

SUPPLEMENT TO “TRADE DYNAMICS IN THE MARKET FOR
FEDERAL FUNDS”: ONLINE APPENDICES
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APPENDIX B: EFFICIENCY

IN THIS SECTION, WE USE OUR THEORY to characterize the optimal process of reallocation of reserve balances in the fed funds market. The spirit of the exercise is to take as given the market structure, including the contact rate α and the regulatory variables $\{u_k, U_k\}_{k \in \mathbb{K}}$, and to ask whether decentralized trade in the over-the-counter market structure reallocates reserve balances efficiently, given these institutions. To this end, we study the problem of a social planner who solves

$$(59) \quad \max_{\{\mathbf{x}(t)\}_{t=0}^T} \left[\int_0^T \sum_{k \in \mathbb{K}} m_k(t) u_k e^{-rt} dt + e^{-rT} \sum_{k \in \mathbb{K}} m_k(T) U_k \right]$$

$$\text{s.t. } \dot{m}_k(t) = -f[\mathbf{m}(t), \boldsymbol{\chi}(t)],$$

$$\chi_{ij}^{ks}(t) \in [0, 1], \quad \text{with } \chi_{ij}^{ks}(t) = 0 \quad \text{if } (k, s) \notin \Pi(i, j),$$

$$\chi_{ij}^{ks}(t) = \chi_{ji}^{sk}(t), \quad \text{and } \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \chi_{ij}^{ks}(t) = 1,$$

for all $t \in [0, T]$, and all $i, j, k, s \in \mathbb{K}$. We have formulated the planner’s problem in chronological time, so $m_k(t)$ denotes the measure of banks with balance k at time t . Since $\tau \equiv T - t$, we have $m_k(t) = m_k(T - \tau) \equiv n_k(\tau)$, and therefore $\dot{m}_k(t) = -\dot{n}_k(\tau)$. Hence the flow constraint is the real-time law of motion for the distribution of balances implied by the bilateral stochastic trading process. The control variable, $\boldsymbol{\chi}(t) = \{\chi_{ij}^{ks}(t)\}_{i, j, k, s \in \mathbb{K}}$, represents the planner’s choice of reallocation of balances between any pair of banks that have contacted each other at time t . The first, second, and fourth constraints on $\boldsymbol{\chi}(t)$ ensure that $\{\chi_{ij}^{ks}(t)\}_{k, s \in \mathbb{K}}$ is a probability distribution for each $i, j \in \mathbb{K}$, and that the planner only chooses among feasible reallocations of balances between a pair of banks. We look for a solution that does not depend on the identities or “names” of banks, so the third constraint on $\boldsymbol{\chi}(t)$ recognizes the fact that $\chi_{ij}^{ks}(t)$ and $\chi_{ji}^{sk}(t)$ represent the same decision for the planner. That is, $\chi_{ij}^{ks}(t)$ and $\chi_{ji}^{sk}(t)$ both represent the probability that a pair of banks with balances i and j who contact each other at time t exit the meeting with balances k and s , respectively.

PROPOSITION 3: *A solution to the planner’s problem is a path for the distribution of balances, $\mathbf{n}(\tau)$, a path for the vector of co-states associated with the law*

of motion for the distribution of balances, $\boldsymbol{\lambda}(\tau) = \{\lambda_k(\tau)\}_{k \in \mathbb{K}}$, and a path for the distribution of trading probabilities, $\boldsymbol{\psi}(\tau) = \{\psi_{ij}^{ks}(\tau)\}_{i,j,k,s \in \mathbb{K}}$. The necessary conditions for optimality are

$$(60) \quad r\lambda_i(\tau) + \dot{\lambda}_i(\tau) \\ = u_i + \alpha \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} n_j(\tau) \psi_{ij}^{ks}(\tau) [\lambda_k(\tau) + \lambda_s(\tau) - \lambda_i(\tau) - \lambda_j(\tau)]$$

for all $(i, \tau) \in \mathbb{K} \times [0, T]$, with

$$(61) \quad \lambda_i(0) = U_i \quad \text{for all } i \in \mathbb{K},$$

with the path for $\mathbf{n}(\tau)$ given by $\dot{\mathbf{n}}(\tau) = f[\mathbf{n}(\tau), \boldsymbol{\psi}(\tau)]$, and with

$$(62) \quad \psi_{ij}^{ks}(\tau) \begin{cases} \geq 0 & \text{if } (k, s) \in \Omega_{ij}[\boldsymbol{\lambda}(\tau)], \\ = 0 & \text{if } (k, s) \notin \Omega_{ij}[\boldsymbol{\lambda}(\tau)], \end{cases}$$

for all $i, j, k, s \in \mathbb{K}$ and all $\tau \in [0, T]$, where $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \psi_{ij}^{ks}(\tau) = 1$.

PROOF: The planner's current-value Hamiltonian can be written as

$$L = \sum_{k \in \mathbb{K}} m_k(t) u_k + \alpha \sum_{i \in \mathbb{K}} \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} m_i(t) m_j(t) \chi_{ij}^{ks}(t) [\mu_k(t) - \mu_i(t)],$$

where $\boldsymbol{\mu}(t) = \{\mu_k(t)\}_{k \in \mathbb{K}}$ is the vector of co-states associated with the law of motion for the distribution of banks across reserve balances. In an optimum, the co-states and the controls must satisfy $\frac{\partial L}{\partial m_i(t)} = r\mu_i(t) - \dot{\mu}_i(t)$, and

$$\chi_{ij}^{ks}(t) \begin{cases} = 1 & \text{if } \left. \frac{\partial L}{\partial \chi_{ij}^{ks}(t)} \right|_{\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)} > 0, \\ \in [0, 1] & \text{if } \left. \frac{\partial L}{\partial \chi_{ij}^{ks}(t)} \right|_{\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)} = 0, \\ 0 & \text{if } \left. \frac{\partial L}{\partial \chi_{ij}^{ks}(t)} \right|_{\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)} < 0. \end{cases}$$

Notice that

$$\left. \frac{\partial L}{\partial \chi_{ij}^{ks}(t)} \right|_{\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)} = \alpha m_i(t) m_j(t) [\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)],$$

and that, given $\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)$,

$$\frac{\partial L}{\partial m_i} = u_i + \alpha \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} m_j(t) \chi_{ij}^{ks}(t) [\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)].$$

Thus, the necessary conditions for optimality are

$$(63) \quad \chi_{ij}^{ks}(t) \begin{cases} \geq 0 & \text{if } (k, s) \in \Omega_{ij}[\boldsymbol{\mu}(t)], \\ = 0 & \text{if } (k, s) \notin \Omega_{ij}[\boldsymbol{\mu}(t)], \end{cases}$$

for all $i, j, k, s \in \mathbb{K}$ and all $t \in [0, T]$, where $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \chi_{ij}^{ks}(t) = 1$, the Euler equations,

$$(64) \quad r\mu_i(t) - \dot{\mu}_i(t) \\ = u_i + \alpha \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} m_j(t) \chi_{ij}^{ks}(t) [\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)]$$

for all $i \in \mathbb{K}$, with the path for $\mathbf{m}(t)$ given by (59), and

$$(65) \quad \mu_i(T) = U_i \quad \text{for all } i \in \mathbb{K}.$$

In summary, the necessary conditions are (59), (63), (64), and (65). Next, we use the fact that $\tau \equiv T - t$ to define $m_k(t) = m_k(T - \tau) \equiv n_k(\tau)$, $\chi_{ij}^{ks}(t) = \chi_{ij}^{ks}(T - \tau) \equiv \psi_{ij}^{ks}(\tau)$, and $\mu_i(t) = \mu_i(T - \tau) \equiv \lambda_i(\tau)$. With these new variables, (64) leads to (60), (59) leads to $\dot{\mathbf{n}}(\tau) = f[\mathbf{n}(\tau), \boldsymbol{\psi}(\tau)]$, (65) leads to (61), and (63) leads to (62). Q.E.D.

The following result provides a full characterization of the solution to the planner's problem under Assumption A.

PROPOSITION 4: *Let the payoff functions satisfy Assumption A. Then:*

(i) *The optimal path for the distribution of trading probabilities, $\boldsymbol{\psi}(\tau) = \{\psi_{ij}^{ks}(\tau)\}_{i,j,k,s \in \mathbb{K}}$, is given by*

$$(66) \quad \psi_{ij}^{ks}(\tau) \begin{cases} \geq 0 & \text{if } (k, s) \in \Omega_{ij}^*, \\ = 0 & \text{if } (k, s) \notin \Omega_{ij}^*, \end{cases}$$

for all $i, j, k, s \in \mathbb{K}$ and all $\tau \in [0, T]$, where $\sum_{(k,s) \in \Omega_{ij}^*} \psi_{ij}^{ks}(\tau) = 1$.

(ii) *Along the optimal path, the shadow value of a bank with i reserve balances is given by (60) and (61), with the path for $\boldsymbol{\psi}(t)$ given by (66), and the path for $\mathbf{n}(\tau)$ given by $\dot{\mathbf{n}}(\tau) = f[\mathbf{n}(\tau), \boldsymbol{\psi}(\tau)]$.*

PROOF: The function $\boldsymbol{\lambda} \equiv [\lambda(\tau)]_{\tau \in [0, T]}$ satisfies (60) and (61) if and only if it satisfies

$$\begin{aligned} \lambda_i(\tau) &= v_i(\tau) + \alpha \int_0^\tau \lambda_i(z) e^{-(r+\alpha)(\tau-z)} dz \\ &+ \alpha \int_0^\tau \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} n_j(z) \psi_{ij}^{ks}(z) \\ &\times [\lambda_k(z) + \lambda_s(z) - \lambda_i(z) - \lambda_j(z)] e^{-(r+\alpha)(\tau-z)} dz. \end{aligned}$$

The right side of this functional equation defines a mapping $\mathcal{P} : \mathbf{B} \rightarrow \mathbf{B}$; that is, for any $w \in \mathbf{B}$,

$$\begin{aligned} (\mathcal{P}w)(i, \tau) &= v_i(\tau) + \alpha \int_0^\tau w(i, z) e^{-(r+\alpha)(\tau-z)} dz \\ &\quad + \alpha \int_0^\tau \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} n_j(z) \psi_{ij}^{ks}(z) [w(k, z) + w(s, z) \\ &\quad - w(i, z) - w(j, z)] e^{-(r+\alpha)(\tau-z)} dz, \end{aligned}$$

for all $(i, \tau) \in \mathbb{K} \times [0, T]$. Hence a function λ satisfies (60) and (61) if and only if it satisfies $\lambda = \mathcal{P}\lambda$. Rewrite the mapping \mathcal{P} as

$$\begin{aligned} (67) \quad (\mathcal{P}w)(i, \tau) &= v_i(\tau) + \alpha \int_0^\tau w(i, z) e^{-(r+\alpha)(\tau-z)} dz \\ &\quad + \alpha \int_0^\tau \sum_{j \in \mathbb{K}} n_j(z) \max_{(k,s) \in \Pi(i,j)} [w(k, z) + w(s, z) \\ &\quad - w(i, z) - w(j, z)] e^{-(r+\alpha)(\tau-z)} dz, \end{aligned}$$

and for any $w, w' \in \mathbf{B}$, define the metric $D^* : \mathbf{B} \times \mathbf{B} \rightarrow \mathbb{R}$ by

$$D^*(w, w') = \sup_{(i, \tau) \in \mathbb{K} \times [0, T]} [e^{-\kappa\tau} |w(i, \tau) - w'(i, \tau)|],$$

where $\kappa \in \mathbb{R}$ satisfies

$$(68) \quad \max\{0, 5\alpha - r\} < \kappa < \infty.$$

The metric space (\mathbf{B}, D^*) is complete (by the same argument used to argue that (\mathbf{B}, D) is complete, in the proof of Lemma 4). For any $w, w' \in \mathbf{B}$, and any $(i, \tau) \in \mathbb{K} \times [0, T]$, the same steps that led to (39) now lead to

$$D^*(\mathcal{P}w, \mathcal{P}w') \leq \frac{5\alpha}{r + \alpha + \kappa} D^*(w, w') \quad \text{for all } w, w' \in \mathbf{B}.$$

Notice that (68) implies $\frac{5\alpha}{r + \alpha + \kappa} \in (0, 1)$, so \mathcal{P} is a contraction mapping on the complete metric space (\mathbf{B}, D^*) . By the Contraction Mapping Theorem (Theorem 3.2 in [Stokey and Lucas \(1989\)](#)), for any given path $\mathbf{n}(\tau)$, there exists a unique $\lambda \in \mathbf{B}$ that satisfies $\lambda = \mathcal{P}\lambda$.

Consider the sets \mathbf{B}'' and \mathbf{B}''' defined in the proof of Proposition 2. By following the same steps as in the first part of that proof, it can be shown that \mathbf{B}'' is closed under D^* . Next, we show that the mapping \mathcal{P} defined in (67) preserves property (EP), that is, that $\mathcal{P}(\mathbf{B}'') \subseteq \mathbf{B}''$. That is, we wish to show that, for any

$w \in \mathbf{B}''$, $w' = \mathcal{P}w \in \mathbf{B}''$, or equivalently, that

$$w\left(\left\lceil \frac{i+j}{2} \right\rceil, \tau\right) + w\left(\left\lfloor \frac{i+j}{2} \right\rfloor, \tau\right) \geq w(k, \tau) + w(s, \tau)$$

for all $(k, s) \in \Pi(i, j)$,

for any $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$, implies that

$$(69) \quad w'\left(\left\lceil \frac{i+j}{2} \right\rceil, \tau\right) + w'\left(\left\lfloor \frac{i+j}{2} \right\rfloor, \tau\right) - w'(k, \tau) - w'(s, \tau) \geq 0$$

for all $(k, s) \in \Pi(i, j)$,

for any $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$. Since $w \in \mathbf{B}''$,

$$\begin{aligned} (\mathcal{P}w)(i, \tau) &= v_i(\tau) + \alpha \int_0^\tau w(i, z) e^{-(r+\alpha)(\tau-z)} dz \\ &\quad + \alpha \int_0^\tau \sum_{q \in \mathbb{K}} n_q(z) \left[w\left(\left\lceil \frac{i+q}{2} \right\rceil, z\right) + w\left(\left\lfloor \frac{i+q}{2} \right\rfloor, z\right) \right. \\ &\quad \left. - w(i, z) - w(q, z) \right] e^{-(r+\alpha)(\tau-z)} dz, \end{aligned}$$

for any $(i, \tau) \in \mathbb{K} \times [0, T]$. For any $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$ and $(k, s) \in \Pi(i, j)$, let $G'(i, j, k, s, \tau)$ denote the left side of inequality (69). Then,

$$\begin{aligned} G'(i, j, k, s, \tau) &= \frac{1 - e^{-(r+\alpha)\tau}}{r + \alpha} (u_{\lceil (i+j)/2 \rceil} + u_{\lfloor (i+j)/2 \rfloor} - u_k - u_s) \\ &\quad + e^{-(r+\alpha)\tau} [U_{\lceil (i+j)/2 \rceil} + U_{\lfloor (i+j)/2 \rfloor} - U_k - U_s] \\ &\quad + \alpha \int_0^\tau \sum_{q \in \mathbb{K}} n_q(z) \left\{ w\left(\left\lceil \frac{\lceil \frac{i+j}{2} \rceil + q}{2} \right\rceil, z\right) \right. \\ &\quad + w\left(\left\lfloor \frac{\lfloor \frac{i+j}{2} \rfloor + q}{2} \right\rfloor, z\right) - w\left(\left\lceil \frac{k+q}{2} \right\rceil, z\right) \\ &\quad \left. - w\left(\left\lfloor \frac{s+q}{2} \right\rfloor, z\right) - w\left(\left\lfloor \frac{k+q}{2} \right\rfloor, z\right) \right\} e^{-(r+\alpha)(\tau-z)} dz \end{aligned}$$

$$\begin{aligned}
& -w\left(\left[\frac{s+q}{2}\right], z\right) + w\left(\left[\frac{\lfloor \frac{i+j}{2} \rfloor + q}{2}\right], z\right) \\
& + w\left(\left[\frac{\lfloor \frac{i+j}{2} \rfloor + q}{2}\right], z\right) \Big\} e^{-(r+\alpha)(\tau-z)} dz.
\end{aligned}$$

What needs to be shown is that $w \in \mathbf{B}'$ implies that, for any $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$, $G'(i, j, k, s, \tau) \geq 0$ for all $(k, s) \in \Pi(i, j)$. By Lemma 5, $w \in \mathbf{B}'$ implies that the integral in the last expression is nonnegative. Together with Assumption A and Corollary 1, this implies

$$\begin{aligned}
0 & < \frac{1 - e^{-(r+\alpha)\tau}}{r + \alpha} (u_{\lceil(i+j)/2\rceil} + u_{\lfloor(i+j)/2\rfloor} - u_k - u_s) \\
& + e^{-(r+\alpha)\tau} [U_{\lceil(i+j)/2\rceil} + U_{\lfloor(i+j)/2\rfloor} - U_k - U_s] \\
& \leq G'(i, j, k, s, \tau),
\end{aligned}$$

so $\mathcal{M}(\mathbf{B}'') \subseteq \mathbf{B}''' \subseteq \mathbf{B}''$.

At this point, we have shown that \mathcal{P} is a contraction mapping on the complete metric space (\mathbf{B}, D^*) , so it has a unique fixed point $\boldsymbol{\lambda} \in \mathbf{B}$. We have also established that \mathbf{B}'' is a closed subset of \mathbf{B} and that $\mathcal{M}(\mathbf{B}'') \subseteq \mathbf{B}''' \subseteq \mathbf{B}''$. Therefore, by Corollary 1 in Stokey and Lucas (1989, p. 52), $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda} \in \mathbf{B}'''$, that is, the unique fixed point $\boldsymbol{\lambda}$ satisfies (SEP). This implies that the set $\Omega_{ij}[\boldsymbol{\lambda}(\tau)]$ in (62) reduces to Ω_{ij}^* (as defined in (11)) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$, and consequently, that (62) reduces to (66). This establishes part (i) in the statement of the proposition.

Given the initial condition $\{n_k(T)\}_{k \in \mathbb{K}}$, and given that the path $\boldsymbol{\psi}(\tau)$ satisfies (66), the system of first-order ordinary differential equations, $\dot{\mathbf{n}}(\tau) = f[\mathbf{n}(\tau), \boldsymbol{\psi}(\tau)]$, is identical to the one in part (iii) of Proposition 2 and therefore also has a unique solution. Given the resulting path $\mathbf{n}(\tau)$, according to Proposition 3, the path for the vector of co-states must satisfy the necessary condition $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda}$, or equivalently, (60) and (61), which establishes part (ii) in the statement of the proposition. *Q.E.D.*

Notice the similarity between the equilibrium conditions and the planner's optimality conditions. First, from (8) and (62), we see that the equilibrium loan sizes are *privately efficient*. That is, given the value function \mathbf{V} , the equilibrium distribution of trading probabilities is the one that would be chosen by the planner. Second, the path for the equilibrium values, $\mathbf{V}(\tau)$, satisfies (6) and (7), while the path for the planner's shadow prices satisfies (60) and (61). These pairs of conditions would be identical were it not for the fact that the planner imputes to each agent gains from trade with frequency 2α , rather than α , which

is the frequency with which the agent generates gains from trade for himself in the equilibrium. This reflects a composition externality typical of random matching environments. The planner's calculation of the value of a marginal agent in state i includes not only the expected gains from trade to this agent, but also the expected gains from trade that having this marginal agent in state i generates for all other agents by increasing their contact rates with agents in state i . In the equilibrium, the individual agent in state i internalizes the former, but not the latter.²⁹

Under Assumption A, however, condition (10) is identical to (66), so the equilibrium paths for the distribution of balances and trading probabilities coincide with the optimal paths. This observation is summarized in the following proposition.

PROPOSITION 5: *Let the payoff functions satisfy Assumption A. Then, the equilibrium supports an efficient allocation of reserve balances.*

APPENDIX C: DATA

C.1. *Treatment of Outliers*

From the histogram of the variable of interest (that is, the 4:00 pm imputed balances over required operating balances, averaged over the two-week maintenance period), we looked for observations that deviate markedly from the other members of the sample. There are no such observations in the 2007 sample. In the 2011 sample, the three largest observations deviate markedly from the rest of the sample. We also implemented a modified version of Grubbs's test that assumes the data can be fitted by a mixture of Gaussians and detects outliers with respect to the Gaussian distribution with the largest variance. This procedure identifies no outliers in 2007 and the same three outliers in 2011.

C.2. *Estimation of Initial Distribution of Balances*

In Section 6.1, we described the procedure to estimate the initial distribution of balances that we used in the 2007 calibration to run the baseline simulations presented in Section 6.2. This procedure is straightforward: it essentially consists of using the data to construct the histogram that we employ as the initial condition for the distribution of reserve balances in the model. In order to conduct policy experiments such as those in Section 6.3 or counterfactual experiments such as those in Appendix D, however, it is convenient to work with a parametric initial distribution of balances rather than an empirical histogram, as this allows one to easily change the mean or the standard deviation

²⁹In a labor market context, a similar composition externality arises in the competitive matching equilibrium of Kiyotaki and Lagos (2007).

of the initial distribution of balances. For this reason, the initial distributions used for the exercises in Section 6.3 and Appendix D were estimated using the following procedure.

C.2.1. Estimation Procedure for 2011 Policy Experiments

As mentioned in the body of the paper, we think of $n_k(T)$ as the model counterpart of the empirical proportion of commercial banks whose balances at the beginning of the trading session are k/\bar{k} times larger than their average daily reserve requirement over a two-week holding period. In order to conduct the policy experiments reported in Section 6.3, we estimate $\{n_k(T)\}_{k \in \mathbb{K}}$ from data using the following procedure.

First, we identified 135 commercial banks that traded fed funds during the first quarter of 2011 (according to their FR Y9-C regulatory filings), for which we have been able to obtain information on their required operating balance, and that are not subject to special analysis (according to item 9425 Bank Type Analysis Code of their regulatory filings). Second, we obtained the empirical cross-sectional distribution of closing balances of these 135 banks for each day of a two-week maintenance period in the same quarter. Third, for every day in the sample, we constructed a measure of each bank's *imputed reserve balance* at 4:00 pm, as follows. Given each bank's closing balance on a given day, we subtracted the bank's net payments activity from 4:00 pm until Fedwire Funds Service closing time (typically 6:30 pm) as well as the discount window activity for that day. Fourth, for each bank, we calculated the *average* (over days in the two-week maintenance period) *imputed reserve balance* at 4:00 pm and normalized it by dividing it by the bank's daily average required operating balance over the same maintenance period. At this stage, we detected three outliers (as described in Section C.1) and removed them from the sample to obtain the final sample of 132 banks. We then used this sample to compute maximum likelihood estimates of the parameters of a Gaussian mixture model with two components. The estimated parameters are $\hat{\mu}_1 = 4.51$ and $\hat{\mu}_2 = 57.19$ (the means), $\sigma_1 = 4.78$ and $\sigma_2 = 44.93$ (the standard deviations), and $p_1 = 1 - p_2 = 0.67$ (the probability of drawing from the first component).³⁰ We discuss the goodness of fit in Section C.2.3 below.

Notice that the mean of the estimated distribution of average normalized imputed reserve balances for the 132 banks in the sample is $p_1 \hat{\mu}_1 + p_2 \hat{\mu}_2 = 21.8$. In order for the calibrated model to capture typical overall market conditions during the first quarter of 2011, we translate the estimated Gaussian mixture by choosing its mean to match the empirical mean of the ratio of total seasonally adjusted reserves of depository institutions to total required reserves reported in the H.3 Federal Reserve Statistical Release during the first quarter of 2011,

³⁰The corresponding standard errors (bootstrap, based on 10,000 iterations) for $\hat{\mu}_1$, $\hat{\mu}_2$, σ_1 , σ_2 , and p_1 are 0.92, 9.43, 0.95, 4.46, and 0.055, respectively.

which equals 17.68. This is done by considering a Gaussian mixture with the same p_1 , p_2 , σ_1 , and σ_2 that were estimated from the sample of 132 banks, but replacing the estimated means, $\hat{\mu}_1$ and $\hat{\mu}_2$, with $\mu_i = 0.81\hat{\mu}_i$, for $i = 1, 2$, that is, $\mu_1 = 3.65$ and $\mu_2 = 46.32$.³¹ This leads to the Gaussian mixture, Φ , with parameters $\mu_1 = 3.65$, $\mu_2 = 46.32$, $\sigma_1 = 4.78$, $\sigma_2 = 44.93$, and $p_1 = 0.67$ used in Section 6.3.

C.2.2. Estimation Procedure for 2007 Counterfactuals

In order to conduct the counterfactual policy experiments for 2007 reported in Appendix D (Section D.1), we estimate $\{n_k(T)\}_{k \in \mathbb{K}}$ from data using the same procedure used for the policy experiments reported in Section 6.3. Below we describe the full procedure for completeness.

First, we identified 134 commercial banks that traded fed funds at the end of the second quarter of 2007 (according to their FR Y9-C regulatory filings), for which we have been able to obtain information on their required operating balance, and that are not subject to special analysis (according to item 9425 Bank Type Analysis Code of their regulatory filings). Second, we obtained the empirical cross-sectional distribution of closing balances of these 134 banks for each day of a two-week maintenance period in the same quarter. Third, for every day in the sample, we constructed a measure of each bank's *imputed reserve balance* at 4:00 pm, as follows. Given each bank's closing balance on a given day, we subtracted the bank's net payments activity from 4:00 pm until Fedwire Funds Service closing time (typically 6:30 pm) as well as the discount window activity for that day. Fourth, for each bank, we calculated the *average* (over days in the two-week maintenance period) *imputed reserve balance* at 4:00 pm and normalized it by dividing it by the bank's daily average required operating balance over the same maintenance period. We then used this sample to compute maximum likelihood estimates of the parameters of a Gaussian mixture model with two components. The estimated parameters are $\hat{\mu}_1 = 0.50$ and $\hat{\mu}_2 = 10.59$ (the means), $\sigma_1 = 2.9$ and $\sigma_2 = 31.1$ (the standard deviations), and $p_1 = 1 - p_2 = 0.73$ (the probability of drawing from the first component).³² We discuss the goodness of fit in Section C.2.3 below. The mean of the estimated distribution of average normalized imputed reserve balances for the 134 banks in the sample is $p_1\hat{\mu}_1 + p_2\hat{\mu}_2 = 3.22$. In order for the calibrated model to capture typical overall market conditions during the second quarter of 2007, we translate the estimated Gaussian mixture by choosing its mean to match the empirical mean

³¹The standard deviation of the Gaussian mixture is a function of the means of the two components, so changes in μ_1 affect the variance of the mixture. As a robustness check, we have also conducted experiments changing σ_1 along with μ_1 so as to keep the variance constant and found no significant difference in our results.

³²The corresponding standard errors (bootstrap, based on 10,000 iterations) for $\hat{\mu}_1$, $\hat{\mu}_2$, σ_1 , σ_2 , and p_1 are 0.37, 7.91, 0.84, 8.90, and 0.07, respectively.

of the ratio of total seasonally adjusted reserves of depository institutions to total required reserves reported in the H.3 Federal Reserve Statistical Release during the second quarter of 2007, which equals 1.04. This is done by considering a Gaussian mixture with the same p_1 , p_2 , σ_1 , and σ_2 that were estimated from the sample of 134 banks, but replacing the estimated means, $\hat{\mu}_1$ and $\hat{\mu}_2$, with $\mu_i = 0.32\hat{\mu}_i$, for $i = 1, 2$. This leads to the Gaussian mixture, Φ , with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, p_1)$. In order to feed this distribution into the model, we let $\bar{k} = 1$ (so k can be interpreted as a multiple of the reserve requirement), $\mathbb{K} = \{0, \dots, 250\}$, $\bar{k}_0 = 100$, and $n_k(T) = \Phi(k - \bar{k}_0 + 1) - \Phi(k - \bar{k}_0)$ for $k = 1, \dots, 249$, $n_0(T) = \Phi(-100)$, and $n_{250}(T) = 1 - \Phi(150)$. By construction, $Q \equiv \sum_{k=0}^{250} (k - \bar{k}_0)n_k(T) \approx 1.04$.

C.2.3. Goodness of Fit

As described above, for our policy experiments and counterfactual exercises, we use a Gaussian mixture with parameters estimated by maximum likelihood. In this section, we describe the process that led us to choose a Gaussian mixture. We estimated four parametric distributions as well as a mixture of two Gaussians to our initial distribution of balances and used several methods to assess goodness of fit.

The 2007 Initial Distribution of Balances. The Kolmogorov–Smirnov goodness-of-fit test does not reject the null hypothesis that the 2007 sample has been drawn from the Gaussian mixture with two components at the 90 percent confidence level. We have also fitted a Gaussian, a Logistic, and a Generalized Extreme Value distribution, but the null hypothesis is rejected by the Kolmogorov–Smirnov goodness-of-fit test at the 1 percent significance level. At the 90 percent confidence level, the test does not reject the null hypothesis that the data have been drawn from a t -Location Scale distribution.

The Chi-squared goodness-of-fit test does not reject the null hypothesis that the 2007 sample has been drawn from a Gaussian mixture with two components at the 99 percent confidence level. However, it rejects, at the 1 percent significance level, the null hypothesis that the data have been drawn from a Gaussian, a Logistic, or a Generalized Extreme Value distribution. At the 99 percent confidence level, the test does not reject the null hypothesis that the data have been drawn from a t -Location Scale distribution.

We also constructed quantile–quantile plots of the sample quantiles of our distribution of initial balances versus theoretical quantiles from a Gaussian, a Logistic, a Generalized Extreme Value, a t -Location Scale distribution, and Gaussian mixture with two components. Visually, the Q–Q plot of the Gaussian mixture with two components closely follows a linear trend line, suggesting that the mixture of two Gaussians is a reasonably good fit to the data.

The 2011 Initial Distribution of Balances. The Kolmogorov–Smirnov goodness-of-fit test does not reject the null hypothesis that the 2011 sample has

been drawn from the Gaussian mixture with two components at the 90 percent confidence level. We have fitted a Gaussian, a Logistic, and a t -Location Scale distribution, but the null hypothesis is rejected by the Kolmogorov–Smirnov goodness-of-fit test at the 99 percent confidence level. The test does not reject the null hypothesis that the data have been drawn from a Generalized Extreme Value distribution at the significance level 0.1.

The Chi-squared goodness-of-fit test rejects the null hypothesis that the data have been drawn from the Gaussian mixture with two components, a Gaussian, a Logistic, a Generalized Extreme Value, or a t -Location Scale distribution at the 99 percent confidence level.

The Q–Q plot of the Gaussian mixture with two components is relatively close to linear, suggesting that the mixture of two Gaussians is a relatively good fit to the data.

APPENDIX D: QUANTITATIVE EXERCISES

D.1. *Policy Counterfactual for 2007*

In this section, we use the model calibrated to mimic the salient features of a typical day in 2007 to conduct the types of policy experiments conducted in Section 6.3. Table IV reports the equilibrium values of $\bar{\rho}$ that result from varying i_f from 0 to 6 percent in 1 percent increments (as before, each column corresponds to a different value of Q/\bar{k}). All other parameter values are as in Section 6.1. Table V reports the equilibrium values of $\bar{\rho}$ that result from varying i_f^w from 575 basis points to 700 basis points in 25 basis point increments, while keeping all other parameter values as in Section 6.1.

D.2. *Sensitivity Analysis*

In this section, we carry out additional quantitative experiments to assess the sensitivity of the model predictions to changes in the contact rate, α , and the standard deviation of the initial distribution of reserve balances. These exercises show how the results of our policy experiments vary with the values of the key calibrated parameters (notice that Tables I–V already show how the equilibrium fed funds rate varies with the mean of the initial distribution of balances).

D.2.1. *Changes in the Contact Rate*

The top row of Figure 5 corresponds to the 2007 calibration with the initial distribution of reserves estimated by a Gaussian mixture as described in Section C.2.2. The top left panel plots the equilibrium value-weighted fed funds rate, $\bar{\rho}$, as a function of the aggregate normalized level of reserves, Q/\bar{k} , corresponding to five values of α . Notice that for any given level of Q/\bar{k} , the equilibrium rate $\bar{\rho}$ increases with α if $Q/\bar{k} < 1$ and decreases with α if $Q/\bar{k} > 1$. The

TABLE IV
EFFECTS OF i_f AND Q/\bar{k} ON THE FED FUNDS RATE (2007 COUNTERFACTUAL)

i_f	$Q/\bar{k} = 0.01$	$Q/\bar{k} = 0.25$	$Q/\bar{k} = 0.5$	$Q/\bar{k} = 1$	$Q/\bar{k} = 5$	$Q/\bar{k} = 10$	$Q/\bar{k} = 15$
0	0.0856	0.0799	0.0725	0.0543	0.0027	0.0003	0.0001
0.01	0.0879	0.0826	0.0759	0.0594	0.0125	0.0103	0.0101
0.02	0.0901	0.0854	0.0793	0.0644	0.0222	0.0202	0.0200
0.03	0.0923	0.0881	0.0827	0.0695	0.0320	0.0302	0.0300
0.04	0.0945	0.0909	0.0861	0.0746	0.0417	0.0402	0.0400
0.05	0.0968	0.0936	0.0896	0.0797	0.0515	0.0502	0.0500
0.06	0.0990	0.0964	0.0930	0.0847	0.0613	0.0601	0.0600

interest rate is independent of α in the “balanced market” with $Q/\bar{k} = 1$. The top right panel plots \bar{p} as a function of α keeping all other parameters (including Q/\bar{k}) at their baseline values for the 2007 calibration. These results are in line with the discussion at the end of Section 6.3. The bottom row of Figure 5 does the same exercise for the 2011 calibration.

D.2.2. Mean-Preserving Spreads of the Initial Distribution of Balances

Consider a Gaussian mixture with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, p_1)$. We parameterize a family of mean-preserving spreads of this distribution as follows. For any $\bar{\sigma} \in \mathbb{R}_+$, define $\tilde{\mu}_1 = \mu_1 + \delta p_2$, $\tilde{\mu}_2 = \mu_2 - \delta p_1$, and $\tilde{\sigma}_i = \bar{\sigma} \sigma_i$ for $i = 1, 2$, with $\delta \equiv (1 - \bar{\sigma})(\mu_2 - \mu_1)$. Then it is easy to see that $\bar{\sigma}$ indexes a family of Gaussian mixtures with parameters $(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, p_1)$, where each member of the family has the same mean, $p_1 \mu_1 + p_2 \mu_2$, and a standard deviation proportional to $\bar{\sigma}$. Thus, by varying $\bar{\sigma}$, we can generate mean-preserving spreads of the original Gaussian mixture. Clearly, the special case $\bar{\sigma} = 1$ corresponds to the original distribution with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, p_1)$.

Figure 6 shows the effects of a mean-preserving spread of the initial distribution of balances. The top row corresponds to the 2007 calibration, with the initial distribution of reserves given by a Gaussian mixture with parameters

TABLE V
EFFECTS OF i_f^w AND Q/\bar{k} ON THE FED FUNDS RATE (2007 COUNTERFACTUAL)

i_f^w	$Q/\bar{k} = 0.01$	$Q/\bar{k} = 0.25$	$Q/\bar{k} = 0.5$	$Q/\bar{k} = 1$	$Q/\bar{k} = 5$	$Q/\bar{k} = 10$	$Q/\bar{k} = 15$
0.0575	0.0817	0.0762	0.0692	0.0518	0.0026	0.0003	0.0001
0.0600	0.0837	0.0780	0.0708	0.0531	0.0027	0.0003	0.0001
0.0625	0.0856	0.0799	0.0725	0.0543	0.0027	0.0003	0.0001
0.0650	0.0876	0.0817	0.0741	0.0555	0.0028	0.0003	0.0001
0.0675	0.0895	0.0835	0.0757	0.0568	0.0028	0.0003	0.0001
0.0700	0.0915	0.0853	0.0774	0.0580	0.0029	0.0003	0.0001

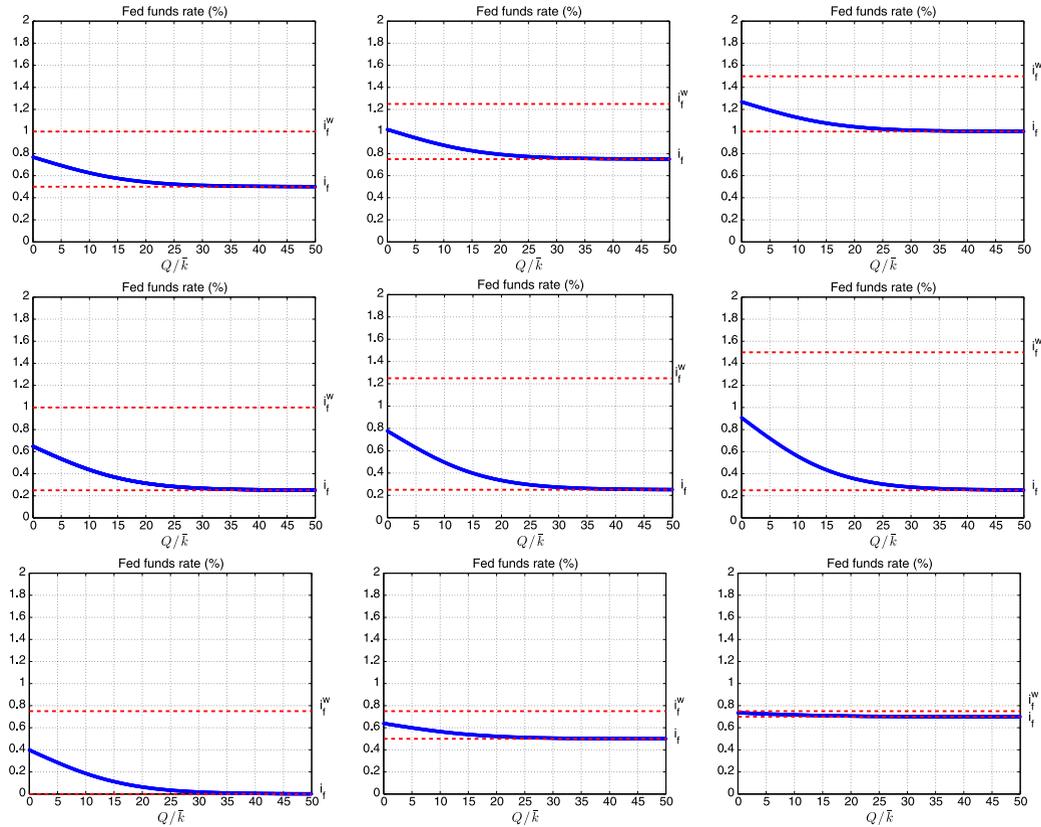


FIGURE 4.—Equilibrium fed funds rate ($\bar{\rho}$) as a function of the consolidated level of reserves (relative to required reserves) in the banking sector for different policies (i_f, i_f^w).

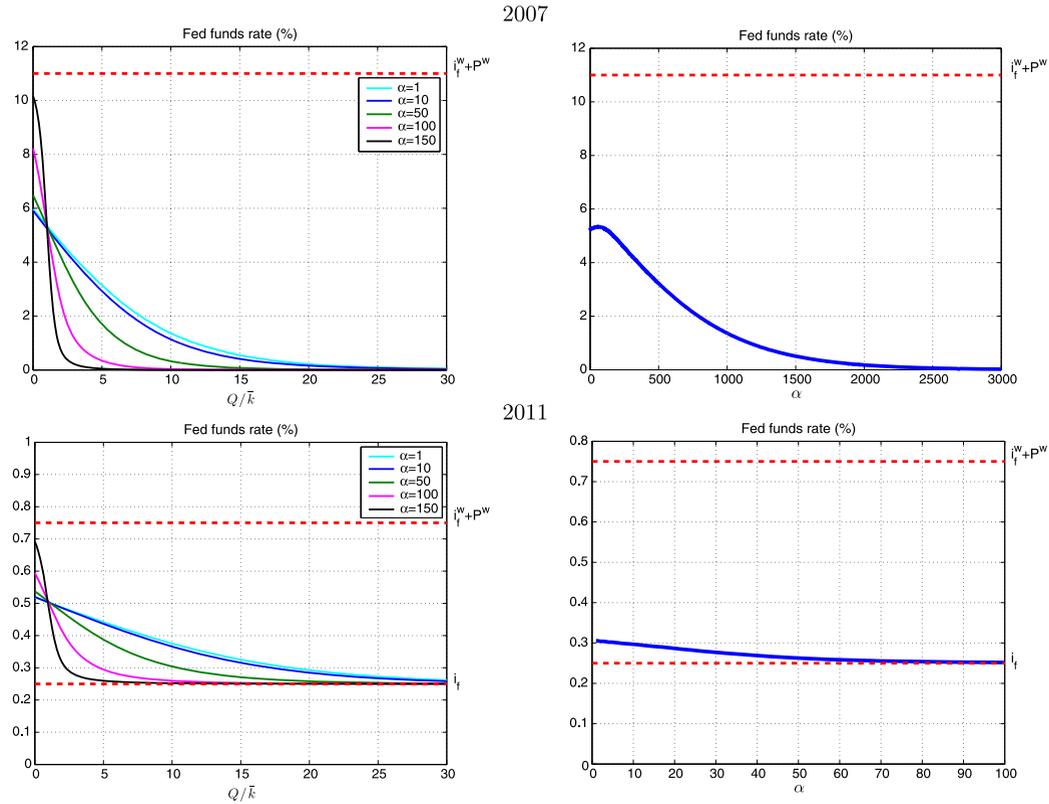


FIGURE 5.—Sensitivity analysis with respect to the contact rate α . Equilibrium fed funds rate (\bar{r}) as a function of the consolidated level of reserves (relative to required reserves) for different values of α (top left for 2007, bottom left for 2011). Equilibrium fed funds rate (\bar{r}) as a function of α (top right for 2007, bottom right for 2011).

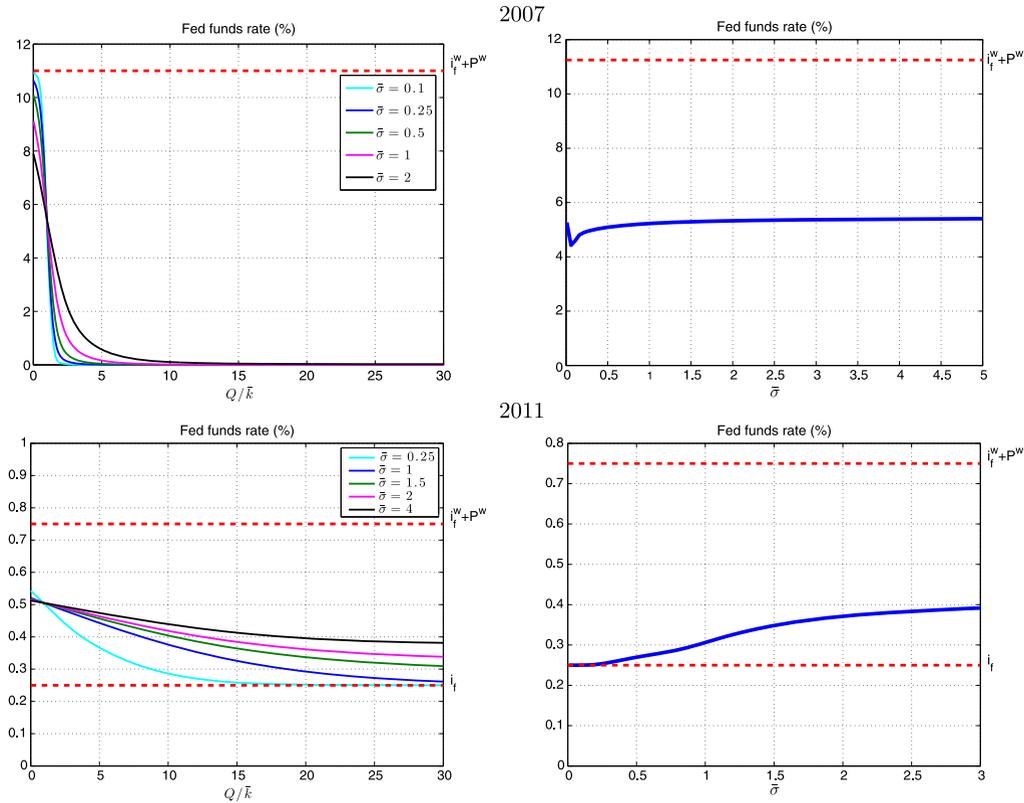


FIGURE 6.—Sensitivity analysis with respect to the standard deviation of the initial distribution of balances. Equilibrium fed funds rate ($\bar{\rho}$) as a function of the consolidated level of reserves (relative to required reserves) for different values of $\bar{\sigma}$ (top left for 2007, bottom left for 2011). Equilibrium fed funds rate ($\bar{\rho}$) as a function of $\bar{\sigma}$ (top right for 2007, bottom right for 2011).

$(\mu_1, \mu_2, \sigma_1, \sigma_2, p_1)$ estimated as described in Section C.2.2. The top left panel plots the equilibrium value weighted fed funds rate, $\bar{\rho}$, as a function of the aggregate normalized level of reserves, Q/\bar{k} , corresponding to five values of $\bar{\sigma}$. Again, we confirm that the distribution is neutral if the market is balanced, that is, if $Q/\bar{k} = 1$ (notice that $\bar{\rho}$ is invariant to $\bar{\sigma}$ when $Q/\bar{k} = 1$). The top right panel plots $\bar{\rho}$ as a function of $\bar{\sigma}$ keeping all other parameters (including Q/\bar{k}) at their baseline values for the 2007 calibration. The bottom row of Figure 6 does the same exercise for the 2011 calibration.

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