we show here that in any stochastic game of incomplete information $(\delta, \pi, g, P)$, if $\pi$ is an invariant distribution of $P$, then making information less persistent by "adding noise" from the invariant distribution $\pi$ is good for the informed player (i.e., it weakly increases the value).

Recall that $D_\pi$ denotes the i.i.d. operator that maps every belief $p \in \Delta S$ to $\pi$.

THEOREM S.5: *Fix any operator $P$. Then, for every discount factor $\delta$, every invariant distribution $\pi$ of $P$, and every game $g$,*

$$v^\delta(\pi; g, P) \le v^\delta\big(\pi; g, \lambda P + (1-\lambda)D_\pi\big) \quad \text{for all } \lambda \in [0, 1].$$

Some remarks are in order. First, notice that $P$ is not assumed to be ergodic. Indeed, when $\delta < 1$, the value will typically depend on the initial distribution even if $P$ is ergodic. Instead, it is the requirement that the game be started from some invariant distribution of $P$ that provides just enough stationarity for the proof.

Second, for the limit value of the game under an ergodic operator $P$, a result analogous to Theorem S.5 is given by Corollary 2(a) from the main text, which implies that $v(g, P) \le v(g, \lambda P + (1-\lambda)D_{\pi_P})$ for every game $g$ and every $\lambda \in [0, 1]$.

Third, we note that setting $\lambda = 0$ in Theorem S.5 shows, as expected, that an i.i.d. operator is the best case for the informed player even with discounting.

With only two states, the seemingly special case covered by Theorem S.5 allows us to rank all operators with positive eigenvalues and a common invariant distribution.

COROLLARY S.1: *Let $|S| = 2$. Suppose operators $P$ and $Q$ have a common invariant distribution $\pi$ and nonnegative smallest eigenvalues $\phi_P$ and $\phi_Q$. Then, for every discount factor $\delta$ and every game $g$, $v^\delta(\pi; g, P) \le v^\delta(\pi; g, Q)$ if and only if $\phi_P \ge \phi_Q$.*

PROOF: It is easy to verify that if $\phi_P \ge \phi_Q \ge 0$, then we have $Q = \lambda P + (1-\lambda)D_\pi$ for some $\lambda \in [0, 1]$. Similarly, $\phi_P < \phi_Q$ implies $P = \lambda Q + (1-\lambda)D_\pi$ for some $\lambda \in [0, 1)$. The claim thus follows from Theorem S.5. *Q.E.D.*

EXAMPLE S.3: Consider Example 1 from the main text. The above corollary allows us to rank all operators such that $\rho \ge \frac{1}{2}$. In particular, the value is decreasing in $\rho$ on $[\frac{1}{2}, 1]$ for any discount factor, provided that the initial distribution $\pi$ is the invariant distribution $(\frac{1}{2}, \frac{1}{2})$.

### S.4.1. *Proof of Theorem S.5*

Fix a stochastic game of incomplete information $(\delta, \pi, g, P)$, where $\pi$ is an invariant distribution of $P$, and fix $\lambda \in [0, 1]$.

LEMMA S.11: *For every $\mu \in \Delta^2 S$ such that $\mathcal{E}\mu = \pi$,*

$$\int v^\delta\big(\lambda p + (1-\lambda)\pi; g, P\big)\, d\mu(p) \ge \int v^\delta(p; g, P)\, d\mu(p).$$

PROOF: Since $v^\delta(p; g, P)$ is concave in $p$ (by Lemma 1 in the main text), the result follows by noting that the distribution of priors on the right-hand side is a mean-preserving spread of that on the left-hand side. *Q.E.D.*

LEMMA S.12: *For every* $\mu \in \Delta^2 S$ *such that* $\mathcal{E}\mu = \pi$, *and every Markov strategy* $\alpha : \Delta S \to (\Delta A)^S$,

$$\int v^\delta\big(p; g, \lambda P + (1-\lambda)D_\pi\big)\,d\mu(p) \geq (1-\delta)\int \min_\beta g\big(\alpha(p), \beta, p\big)\,d\mu(p) + \delta \int v^\delta\big(Pq^{\alpha(p),p}; g, \lambda P + (1-\lambda)D_\pi\big)\,d\mu(p).$$

PROOF: The recursive formula in Lemma 1 in the main text implies that

$$\int v^\delta\big(p; g, \lambda P + (1-\lambda)D_\pi\big)\,d\mu(p) \geq (1-\delta)\int \min_\beta g\big(\alpha(p), \beta, p\big)\,d\mu(p) + \delta \int v^\delta\big(\big(\lambda P + (1-\lambda)D_\pi\big)q^{\alpha(p),p}; g, \lambda P + (1-\lambda)D_\pi\big)\,d\mu(p).$$

Since $D_\pi q = \pi$ for every $q$, the second line above equals

$$\delta \int v^\delta\big(\lambda P q^{\alpha(p),p} + (1-\lambda)\pi; g, \lambda P + (1-\lambda)D_\pi\big)\,d\mu(p).$$

Now, $\mathcal{E}\mu = \pi$ implies via Bayes's rule that $\int q^{\alpha(p),p}\,d\mu(p) = \pi$. Because $\pi$ is an invariant distribution of $P$, this implies that $\int Pq^{\alpha(p),p}\,d\mu(p) = P\int q^{\alpha(p),p}\,d\mu(p) = P\pi = \pi$. Thus, the above display is not smaller than

$$\delta \int v^\delta\big(Pq^{\alpha(p),p}; g, \lambda P + (1-\lambda)D_\pi\big)\,d\mu(p)$$

by Lemma S.11. *Q.E.D.*

We can move to the proof of the theorem. Let $H = \bigcup_t A^t$ be the space of histories of the informed player's actions. Let $\alpha^* : S \times H \to \Delta A$ be an optimal strategy for the maximizer in the game $(\delta, \pi, g, P)$ we fixed in the beginning of the proof. Let $p(h)$ be the associated prior after history $h \in H$. For each history $h = (a_1, a_2, \dots, a_t) \in A^t$, let $\gamma(h)$ be the ex ante probability of history $h$. It is straightforward to verify that $\sum_{h\in A^t} \gamma(h)p(h) = \pi$. Repeated application of Lemma S.12 then gives the following sequence of inequalities:

$$\begin{aligned}
&v^\delta\big(\pi; g, \lambda P + (1-\lambda)D_\pi\big) \\
&\quad\geq \min_{\beta: H\to\Delta B} (1-\delta)g\big(\alpha^*(\emptyset), \beta, \pi\big) + \delta\sum_{a_1}\gamma(a_1)v^\delta\big(p(a_1); g, \lambda P + (1-\lambda)D_\pi\big) \\
&\quad\geq \min_{\beta: H\to\Delta B} (1-\delta)g\big(\alpha^*(\emptyset), \beta, \pi\big) + \delta(1-\delta)\sum_{a_1}\gamma(a_1)g\big(\alpha^*(a_1), \beta(a_1), p(a_1)\big) \\
&\qquad + \delta^2\sum_{a_1,a_2}\gamma(a_1,a_2)v^\delta\big(p(a_1,a_2); g, \lambda P + (1-\lambda)D_\pi\big) \\
&\quad\geq \min_{\beta: H\to\Delta B} (1-\delta)g\big(\alpha^*(\emptyset), \beta, \pi\big) + \delta(1-\delta)\sum_{a_1}\gamma(a_1)g\big(\alpha^*(a_1), \beta(a_1), p(a_1)\big) \\
&\qquad + (1-\delta)\delta^2\sum_{a_1,a_2}\gamma(a_1,a_2)g\big(\alpha^*(a_1,a_2), \beta(a_1,a_2), p(a_1,a_2)\big)
\end{aligned}$$

$$+\delta^3 \sum_{a_1,a_2,a_3} \gamma(a_1,a_2,a_3) v^\delta\big(p(a_1,a_2,a_3); g, \lambda P + (1-\lambda)D_\pi\big)$$

$$\geq \cdots.$$

This gives a decreasing sequence of numbers $v_t$ such that $v^\delta(\pi; g, \lambda P + (1-\lambda)D_\pi) \geq v_1$ and $v_t \to v^\delta(\pi; g, P)$. Hence, $v^\delta(\pi; g, \lambda P + (1-\lambda)D_\pi) \geq v^\delta(\pi; g, P)$ as desired.

## S.5. OPERATORS WITH COMPLEX EIGENVALUES

In this section, we show that the characterization given by conditions (a) and (c) of Corollary 2 does not extend to operators with complex eigenvalues.

PROPOSITION S.2: *Let $S = \{1, 2, 3\}$. There exist ergodic operators $P$ and $Q$ such that $P \preceq Q$ and $Q \notin \{P(\lambda) : \lambda \in \Lambda\}$.*

### S.5.1. *Operator P*

Let $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \Delta S$. For each $q \in \Delta S$, let

$$d(q) = \|q - \pi\|$$

be the distance from $\pi$. Define the operator

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

which defines a rotation of the space $\Delta S$. Note that $d(q) = d(Rq)$ for each $q$.

Fix $\alpha = \frac{1}{2}$ and define the operator $P$ by

$$P = \alpha R + (1-\alpha)D_\pi.$$

Then, $P$ is ergodic with invariant distribution $\pi$. It is easy to check that for each $q$,

$$d(Pq) = \big\|\alpha Rq + (1-\alpha)D_\pi q - \pi\big\| = \|\alpha Rq - \alpha\pi\| = \alpha\|Rq - \pi\| = \alpha d(Rq) = \alpha d(q).$$

LEMMA S.13: *There exists $\kappa > 0$ such that for every $p$ and $q$ in $\Delta S$, if $d(q) \leq \kappa p$, then $q \in \operatorname{con}\{Pp, P^2p, P^3p\}$.*

PROOF: We sketch the proof. Note that $\operatorname{con}\{P^3p, RP^3p, R^2P^3p\} \subseteq \operatorname{con}\{Pp, P^2p, P^3p\}$, where $\operatorname{con}\{P^3p, RP^3p, R^2P^3p\}$ is an equilateral triangle whose sides are of length $ad(p)$, with $a = \frac{1}{2^4}\sqrt{2}$. Let $\kappa = \frac{1}{3}\frac{\sqrt{3}}{2}a$. Then $\kappa d(p)$ is the radius of the largest ball that fits into this triangle. (The argument is best seen by drawing a picture.) *Q.E.D.*

LEMMA S.14: *For every $\lambda \in \Lambda$, there exist numbers $a$, $b$, and $c$ such that*

$$P(\lambda) = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}. \tag{S.7}$$

PROOF: Recall that, by definition, $D_\pi q = \pi$ for every $q$. Thus, $RD_\pi = D_\pi R = D_\pi$. It follows that for each $k$,

$$P^k = \big(\alpha R + (1-\alpha)D_\pi\big)^k = \alpha^k R^k + \big(1-\alpha^k\big)D_\pi.$$

Thus, $P(\lambda) = \sum_k \lambda_k P^k$ is a convex combination of $D_\pi$ and rotations $R^k$, $k = 1, 2, \dots$, each of which is equal to either $R$, $R^2$, or $R^3$. The claim now follows by verifying that any such combination has the form (S.7) for some numbers $a$, $b$, and $c$. *Q.E.D.*

### S.5.2. *Operator Q*

We then construct the operator $Q$ with invariant distribution $\pi$.

LEMMA S.15: *There exists an ergodic operator $Q$ such that $\pi$ is the invariant distribution of $Q$ and $d(Qp) \le \frac{1}{3}\kappa d(p)$ for every $p \in \Delta S$, but $Q$ is not equal to the right-hand side of* (S.7) *for any $a$, $b$, and $c$.*

PROOF: Take

$$Q = \begin{bmatrix} \frac{1}{3} + x & \frac{1}{3} - x & \frac{1}{3} \\ \frac{1}{3} - x & \frac{1}{3} + x & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

for sufficiently small $x > 0$. *Q.E.D.*

### S.5.3. *Proof of Proposition S.2*

The operators $P$ and $Q$ constructed above satisfy $Q \notin \{P(\lambda) : \lambda \in \Lambda\}$ by Lemmas S.14 and S.15. It remains to show that $P \preceq Q$.

Take any $\nu \in \Delta^2 S$ such that $P\nu \le^B \nu$. It follows[5] that

$$P^2\nu = P(P\nu) \le^B P\nu,$$

and similarly, that $P^3\nu \le^B P\nu$. Thus,

$$\frac{1}{3}\big(P\nu + P^2\nu + P^3\nu\big) \le^B P\nu. \tag{S.8}$$

By Theorem 2(c), $P \preceq Q$ follows if (and only if) we show that $Q\nu \le^B P\nu$, or equivalently, that for any convex function $f : \Delta S \to R$, we have

$$\int f(p)\, dQ\nu(p) \le \int f(p)\, dP\nu(p). \tag{S.9}$$

[5]We use the following easy-to-establish facts about the Blackwell order. For any $\mu, \psi, \tau \in \Delta^2 S$ and any operator $P$,

- if $\mu \le^B \psi$, then $P\mu \le^B P\psi$;
- if $\mu, \psi \le \tau$, then for each $\alpha \in (0,1)$, $\alpha\mu + (1-\alpha)\psi \le^B \tau$. (It is enough to check that for each concave $f$, $\int f(p)\, d(\alpha\mu + (1-\alpha)\psi)(p) \ge \int f(p)\, d\tau(p)$, which follows from the hypothesis.)

So fix any convex function $f$. Let $l : \Delta S \to R$ be an affine function such that $l(\pi) = f(\pi)$ and such that for each $p$, $l(p) \leq f(p)$. Let $f_0(p) = f(p) - l(p)$. Then (S.9) is equivalent to

$$\int f(p)\,dP\nu(p) - \int f(p)\,dQ\nu(p) = \int f_0(p)\,dP\nu(p) - \int f_0(p)\,dQ\nu(p) + \int l(p)\,dP\nu(p) - \int l(p)\,dQ\nu(p).$$

The last two terms cancel each other out. To see this, note that $P\nu \leq^B \nu$ implies that $\mathcal{E}\nu = \pi$. But $\pi$ is also the invariant distribution of $Q$, and so we have

$$\int l(p)\,dP\nu(p) = l(\mathcal{E}P\nu) = l(P\mathcal{E}\nu) = l(\pi) = l(Q\mathcal{E}\nu) = l(\mathcal{E}Q\nu) = \int l(p)\,dQ\nu(p).$$

Thus, it is enough to prove (S.9) for convex functions $f$ such that

$$f(\pi) = 0 \leq f(p) \quad \text{for every } p \in \Delta S. \tag{S.10}$$

Further, inequality (S.8) implies that it is enough to show that

$$\begin{aligned} 0 &\leq \int f(p)\,d\left(\frac{1}{3}\left(P\nu + P^2\nu + P^3\nu\right)\right)(p) - \int f(p)\,dQ\nu(p) \\ &= \int \left[\frac{1}{3}\left(f(Pp) + f\left(P^2 p\right) + f\left(P^3 p\right)\right) - f(Qp)\right] d\nu(p). \end{aligned}$$

This follows from the following lemma:

LEMMA S.16: *Suppose that $f$ satisfies* (S.10). *Then for every $p \in \Delta S$,*

$$\frac{1}{3}\left(f(Pp) + f\left(P^2 p\right) + f\left(P^3 p\right)\right) \geq f(Qp).$$

PROOF: Fix $p$ and let $q$ be a vector so that

$$Qp = \frac{1}{3}q + \frac{2}{3}\pi.$$

Then $d(q) = 3d(Qp) \leq \kappa d(p)$ by the choice of operator $Q$. Lemma S.13 then implies that $q \in \mathrm{con}\{Pp, P^2 p, P^3 p\}$. Let $q'$ be a vector on the boundary of the set $\mathrm{con}\{Pp, P^2 p, P^3 p\}$ such that $q = \lambda q' + (1-\lambda)\pi$ for some $\lambda \in (0,1)$. Assume without loss of generality that $q'$ belongs to the interval $[Pp, P^2 p]$. (The other boundaries of $\mathrm{con}\{Pp, P^2 p, P^3 p\}$ are handled analogously.) Because $f$ is convex and $f(\pi) = 0$,

$$f(Qp) \leq \frac{1}{3}f(q) \leq \frac{1}{3}f\left(q'\right) \leq \frac{1}{3}\max\left\{f(Pp), f\left(P^2 p\right)\right\} \leq \frac{1}{3}\left(f(Pp) + f\left(P^2 p\right) + f\left(P^3 p\right)\right),$$

as desired. *Q.E.D.*

## S.6. COMPARATIVE STATICS ON INFORMATION

A natural interesting question is what does our notion of persistence, $\preceq$, imply about behavior. A possible intuition is that a less persistent operator allows the maximizer in some average sense to reveal more information. One way to measure this is to consider the dispersion of the minimizer's beliefs. If beliefs are more disperse (in the Blackwell sense), then the minimizer's information is better.

More formally, fix a game $g$. For each ergodic operator $P$, let $(\mu_P, m_P) \in \Delta^2 S \times \mathcal{M}$ be the distribution of priors and the revelation policy that solve the value problem in Theorem 1. Then $\mu_P$ is the (time-)average distribution of the minimizer's priors and $\mu_P * m_P$ is the average distribution of his posteriors. Suppose that $P \preceq Q$. The above intuition suggests that $\mu_P * m_P \leq^B \mu_Q * m_Q$, or perhaps $\mu_P \leq^B \mu_Q$.

As it turns out, this intuition is not correct.

PROPOSITION S.3: *There exist a game $g$ and ergodic operators $P$ and $Q$ such that $P \preceq Q$ and for every $(\mu_P, m_P)$ and $(\mu_Q, m_Q)$ that solve the value problems, neither $\mu_P \leq^B \mu_Q$ nor $\mu_P * m_P \leq^B \mu_Q * m_Q$.*

### S.6.1. *Proof of Proposition S.3*

Let $S = \{X_0, X_1\} \times \{Y_0, Y_1\}$. Thus, the state space has four elements.[6] Let $P_\alpha$ be an ergodic operator on $\Delta S$ such that each of the two dimensions changes independently with probability $\alpha$. That is,

$$P_\alpha = \begin{bmatrix} (1-\alpha)^2 & \alpha(1-\alpha) & \alpha(1-\alpha) & \alpha^2 \\ \alpha(1-\alpha) & (1-\alpha)^2 & \alpha^2 & \alpha(1-\alpha) \\ \alpha(1-\alpha) & \alpha^2 & (1-\alpha)^2 & \alpha(1-\alpha) \\ \alpha^2 & \alpha(1-\alpha) & \alpha(1-\alpha) & (1-\alpha)^2 \end{bmatrix},$$

where the rows and columns correspond to states $X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1$.

LEMMA S.17: *For each $\alpha \in (0, \frac{1}{2})$, $P_\alpha \preceq P_\alpha^2 = P_{2\alpha(1-\alpha)}$.*

PROOF: The inequality is by Corollary 2(a). The equality is by direct calculations. *Q.E.D.*

Construct a game $(A, B, g)$ with action sets

$$A = \{x_0, x_1, y_0, y_1\} \quad \text{and} \quad B = \{\xi_0, \xi_1\} \times \{\psi_0, \psi_1\},$$

and payoff function

$$g(a, b, s) = g_{\max}(a, s) - g_{\min}(b, s),$$

with

$$g_{\max}(a, X_k Y_l) = \begin{cases} 2\delta_{ik} - 1 & \text{if } a = x_i, \\ 10(2\delta_{jl} - 1) & \text{if } a = y_j, \end{cases}$$

[6]It is important for our example that $\Delta S$ has more than one dimension. In our example, $\Delta S$ is three-dimensional. It should be possible to construct a similar example with only three states, but we suspect that the comparative statics of revealed information hold when there are only two states.

and

$$g_{\min}(\xi_i\psi_j, X_kY_l) = \frac{1}{2}(2\delta_{ik}-1) + 20(2\delta_{jl}-1),$$

where $\delta_{mn}$ denotes the Dirac measure, that is, $\delta_{mn}=1$ if $m=n$ and $\delta_{mn}=0$ otherwise.

LEMMA S.18: *For every $\alpha \in (0, \frac{1}{2})$, there exists a unique maximizer $(\mu_{P_\alpha}, m_{P_\alpha})$ to the value problem. Moreover, there exists $\alpha_0 \in (0, \frac{1}{2})$ such that for every $\alpha < \alpha_0 < \alpha'$, neither $\mu_{P_\alpha} \leq^B \mu_{P_{\alpha'}}$ nor $\mu_{P_\alpha} * m_{P_\alpha} \leq^B \mu_{P_{\alpha'}} * m_{P_{\alpha'}}$.*

Proposition S.3 follows from the above lemmas, since $\alpha \leq 2\alpha(1-\alpha)$ for $\alpha \in (0, \frac{1}{2})$.

*Proof of Lemma S.18*

For each belief $q \in \Delta S$, let

$$X_q = \left|q(X_1Y_0) + q(X_1Y_1) - q(X_0Y_0) - q(X_0Y_1)\right|,$$
$$Y_q = \left|q(X_0Y_1) + q(X_1Y_1) - q(X_0Y_0) - q(X_1Y_0)\right|.$$

Heuristically, $X_q$ and $Y_q$ measure the quality of information, respectively, about the $X$-dimension and the $Y$-dimension of the state. Notice that for each $\alpha$, we have $X_{P_\alpha q} = (1-2\alpha)X_q$ and $Y_{P_\alpha q} = (1-2\alpha)Y_q$.

Recalling the definition of $\hat{g}$, we have for any $\nu \in \Delta^2 S$,

$$\begin{aligned}\hat{g}(\nu) &= \min_\beta \int \max_a g(a, \beta, q)\, d\nu(q) \\ &= \int \max_a g_{\max}(a, q)\, d\nu(q) - \max_\beta \int g_{\min}(\beta, q)\, d\nu(q) \\ &= \int \max_a g_{\max}(a, q)\, d\nu(q) - \max_\beta g_{\min}(\beta, \mathcal{E}\nu) \\ &= \int h_{\max}(q)\, d\nu(q) - h_{\min}(\mathcal{E}\nu),\end{aligned}$$

where for each $q \in \Delta S$,

$$h_{\max}(q) = \max_a \sum_s q(s) g_{\max}(a, s) = \max\{X_q, 10Y_q\},$$

$$h_{\min}(q) = \max_b \sum_s q(s) g_{\min}(b, s) = \frac{1}{2}X_q + 20Y_q.$$

(No mixing is necessary because the players' actions do not directly interact.)

By Theorem 1 (and the remarks afterwards), we then have

$$\begin{aligned}v(g, P_\alpha) &= \max_{(\mu,m):P_\alpha(\mu * m)=\mu} \int \left(\int h_{\max}(q)\, dm(q|p) - h_{\min}\big(\mathcal{E}m(p)\big)\right) d\mu(p) \\ &= \max_{(\mu,m):P_\alpha(\mu * m)=\mu} \int \int h_{\max}(q)\, dm(q|p)\, d\mu(p) - \int h_{\min}(p)\, d\mu(p) \qquad \text{(S.11)}\end{aligned}$$

$$= \max_{(\mu,m):P_\alpha(\mu*m)=\mu} \int h_{\max}(q)\,d(\mu*m)(q) - \int h_{\min}(p)\,d\mu(p)$$

$$= \max_{(\mu,m):P_\alpha(\mu*m)=\mu} \int h_{\max}(q)\,d(\mu*m)(q) - \int h_{\min}(p)\,dP_\alpha(\mu*m)(p)$$

$$= \max_{\nu:P_\alpha\nu\leq^B\nu} \int h_{\max}(q)\,d\nu(q) - \int h_{\min}(q)\,dP_\alpha\nu(q)$$

$$= \max_{\nu:P_\alpha\nu\leq^B\nu} \int \big(h_{\max}(q) - h_{\min}(P_\alpha q)\big)\,d\nu(q)$$

$$= \max_{\nu:P_\alpha\nu\leq^B\nu} \int \left(\max\{X_q, 10Y_q\} - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right)\right) d\nu(q),$$

where the fifth line replaces $\mu * m$ with $\nu$.

Define the four beliefs $q_{X0}, q_{X1}, q_{Y0}q_{Y1} \in \Delta S$ by

$$q_{X0}(X_0Y_0) = q_{X0}(X_0Y_1) = \frac{1}{2}, \qquad q_{X1}(X_1Y_0) = q_{X1}(X_1Y_1) = \frac{1}{2},$$

$$q_{Y0}(X_0Y_0) = q_{Y0}(X_1Y_0) = \frac{1}{2}, \qquad q_{Y1}(X_0Y_1) = q_{Y1}(X_1Y_1) = \frac{1}{2}.$$

For instance, under $q_{X0}$ the minimizer knows that the $X$-dimension of the state is equal to $X_0$, but she does not know the $Y$-dimension and believes both values to be equally likely. Consider the two distributions $\nu_X, \nu_Y \in \Delta^2 S$ defined by

$$\nu_X(q_{X0}) = \nu_X(q_{X1}) = \frac{1}{2} \quad \text{and} \quad \nu_Y(q_{Y0}) = \nu_Y(q_{Y1}) = \frac{1}{2}.$$

In particular, $\nu_X$ assigns positive probability to two beliefs, for both of which the $X$-dimension of the state is perfectly known and the $Y$-dimension is believed to be equal to $Y_0$ with probability $\frac{1}{2}$. In $\nu_Y$, the $Y$-dimension is known. We note that both are feasible in the maximization problem in (S.11):

CLAIM S.1: *For any $\alpha \in (0, \frac{1}{2})$, $P_\alpha\nu_X \leq \nu_X$ and $P_\alpha\nu_Y \leq \nu_Y$. Moreover, neither $\nu_X \leq^B \nu_Y$ nor $P_\alpha\nu_X \leq^B P_\alpha\nu_Y$.*

PROOF: Direct verification. *Q.E.D.*

CLAIM S.2: *If $1 - \frac{1}{2}(1-2\alpha) < 10(1-2(1-2\alpha))$, then the problem (S.11) is uniquely maximized by $\nu_X$.*

PROOF: Notice first that for each $q \in \{q_{X0}, q_{X1}\}$, $X_q = 1$, $Y_q = 0$, and

$$\max\{X_q, 10Y_q\} - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) = 1 - \frac{1}{2}(1-2\alpha).$$

Moreover, for any $q \notin \{q_{X0}, q_{X1}\}$,

$$\max\{X_q, 10Y_q\} - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) < 1 - \frac{1}{2}(1-2\alpha).$$

(Indeed, for any $q \notin \{q_{X0}, q_{X1}\}$, either $X_q < 1$, or $Y_q > 0$. Moreover, either $X_q < 10Y_q$, or $X_q \geq 10Y_q$. If $X_q < 10Y_q$, then

$$\max\{X_q, 10Y_q\} - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) = 10Y_q - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) \leq 10\big(1 - 2(1-2\alpha)\big)Y_q < 1 - \frac{1}{2}(1-2\alpha).$$

If $X_q \geq 10Y_q$, and $X_q < 1$, then

$$\max\{X_q, 10Y_q\} - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) = X_q - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) \leq \big(1 - 2(1-2\alpha)\big)X_q < 1 - \frac{1}{2}(1-2\alpha).$$

If $X_q \geq 10Y_q$, and $Y_q > 0$, then

$$\max\{X_q, 10Y_q\} - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) = X_q - (1-2\alpha)\left(\frac{1}{2}X_q + 20Y_q\right) < \left(1 - \frac{1}{2}(1-2\alpha)\right)X_q \leq 1 - \frac{1}{2}(1-2\alpha).$$

Thus, $\nu_X$ achieves the maximum in (S.11) and any distribution $\nu$ that achieves the maximum must have its support concentrated on $\{q_{X0}, q_{X1}\}$. It is easy to check that $\nu_X$ is the only such distribution that satisfies the constraint $P_\alpha \nu \leq \nu$. *Q.E.D.*

CLAIM S.3: *If* $1 - \frac{1}{2}(1-2\alpha) > 10(1 - 2(1-2\alpha))$, *then the problem* (S.11) *is uniquely maximized by* $\nu_Y$.

PROOF: The claim is proven in similar fashion as the previous claim. *Q.E.D.*

Finally, notice that if $\alpha_0 = \frac{10.5}{39}$, then for $\alpha < \alpha_0 < \alpha'$ we have

$$1 - \frac{1}{2}(1-2\alpha) < 10\big(1 - 2(1-2\alpha)\big) \quad \text{and} \quad 1 - \frac{1}{2}\big(1-2\alpha'\big) > 10\big(1 - 2\big(1-2\alpha'\big)\big).$$

Lemma S.18 thus follows from the above two claims.

## S.7. CONNECTION TO RENAULT AND VENEL (2017)

As discussed at the end of Section 3, Renault and Venel (2017) (RV henceforth) studied the long-run values of Markov decision problems (MDPs) and repeated games in a setting general enough to include our model as a special case. Their results can be used to derive an alternative formula for the value $v(\pi; g, P)$, which is different from, but equivalent to, that in Theorem 1. We state here Renault and Venel's formula, specialized to our setting, and sketch how to connect it to Theorem 1 in the case of an ergodic operator.

We use the notation from the main text. (Some of the notation used in this section is introduced in the beginning of the proof of Theorem 1 in Section 6.) In order to avoid confusion, we label the additional notation introduced for the RV characterization with the superscript "RV".

To state the RV formula, we introduce the following auxiliary MDP, which is suggested by the well-known Lemma 1 in the main text. Let $X^{RV} = \Delta S$ be the state space of the MDP; this is the space of the minimizer's belief about the state in the game. Let $A^{RV} = (\Delta A)^S$ be the action space of the MDP; this is the space of the maximizer's stage-game mixed strategies in the game. Define the per-period payoff function $g^{RV} : X^{RV} \times A^{RV} \to \mathbb{R}$ for the MDP by

$$g^{RV}(p, \alpha) = \min_{\beta \in \Delta B} \sum_a g\big(a, \beta, q^{\alpha,p}(a)\big)\overline{\alpha}(a);$$

this is the maximizer's expected stage-game payoff in the game when the minimizer is assumed to best-respond to the maximizer's strategy $\alpha$ given his belief $p$ about the state. Finally, define the transition function $q^{RV} : X^{RV} \times A^{RV} \to \Delta X^{RV}$ for the MDP by setting

$$q^{RV}(p, \alpha) = Pq^{\alpha,p} \in \Delta X^{RV} = \Delta^2 S;$$

this is the distribution of the next period's state as a function of the current state $p$ and current action $\alpha$. Note that the transition function $q^{RV}$ in the MDP captures both the revelation of information and the effect of the exogenous state transition in the game. To see this, recall that in the game, $q^{\alpha,p}$ is the distribution of the minimizer's posteriors at the end of the period resulting from the information revealed by the maximizer's action under the strategy $\alpha$ given prior $p$. Then $Pq^{\alpha,p}$ is the distribution of the next period's priors after the state has evolved according to the transition described by the operator $P$.

Let $v^{RV,\delta}(x)$ be the discounted-average value of the MDP $(X^{RV}, A^{RV}, g^{RV}, q^{RV})$ starting at state $x$ given discount factor $\delta < 1$. RV showed that the limit (or long-run) value $v^{RV}(x) = \lim_{\delta \to 1} v^{RV,\delta}(x)$ exists.

It is well-known that our repeated zero-sum game with one-sided incomplete information can be reduced to the above MDP. This can be seen from Lemma 1 in the main text by applying the minmax theorem and using the above definitions. This equivalence implies that for each $p \in \Delta S$, we have

$$v^{RV,\delta}(p) = v^\delta(p; g, P) \quad \text{and} \quad v^{RV}(p) = v(p; g, P),$$

where the right-hand sides are, respectively, the discounted value and the limit value of the game as defined in the main text. Note, in particular, that any formula for the limit value $v^{RV}(p)$ thus gives a formula for the value $v(p; g, P)$.

In order to state RV's formula, define $RR^{RV} \subseteq (\Delta^2 S) \times \mathbb{R}$ as the set of distribution-payoff pairs $(\mu, y)$ for which there exists a Markov strategy $\alpha : \Delta S \to (\Delta A)^S$ such that

$$\mu = \int Pq^{\alpha(p),p}\, d\mu(p) \quad \text{and} \quad y = \int g^{RV}\big(p, \alpha(p)\big)\, d\mu(p).$$

The interpretation is that if the maximizer plays $\alpha$, then $\mu$ is an invariant distribution of the state (i.e., beliefs) and $y$ is the associated expected long-run payoff.

RV's characterization of the limit value is the following:

RV'S THEOREM: *For every* $\pi \in \Delta S$,

$$v^{RV}(\pi) = \inf \left\{ w(\pi) \text{ s.t. } \begin{array}{c} w : \Delta^2 S \to \mathbb{R} \textit{ is affine and continuous}, \\ \forall p \in \Delta S, w(p) \geq \max_\alpha w\big(q^{RV}(p, \alpha)\big), \\ \forall (\mu, y) \in RR^{RV}, w(\mu) \geq y \end{array} \right\}. \tag{S.12}$$

(Here $w(p)$ denotes the value of $w : \Delta^2 \to \mathbb{R}$ at the Dirac measure $e_p \in \Delta^2 S$.)

On the other hand, for a fixed game $g$ and operator $P$, our result is the following:

THEOREM 1: *For every* $\pi \in \Delta S$,

$$v(\pi; g, P) = \max_{(\mu,m)\in\Delta^2 S\times\mathcal{M}:P(\mu*m)\leq^B \mu \text{ and } \psi_{\pi,P}\leq^B \mu} \int \hat{g}\big(m(p)\big)\, d\mu(p).$$

Our proof of Theorem 1 and RV's proof of their result are unrelated: ours is direct and elementary, RV's relies on some new facts about a certain metric on probability spaces.

The key advantage of Theorem 1 over RV's Theorem for the comparison of operators is that it separates the choice of a stationary information structure $(\mu, m)$ from the choice of actions, and the operator $P$ only enters the constraints. In contrast, it is unclear how to derive Theorem 2 or 2′ starting from RV's formula.

Given that the equivalence of the formulas is not readily apparent, it is natural to wonder about their precise connection. We sketch here how to directly verify their equivalence when the operator $P$ is ergodic.[7] We conjecture that a similar argument can be given for general $P$.

We start by restating our formula in a form closer to RV's result. For this, we need to essentially undo Lemma 2 in the main text, and rewrite the problem with the maximizer choosing a Markov strategy $\alpha : \Delta S \to (\Delta A)^S$ rather than an information revelation policy $m \in \mathcal{M}$. Omitting any technical verifications, we have

$$\begin{aligned}
\hat{g}\big(m(p)\big) &= \min_{\beta\in\Delta B} \int \max_a g(a, \beta, q)\, dm(q|p) \\
&= \min_{\beta\in\Delta B} \max_{\alpha\in(\Delta A)^S:\nu^{\alpha,p}\leq^B m(p)} \sum_a g\big(a, \beta, q^{\alpha,p}(a)\big)\overline{\alpha}(a) \\
&= \max_{\alpha\in(\Delta A)^S:\nu^{\alpha,p}\leq^B m(p)} \min_{\beta\in\Delta B} \sum_a g\big(a, \beta, q^{\alpha,p}(a)\big)\overline{\alpha}(a) \\
&= \max_{\alpha\in(\Delta A)^S:\nu^{\alpha,p}\leq^B m(p)} g^{RV}(p, \alpha).
\end{aligned}$$

Recall that in the ergodic case, Theorem 1 simplifies to

$$v(g, P) = \max_{(\mu,m)\in\Delta^2 S\times\mathcal{M}:P(\mu*m)=\mu} \int \hat{g}\big(m(p)\big)\, d\mu(p),$$

where we have also used the fact that the maximum is achieved by some $(\mu, m)$ that satisfies the constraint as an equality. Substituting for $\hat{g}(m(p))$ using the expression from above then yields, after some further manipulations (omitting again any details), a formula that starts to bear some resemblance to RV's result:

$$\begin{aligned}
v(g, P) &= \max_{(\mu,m)\in\Delta^2 S\times\mathcal{M}:P(\mu*m)=\mu} \int \max_{\alpha\in(\Delta A)^2:\nu^{\alpha,p}\leq^B m(p)} g^{RV}(p, \alpha)\, d\mu(p) \\
&= \max_{(\mu,m)\in\Delta^2 S\times\mathcal{M}:P(\mu*m)=\mu}\ \max_{\alpha:\Delta S\to(\Delta A)^S:\forall p,\nu^{\alpha,p}\leq^B m(p)} \int g^{RV}\big(p, \alpha(p)\big)\, d\mu(p) \qquad \text{(S.13)}
\end{aligned}$$

[7]We thank an anonymous referee for supplying some of the steps for the argument.

$$= \max_{\alpha:\Delta S\to(\Delta A)^S} \max_{\mu\in\Delta^2 S:P(\mu*\nu^{\alpha,\cdot})=\mu} \int g^{RV}\big(p,\alpha(p)\big)\,d\mu(p)$$
$$= \max\big\{y:(\mu,y)\in RR^{RV} \text{ for some } \mu\big\}.$$

We now conclude the argument by showing that RV's formula (S.12) is in fact equivalent to (S.13) when $P$ is ergodic. Note that since $v^{RV}$ is concave, the functions $w$ in (S.12) can be taken to be such that their restriction to $\Delta S$ is concave. Moreover, the condition $\forall p\in\Delta S, w(p)\geq \max_\alpha w(q^{RV}(p,\alpha))$ implies that $\forall p\in\Delta S, w(p)\geq w(\pi_P)$. To see this, note that one feasible strategy $\alpha$ is to not reveal anything about the state by always playing the same action. For any such $\alpha$, $q^{RV}(p,\alpha)$ is just a point mass at $Pp$. Thus we have $w(p)\geq w(Pp)\geq w(P^2p)\geq\cdots\geq w(\pi_P)$ by ergodicity of $P$ and continuity of $w$. But $\pi_P$ is in the interior of $\Delta S$ by ergodicity, so any $w$ satisfying the above conditions is just a constant function. Therefore, (S.12) simplifies to

$$v^{RV} = \inf\big\{w\in\mathbb{R}:\forall(\mu,y)\in RR^{RV}, w\geq y\big\}. \tag{S.14}$$

Equations (S.13) and (S.14) are equivalent by inspection.

The above argument suggests that at least in the ergodic case, our formula can be viewed, in a sense, to be the dual of RV's formula. However, it is important to notice that this duality is apparent only after we transform our formula to the version stated as (S.13). This transformation amounts to undoing the very step in the proof of Theorem 1 that allows separating the choice of a stationary belief distribution from the choice of actions (i.e., Lemma 2), which is the key novel feature of our characterization.

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