SUPPLEMENTARY MATERIAL FOR "LARGE MATCHING MARKETS AS TWO-SIDED DEMAND SYSTEMS"
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This supplement contains the proofs for lemmas and theorems that were omitted in the main paper. Equation numbers refer to formulas and expressions in the main text.

APPENDIX A: PROOF OF LEMMA 2.1

TO VERIFY THAT THE STATEMENT IN LEMMA 2.1 is indeed equivalent to the usual definition of pairwise stability, notice that if μ is not pairwise stable, there exists a pair $i, j \neq \mu_w(i)$ such that $U_{ij} > U_{i\mu_w(i)}$ and $V_{ji} > V_{j\mu_m(j)}$. In particular, j is available to i under μ, that is, $j \in M_i[\mu]$, so that $U_i^*(M_i[\mu]) \geq U_{ij} > U_{i\mu_w(i)}$, which violates the first part of the condition. Conversely, if the condition in the lemma does not hold for woman i, then there exists $j \in W_i^*[\mu]$ such that $U_i^*(M_i[\mu]) \geq U_{ij} > U_{i\mu_w(i)}$. On the other hand, $j \in M_i[\mu]$ implies that $V_{ji} \geq V_{j\mu_m(j)}$, and that inequality is strict in the absence of ties since, by assumption, $i \neq \mu_m(j)$. On the other hand, if the condition is violated for a man j, we can find a blocking pair consisting of j and a woman $i \in W_j[\mu]$ using an analogous argument. *Q.E.D.*

APPENDIX B: PROOFS FOR SECTION 3

B.1. *Proof of Theorem 3.1*

Without loss of generality, let $\gamma^* = 0$. The proof consists of three steps: We first show that under Assumption 2.1, any solution to the fixed-point problem in (3.5) is differentiable, so that we can restrict the problem to fixed points in a Banach space of continuous functions. We then show that the mapping $(\log \Gamma_w, \log \Gamma_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])$ is a contraction, so that the conclusions of the theorem follow from Banach's fixed-point theorem. Without loss of generality, we only consider the case in which all observable characteristics are continuously distributed, $x_{1i} = x_i$ and $z_{1i} = z_i$.

Bounds on Solutions. We first establish that any pair of functions $(\Gamma_w^*(x), \Gamma_m^*(z))$ solving the fixed-point problem in (3.5) are bounded from above: Assuming the solutions exist, and noticing that $\Gamma_m(z) \geq 0$ for all $z \in \mathcal{Z}$, we have that

$$(B.1) \qquad \Gamma_w^*(x) = \Psi_w[\Gamma_m^*](x) = \int \frac{\exp\{U(x,s) + V(s,x)\}m(s)}{1 + \Gamma_m^*(s)} \, ds$$

$$\leq \int \exp\{U(x,s) + V(s,x)\}m(s) \, ds \leq \exp\{\bar{U} + \bar{V}\},$$

which is finite by Assumption 2.1. Similarly, we can see that

$$\text{(B.2)} \qquad \Gamma_m^*(z) \le \exp\{\bar{U} + \bar{V}\}$$

if a solution to the fixed-point problem exists.

Continuity of Solutions. In order to establish continuity, notice that any fixed point (Γ_w, Γ_m) of (Ψ_w, Ψ_m) has to satisfy

$$\Gamma_w = \Psi_w\big[\Psi_m[\Gamma_w]\big].$$

Now, consecutive application of Ψ_w and Ψ_m gives

$$\Psi_w\big[\Psi_m[\Gamma_w]\big](x) = \int \frac{\exp\{U(x,t) + V(t,x)\}}{1 + \int \dfrac{\exp\{U(s,z) + V(z,s)\}w(s)}{1 + \Gamma_w(s)}\,ds} m(t)\,dt$$

for any function Γ_w. Since $\exp\{U(x,z)\}$ and $\exp\{V(z,x)\}$ are also continuous in z, x, and the integrals are all nonnegative, $\Psi_w[\Psi_m[\Gamma_w]]$ is also bounded and continuous in x for any nonnegative function Γ_w. Similarly, $\Psi_m[\Psi_w[\Gamma_m]]$ is also bounded and continuous, so that any solution of the fixed-point problem in (3.5), if one exists, must be continuous.

Hence, the range of the operators $\Psi_w \circ \Psi_m$ and $\Psi_m \circ \Psi_w$ is restricted to a set of bounded continuous functions, so that we can w.l.o.g. restrict the fixed-point problem to the space of continuous functions satisfying the bounds derived before. Existence of bounded derivatives up to the pth order follows by induction using the product rule and existence of bounded partial derivatives of the functions $U(x,z)$ and $V(z,x)$; see Assumption 2.1.

Contraction Mapping. We next show that the mapping $(\log \Gamma_w, \log \Gamma_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])$ is a contraction on a Banach space of functions that includes all potential solutions of the fixed-point problem (3.5). Specifically, let $\mathcal{C}^*$ denote the space of continuous functions on $\mathcal{X} \times \mathcal{Z}$ taking nonnegative values and satisfying (B.2) and (B.1). As shown above, any solution to the fixed-point problem—if a solution exists—is an element of $\mathcal{C}^* \times \mathcal{C}^*$, which is a Banach space.

Consider alternative pairs of functions (Γ_w, Γ_m) and $(\tilde{\Gamma}_w, \tilde{\Gamma}_m)$. Using the definitions of the operators,

$$\log \Psi_w[\tilde{\Gamma}_m](x) - \log \Psi_w[\Gamma_m](x)$$
$$= \log \int \frac{\exp\{U(x,s) + V(s,x)\}m(s)}{1 + \exp\{\log \tilde{\Gamma}_m(s)\}}\,ds$$
$$- \log \int \frac{\exp\{U(x,s) + V(s,x)\}m(s)}{1 + \exp\{\log \Gamma_m(s)\}}\,ds.$$

By the mean-value theorem for real-valued functions of a scalar variable, for every value of x, there exists $t(x) \in [0, 1]$ such that

$$\log \frac{\Psi_w[\tilde{\Gamma}_m](x)}{\Psi_w[\Gamma_m](x)} = -\frac{1}{\Psi_w\big[\Gamma_m^{1-t(x)} \tilde{\Gamma}_m^{t(x)}\big](x)}$$

$$\times \int \frac{\exp\big\{U(x,s) + V(s,x)\big\} \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}}{\big[1 + \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}\big]^2}$$

$$\times \big[\log \tilde{\Gamma}_m(s) - \log \Gamma_m(s)\big] m(s)\, ds.$$

Since we are restricting our attention to functions $\Gamma_m(z)$, $\tilde{\Gamma}_m(z)$ satisfying the bounds in (B.1), we can bound the ratio

$$(B.3) \qquad 0 \leq \frac{\Gamma_m(z)^{1-t(x)} \tilde{\Gamma}_m(z)^{t(x)}}{1 + \Gamma_m(z)^{1-t(x)} \tilde{\Gamma}_m(z)^{t(x)}} \leq \frac{\exp\{\bar{U} + \bar{V}\}}{1 + \exp\{\bar{U} + \bar{V}\}} =: \lambda$$

for all $z \in \mathcal{Z}$. Since all components of the integrand are nonnegative, we can bound the right-hand side in absolute value by

$$\left| \log \frac{\Psi_w[\tilde{\Gamma}_m](x)}{\Psi_w[\Gamma_m](x)} \right| \leq \frac{\lambda}{\Psi_w\big[\Gamma_m^{1-t(x)} \tilde{\Gamma}_m^{t(x)}\big](x)} \int \frac{\exp\big\{U(x,s) + V(s,x)\big\}}{1 + \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}}$$

$$\times \sup_{z \in \mathcal{Z}} \big|\log \tilde{\Gamma}_m(s) - \log \Gamma_m(s)\big| m(s)\, ds$$

$$= \frac{\lambda}{\Psi_w\big[\Gamma_m^{1-t(x)} \tilde{\Gamma}_m^{t(x)}\big](x)} \| \log \tilde{\Gamma}_m - \log \Gamma_m \|_\infty$$

$$\times \int \frac{\exp\big\{U(x,s) + V(s,x)\big\}}{1 + \Gamma_m(s)^{1-t(x)} \tilde{\Gamma}_m(s)^{t(x)}} m(s)\, ds$$

$$= \lambda \| \log \tilde{\Gamma}_m - \log \Gamma_m \|_\infty,$$

since the integral in the second to last line is equal to $\Psi_w[\Gamma_m^{1-t(x)} \tilde{\Gamma}_m^{t(x)}](x)$ by definition of the operator Ψ_w. Since this upper bound does not depend on the value of x, it follows that

$$\big\| \log \Psi_w[\tilde{\Gamma}_m] - \log \Psi_w[\Gamma_m] \big\|_\infty = \sup_{x \in \mathcal{X}} \big|\log \Psi_w[\tilde{\Gamma}_m](x) - \log \Psi_w[\Gamma_m](x)\big|$$

$$\leq \lambda \| \log \tilde{\Gamma}_m - \log \Gamma_m \|_\infty,$$

and by a similar argument,

$$\big\| \log \Psi_m[\tilde{\Gamma}_w] - \log \Psi_m[\Gamma_w] \big\|_\infty \leq \lambda \| \log \tilde{\Gamma}_w - \log \Gamma_w \|_\infty.$$

Since by Assumption 2.1 and the expression in equation (B.3), $\lambda = \frac{\exp\{\bar{U}+\bar{V}\}}{1+\exp\{\bar{U}+\bar{V}\}} <$ 1, the mapping $(\log \Gamma_w, \log \Gamma_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])$ is indeed a contraction.

Existence and Uniqueness of Fixed Point. Since we showed in the first step that the solution (Γ_w^*, Γ_m^*), if it exists, has to be continuous, we can take the fixed-point mapping $(\log \Gamma_w, \log \Gamma_m) \mapsto (\log \Psi_w[\Gamma_m], \log \Psi_m[\Gamma_w])$ to be its restriction to the space of continuous functions $(\mathcal{C}^* \times \mathcal{C}^*, \|\cdot\|_\infty)$ endowed with the supremum norm

$$\big\|(\Gamma_w, \Gamma_m)\big\|_\infty := \max\Big\{\sup_x\big|\log \Gamma_w(x)\big|, \sup_z\big|\log \Gamma_m(z)\big|\Big\}.$$

Since this space is a complete vector space, and $(\log \Psi_w, \log \Psi_m)$ is a contraction mapping, the conclusion follows directly using Banach's fixed-point theorem.   *Q.E.D.*

In the following, denote $\tilde{U}_{ij} := U(x_i, z_j)$ and $\tilde{U}_{ik} := U(x_i, z_k)$. Before proving Lemma 3.1, we are going to establish the following lemma:

LEMMA B.1: *Suppose that Assumptions* 2.1, 2.2, *and* 2.3 *hold, and that the random utilities* $U_{i0}, U_{i1}, \ldots, U_{iJ}$ *are J i.i.d. draws from the model in* (2.1) *where the outside option is given by* (2.2). *Then as* $J \to \infty$,

$$\left| P\big(U_{i0} \geq U_{ik}, k = 0, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ}\big) - \frac{1}{1 + \dfrac{1}{J}\displaystyle\sum_{k=1}^{J} \exp\{\tilde{U}_{ik}\}} \right| \to 0$$

and

$$\left| JP\big(U_{ij} \geq U_{ik}, k = 0, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ}\big) - \frac{\exp\{\tilde{U}_{ij}\}}{1 + \dfrac{1}{J}\displaystyle\sum_{k=1}^{J} \exp\{\tilde{U}_{ik}\}} \right| \to 0$$

for any fixed $j = 1, 2, \ldots, J$.

PROOF: For this proof, denote the J draws for the outside option in (2.2) with random utilities $U_{ij} = \tilde{U}_{ij} + \sigma \eta_{ij}$ for $j = J+1, \ldots, 2J$, where $\tilde{U}_{ij} = 0$. Using independence, the conditional probability that $U_{ij} \geq U_{ik}$ for all $k = 1, \ldots, J$ given η_{ij} is equal to

$$P(U_{ij} \geq U_{ik}, k = 1, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ}, \eta_{ij})$$
$$= \prod_{k \neq j} G\big(\eta_{ij} + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})\big)$$

for any $j = 0, 1, \ldots, J$. By the law of iterated expectations, the unconditional probability is obtained by integrating over the density of η,

$$(B.4) \qquad P(U_{ij} \geq U_{ik}, k = 1, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ})$$

$$= \int_{-\infty}^{\infty} \left[\prod_{k \neq j} G\left(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})\right) \right] g(s) \, ds$$

$$= \int_{-\infty}^{\infty} \exp\left\{ \sum_{k \neq j} \log G\left(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})\right) \right\} g(s) \, ds$$

$$= \int_{-\infty}^{\infty} \exp\left\{ \sum_{k=0}^{2J} \log G\left(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})\right) \right\} \frac{g(s)}{G(s)} \, ds,$$

where the last step follows since $\tilde{U}_{ij} - \tilde{U}_{ij} = 0$. Now we can rewrite the exponent in the last expression as

$$\sum_{k=0}^{2J} \log G\left(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})\right)$$

$$= \frac{1}{J} \sum_{k=0}^{2J} J \log\left(G\left(s + \sigma^{-1}(\tilde{U}_{ij} - \tilde{U}_{ik})\right)\right).$$

We now let the sequences $b_J := G^{-1}(1 - \frac{1}{J}) \to \infty$ and $a_J = a(b_J) = \sigma^{-1}$, where $a(z)$ is the auxiliary function specified in Assumption 2.2. Then, by a change of variables $s = a_J t + b_J$, we can rewrite the integral in (B.4) as

$$P(U_{ij} \geq U_{ik}, k = 1, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ})$$

$$= \int_{-\infty}^{\infty} \exp\left\{ \frac{1}{J} \sum_{k=0}^{2J} J \log G\left(a_J(t + \tilde{U}_{ij} - \tilde{U}_{ik}) + b_J\right) \right\}$$

$$\times \frac{a_J g(a_J t + b_J)}{G(a_J t + b_J)} \, dt.$$

Convergence of the Integrand. We next show that for $j \neq 0$, the integrand converges to a non-degenerate limit as $J \to \infty$. First consider the exponent

$$R_J(t) := \frac{1}{J} \sum_{k=0}^{2J} J \log G\left(a_J(t + \tilde{U}_{ij} - \tilde{U}_{ik}) + b_J\right).$$

Since for $G \to 1$, $-\log G \approx 1 - G$, we obtain

$$R_J(t) = -\frac{1}{J} \sum_{k=0}^{2J} J\big(1 - G\big(b_J + a_J(t + \tilde{U}_{ij} - \tilde{U}_{ik})\big)\big) + o(1)$$

$$= -\frac{1}{J} \sum_{k=0}^{2J} J\big(1 - G\big(b_J + a(b_J)(t + \tilde{U}_{ij} - \tilde{U}_{ik})\big)\big) + o(1),$$

where the last step follows from the choice of a_J. Since $(1 - G(s))^{-1}$ is Γ-varying with auxiliary function $a(s)$, and $b_J \to \infty$,

$$\frac{1 - G\big(b_J + a(b_J)(t + \tilde{U}_{ij} - \tilde{U}_{ik})\big)}{1 - G(b_J)} \to \exp\big\{-t - (\tilde{U}_{ij} - \tilde{U}_{ik})\big\}.$$

Finally, since $G(b_J) = 1 - \frac{1}{J}$,

$$J\big(1 - G\big(b_J + a(b_J)(t + \tilde{U}_{ij} - \tilde{U}_{ik})\big)\big)$$

$$= \frac{\big(1 - G\big(b_J + a(b_J)(t + \tilde{U}_{ij} - \tilde{U}_{ik})\big)\big)}{1 - G(b_J)} \to \exp\big\{-t - (\tilde{U}_{ij} - \tilde{U}_{ik})\big\}.$$

Since $G(a_J t + b_J)$ is also nondecreasing in t, convergence of the integrand is also locally uniform with respect to t and $(\tilde{U}_{ij} - \tilde{U}_{ik})$ by the arguments in Section 0.1 in Resnick (1987). Hence,

$$\text{(B.5)} \qquad R_J(t) = -e^{-t} \frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik} - \tilde{U}_{ij}\} + o(1),$$

where the term $\frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik} - \tilde{U}_{ij}\} \le 1 + \exp\{2\bar{U}\} < \infty$ is uniformly bounded by Assumption 2.1. Next, we turn to the term

$$r_J(t) = J a_J g(b_J + a_J t) = J a(b_J) g(b_J + a_J t).$$

Since $a(z) = \frac{1 - G(z)}{g(z)}$, we can write

$$r_J(t) = J a(b_J) \frac{1 - G(b_J + a_J t)}{a(b_J + a_J t)}.$$

By the same steps as before,

$$J\big(1 - G(b_J + a_J t)\big) \to e^{-t}.$$

Furthermore, by Lemma 1.3 in Resnick (1987), we have

$$\frac{a(b_J)}{a(b_J + a_J t)} \to 1,$$

so that

$$r_J(t) \to e^{-t}.$$

Combining this result with (B.5), we get

$$J \exp\left\{ \frac{1}{J} \sum_{k=0}^{2J} J \log G\big(a_J(t + \tilde{U}_{ij} - \tilde{U}_{ik}) + b_J\big) \right\} a_J g(a_J t + b_J)$$

$$= \exp\{R_J(t)\} r_J(t)$$

$$= \exp\left\{ -t - e^{-t} \frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik} - \tilde{U}_{ij}\} \right\} + o(1)$$

for every $t \in \mathbb{R}$.

Convergence of the Integral. Let $h_J^*(t) := \exp\{-t - e^{-t} \frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik} - \tilde{U}_{ij}\}\}$. Since the function $h_J(t) := \exp\{R_J(t)\} r_J(t)$ is bounded uniformly in J, and $|h_J(t) - h_J^*(t)| \to 0$ pointwise, it follows that

$$\left| JP\big(U_{ij} \geq U_{ik}, k = 0, \dots, J | \tilde{U}_{i1}, \dots, \tilde{U}_{iJ}\big) - \int_{-\infty}^{\infty} h_J^*(t) \, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} \big(h_J(t) - h_J^*(t)\big) \, dt \right| \to 0$$

using dominated convergence. From a change in variables $\psi := -e^{-t}$, we can evaluate the integral

$$\int_{-\infty}^{\infty} h_J^*(t) \, dt = \int_{-\infty}^{\infty} \exp\left\{ -e^{-t} \frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik} - \tilde{U}_{ij}\} \right\} e^{-t} \, dt$$

$$= \left(\frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik} - \tilde{U}_{ij}\} \right)^{-1} = \frac{\exp\{\tilde{U}_{ij}\}}{\frac{1}{J} \sum_{k=0}^{2J} \exp\{\tilde{U}_{ik}\}}$$

$$= \frac{\exp\{\tilde{U}_{ij}\}}{1 + \frac{1}{J} \sum_{k=1}^{J} \exp\{\tilde{U}_{ik}\}}$$

since $\sum_{j=J+1}^{2J} \exp\{\tilde{U}_{ij}\} = J$. Hence,

$$\left| JP\big(U_{ij} \geq U_{ik}, k = 0, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ}\big) - \frac{\exp\{\tilde{U}_{ij}\}}{1 + \dfrac{1}{J}\sum_{j=1}^{J} \exp\{\tilde{U}_{ik}\}} \right| \to 0$$

for each $j = 1, 2, \ldots, J$, as claimed in (B.1). Furthermore, it follows that

$$(B.6) \qquad P(U_{i0} \geq U_{ik}, k = 0, \ldots, J | \tilde{U}_{i1}, \ldots, \tilde{U}_{iJ})$$

$$= 1 - \sum_{j=1}^{J} P(U_{ij} \geq U_{ik}, k = 0, \ldots, J)$$

$$= \frac{1}{1 + \dfrac{1}{J}\sum_{j=1}^{J} \exp\{\tilde{U}_{ik}\}} + o(1),$$

which establishes the second assertion.   *Q.E.D.*

B.2. *Proof of Lemma 3.1*

For the main conclusion of the lemma, note that since $z_1, z_2, \ldots$ are a sequence of i.i.d. draws from $M(z)$, Assumption 2.1 and a law of large numbers can be used to establish $\frac{1}{J}\sum_{l=1}^{J} \exp\{U(x_i, z_l)\} \to \int \exp\{U(x_i, s)\}m(s)\,ds$. It follows by the continuous mapping theorem that

$$\frac{\exp\{U(x_i, z_j)\}}{1 + \dfrac{1}{J}\sum_{k=l}^{J} \exp\{U(x_i, z_l)\}} \to \frac{\exp\{U(x_i, z_j)\}}{1 + \displaystyle\int \exp\{U(x_i, s)\}m(s)\,ds}$$

almost surely, so that the conclusion follows from Lemma B.1 and the triangle inequality.   *Q.E.D.*

B.3. *Auxiliary Lemmas for the Proof of Theorem 3.2*

In order to prove Theorem 3.2, we start by establishing the main technical steps separately as Lemmata B.2–B.6. The first result concerns the rate at which the number of available potential spouses increases for each individual in the market. For a given stable matching μ^*, we let

$$J_{wi}^* = \sum_{j=1}^{n} \mathbb{1}\big\{V_{ji} \geq V_j^*\big(W_j^*\big)\big\} \quad \text{and} \quad J_{mj}^* = \sum_{i=1}^{n} \mathbb{1}\big\{U_{ij} \geq U_i^*\big(M_i^*\big)\big\}$$

denote the number of men available to woman i, and the number of women available to man j, respectively, where M_i^* and W_j^* denote woman i's and man j's opportunity sets under μ^*, and $U_i^*(M) := \max_{j \in M} U_{ij}$ and $V_j^*(W) := \max_{i \in W} V_{ji}$, where by convention, the outside option $0 \in W_j^*$ and $0 \in M_i^*$. Similarly, we let

$$L_{wi}^* = \sum_{j=1}^n \mathbb{1}\{U_{ij} \geq U_i^*(M_i^*)\} \quad \text{and} \quad L_{mj}^* = \sum_{i=1}^n \mathbb{1}\{V_{ji} \geq V_j^*(W_j^*)\}$$

so that L_{wi}^* is the number of men to whom woman i is available, and L_{mj}^* is the number of women to whom man j is available. Lemma B.2 below establishes that in our setup, the number of available potential matches grows at a root-n rate as the size of the market grows.

LEMMA B.2: *Suppose Assumptions* 2.1, 2.2, *and* 2.3 *hold*; *then* (a) *under any stable matching*,

$$n^{1/2} \frac{\exp\{-\bar{V} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}} \leq J_{wi}^* \leq n^{1/2} \exp\{\bar{V} + \gamma_m\},$$

$$n^{1/2} \frac{\exp\{-\bar{U} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}} \leq J_{mj}^* \leq n^{1/2} \exp\{\bar{U} + \gamma_w\},$$

for each $i = 1, \ldots, n$ *and* $j = 1, \ldots, n$ *with probability approaching* 1 *as* $n \to \infty$. (b) *Furthermore*,

$$n^{1/2} \frac{\exp\{-\bar{U} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}} \leq L_{wi}^* \leq n^{1/2} \exp\{\bar{U} + \gamma_m\},$$

$$n^{1/2} \frac{\exp\{-\bar{V} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}} \leq L_{mj}^* \leq n^{1/2} \exp\{\bar{V} + \gamma_w\},$$

for each $i = 1, \ldots, n$ *and* $j = 1, \ldots, n$ *with probability approaching* 1 *as* $n \to \infty$.

PROOF: First note that since the sets of available spouses W_i^* and M_j^* under the stable matching are endogenous, the taste shifters η_{ij} and ζ_{ji} are, in general, not independent conditional on those choice sets. To circumvent this difficulty, the following argument only relies on lower and upper bounds on U_i^* and V_i^* that are implied by the respective utilities of the outside option U_{i0}, V_{j0}, and unconditional independence of taste shocks.

Rate for Expectation of Upper Bound $\bar{J}_{mj}$. In the following, we denote the set of women that prefer man j to their outside option by $\bar{W}_j$. Since every woman

can choose to remain single, we can bound J^*_{mj} by

$$J^*_{mj} = \sum_{i=1}^{n_w} \mathbb{1}\{i \in W^*_j\} = \sum_{i=1}^{n_w} \mathbb{1}\{U_{ij} \geq U^*_i(M^*_i)\}$$

$$\leq \sum_{i=1}^{n_w} \mathbb{1}\{U_{ij} \geq U_{i0}\} = \sum_{i=1}^{n_w} \mathbb{1}\{i \in \bar{W}_j\} =: \bar{J}_{mj}.$$

By Assumption 2.3 and Lemma B.1,

$$JP(U_{ij} \geq U_{i0}|x_i, z_j) \to \frac{\exp\{\tilde{U}_{ij}\}}{1 + \dfrac{1}{J}\exp\{\tilde{U}_{ij}\}}.$$

Hence, we can obtain the expectation of the upper bound J^*_{mj},

$$\mathbb{E}[\bar{J}_{mj}|z_j, x_1, \dots, x_n] \to \frac{1}{J}\sum_{i=1}^{n_w} \frac{\exp\{\tilde{U}_{ij}\}}{1 + \dfrac{1}{J}\exp\{\tilde{U}_{ij}\}} \leq \frac{n_w}{J}\exp\{\bar{U}\},$$

where $\bar{U} < \infty$ was given in Assumption 2.1. Since by Assumption 2.3, $J = [n^{1/2}]$ and the bound on the right-hand side does not depend on $z_j, x_1, \dots, x_n$, we have, by the law of iterated expectations, that

$$(B.7) \qquad \mathbb{E}[\bar{J}_{mj}] \leq n^{1/2}\big(\exp\{\bar{U} + \gamma_w\} + o(1)\big),$$

where the remainder term $o(1)$ can be shown to converge uniformly for $j = 1, 2, \dots$.

Rate for Variance of $\bar{J}_{mj}$. Let $p_{ijn} := \frac{\exp\{\tilde{U}_{ij}\}}{J + \exp\{\tilde{U}_{ij}\}}$ and $\bar{v}_{jn} := \frac{1}{n}\sum_{i=1}^{n_w} p_{ijn}(1 - p_{ijn})$. Since by Assumption 2.3, $\frac{\exp\{-\bar{U}\}}{\sqrt{n}+1+\exp\{-\bar{U}\}} \leq p_{ijn} \leq \frac{\exp\{\bar{U}\}}{\sqrt{n}+1+\exp\{\bar{U}\}}$, we have that $(n^{1/2} + 2)^{-1}\exp\{-\bar{U} + \gamma_w\} \leq \bar{v}_{jn} \leq n^{-1/2}\exp\{\bar{U} + \gamma_w\}$. Hence, $\bar{v}_{jn} \to 0$ and $n\bar{v}_{jn} \to \infty$.

Since $\eta_{i0,k}$, $k = 1, \dots, J$ are i.i.d. draws from the distribution $G(\eta)$, we can apply a CLT for independent heterogeneously distributed random variables to the upper bound $\bar{J}_{mj}$,

$$\frac{\bar{J}_{mj} - \mathbb{E}[\bar{J}_{mj}]}{\sqrt{n\bar{v}_{jn}}} = \frac{1}{\sqrt{n\bar{v}_{jn}}}\sum_{i=1}^{n_w}\big(\mathbb{1}\{U_{ij} \geq U_{i0}\} - p_{ijn}\big) \xrightarrow{d} N(0, 1),$$

where the Lindeberg condition holds since the random variables $\mathbb{1}\{U_{ij} \geq U_{i0}\}$ are bounded, and $n\bar{v}_{jn} \to \infty$. Since $\bar{v}_{jn} \to 0$ uniformly in $j = 1, 2, \ldots$, we obtain that

$$\frac{\bar{J}_{mj} - \mathbb{E}[\bar{J}_{mj}]}{\sqrt{n}} \xrightarrow{p} 0$$

uniformly in $j = 1, 2, \ldots$.

Rate for Expectation of Lower Bound J°_{wi}. Next, we denote the set of men j that prefer woman i to their outside option or any woman in $\bar{W}_j$ by M°_i. Since by construction, $\bar{W}_j$ is a superset of (i.e., contains) W^*_j, $M^{\circ}_i \subset M^*_i$. Hence, we can bound J^*_{wi} by

$$J^*_{wi} = \sum_{j=1}^{n_m} \mathbb{1}\{j \in M^*_i\} = \sum_{j=1}^{n_m} \mathbb{1}\{V_{ji} \geq V^*_j(W^*_j)\}$$

$$\geq \sum_{j=1}^{n_m} \mathbb{1}\{V_{ji} \geq V^*_j(\bar{W}_j)\} = \sum_{j=1}^{n_m} \mathbb{1}\{j \in M^{\circ}_i\} =: J^{\circ}_{wi}.$$

Applying Lemma B.1 again, we obtain

$$JP\left(V_{ji} \geq \max_{k \in \bar{W}_j} V_{jk} \,\middle|\, x_i, z_j, \bar{W}_j\right) \to \frac{\exp\{\tilde{V}_{ji}\}}{1 + \dfrac{1}{J}\sum_{k \in \bar{W}_j} \exp\{\tilde{V}_{jk}\}} \geq \frac{J\exp\{-\bar{V}\}}{J + \bar{J}_{mj}\exp\{\bar{V}\}},$$

where $\bar{V} < \infty$ was defined in Assumption 2.1.

Finally, note that this lower bound is a convex function of $\bar{J}_{mj}$, so that we can use our previous bound in (B.8) together with Jensen's inequality to obtain

$$JP\left(V_{ji} \geq \max_{k \in \bar{W}_j} V_{jk} \,\middle|\, x_i, z_j\right)$$

$$\geq \frac{J\exp\{-\bar{V}\}}{J + \mathbb{E}[\bar{J}_{mj}]\exp\{\bar{V}\}} \geq \frac{J\exp\{-\bar{V}\}}{J + n^{1/2}\exp\{\bar{U} + \gamma_w + \bar{V}\}},$$

which is bounded for all values of J since $J = [\sqrt{n}]$. Hence, by the law of iterated expectations, we can obtain the expectation of the lower bound J°_{wi},

$$(\text{B.8}) \qquad \mathbb{E}[J^{\circ}_{wi}] = \sum_{j=1}^{n_m} P\left(V_{ji} \geq \max_{k \in \bar{W}_j} V_{jk} \,\middle|\, x_i, z_j\right)$$

$$\geq n^{1/2}\left(\frac{\exp\{-\bar{V} + \gamma_m\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}} + o(1)\right),$$

where the remainder term $o(1)$ can be shown to converge uniformly for $i = 1, 2, \ldots$.

Rate for Variance J_{wi}°. Let $p_{jin} := \frac{\exp\{\bar{V}_{ij}\}}{J + \sum_{k \in \bar{W}_j} \exp\{\bar{V}_{ik}\}}$ and $\bar{v}_{in} := \frac{1}{n} \sum_{j=1}^{n_m} p_{jin}(1 - p_{jin})$. Using the corresponding bounds derived above and similar steps as for $\bar{v}_{jn}$, we obtain $\bar{v}_{in} \to 0$ and $n\bar{v}_{in} \to \infty$. Since $\zeta_{j0,k}$, $k = 1, \ldots, J$, and ζ_{ji}, $i = 1, \ldots, n$ are i.i.d. draws from the distribution $G(\eta)$ and independent of $\bar{W}_j$, we can again apply the Lindeberg–Lévy CLT to obtain that

$$\frac{J_{wi}^\circ - \mathbb{E}[J_{wi}^\circ]}{\sqrt{n}} \xrightarrow{p} 0$$

uniformly in $i = 1, 2, \ldots$.

Symmetry: Bounds for Both Sides. If we reverse the role of the male and female sides of the market, we can repeat the same sequence of steps and obtain a lower bound $J_{mj}^\circ \leq J_{mj}^*$ and an upper bound $\bar{J}_{wi} \geq J_{wi}^*$ satisfying

$$\mathbb{E}[\bar{J}_{wi}] \leq n^{1/2}\big(\exp\{\bar{V} + \gamma_m\} + o(1)\big),$$

$$\mathbb{E}[J_{mj}^\circ] \geq n^{1/2}\bigg(\frac{\exp\{-\bar{U} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_m\}} + o(1)\bigg),$$

where

$$\frac{\bar{J}_{wi} - \mathbb{E}[\bar{J}_{wi}]}{\sqrt{n}} \xrightarrow{p} 0 \quad \text{and} \quad \frac{J_{mj}^\circ - \mathbb{E}[J_{mj}^\circ]}{\sqrt{n}} \xrightarrow{p} 0,$$

which concludes the proof of part (a). The proof of part (b) is completely analogous.   *Q.E.D.*

We now show that an exogenous change to an arbitrarily chosen availability indicator affects a given individual's opportunity set with a probability that converges to zero as n grows. In the following, we use indices i and k to denote a specific (generic, respectively) woman in the market, and j and l to denote a specific (generic) man. The indicator variable $D_{il}^* := \mathbb{1}\{l \in M_i^*\}$ is equal to 1 if man l is available to woman i under the stable matching μ^*, and zero otherwise. Similarly, we let $E_{jk} := \mathbb{1}\{k \in W_j^*\}$ be an indicator variable that is equal to 1 if woman k is available to j, and zero otherwise. We can stack the indicator variables whether men $l = 1, \ldots, n_m$ are available to woman i under the stable matching μ^* to form the vector $D_i^* := (D_{i1}^*, \ldots, D_{in_m}^*)'$, and dummies whether women $k = 1, \ldots, n_w$ are available to man j to form the vector $E_j^* := (E_{j1}^*, \ldots, E_{jn_w}^*)'$. In the following, we also use D_i^W, E_j^W, D_i^M, and E_j^M to denote the corresponding vectors of availability indicators under the W- and M-preferred matchings.

The following lemma gives a bound on the probability that changing one arbitrarily chosen availability indicator from 1 to zero (or vice versa) alters another agent's opportunity set, where the bound converges to zero at a root-n rate.

LEMMA B.3: *Suppose Assumptions 2.1, 2.2, and 2.3 hold, and let $\{D_{kl}^*, E_{lk}^* : k = 1, \ldots, n_w, l = 1, \ldots, n_m\}$ be the availability indicators arising from a stable matching. Suppose we change E_{ji} exogenously to $\tilde{E}_{ji} = 1 - E_{ji}^*$ for some woman i and man j and then iterate the deferred acceptance algorithm from that starting point to convergence. Denoting the resulting availability indicators with $\{\tilde{D}_{kl}, \tilde{E}_{lk}^* : k = 1, \ldots, n_w, l = 1, \ldots, n_m\}$, for any woman $\bar{k}$ and man $\bar{l} \neq j$ we have that* (a)

$$P\big(\tilde{D}_{\bar{k}} \neq D_{\bar{k}}^* | D_{\bar{k}}^*, D_{ij}^* = 0\big) = P\big(\tilde{E}_{\bar{l}} \neq E_{\bar{l}}^* | E_{\bar{l}}^*, D_{ij}^* = 0\big) = 0,$$

and (b) *there exist constants $\bar{q} < \infty$ and $0 < \lambda < 1$ such that we can bound the conditional probabilities*

$$P\big(\tilde{D}_{\bar{k}} \neq D_{\bar{k}}^* | D_{\bar{k}}^*, D_{ij}^* = 1\big) \leq \frac{n^{-1/2} \bar{q}}{1 - \lambda},$$

$$P\big(\tilde{E}_{\bar{l}} \neq E_{\bar{l}}^* | E_{\bar{l}}^*, D_{ij}^* = 1\big) \leq \frac{n^{-1/2} \bar{q}}{1 - \lambda}.$$

The analogous result holds for an exogenous change of an availability indicator D_{ij} exogenously to $\tilde{D}_{ij} = 1 - D_{ij}^*$.

PROOF: In the following, we analyze the sequence of adjustments to a stable matching and opportunity sets resulting from an arbitrary change in i's taste shifters. We let $\tilde{D}_{kl}^{(s)}$ denote the indicator whether man l is available to woman k after the sth iteration, $\tilde{E}_{lk}^{(s)}$ an indicator whether woman k is available to man l after the sth iteration of the algorithm, and $\tilde{U}_k^{*(s)} := \max_{l:\tilde{D}_{kl}^{(s)}=1}\{U_{kl}\}$ and $\tilde{V}_l^{*(s)} := \max_{k:\tilde{E}_{lk}^{(s)}=1}\{V_{lk}\}$ woman k's and man l's respective indirect utility given their opportunity sets at the sth stage.

We first show that switching one of i's availability indicators and then following the resulting proposals and rejections in the Gale–Shapley algorithm starts a "chain" of subsequent changes, where at each iteration, there is at most one element in each of the two sets of dummies $\{\tilde{D}_{kl}^{(s)}\}_{k,l}$ and $\{\tilde{E}_{kl}^{(s)}\}_{k,l}$ that will be changed at the sth stage and has an impact on subsequent rounds. Furthermore, at each iteration, there is a nontrivial probability that the shift in the previous iteration only affects the outside option, in which case the chain will be terminated at that stage.

Base Case. First, note that changing E_{ji} exogenously to $\tilde{E}_{ji}^{(1)} = 1 - E_{ji}^*$, and leaving the indicators $\tilde{E}_{li}^{(1)} = E_{li}^*$ unchanged for all other men $l \neq j$,

changes only man j's indirect utility from $V_j^* := \max_{k:E_{jk}^*=1}\{V_{jk}\}$ to $\tilde{V}_j^{*(1)} = \max_{k:\tilde{E}_{jk}^{(1)}=1}\{V_{jk}\}$. In particular, if $D_{ij} = 0$, then $V_{ji} < V_j^*$, so that $V_j^* = \tilde{V}_j^{*(1)}$. Then any changes to E_{ij} do not have any subsequent effects on j's choices and can therefore be ignored, which establishes part (a) of the lemma. On the other hand, if $D_{ij} = 1$, then $V_{ji} \geq V_j^*$, so that a change from $E_{ji} = 0$ to $\tilde{E}_{ji}^{(1)} = 1$ increases j's indirect utility. Hence if for woman k, $V_{ji} > V_{jk} > V_j^*$, we have that $D_{kj} = 1$ and $\tilde{D}_{kj}^{(2)} = 0$. Hence it is sufficient to consider shifts in E_{ji} for men j such that $D_{ij} = 1$.

Inductive Step. We now use induction to show that there is at most one such adjustment at each subsequent round $s = 2, 3, \ldots$: suppose that after s iterations of one of the two chains, the availability indicators are given by $\tilde{D}_{kl}^{(s)}$ and $\tilde{E}_{lk}^{(s)}$, where $k = 1, \ldots, n_w$, and $l = 1, \ldots, n_m$. Under the inductive hypothesis, for the sth stage there is at most one woman k such that $\tilde{E}_k^{(s)} \neq \tilde{E}_k^{(s-1)}$. Furthermore, among all men $l = 1, \ldots, n_m$ which were available to k at stage s (i.e., $\tilde{D}_{il}^{(s)} = 1$), there was at most one change of an indicator $\tilde{E}_{lk}^{(s-1)}$ to a new value $\tilde{E}_{lk}^{(s)}$.

Consider first the last change from $\tilde{E}_{lk}^{(s-1)} = 1$ to $\tilde{E}_{lk}^{(s)} = 0$. It follows that $\tilde{V}_l^{*(s)} =: V_{lk'}$ for some k' such that $\tilde{E}_{lk'}^{(s)} = 1$, where k' is unique with probability 1. Hence, at the $(s+1)$st iteration, there is a shift from $\tilde{D}_{k'l}^{(s)} = 0$ to $\tilde{D}_{k'l}^{(s+1)} = 1$, that is, l becomes available to k'.

Note that man l also becomes available to any woman $\tilde{k}$ for whom $V_l^{*(s-1)} > V_{l\tilde{k}} \geq \tilde{V}_l^{*(s)}$. However, by definition of k', any such $\tilde{k}$ would not have been available to l, that is, $\tilde{E}_{l\tilde{k}}^{(s)} = 0$. Hence for $\tilde{k}$, $U_{\tilde{k}l} < \tilde{U}_{\tilde{k}}^{*(s)}$ so that this change has no effect on subsequent iterations. Note that this includes, in particular, woman k who became unavailable to j at the previous stage. Next, consider a change from $\tilde{E}_{lk}^{(s-1)} = 0$ to $\tilde{E}_{lk}^{(s)} = 1$, where $\tilde{D}_{kl}^{(s)} = 1$ and man l's indirect utility in the previous round was $\tilde{V}_l^{*(s-1)} =: V_{lk'}$ for some k' with $\tilde{E}_{lk'}^{(s-1)} = 1$. Since $\tilde{D}_{kl}^{(s)} = 1$, it must be true that $V_{lk} \geq \tilde{V}_l^{*(s-1)}$, so that l may become unavailable to woman k', $\tilde{D}_{k'l}^{(s+1)} = 0$. On the other hand, for any $\tilde{k}$ such that $V_{lk} = \tilde{V}_l^{*(s)} > V_{l\tilde{k}} > \tilde{V}_l^{*(s-1)}$, we must have had $\tilde{E}_{l\tilde{k}}^{(s)} = 0$ by definition of $\tilde{V}_l^{*(s-1)}$. Hence with probability 1, the change in the sth round affects at most one woman k' with $\tilde{E}_{lk'}^{(s)} = 1$, whereas for women $\tilde{k}$ with $\tilde{E}_{l\tilde{k}}^{(s)} = 0$, indirect utility does not depend on whether l is available at round $s+1$, so that there is no effect on subsequent iterations.

Hence there is at most one indicator corresponding to a woman k' with $\tilde{E}_{lk'}^{(s)} = 1$ that changes in the sth round. Interchanging the roles of men and women, an analogous argument yields that there is at most one indicator corresponding to a man l' with $\tilde{D}_{k'l'}^{(s)} = 1$ that changes in the second part of the sth round, confirming the inductive hypothesis.

Probability of Terminating Events. Each of the two chains of adjustments can terminate at any given stage s if the change in the previous round only affects the outside option, that is, if $\tilde{V}_l^{*(t)} = V_{l0}$ or $\tilde{U}_k^{*(t)} = U_{k0}$ at $t = s$ or $t = s - 1$. On the other hand, if the chain results in a change of $D_{\bar{k}1}, \ldots, D_{\bar{k}n_m}$ at a given stage, we ignore any subsequent adjustments and treat such a change as the second terminating event. In the following, we bound the conditional probability for each of these two terminating events given $D_{\bar{k}}^*$ and the chain not having terminated before the sth stage.

We first derive a lower bound for the probability that the chain is terminated by the outside option at stage s: By the same reasoning as in the proof of Lemma B.2, man l's opportunity set is contained in the set W_l°, where the taste shifters ζ_{lk} are jointly independent of W_l°, and the size of W_l° is bounded from above by $n^{1/2} \exp\{\bar{U} + \gamma_w\}$ with probability approaching 1. Hence, by Lemma B.1, we have

$$n^{-1/2} \exp(\bar{V}) \geq \frac{\exp(V(z_l, x_k))(1 + \exp(\bar{V} - \bar{U} + \gamma_w))}{n^{1/2}(1 + 2\exp(\bar{V} - \bar{U} + \gamma_w))}$$

$$\geq P(V_{lk} > V_l^* | x_k, z_l) \geq \frac{\exp(V(z_l, x_k))}{n^{1/2}(1 + \exp(\bar{U} + \bar{V} + \gamma_w))}$$

for any k, l if n is sufficiently large.

This implies that as n grows large, the (unconditional) share of men remaining single is bounded from below by $\frac{1}{1+\exp(\bar{U}+\bar{V}+\gamma_w)} =: p_s$ with probability approaching 1. By the law of total probability, that bound also holds conditional on i's opportunity set, $(D_{i1}^*, \ldots, D_{in_m}^*)$, with probability arbitrarily close to 1 as n grows. Specifically, for the outside option, $P(V_{l0} > V_l^* | D_{ik}^*, x_k, z_l) \geq \frac{1}{1+\exp(\bar{U}+\bar{V}+\gamma_w)}$. Furthermore, by Lemma B.2 part (b), the number of women to whom man l is available is bounded by

$$\underline{L} := n^{1/2} \frac{\exp\{-\bar{V} + \gamma_w\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma_w\}} \leq L_{ml} \leq n^{1/2} \exp\{\bar{V} + \gamma_w\} =: \bar{L}$$

with probability approaching 1.

In order to construct a lower bound on the conditional probability that man l is unmatched given $\tilde{D}_{kl}^{(s)} = 1$, we can assume the lower bound for L_j^* if j is unmatched, and the upper bound if j is matched. Then, by Bayes' law,

$$P(V_{l0} > V_l^* | D_{\bar{k}}^*, \tilde{D}_{lk}^{(s)} = 1, x_k, z_l)$$

$$\geq \frac{\underline{L} p_s}{\bar{L}(1 - p_s) + \underline{L} p_s} = \frac{n^{-1/2} \underline{L}}{n^{-1/2} \bar{L} \exp(\bar{U} + \bar{V} + \gamma_w) + n^{-1/2} \underline{L}},$$

which is strictly greater than zero by Assumptions 2.1 and 2.3. Hence, the probability that the shift is not absorbed by the outside option in the sth step is less than or equal to

$$1 - P\big(V_{l0} > V_l^* | D_{\bar{k}}^*, \tilde{D}_{lk}^{(s)} = 1, x_k, z_l\big)$$

$$\leq \frac{\bar{L} \exp(\bar{U} + \bar{V} + \gamma_w)}{\bar{L} \exp(\bar{U} + \bar{V} + \gamma_w) + \underline{L}} =: \lambda < 1,$$

where the bound on the right-hand side does not depend on s.

Finally, we construct an upper bound for the probability that the chain leads to a change in the availability indicators $D_{\bar{k}1}, \ldots, D_{\bar{k}n_m}$ at stage s. To this end, we can follow the same reasoning as for the choice probability for the outside option, where we use the lower bound on the size of the opportunity set from Lemma B.2. Applying Lemma B.1, we then have

$$P\big(V_{l\bar{k}} > V_l^* | D_{\bar{k}}^*, x_k, z_l\big)$$

$$\leq n^{-1/2} \frac{\exp\big(V(z_l, x_k)\big)\big(1 + \exp(\bar{V} - \bar{U} + \gamma_m)\big)}{1 + \exp(\bar{V} - \bar{U} + \gamma_m) + \exp(-\bar{U} - \bar{V} + \gamma_w)}$$

$$\leq n^{-1/2} \exp\{\bar{V}\}$$

for n sufficiently large. Hence, the conditional probability that one of the indicators $D_{\bar{k}l}$, $l = 1, \ldots, n_m$ is switched given that the process is still active at the sth stage can be bounded by

$$P\big(\tilde{D}_{\bar{k}}^{(s)} \neq D_{\bar{k}}^* | D_{\bar{k}}^*, \tilde{D}_{lk}^{(s)} = 1, z_l, x_k\big)$$

$$\leq \frac{n^{-1/2} \exp\{\bar{V}\}\bar{L}}{n^{-1/2} \exp\{\bar{V}\}\bar{L} + \underline{L}} \leq n^{-1/2} \exp\{\bar{V}\}\frac{\bar{L}}{\underline{L}} =: n^{-1/2}\bar{q},$$

where $\bar{q} < \infty$. Clearly, this upper bound becomes arbitrarily small as n gets large.

By the law of total probability, the conditional probability that $\tilde{D}_{\bar{k}}^W \neq D_{\bar{k}}^W$ can therefore be bounded almost surely by

$$P\big(\tilde{D}_{\bar{k}} \neq D_{\bar{k}}^* | D_{\bar{k}}^*\big) \leq \sum_{s=1}^{\infty} \lambda^s n^{-1/2}\bar{q} \leq \frac{n^{-1/2}\bar{q}}{1 - \lambda},$$

which establishes part (b) of the lemma   Q.E.D.

Next, we show that the dependence between taste shifters η_{ij}, ζ_{ji}, and opportunity sets becomes small as n increases. We consider the joint distribution of $\eta_i := (\eta_{i1}, \ldots, \eta_{in_m})'$, $\zeta_j = (\zeta_{j1}, \ldots, \zeta_{jn_w})'$ and the availability indicators D_i^W,

E_j^W, D_i^M, and E_j^M corresponding to the W- and M-preferred matchings, respectively. We also let $D_{i,-j}^W := (D_{i1}^W, \ldots, D_{i(j-1)}^W, D_{i(j+1)}^W, \ldots, D_{in_m}^W)$ and $E_{j,-i}^W := (E_{j1}^W, \ldots, E_{j(i-1)}^W, E_{j(i+1)}^W, \ldots, E_{jn_w}^W)$ for the W-preferred matching, and use analogous notation for the M-preferred matching. Then for any vectors of indicator variables $\mathbf{d} = (d_1, \ldots, d_{n_m-1}) \in \{0,1\}^{n_m-1}$ and $\mathbf{e} = (e_1, \ldots, e_{n_w-1}) \in \{0,1\}^{n_w-1}$, we denote the conditional c.d.f.s

$$G_{\eta|D}^W(\eta|\mathbf{d}) := P\big(\eta_i \le \eta | D_i^W = \mathbf{d}\big), \quad G_{\eta|D}^M(\eta|\mathbf{d}) := P\big(\eta_i \le \eta | D_i^M = \mathbf{d}\big),$$

$$G_{\eta,\zeta|D,E}^W(\eta, \zeta|\mathbf{d}, \mathbf{e}) := P\big(\eta_i \le \eta, \zeta_j \le \zeta | D_{i,-j}^W = \mathbf{d}, E_{j,-i}^W = \mathbf{e}\big), \quad \text{and}$$

$$G_{\eta,\zeta|D,E}^M(\eta, \zeta|\mathbf{d}, \mathbf{e}) := P\big(\eta_i \le \eta, \zeta_j \le \zeta | D_{i,-j}^M = \mathbf{d}, E_{j,-i}^M = \mathbf{e}\big)$$

with the associated p.d.f.s $g_{\eta|D}^W(\eta|\mathbf{d})$, $g_{\eta|D}^M(\eta|\mathbf{d})$, and $g_{\eta,\zeta|D,E}^W(\eta, \zeta|\mathbf{d}, \mathbf{e})$, respectively. We also use the analogous notation for the conditional distribution of ζ_j given E_j^W and E_j^M, respectively.

We can now state the following lemma characterizing the conditional distribution of taste shifters given an agent's opportunity set.

LEMMA B.4: *Suppose Assumptions* 2.1, 2.2, *and* 2.3 *hold. Then* (a) *the conditional distributions for η given D_i^W and D_i^M, respectively, satisfy*

$$\lim_n \left| \frac{g_{\eta|D}^W(\eta|D_i^W)}{g_\eta(\eta)} - 1 \right| = \lim_n \left| \frac{g_{\eta|D}^M(\eta|D_i^M)}{g_\eta(\eta)} - 1 \right| = 0$$

with probability approaching 1 *as $n \to \infty$. The analogous results hold for the male side of the market. Furthermore,* (b) *the conditional distributions for (η, ζ) given $D_{i,-j}^W, E_{j,-i}^W$ (given $D_{i,-j}^M, E_{j,-i}^M$, respectively) satisfy*

$$\lim_n \left| \frac{g_{\eta,\zeta|D,E}^W\big(\eta, \zeta|D_{i,-j}^W, E_{j,-i}^W\big)}{g_{\eta,\zeta}(\eta, \zeta)} - 1 \right|$$
$$= \lim_n \left| \frac{g_{\eta,\zeta|D,E}^M\big(\eta, \zeta|D_{i,-j}^M, E_{j,-i}^M\big)}{g_{\eta,\zeta}(\eta, \zeta)} - 1 \right| = 0$$

with probability approaching 1 *as $n \to \infty$.* (c) *The analogous conclusion holds for any fixed finite subset of men $M_0 \subset \{1, \ldots, n_m\}$ and women $W_0 \subset \{1, \ldots, n_w\}$, where the conditioning set excludes the availability indicators between any pair $k \in W_0$ and $l \in M_0$.*

PROOF: Without loss of generality, let $\gamma_w = \gamma_m = 0$. We first prove part (a) for the W-preferred matching, where we need to establish that the conditional distribution of η_i given D_i^W converges to the unconditional distribution at a

sufficiently fast rate: Let $g_{\eta,D}^W(\cdot)$ denote the p.d.f. of the joint distribution of D_i^W with η. By the definition of conditional densities, we can write

$$
\text{(B.9)} \quad \frac{g_{\eta|D}^W(\eta|D_i^W)}{g_\eta(\eta)} = \frac{g_{\eta,D}^W(\eta, D_i^W)}{g_\eta(\eta) P(D_i^W)}
$$

$$
= \frac{P(D_i^W|\eta_i = \eta) g_\eta(\eta)}{P(D_i^W) g_\eta(\eta)}
$$

$$
= \frac{P(D_i^W|\eta_i = \eta)}{P(D_i^W)},
$$

where the last step follows since the marginal distributions of η_i has p.d.f. $g_\eta(\eta)$ by assumption.

The remainder of the proof then derives a common bound on the relative change in the conditional probability of D_i^W given η_i that does not depend on D_i^W and η_i, and applies that bound to the event D_i^W to establish that $|\frac{P(D_i^W|\eta_i)}{P(D_i^W)} - 1| \to 0$ almost surely. Hence, as a final step it follows from (B.9) that $|\frac{g_{\eta|D}^W(\eta|D_i^W)}{g_\eta(\eta)} - 1| \to 0$.

Specifically, let $\bar{\eta} = (\eta_1', \ldots, \eta_{n_w}')'$, $\bar{\zeta} = (\zeta_1', \ldots, \zeta_{n_m}')'$, $\bar{\eta}_{-i} = (\eta_1', \ldots, \eta_{i-1}', \eta_{i+1}', \ldots, \eta_{n_w}')'$, and define the random variable

$$
I(\eta, \mathbf{d}^W) := \mathbb{1}\{\bar{\eta}_{-i}, \eta_i = \eta, \bar{\zeta} \text{ imply } D_i^W = \mathbf{d}^W\}
$$

an indicator whether D_i^W results from the W-preferred stable matching given the realizations of taste shifters. We can then write

$$
P(D_i^W = \mathbf{d}^W|\eta_i = \eta_1) = \int I(\eta_1, \mathbf{d}^W)\, dG(\bar{\eta}_{-i}, \bar{\zeta}|\eta_i = \eta_1)
$$

$$
= \int I(\eta_i, \mathbf{d}^W)\, dG(\bar{\eta}_{-i}, \bar{\zeta}),
$$

since η_i and $\bar{\eta}_{-i}, \bar{\zeta}$ are (unconditionally) independent by assumption.

Now for any pair of alternative values η_1, η_2, we can then bound

$$
\text{(B.10)} \quad \frac{P(D_i^W = \mathbf{d}^W|\eta_i = \eta_1) - P(D_i^W = \mathbf{d}^W|\eta_i = \eta_2)}{P(D_i^W = \mathbf{d}^W|\eta_i = \eta_1)}
$$

$$
= \frac{\int \left(I(\eta_1, \mathbf{d}^W) - I(\eta_2, \mathbf{d}^W) \right) dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D_i^W = \mathbf{d}^W|\eta_i = \eta_1)}
$$

$$\leq \frac{\int I(\eta_1, \mathbf{d}^W)(1 - I(\eta_2, \mathbf{d}^W)) \, dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D_i^W = \mathbf{d}^W | \eta_i = \eta_1)}$$

$$= \int (1 - I(\eta_2, \mathbf{d}^W)) \, dG(\bar{\eta}_{-i}, \bar{\zeta} | D_i^W = \mathbf{d}^W, \eta_i = \eta_1),$$

where the last equality follows from the definition of a conditional density. Similarly,

(B.11) $\quad \dfrac{P(D_i^W = \mathbf{d}^W | \eta_i = \eta_2) - P(D_i^W = \mathbf{d}^W | \eta_i = \eta_1)}{P(D_i^W = \mathbf{d}^W | \eta_i = \eta_2)}$

$$= \frac{\int (I(\eta_2, \mathbf{d}^W) - I(\eta_1, \mathbf{d}^W)) \, dG(\bar{\eta}_{-i}, \bar{\zeta})}{P(D_i^W = \mathbf{d}^W | \eta_i = \eta_2)}$$

$$\leq \int (1 - I(\eta_1, \mathbf{d}^W)) \, dG(\bar{\eta}_{-i}, \bar{\zeta} | D_i^W = \mathbf{d}^W, \eta_i = \eta_2).$$

It therefore only remains to be shown that the bounds on the right-hand side of (B.10) and (B.11) both converge to zero as n grows large. We can then combine those two bounds to conclude that

$$\left| \frac{P(D_i^W = \mathbf{d}^W | \eta_i)}{P(D_i^W = \mathbf{d}^W)} - 1 \right| \leq \sup_{\eta_1, \eta_2} \left| \frac{P(D_i^W = \mathbf{d}^W | \eta_i = \eta_1)}{P(D_i^W = \mathbf{d}^W | \eta_i = \eta_2)} - 1 \right| \to 0,$$

so that claim (a) of the lemma follows (B.9).

Deferred Acceptance Algorithm. We now consider the direct effect of a change η_i from $\eta_i =: \eta_1 = (\eta_{11}, \ldots, \eta_{1n_m})'$ to $\eta_2 = (\eta_{21}, \ldots, \eta_{2n_m})'$ on her opportunity sets under the two extremal matchings, holding all other agents' taste shifters fixed: By Theorem 2.12 in Roth and Sotomayor (1990), the W- and M-preferred matchings coincide with the solutions of the Gale–Shapley (deferred acceptance) algorithm with the female (male, respectively) side proposing under the assumptions of this paper (see Section 2 in their monograph for a detailed description of the algorithm). It is now easy to verify that the result of the deferred acceptance algorithm only depends on which proposals are eventually made and/or rejected, but not their particular order, which may only change the number of iterations needed for the algorithm to converge. Specifically, if i makes a proposal to a man who is not available to her under the W-preferred matching, that proposal will be rejected at some stage of the algorithm and does not affect the resulting matching.

Among the men who received a proposal from woman i under the original W-preferred matching but not after the change to i's preferences, there is exactly one man j who was available under the initial matching, that is, $D_{ij}^W = 1$

and $D_{il}^W = 0$ for all l such that $E_{li}^W = 1$ and $\tilde{E}_{li}^W = 0$. Similarly, there is exactly one man $\tilde{j}$ among those receiving a proposal after the change who is also available to i under the new matching, that is, $\tilde{D}_{i\tilde{j}}^W = 1$ and $\tilde{D}_{il}^W = 0$ for all l such that $\tilde{E}_{li}^W = 1$ and $E_{li}^W = 0$. If man l is unavailable to i under both matchings, then by Lemma B.3 part (a), a proposal by i to l does not alter the resulting stable matching. Any other men who were initially unavailable may have become available under the new matching only as a consequence of changing D_{ij} and $D_{i\tilde{j}}$, respectively.

Conditional and Unconditional Probability of D_i^W. Hence, in order to verify whether a change of η_i from η_1 to η_2 results in a different opportunity set for woman i under either of the extremal matchings, it is sufficient to verify whether a proposal by i to her respective spouses under the W-preferred matching given η_1 and η_2, respectively, has an effect on her opportunity set. If such a change does not alter the indicator variables $D_{i1}, \ldots, D_{in_m}$, then the same opportunity set is supported by the W-preferred (M-preferred, respectively) matching given the new realization $\tilde{\eta}_i$ of woman i's taste shocks.

Now denote the availability indicators for the W-preferred matching resulting from replacing $\eta_i = \eta_1$ with $\tilde{\eta}_i = \eta_2$ by $\tilde{D}_i^W := (\tilde{D}_{i1}^W, \ldots, \tilde{D}_{in_m}^W)'$. It then follows from Lemma B.3 part (b) that the conditional probability for $\tilde{D}_i^W \neq D_i^W$ given η_k and ζ_l can be bounded by

$$\frac{P(\tilde{D}_i^W \neq D_i^W \mid \eta_i = \eta_1)}{P(D_i^W \mid \eta_i = \eta_1)} \leq \sum_{s=1}^{\infty} \lambda^s n^{-1/2} \bar{q} \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}.$$

It follows that

$$\frac{P(D_i^W \mid \eta_i = \eta_2)}{P(D_i^W \mid \eta_i = \eta_1)} - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda},$$

which converges to zero as $n \to \infty$. Similarly, exchanging the roles of η_1 and η_2, as well as D_i^W and $\tilde{D}_i^W$, and repeating these steps, we can bound

$$\frac{P(D_i^W \mid \eta_i = \eta_1)}{P(D_i^W \mid \eta_i = \eta_2)} - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}.$$

In order to show that these two inequalities imply the desired bound, we have to distinguish two cases: If $P(D_i^W \mid \eta_i = \eta_2) \geq P(D_i^W \mid \eta_i = \eta_1)$, then $\frac{P(D_i^W \mid \eta_i = \eta_2)}{P(D_i^W \mid \eta_i = \eta_1)} - 1 > 0$ so that by first inequality,

$$\left| \frac{P(D_i^W \mid \eta_i = \eta_2)}{P(D_i^W \mid \eta_i = \eta_1)} - 1 \right| = \frac{P(D_i^W \mid \eta_i = \eta_2)}{P(D_i^W \mid \eta_i = \eta_1)} - 1 \leq 2 \frac{n^{-1/2} \bar{q}}{1 - \lambda}.$$

If, on the other hand, $P(D_i^W|\eta_i = \eta_2) \leq P(D_i^W|\eta_i = \eta_1)$, then $\frac{P(D_i^W|\eta_i=\eta_1)}{P(D_i^W|\eta_i=\eta_2)} \geq 1$ so that the second inequality also holds in absolute values. In that case we also have

$$\left| \frac{P(D_i^W|\eta_i = \eta_2)}{P(D_i^W|\eta_i = \eta_1)} - 1 \right| \leq \frac{P(D_i^W|\eta_i = \eta_1)}{P(D_i^W|\eta_i = \eta_2)} \left| \frac{P(D_i^W|\eta_i = \eta_2)}{P(D_i^W|\eta_i = \eta_1)} - 1 \right|$$

$$= \left| \frac{P(D_i^W|\eta_i = \eta_1)}{P(D_i^W|\eta_i = \eta_2)} - 1 \right| \leq 2\frac{n^{-1/2}\bar{q}}{1 - \lambda}.$$

Hence the upper bound is the same in both cases, so that

$$\left| \frac{P(D_i^W|\eta_i = \eta_2)}{P(D_i^W|\eta_i = \eta_1)} - 1 \right| \leq 2\frac{n^{-1/2}\bar{q}}{1 - \lambda},$$

which converges to zero. Combining the two bounds with (B.9) yields the conclusion of part (a) for the W-preferred matching. The argument for the M-preferred matching is completely analogous.

For parts (b) and (c), note that the argument in part (a) can be extended directly from one to any finite number of individuals. Specifically, if we change the values of η_i and ζ_j in an arbitrary manner, we generate four rather than two chains of adjustments, whereas at any iteration, each chain can affect either i's or j's opportunity set. Hence, we can bound the probability of a shift by a multiple of the bound in part (a), $\frac{4n^{-1/2}\bar{q}}{1-\lambda}$, which can in turn be made arbitrarily small by choosing n large enough. Part (c) can be established in a similar fashion.   *Q.E.D.*

In the following, let $I_{wi}^M = I_{wi}[M_i^M]$ and $I_{wi}^W = I_{wi}[M_i^W]$ denote the inclusive values for woman i under the two extremal matchings, so that for any other stable matching, $I_{wi}^M \leq I_{wi}[M_i^*] \leq I_{wi}^W$. Also, let $\Gamma_w^M(x)$ and $\Gamma_w^W(x)$ be the corresponding expected inclusive value functions. Similarly, we let $I_{mj}^M = I_{mj}[W_j^M]$ and $I_{mj}^W = I_{mj}[W_j^W]$ denote the men's inclusive values, and $\Gamma_m^M(z)$ and $\Gamma_m^W(z)$ be the corresponding expected inclusive value functions.

LEMMA B.5: *Suppose Assumptions 2.1, 2.2, and 2.3 hold. Then*, (a) *for the M-preferred stable matching,*

$$I_{wi}^M \geq \hat{\Gamma}_{wn}^M(x_i) + o_p(1) \quad and \quad I_{mj}^M \leq \hat{\Gamma}_{mn}^M(z_j) + o_p(1)$$

for all $i = 1, \ldots, n_w$ and $j = 1, \ldots, n_m$. Furthermore, (b) *if the weight functions $\omega(x, z) \geq 0$ are bounded and form a Glivenko–Cantelli class in x, then*

$$\sup_{x \in \mathcal{X}} \frac{1}{n} \sum_{j=1}^{n_m} \omega(x, z_j)\left(I_{mj}^M - \hat{\Gamma}_m^M(z_j)\right) \leq o_p(1)$$

and

$$\inf_{z \in \mathcal{Z}} \frac{1}{n} \sum_{j=1}^{n_m} \omega(x_i, z)\big(I_{wi}^M - \hat{\Gamma}_w^M(x_i)\big) \geq o_p(1).$$

The analogous conclusions hold for the W-preferred stable matching with the inequalities reversed, and if weights $\omega(x, z) \geq 0$ form a Glivenko–Cantelli class in z.

PROOF OF LEMMA B.5: First, note that we can bound conditional choice probabilities given an opportunity set from a pairwise stable matching using the extremal matchings: Specifically, we define

$$\Lambda_w^M\big(x, z; M^M\big) := P\big(U_{ij} \geq U_i^*\big(M_i^M\big)|M_i^M = M^M, x_i = x, z_j = z\big)$$

as the conditional choice probability given the realization of the opportunity set M^M from the male-preferred matching, where indirect random utility $U_i^*(M)$ for the opportunity set M was defined in (2.3). Also, we let the function $\Lambda_w(x, z, W)$ be the conditional choice probability for an exogenously fixed opportunity set M,

$$\Lambda_w(x, z; M) := P\big(U_{ij} \geq U_i^*(M)|x_i = x, z_j = z\big).$$

By Lemma B.4, the conditional distribution of taste shifters η_i given W_i^M is approximated by its marginal distribution as n grows large. Hence, combining Lemmas B.1 and B.4, there exists a selection of stable matchings μ^* such that $M_i^* = M_i^M$ w.p.a.1, and taste shifters are independent of M_i^*. In particular, we have

$$J\Lambda_w^M\big(x_i, z; M_i^M\big) \leq J\Lambda_w\big(x_i, z; M_i^M\big) + o_p(1).$$

Furthermore, the conditional success probabilities $\Lambda_w(\cdot; I_w)$ and $\Lambda_m(\cdot; I_m)$ are of the order $J^{-1} = n^{-1/2}$, whereas the approximation error in Lemma B.4 is multiplicative. Hence, by Lemma B.4 part (b),

$$\mathbb{E}\big[J\big(D_{il_1}^M - \Lambda_m(z_{l_1}, x_i; I_{ml_1}^M)\big)|I_{ml_1}^M, I_{ml_2}^M, x_i, z_j\big] \to 0,$$

and

$$\mathbb{E}\big[J^2\big(D_{il_1}^M - \Lambda_m\big(z_{l_1}, x_i; I_{ml_1}^M\big)\big)\big(D_{il_2}^M - \Lambda_m\big(z_{l_2}, x_i; I_{ml_2}^M\big)\big)|I_{ml_1}^M, I_{ml_2}^M, x_i, z_j\big]$$
$$\to 0$$

with probability approaching 1, and for all $l_1 = 1, \ldots, n_m$ and $l_2 \neq l_1$. Therefore by the law of iterated expectations, and the conditional variance identity, we have that for any two men $l_1 \neq l_2$, the unconditional pairwise covariance

$$J^2 \operatorname{Cov}\big(\big(D_{il_1}^M - \Lambda_m\big(z_{l_1}, x_i; I_{ml_1}^M\big)\big), \big(D_{il_2}^M - \Lambda_m\big(z_{l_2}, x_i; I_{ml_2}^M\big)\big)\big) \to 0$$

with probability approaching 1. Since by Assumption 2.1, $\exp\{U(x, z)\}$ is bounded by a constant, we have that $\mathrm{Var}(I_{wi}^M - \hat{I}_{wn}^M(x_i)) \to 0$ for each $i = 1, \ldots, n_w$, so that part (a) follows from Chebyshev's inequality.

For part (b), it is sufficient to notice that part (a) and boundedness of $\omega(x, z)$ imply joint convergence in probability for any finite grid of values $x^{(1)}, \ldots, x^{(k)} \in \mathcal{X}$, so that uniform convergence follows from the Glivenko–Cantelli condition on $\omega(x, z)$ following standard arguments.   *Q.E.D.*

Next, we establish uniform convergence of the fixed-point mapping $\hat{\Psi}$ in equation (3.1). We consider uniformity with respect to $\Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$, where $\mathcal{T}_w$ and $\mathcal{T}_m$ denote the space of bounded continuous real-valued functions on $\mathcal{X}$ and $\mathcal{Z}$, respectively, whose values and first p partial derivatives are uniformly bounded by constants corresponding to the bounds in Theorem 3.1.

Recall that $\hat{\Psi}_w[\Gamma_m](x)$ as defined in (3.1) is the sample average

$$\hat{\Psi}_{wn}[\Gamma_m](x) = \frac{1}{n} \sum_{j=1}^{n_m} \psi_w(z_j, x; \Gamma_m),$$

where

$$\psi_w(z_j, x; \Gamma_m) := \frac{\exp\{U(x, z_j) + V(z_j, x)\}}{1 + \Gamma_m(z_j)}.$$

Similarly, we denote

$$\psi_m(x_i, z; \Gamma_w) := \frac{\exp\{U(x_i, z) + V(z, x_i)\}}{1 + \Gamma_w(x_i)},$$

and define the classes of functions $\mathcal{F}_w : \{\psi_w(\cdot, x; \Gamma_m) : x \in \mathcal{X}, \Gamma_m \in \mathcal{T}_m\}$ and $\mathcal{F}_m : \{\psi_m(\cdot, z; \Gamma_w) : z \in \mathcal{Z}, \Gamma_w \in \mathcal{T}_w\}$.

LEMMA B.6: *Suppose Assumption* 2.1 *holds. Then* (i) *the classes $\mathcal{F}_w$ and $\mathcal{F}_m$ are Glivenko–Cantelli, and* (ii) *the mapping*

$$\left(\hat{\Psi}_w[\Gamma_m](x), \hat{\Psi}_m[\Gamma_w](x)\right) \xrightarrow{p} \left(\Psi_w[\Gamma_m](x), \Psi_m[\Gamma_w](z)\right)$$

uniformly in $\Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$ and $(x', z')' \in \mathcal{X} \times \mathcal{Z}$ as $n \to \infty$.

PROOF: The Glivenko–Cantelli property follows from fairly standard arguments: By Assumption 2.1, the function $\exp\{U(x, z) + V(z, x)\}$ is Lipschitz in x so that, following Example 19.7 in van der Vaart (1998), $\mathcal{G} := \{\exp\{U(x, z) + V(z, x)\} : x \in \mathcal{X}\}$ is a Glivenko–Cantelli class with respect to the distribution $m(z)$. Since by definition of $\mathcal{T}_w, \mathcal{T}_m, \Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$ also have at least $p \geq 1$ bounded derivatives, the class $\mathcal{H} = \{\Gamma_w \in \mathcal{T}_w\} \cup \{\Gamma_m \in \mathcal{T}_m\}$ satisfies the

conditions for Example 19.9 in van der Vaart (1998), and is also Glivenko–Cantelli. Now note that the transformation $\psi(g, h) := \frac{g}{1+h}$ for $g \in \mathcal{G}$ and $h \in \mathcal{H}$ is continuous and bounded on its domain since g and h are bounded, and $h \geq 0$. It then follows from Theorem 3 in van der Vaart and Wellner (2000) that the class $\{\psi(g, h) := \frac{g}{1+h} | g \in \mathcal{G}, h \in \mathcal{H}\}$ is also Glivenko–Cantelli.

For (ii), the Glivenko–Cantelli property of $\mathcal{F}_w, \mathcal{F}_m$ immediately implies uniform convergence of $\hat{\Psi}_w$ and $\hat{\Psi}_m$ to their respective population expectations.   *Q.E.D.*

B.4. *Proof of Theorem 3.2*

We now turn to the proof of the main theorem, starting with part (a).

Fixed-Point Representation. By Lemma B.5, we have that for the M-preferred matching, $I_{wi}^M \geq \hat{\Gamma}_{wn}^M(x_i) + o_p(1)$ and $I_{mj}^M \leq \hat{\Gamma}_{mn}^M(z_j) + o_p(1)$ for all $i = 1, \ldots, n_w$ and $j = 1, \ldots, n_m$. Note that, by construction, $I_{mj}^M \geq 0$ a.s., and $\exp\{U(x, z) + V(z, x)\} \leq \exp\{\bar{U} + \bar{V}\} < \infty$ is bounded by Assumption 2.1, and is a Glivenko–Cantelli class of functions in x, z. Hence we can apply Lemma B.5 part (b) to conclude that

$$\hat{\Gamma}_w^M(x) = \frac{1}{n} \sum_{j=1}^{n_m} \frac{\exp\{U(x, z_j) + V(z_j, x)\}}{1 + I_{mj}^M}$$

$$\geq \frac{1}{n} \sum_{j=1}^{n_m} \frac{\exp\{U(x, z_j) + V(z_j, x)\}}{1 + \hat{\Gamma}_m^M(z_j)} + o_p(1),$$

where the remainder converges to zero in probability uniformly in x. We obtain similar expressions for $\hat{\Gamma}_m^M(z)$, $\hat{\Gamma}_w^W(x)$, and $\hat{\Gamma}_m^W(z)$. Hence, the inclusive value functions satisfy

$$\hat{\Gamma}_w^M \geq \hat{\Psi}_w^M[\hat{\Gamma}_m^M] + o_p(1) \quad \text{and} \quad \hat{\Gamma}_m^M \leq \hat{\Psi}_m^M[\hat{\Gamma}_w^M] + o_p(1),$$

$$\hat{\Gamma}_w^W \leq \hat{\Psi}_w^W[\hat{\Gamma}_m^W] + o_p(1) \quad \text{and} \quad \hat{\Gamma}_m^W \geq \hat{\Psi}_m^W[\hat{\Gamma}_w^W] + o_p(1),$$

where inequalities are component-wise, that is, for $\hat{\Gamma}^M(x)$ and $\hat{\Gamma}^W(z)$ evaluated at any value of $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, respectively. Noting that $\hat{\Psi}_w[\Gamma_m]$ and $\hat{\Psi}_m[\Gamma_w]$ are nonincreasing and Lipschitz continuous in Γ_m and Γ_w, respectively, we have

$$\hat{\Gamma}_w^M \geq \hat{\Psi}_w^M[\hat{\Gamma}_m^M] + o_p(1) \geq \hat{\Psi}_w^M[\hat{\Psi}_m^M[\hat{\Gamma}_w^M]] + o_p(1)$$

from the first two inequalities. Hence, for any functions (Γ_w^*, Γ_m^*) solving the fixed-point problem

$$\Gamma_w^* = \hat{\Psi}_w[\Gamma_m^*] + o_p(1) \quad \text{and} \quad \Gamma_m^* = \hat{\Psi}_m[\Gamma_w^*] + o_p(1)$$

with equality, we have

$$\hat{\Gamma}_w^M \geq \Gamma_w^* + o_p(1) \quad \text{and} \quad \hat{\Gamma}_m^M \leq \Gamma_m^* + o_p(1)$$

and, from the second set of inequalities,

$$\hat{\Gamma}_w^W \leq \Gamma_w^* + o_p(1) \quad \text{and} \quad \hat{\Gamma}_m^W \geq \Gamma_m^* + o_p(1).$$

However, since the mapping $\hat{\Psi}$ is a contraction in logs, the fixed point (Γ_w^*, Γ_m^*) is unique up to a term converging to zero in probability. Furthermore, since $M_i^M \subset M_i^W$ and $W_j^W \subset W_j^W$ almost surely, we also have

$$\hat{\Gamma}_w^M \leq \hat{\Gamma}_w^W \quad \text{and} \quad \hat{\Gamma}_m^M \geq \hat{\Gamma}_m^W.$$

It therefore follows that

$$\hat{\Gamma}_w^M = \Gamma_w^* + o_p(1) \quad \text{and} \quad \hat{\Gamma}_m^M = \Gamma_m^* + o_p(1),$$

and the same condition also holds for the inclusive values from the W-preferred matching. Note that this argument does not require uniformity with respect to (any random selection from) the full set of stable matchings, but only joint convergence for the two extremal matchings.

This establishes the fixed-point representation for $\hat{\Gamma}_w^W$ and $\hat{\Gamma}_w^M$ in equations (3.1) and (3.2). Similarly, we can also establish the fixed-point characterization for the inclusive value function $\hat{\Gamma}_m^W$ and $\hat{\Gamma}_m^M$ for the male side of the market. Since for any other stable matching, $\hat{\Gamma}_w^M \leq \hat{\Gamma}_w^* \leq \hat{\Gamma}_w^W$ and $\hat{\Gamma}_m^W \leq \hat{\Gamma}_m^* \leq \hat{\Gamma}_m^M$, and furthermore by Theorem 3.1, the solution to the exact fixed-point problem $\Gamma = \hat{\Psi}[\Gamma]$ is unique with probability 1, it follows that (3.2) is also valid for the inclusive value functions under any other stable matching.

In order to prove part (b), we will proceed by the following steps: we first show existence and smoothness of the solutions to the fixed-point problem in the finite economy (3.2), and then show that the solution to the fixed-point problem of the limiting market in (3.5) is well separated, so that uniform convergence of the mapping $\log \hat{\Psi}$ to $\log \Psi$ implies convergence of $\hat{\Gamma}$ to Γ^*.

Existence and Smoothness Conditions for $\hat{\Gamma}$. First, note that existence and differentiability of $\hat{\Gamma}_w$ and $\hat{\Gamma}_m$ solving the fixed-point problem in (3.2) follow from Theorem 3.1: Since the conditions of the theorem do not make any assumptions on the distribution of x_i and z_j, it applies to the case in which $w(x)$ and $m(z)$ are the p.m.f.s corresponding to the empirical distributions of x_i and z_j, respectively. Hence, Assumption 2.1 and Theorem 3.1 imply uniqueness and differentiability to pth order with uniformly bounded partial derivatives conditional on any realization of the empirical distribution of observable characteristics. Since the bounds on the contraction constant λ and on partial derivatives

of $\hat{\Gamma}_w$, $\hat{\Gamma}_m$ do not depend on the marginal distributions, they also hold almost surely with respect to realizations of the empirical distribution.

Local Uniqueness. Next, we verify that, for all $\delta > 0$, we can find $\eta > 0$ such that, for any pair $\Gamma, \tilde{\Gamma}$ with $\|\log \tilde{\Gamma} - \log \Gamma\|_\infty > \delta$, we have $\|(\log \tilde{\Gamma} - \log \Psi[\tilde{\Gamma}]) - (\log \Gamma - \log \Psi[\Gamma])\|_\infty > \eta$: First, note that by Theorem 3.1, the mapping $(\log \Gamma) \mapsto (\log \Psi[\Gamma])$ is a contraction with constant $\lambda := \frac{\exp\{\bar{U} + \bar{V} + \gamma^*\}}{1 + \exp\{\bar{U} + \bar{V} + \gamma^*\}} < 1$, where we let $\gamma^* := \max\{\gamma_w, \gamma_m\}$. Then, using the triangle inequality, we can bound

$$\left\| \left(\log \tilde{\Gamma} - \log \Psi[\tilde{\Gamma}]\right) - \left(\log \Gamma - \log \Psi[\Gamma]\right) \right\|_\infty$$

$$\geq \| \log \tilde{\Gamma} - \log \Gamma \|_\infty - \left\| \log \Psi[\tilde{\Gamma}] - \log \Psi[\Gamma] \right\|_\infty$$

$$\geq \| \log \tilde{\Gamma} - \log \Gamma \|_\infty - \lambda \| \log \tilde{\Gamma} - \log \Gamma \|_\infty$$

$$> (1 - \lambda)\delta > 0,$$

so that we can choose $\eta = \eta(\delta) := (1 - \lambda)\delta$.

Convergence of $(\hat{\Gamma} - \Gamma^*)$. Finally, Lemma B.6 implies that the fixed-point mapping $\hat{\Psi}$ converges to Ψ_0 uniformly in x, z and $\Gamma_w \in \mathcal{T}_w$, and $\Gamma_m \in \mathcal{T}_m$. Since $\hat{\Psi} > 0$ is bounded away from zero almost surely, it follows that $|\log \hat{\Psi} - \log \Psi_0|$ converges to zero in outer probability and uniformly in x, z and $\Gamma_w \in \mathcal{T}_w$, and $\Gamma_m \in \mathcal{T}_m$ as well. Hence, for any $\varepsilon > 0$ and n large enough, we have

$$P\left(\sup_{\Gamma \in \mathcal{T}} \left\| \log \hat{\Psi}[\Gamma] - \log \Psi_0[\Gamma] \right\|_\infty > \frac{\eta}{2}\right) \leq 1 - \varepsilon.$$

It follows from the choice of η above that

$$P\left(\left\| \log \hat{\Gamma} - \log \Gamma^* \right\|_\infty > \delta\right) \leq 1 - \varepsilon,$$

so that convergence of $\hat{\Gamma}$ to Γ^* in probability under the sup norm follows from the continuous mapping theorem.   *Q.E.D.*

B.5. *Proof of Corollary 3.1*

As shown in Section 2, the event that woman i and man j are matched under a stable matching requires that woman i prefers j over any man l in her opportunity set M_i^* given that matching, and that man j prefers i over any woman k in his opportunity set W_j^*. Now, by Lemmata B.1 and B.4 part (a), the conditional probability that i prefers j over any $l \in M_i^*$ given her inclusive value satisfies

$$JP\left(U_{ij} \geq U_i^*\left(M_i^*\right)|I_{wi}, x_i, z_j\right) = J\Lambda_w(x_i, z_j; I_{wi}) + o(1)$$

with probability approaching 1, where $\Lambda_w(\cdot)$ is as defined in Section 2.3. Also, by Theorem 3.2(b), the inclusive values I_{wi} and I_{mj} converge in probability to $\Gamma_w(x_i)$ and $\Gamma_m(z_j)$, respectively, so that, by the continuous mapping theorem,

$$J\Lambda_w(x_i, z_j; I_{wi}) = J\Lambda_w\big(x_i, z_j; \Gamma_w(x_i)\big) + o_p(1).$$

Similarly, the conditional probability that man j chooses i over every $k \in W_j^*$ converges according to

$$JP\big(V_{ji} \geq V_j^*\big(W_i^*\big)|I_{mj}, z_j, x_i\big) = J\Lambda_m\big(z_j, x_i; \Gamma_m(z_j)\big) + o_p(1).$$

Finally, by Lemma B.4 part (b) and Assumption 2.3, the joint probability of the two events converges to the product of the marginals,

$$nP\big(U_{ij} \geq U_i^*\big(M_i^*\big), V_{ji} \geq V_j^*\big(W_i^*\big)|I_{wi}, I_{mj}, x_i, z_j\big)$$
$$= J^2 \Lambda_w\big(x_i, z_j; \Gamma_w(x_i)\big)\Lambda_m\big(z_j, x_i; \Gamma_m(z_j)\big),$$

so that the conclusion of this corollary follows from a LLN using Lemma B.4, part (c), together with Assumptions 2.1 and 2.3, via an argument analogous to the proof of Lemma B.5.   *Q.E.D.*

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