Econometrica Supplementary Material

SUPPLEMENT TO "A FAIRNESS JUSTIFICATION OF UTILITARIANISM"

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S1. DOMAIN MAXIMALITY

FOLLOWING THE APPROACH OF Barberà, Sonnenschein, and Zhou (1991) and Ching and Serizawa (1998), we show that our domain of preferences cannot be enlarged. If we were to allow individuals to have strongly monotonic, continuous, and convex preferences that do not admit a concave representation, an impossibility result would emerge.

Let $R \equiv (R_i)_{i \in N}$ be a preference profile. Let $\mathcal{R}$ be the domain of preference profiles adopted. For memory, each preference profile $R \in \mathcal{R}$ satisfies: (a) for each $i \in N$, R_i is strongly monotonic, continuous, and admits a concave representation; (b) for each $i \in N$, there exists $j \in N \backslash \{i\}$ such that $R_i = R_j$. A different domain of preferences $\bar{\mathcal{R}}$ can be defined. Each preference profile $R \in \bar{\mathcal{R}}$ satisfies: ($\bar{\text{a}}$) for each $i \in N$, R_i is a weak order on X that is strongly monotonic, continuous, and convex; ($\bar{\text{b}}$) for each $i \in N$, there exists $j \in N \backslash \{i\}$ such that $R_i = R_j$.

Domain Maximality: A domain $\mathcal{R}^*$ is a maximal domain for a set of axioms if:

(i) $\mathcal{R}^* \subseteq \bar{\mathcal{R}}$;

(ii) there exists a social ranking satisfying the axioms on $\mathcal{R}^*$;

(iii) there exists no domain $\mathcal{R}^+$ such that $\mathcal{R}^* \subset \mathcal{R}^+ \subseteq \bar{\mathcal{R}}$ and such that there exists a social ranking satisfying the axioms on $\mathcal{R}^+$.

The following result shows that our domain of preferences is a maximal domain for the set of axioms characterizing the *opportunity-equivalent utilitarian* criterion. The result holds even without imposing *possibility of trade-offs* and *nondiscrimination*.

THEOREM S1: *The domain $\mathcal{R}^n$ is a maximal domain for efficiency, continuity, separability, and equal-preference transfer.*

PROOF: Condition (a) implies condition ($\bar{\text{a}}$), while condition (b) is identical to condition ($\bar{\text{b}}$). It follows that $\mathcal{R}^* \subset \bar{\mathcal{R}}$ and (i) is satisfied. By Theorem 1, also (ii) holds.

We show next that for each $R \in \bar{\mathcal{R}} \backslash \mathcal{R}$, no social ranking $\succsim$ satisfies the axioms. Let $R \in \bar{\mathcal{R}} \backslash \mathcal{R}$. Then, there exists an individual $i \in N$ whose preferences R_i do not admit a concave representation. By *efficiency*, *continuity*, and *separability*, for each $j \in N$ there exists a numerical representation of preferences R_j, say $v_j : X \to \mathbb{R}$, such that for each pair $x, x' \in X^n$, $x \succsim x' \Leftrightarrow \sum_{j \in N} v_j(x_j) \geq \sum_{j \in N} v_j(x'_j)$ (see Step 1 of Section 3.4). By *equal-preference transfer*, for each $j \in N$, v_j needs to be a concave representation of preferences (see Step 4 of Section 3.4). This is a contradiction. *Q.E.D.*

S2. INDEPENDENCE OF THE AXIOMS

For each axiom imposed in Theorem 1, we provide an example of a social ranking that satisfies only the remaining axioms.

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– ***Efficiency***. For each pair $x, x' \in X^n$, $x \succsim x'$ if and only if $\sum_{i\in N} \|x_i\|_1 \geq \sum_{i\in N} \|x'_i\|_1$, where $\|\cdot\|_1$ denotes the 1-norm.

– ***Continuity***. Let $C \in \mathcal{C}$ and, for each $x \in X^n$ and each $i \in N$, let $r(i, x) \in N$ denote the rank occupied by individual i at x (ties are broken arbitrarily): $r(i, x) = j$ means that i is (one of) the individual with the jth lowest well-being at x. Let $[U^C(x)] \equiv (U_1^C(x_1), \ldots, U_{r(i,x)}^C(x_{r(i,x)}), \ldots, U_n^C(x_n))$ denote the vector of each individual's opportunity-equivalent well-being level at x ordered by increasing rank. Then, for each pair $x, x' \in X^n$, $x \succsim x'$ if and only if $[U^C(x)] \geq_{\text{lex}} [U^C(x')]$, where $\geq_{\text{lex}}$ is the leximin ordering.

– ***Separability***. Let $C \in \mathcal{C}$ and $\phi \in \Phi$. For each pair $x, x' \in X^n$, $x \succsim x'$ if and only if $\sum_{i\in N} \lambda_{r(i,x)} \phi \circ U_i^C(x_i) \geq \sum_{i\in N} \lambda_{r(i,x')} \phi \circ U_i^C(x'_i)$, where $\lambda_1 > \cdots > \lambda_n$.

– ***Equal-preference transfer***. Let $C \in \mathcal{C}$ and $\phi \in \Phi$. For each pair $x, x' \in X^n$, $x \succsim x'$ if and only if $\sum_{i\in N} \exp[\phi \circ U_i^C(x_i)] \geq \sum_{i\in N} \exp[\phi \circ U_i^C(x'_i)]$.

The remaining axioms are independent only if specific richness conditions hold. If, instead, these conditions do not hold, the difficulties of aggregating ordinal preferences are either avoided or significantly reduced. First, if all individuals in society share the same preference, *efficiency*, *continuity*, *separability*, and *equal-preference transfer* imply *possibility of trade-offs* and *nondiscrimination*. Intuitively, the first three requirements force society to measure social welfare by the sum of some representations of preferences (see Step 1 of Section 3.4); by *equal-preference transfer* these representations need to be identical across same-preference individuals and imply that *possibility of trade-offs* and *nondiscrimination* hold. A special case of equal-preference society arises when there is only one commodity. To show independence, we thus need to consider societies where at least two individuals have different preferences.

Second, if for each individual $i \in N$ and each $x_i \in X$, the lower-contour sets at x_i are bounded above, *efficiency*, *continuity*, *separability*, and *nondiscrimination* imply *possibility of trade-offs*. Let $i \in N$ and $x_i \in X \backslash \{0\}$. Let $j \in N$. Since the lower-contour sets are bounded above, there exists a pair $\underline{x}_j, \bar{x}_j \in X \backslash \{0\}$ such that

$$\text{LCS}_j(\underline{x}_j) \cap \text{UCS}_i(x_i) = \emptyset$$

and

$$\text{LCS}_i(x_i) \cap \text{UCS}_j(\bar{x}_j) = \emptyset.$$

By *nondiscrimination*, $e^i(x_i) \succ e^j(\underline{x}_j)$ and $e^j(\bar{x}_j) \succ e^i(x_i)$. By *efficiency*, *continuity*, and *separability*, there exists $x_j \in \text{LCS}_j(\bar{x}_j) \cap \text{UCS}_j(\underline{x}_j)$ such that $e^j(x_j) \sim e^i(x_i)$. Since $\text{LCS}_j(\bar{x}_j)$ is bounded above and $x_j \, P_j \, \underline{x}_j$, $x_j \in X \backslash \{0\}$. As this holds for each pair $i, j \in N$ and each $x_i \in X \backslash \{0\}$, *possibility of trade-offs* holds. Therefore, independence requires us to select preference profiles with at least one individual for which lower-contour sets are not bounded above.

– ***Possibility of trade-offs***. Consider a society with two types of preferences: each $i \in \bar{N} \subset N$ has quasi-linear preferences $\bar{R}$, which can be represented by the function $\bar{U} : X \to \mathbb{R}$ defined by setting, for each $x_i \in X$, $\bar{U}(x_i) \equiv x_i^1 + \sum_{\ell=2}^{L} (x_i^\ell + 1)^{-1} - 1$; each of the remaining individuals $i \in N \backslash \bar{N}$ have preferences R, which can be represented by the function $U : X \to \mathbb{R}$ defined by setting, for each $x_i \in X$, $U(x_i) \equiv \|x_i\|_1$. For each $z \geq 0$, let $g(z) \equiv z$ if $z \in [0, 1]$ and $g(z) \equiv (2 - z)^{-1}$ if $z > 1$; note that this function is continuous and concave. We can now define the ranking: for each pair $x, x' \in X^n$, $x \succsim x'$ if and only if $\sum_{i\in \bar{N}} g \circ \bar{U}(x_i) + \sum_{i\in N\backslash \bar{N}} U(x_i) \geq \sum_{i\in \bar{N}} g \circ \bar{U}(x_i) + \sum_{i\in N\backslash \bar{N}} U(x_i)$.

– ***Nondiscrimination***. Consider a society with two types of preferences: $\bar{R}$ for each $i \in \bar{N} \subset N$ and R for each $i \in N \backslash \bar{N}$. Let $C \in \mathcal{C}$ and $\phi \in \Phi^C$. By strong monotonicity of preferences, there exists a pair $\bar{c}, c \in X$ such that $\text{LCS}_i(\bar{c}) \cap \text{UCS}_j(c) = \emptyset$ and such that i and j have different preferences. Without loss of generality, assume $i \in \bar{N}$ and $j \in N \backslash \bar{N}$; then, $\phi \circ U_i^C(\bar{c}) \equiv u_{\bar{c}} < u_c \equiv \phi \circ U_j^C(c)$. We now construct a transformation of these indices of well-being such that the inequality $u_{\bar{c}} < u_c$ is reversed (while preserving continuity, concavity, and the image of the well-being functions). Let $\varepsilon \in (0, 1)$. For each $z \in \mathbb{R}$, let $\bar{g}(z) \equiv z(u_c/u_{\bar{c}})$ if $z \leq u_{\bar{c}}$ and $\bar{g}(z) \equiv u_c + (z - u_{\bar{c}})(1 - \varepsilon)$ if $z > u_{\bar{c}}$ and let $g(z) \equiv z$ if $z \leq \bar{z} \equiv (u_c - (1 - \varepsilon)u_{\bar{c}})/\varepsilon$ and $g(z) \equiv \bar{z} + (z - \bar{z})(1 - \varepsilon)$ if $z > \bar{z}$. We can now define the ranking: for each pair $x, x' \in X^n$, $x \succsim x'$ if and only if $\sum_{i \in \bar{N}} \bar{g} \circ \phi \circ U_i^C(x_i) + \sum_{i \in N \backslash \bar{N}} g \circ \phi \circ U_i^C(x_i) \geq \sum_{i \in \bar{N}} \bar{g} \circ \phi \circ U_i^C(x'_i) + \sum_{i \in N \backslash \bar{N}} g \circ \phi \circ U_i^C(x'_i)$.

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