

SUPPLEMENT TO “GAMBLING REPUTATION: REPEATED
BARGAINING WITH OUTSIDE OPTIONS”
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THIS SUPPLEMENT CONTAINS additional results and proofs that were left out of the main paper. Section S.1 presents alternative equilibrium constructions that are outcome-equivalent to the reputation equilibrium of Theorem 1 in the main paper. Section S.2 demonstrates multiple reputation equilibria in the non-generic symmetric binary case. Section S.3 offers examples of non-reputation equilibria. Section S.4 formally compares our analysis to that of the reputation literature.

S.1. OUTCOME-EQUIVALENT EQUILIBRIUM PROFILES

We offer and verify alternative equilibrium strategy/belief profiles that induce the same outcome paths and payoffs as the equilibrium constructed for Theorem 1 in the main paper (Appendix C). We say that two strategy profiles are outcome-equivalent at belief p if the two profiles generate the same probability distribution of outcomes (transfers, outside options, and posterior beliefs).

S.1.1. *Randomization by the Short-Run Player at $p < p^*$*

Consider the following modification to the strategy profile and beliefs stated in Appendix C.2.2:

1. Player 2's strategy is the same except at $p \in (0, p^*)$, it offers some $s' > E^B[v] - c$ with probability $r(p) = \frac{p}{p^*} \frac{1-p^*}{1-p}$ and $E^B[v] - c$ with probability $1 - r(p)$.
2. Type G 's strategy is the same.
3. Type B 's strategy is the same except at $p \in (0, p^*)$, it accepts s if and only if $s \leq E^B[v] - c$.
4. Beliefs are the same.

It is straightforward to see that the arguments for Lemma 15 also apply here to verify the above profile, which is outcome-equivalent to the original equilibrium at all p .

S.1.2. *Type G 's Response*

Next, we consider an equilibrium in which type G 's response to the cutoff demand $E^G[v]$ depends on p . Consider the following modification to the strategy profile and beliefs stated in Appendix C.2.2:

1. Player 2's strategy is the same.
2. Type G 's strategy is if $p < p^{**}$, it accepts s if and only if $s < E^G[v]$; if $p \geq p^{**}$, it accepts s if and only if $s \leq E^G[v]$.
3. Type B 's strategy is the same except at any $p \in (p^*, p^{**})$, it accepts s if and only if $s < \max\{\xi(p), E^G[v]\}$, where $\xi(p) = \frac{S_\alpha(p) - \delta E^B[v]}{1 - \delta}$ and $S_\alpha(p)$ is the fixed point of T_α for $\alpha \in [0, 1]$ as in Appendix C.
4. Beliefs:
 - (a) The belief is updated by Bayes' rule whenever possible.
 - (b) At $p < p^{**}$, the posterior belief assigns probability 1 to type B after acceptance of a demand higher than or equal to $E^G[v]$; there is no change of belief after acceptance of a demand lower than $E^G[v]$.
 - (c) At $p \geq p^{**}$, the posterior belief assigns probability 1 to type B after acceptance of a demand strictly higher than $E^G[v]$; there is no change of belief after acceptance of a demand lower than or equal to $E^G[v]$.
 - (d) At any $p \in (p^{**}, 1)$, the posterior belief assigns probability 1 to type G after rejection (which is off-path).

Note that the only substantive difference between the above profile and the equilibrium profile in the main paper lies in type B 's response at $p \in (p^*, p^{**})$ when the offer is exactly $\max\{\xi(p), E^G[v]\}$; instead of accepting, type B now rejects it. The beliefs are rewritten to take account of this modification. We define $\xi(p)$ precisely as the demand whose acceptance and subsequent revelation generate the continuation payment $S_\alpha(p)$. Thus, the proposed profile can be verified similarly as in the proof of Lemma 15.

S.2. MULTIPLE EQUILIBRIA IN THE NON-GENERIC SYMMETRIC BINARY CASE

In the symmetric binary model considered in Section 4 and Appendix E, the equilibrium value function is given explicitly by the second-order difference equation for positive integer n :

$$(S.1) \quad S_n = (1 - \delta)E^B[v] + \delta q S_{n-1} + \delta(1 - q)S_{n+1}.$$

In the parametrized model, $\delta = e^{-r\Delta}$ and $q = \frac{1 + \mu\sqrt{\Delta}}{2}$. Starting from the initial conditions $S_{-1} = S_0 = \bar{S} = E^B[v] - (1 - \delta)c$, we define $N = \sup\{n : S_n > E^G[v]\}$. In Appendix E, we showed that, generically, $S_{N+1} < E^G[v]$.

Suppose the non-generic case in which $S_{N+1} = E^G[v]$. In this case, two equilibrium outcomes are possible. The first equilibrium is as reported above such that $p^* = \phi^{-N}(p^{**})$ (i.e., N consecutive unfavorable signals from p^{**}) and at p^{**} , player 2 makes a losing demand for sure, and, hence, $S^B(p^{**}) = S_N$. The other equilibrium is identical to the first equilibrium except for the following situations:

- At any $p \in (0, \phi^{-1}(p^*))$, type B rejects player 1's demand in a way that immediately after rejection moves posterior to $\phi^{-1}(p^*)$ (and not to p^*).
- At any $p \in [\phi^{-1}(p^*), p^*)$, type B rejects player 1's demand for sure.

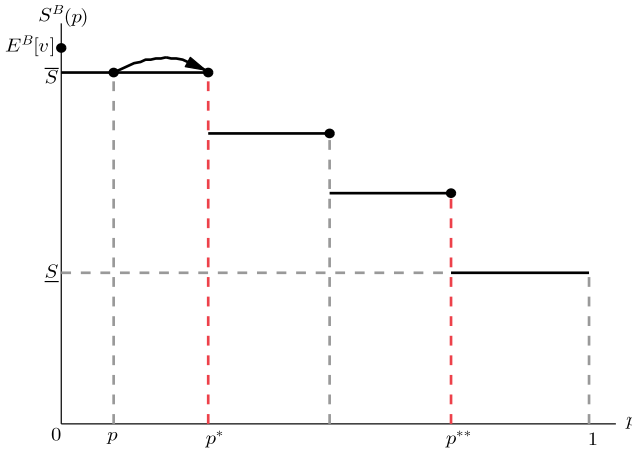


FIGURE S.1.—Non-generic equilibrium I.

- At p^{**} , player 2 demands $E^G[v]$ for sure and type B accepts it for sure. Thus, in this equilibrium, $S^B(p^{**}) = E^G[v]$.

Figures S.1 and S.2 illustrate these two equilibria.

Note that we can construct many other equilibrium strategies that result in the two value functions depicted in Figures S.1 and S.2. For instance, take the second case (Figure S.2) and consider some $p \in [\phi^{-1}(p^*), p^*)$. Here, type B is still indifferent to the demand $E^B[v] - c$ and, hence, may sometimes accept it, as long as the acceptance probability is not too large so that the posterior right after rejection does not jump above p^* .

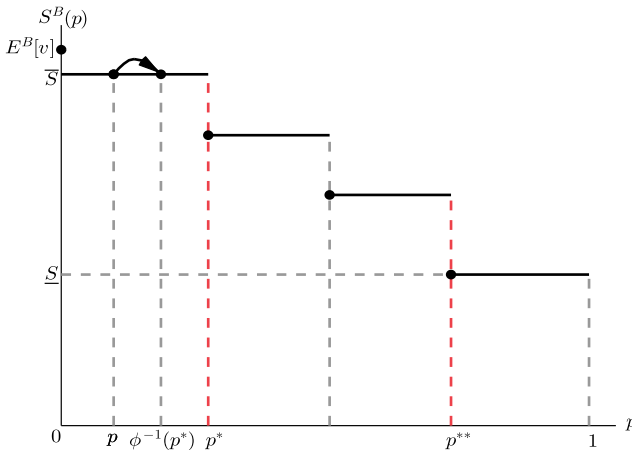


FIGURE S.2.—Non-generic equilibrium II.

S.3. NON-REPUTATION EQUILIBRIA

In this section, we demonstrate non-reputation equilibria of our game by setting $V = \{0, 1\}$ and $f^B(1) = f^G(0) = q \in (\frac{1}{2}, 1)$. Thus, $E^B[v] = q$ and $E^G[v] = 1 - q$; also, $2q - 1 > c$ and $p^{**} = \frac{E^B[v] - E^G[v] - c}{E^B[v] - E^G[v]} = \frac{2q - 1 - c}{2q - 1}$.

S.3.1. A Folk Theorem With Complete Information

First, we relax the restriction to Markov strategies and establish a folk theorem for the case of complete information with $p = 0$.

LEMMA S.1: *Suppose that player 1's type is known to be B. Then the following statements hold:*

(a) *In any subgame perfect equilibrium, player 1's equilibrium expected payment, S , is such that $S \in [q - c, q]$.*

(b) *Fix any $\delta > \frac{1}{2}$. Then any $S \in [q - c, q]$ can be supported as an equilibrium expected payment of player 1.*

PROOF: *Part (a):* Fix any δ and any subgame perfect equilibrium. First, let us show that $S \geq q - c$. Suppose not, so $S < q - c$. Then since rejecting any offer gives player 1 (one-period) expected payment of q , acceptance of an offer strictly below $q - c$ must occur at some history on the equilibrium path. Consider player 2 who makes such an offer. But, clearly, this short-run player can improve his expected payoff by instead making any offer at least $q - c$; player 1's rejection gives him payoff $q - c$.

Next, let us show that $S \leq q$. Suppose not. But the bad type can guarantee himself payment of q by always rejecting.

Part (b): We know that there exists a Markov equilibrium that supports payment q . Consider any $S \in [q - c, q]$ and the following trigger strategy profile:

- At any history in which no deviation from the equilibrium has been observed, player 2 offers S for sure and player 1 accepts an offer if and only if it is less than or equal to S .

- At any history in which acceptance of an offer higher than S has been observed, player 2 offers q for sure and player 1 accepts an offer if and only if it is less than or equal to q .

- At any other history, player 2 offers S for sure and player 1 accepts an offer if and only if it is less than or equal to S .

To establish that the above profile constitutes a subgame perfect equilibrium, it suffices to consider player 1's incentives when facing a deviating offer $S + \varepsilon$ for small $\varepsilon > 0$. Given the above profile, rejecting the offer yields payment $(1 - \delta)q + \delta S$, while accepting leads to $(1 - \delta)(S + \varepsilon) + \delta q$. Since $\delta > \frac{1}{2}$ and $S < q$, it is easily seen that the latter is larger than the former. Thus, player 1 will reject $S + \varepsilon$ for sure. This, in turn, supports optimality of player 2's strategy. *Q.E.D.*

Now fix any $\delta > \bar{\delta}$ as in the proof of Theorem 1. The following strategies describe a non-reputation equilibrium:

- At any history/period t with $p_t > 0$, all players play according to the equilibrium of Theorem 1 for belief p_t .
- At any history with $p_t = 0$, the continuation strategies are given by the equilibrium in which the bad type obtains payment $S^* \in [q - c, q)$ (Lemma S.1 above).

It is straightforward to see that this profile only changes the initial condition for the recursive equation (S.1) above, from $S_0 = \bar{S} = q - (1 - \delta)c$ to $S_0 = (1 - \delta)(q - c) + \delta S^*$.

S.3.2. Non-Monotone Payoffs

In a reputation equilibrium, the long-run player's payoff (or payment) is assumed to be monotone increasing (or decreasing) in reputation $p \in [0, 1]$. A motivation for this restriction is that reputation is often taken to be a valuable asset. In this section, we offer examples of non-reputation equilibrium in which equilibrium payoffs are non-monotone. A common feature of the equilibria constructed below is that type G adopts different cutoffs at different beliefs.

EXAMPLE S.1: In this example, we construct and verify an equilibrium in which the payments are monotone decreasing at interior beliefs $p \in (0, 1)$, but jump up at $p = 1$.

- For any $C \in (1 - q - c, 1 - q)$, define \bar{p}^{**} to be such that

$$C = \bar{p}^{**}(1 - q) + (1 - \bar{p}^{**})q - c$$

or $\bar{p}^{**} = \frac{q-c-C}{2q-1} \in (0, 1)$. Thus, at \bar{p}^{**} , player 2 is indifferent between C accepted for sure and a losing demand. Note that $\bar{p}^{**} \in (p^{**}, 1)$.¹

- Fix any $\bar{p}^* \in (\phi^1(p^{**}), 1)$, that is, \bar{p}^* cannot be reached from p^{**} after a good signal.

- Fix any $C \in (1 - q - c, 1 - q)$ such that $\bar{p}^{**} \in (\bar{p}^*, \phi^1(\bar{p}^*))$. It is easy to see that such a C exists.

Next, consider equilibrium strategies and beliefs. The new equilibrium and its belief system are identical to our main equilibrium with two belief thresholds p^* and p^{**} (Appendix C.2.2) at all levels of p except at the interval $(\bar{p}^*, 1)$. In this region of beliefs, we have the following scenarios:

1. At $p \in (\bar{p}^*, \bar{p}^{**})$, type G accepts s if and only if $s^G \leq \max\{\frac{S^G(p) - \delta(1-q)}{1-\delta}, C\}$ and type B accepts s if and only if $s^B \leq \max\{\frac{S^B(p) - \delta q}{1-\delta}, C\}$.

¹Note that $C > 1 - q - c$. This is because player 2 can guarantee the expected payoff $1 - q - c$ from the outside option and, hence, \bar{p}^{**} is not well defined if $C = 1 - q - c$.

2. At $p \in [\bar{p}^{**}, 1)$, both types accept s if and only if $s \leq C$.
3. Player 2 demands q at $p \in (\bar{p}^*, \bar{p}^{**})$ and C at $p \in [\bar{p}^{**}, 1)$.

4. Beliefs:

(a) Beliefs are updated via Bayes' rule whenever possible.

(b) At any $p \in (\bar{p}^*, 1)$, acceptance of any demand strictly larger than C takes the posterior to 0; there is no change of belief after acceptance of a demand lower than or equal to C .

(c) At any $p \in [\bar{p}^{**}, 1)$, rejection itself does not change the belief.

Thus, in the proposed equilibrium, the long-run players' bargaining postures become tougher at high beliefs above p^{**} , with a new cutoff $C < 1 - q$. In fact, another region of gambling appears, $(\bar{p}^*, \bar{p}^{**})$, where both types of player 1 reject the equilibrium demand in the hope of obtaining a payoff even better than $1 - q = E^G[v]$ at $[\bar{p}^{**}, 1)$. This equilibrium is depicted in Figure S.3 below.

Now, we show that there exists some $\bar{\delta}$ such that, for $\delta > \bar{\delta}$, the proposed profile constitutes an equilibrium. It suffices to consider $p \in (\bar{p}^*, 1)$.

Case I: Suppose $p \in (\bar{p}^*, \bar{p}^{**})$. In this case, clearly, the two types' best responses to C or less are to accept it for sure. In addition, note that by our definitions of \bar{p}^* and \bar{p}^{**} , after rejection, a good signal takes the belief to $(\bar{p}^{**}, 1)$ and a bad signal takes it to (p^{**}, \bar{p}^*) . Also, accepting a demand higher than C takes the belief to 0, where the offer must be q (which type B accepts and type G rejects).

Thus, type G 's reservation one-period payment $s^G = \frac{S^G(p) - \delta(1-q)}{1-\delta}$ is given by

$$\begin{aligned} & (1 - \delta)s^G + \delta(1 - q) \\ &= S^G(p) = (1 - \delta)(1 - q) + \delta(1 - q)^2 + \delta q C \in (C, 1 - q), \end{aligned}$$

while type B 's reservation one-period payment $s^B = \frac{S^B(p) - \delta q}{1-\delta}$ is given by

$$(1 - \delta)s^B + \delta q = S^B(p) = (1 - \delta)q + \delta q(1 - q) + \delta(1 - q)C,$$

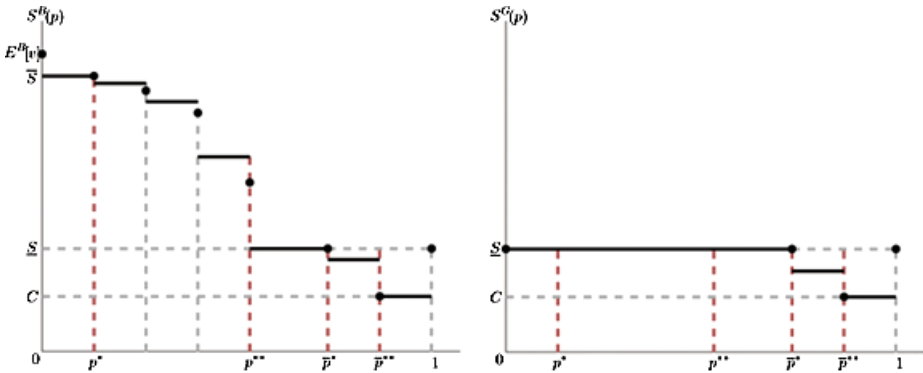


FIGURE S.3.—Non-monotone equilibrium I.

where $S^B(p) \in (C, 1 - q)$ if $\delta > \frac{2q-1}{q^2-(1-q)C} \in (0, 1)$.

Consider player 2. We have that $s^G < 1 - q - c$ if $\delta > \frac{c}{c+(1-q)-(1-q)^2-qC} \in (0, 1)$. Similarly, $s^B < q - c$ with sufficiently large δ . Since $p < \bar{p}^{**}$, player 2 strictly prefers a losing demand to a sure payoff of C , while $1 - q - c$ and $q - c$ are the expected payoffs from an outside option when player 1 is G and B , respectively. This implies that with sufficiently large δ , player 2 finds it optimal to offer a losing demand at $p \in (\bar{p}^*, \bar{p}^{**})$.

Case 2: Suppose $p \in (\bar{p}^{**}, 1)$. Consider type G . Clearly, accepting s is optimal if $s \leq C$. Suppose that $s > C$. Since acceptance takes the belief to 0 and hence, the ensuing demand is q (which he will reject), the continuation payment equals $(1 - \delta)s + \delta(1 - q)$, while the continuation payment from rejection (which does not alter the belief itself) is at most $(1 - \delta)(1 - q) + \delta(1 - q)^2 + \delta qC < 1 - q$. Thus, rejection is the best response with sufficiently large δ . We can handle type B via a similar argument. Since $p \geq \bar{p}^{**}$, player 2's strategy best responds to his opponents' strategies.

EXAMPLE S.2: Our next construction demonstrates the possibility of non-monotone payoff jump below $p = 1$. Fix two cutoff demands $C^1, C^2 \in (1 - q - c, 1 - q)$ such that $C^1 < C^2$. Define $p^{**} = \frac{q-c-C^1}{2q-1}$ and fix $\bar{p} \in (\phi^1(p^{**}), 1)$. The equilibrium strategies are essentially the same as before, except that type G accepts a demand s if and only if $s \leq C^1$ at $p < \bar{p}$, and if and only if $s \leq C^2$ at $p \in [\bar{p}, 1)$. Figure S.4 illustrates type B 's payoffs induced by such a profile.

Up to \bar{p} , the construction is identical to our main equilibrium except that type G 's cutoff is C^1 instead of $1 - q$ and p^{**} is redefined accordingly. To show that it is mutually optimal for C^2 to be offered and accepted for sure

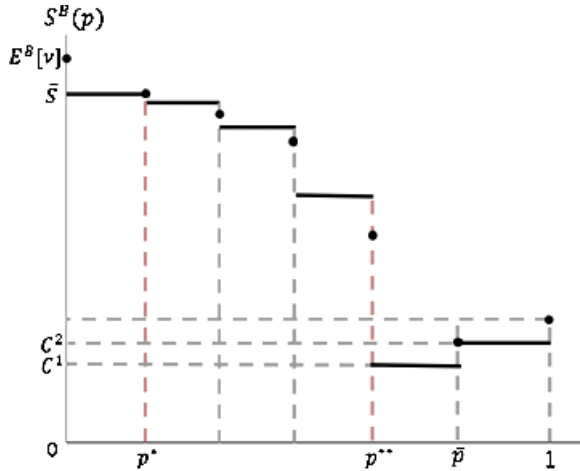


FIGURE S.4.—Non-monotone equilibrium II.

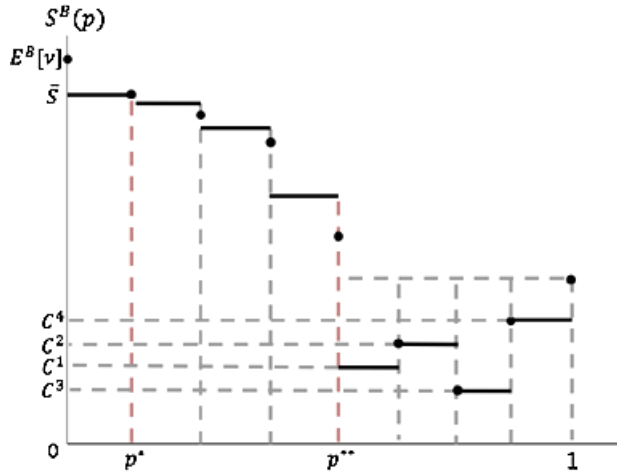


FIGURE S.5.—Further example.

at $p \in (\bar{p}, 1)$, consider type B 's incentive to reject C^2 at such a belief (a similar argument would also apply to type G). Here, rejection yields at best $(1 - \delta)q + \delta[qC^1 + (1 - q)C^2]$, but for given δ , this will not be less than the equilibrium payment C^2 from acceptance as long as $C^2 - C^1$ is sufficiently small.

Now, we see that there are many other non-monotone equilibria with a similar structure: when p is sufficiently close to 1, type G changes the cutoff demand over multiple intervals. For given δ , this kind of non-stationarity can generate multiple non-monotone steps in the equilibrium value function as long as the cutoffs are sufficiently close to one another; see Figure S.5 for an illustration.

S.4. A COMPARISON WITH THE REPUTATION LITERATURE

Our model differs from the behavior-type reputation approach of Fudenberg and Levine (1989, 1992) in two major aspects. First, we have assumed payoff types on the long-run player and derived their equilibrium behavior endogenously in reputation equilibrium. Second, outside options in our model reveal information about payoff types and, at the same time, are directly payoff-relevant.

To bring our analysis closer to Fudenberg and Levine (1989), we consider the following variant of our model:

- Player 1 is either the rational type or an insistent type who accepts a demand if and only if it is no larger than a cutoff C . Let the prior probability on the insistent type be p_0 .
- An outside option is type-independent; that is, we let $f^B = f^G = f$ with a mean $E[v]$.

- Player 2 incurs a cost $c \in (0, E[v])$ upon rejection from player 1.

It is important to note that this new model still differs from the canonical model of [Fudenberg and Levine \(1989\)](#) in that its stage game is one of extensive form and, moreover, not all of the long-run player's strategies are identifiable, since only actual transfers are observable.

S.4.1. Benchmark: A Folk Theorem With $p_0 = 0$

First consider the benchmark case of $p_0 = 0$ in our new model. Using a similar argument employed in the proof of [Lemma S.1](#) above, we obtain the following lemma.

LEMMA S.2: *Suppose that $p_0 = 0$. Then the following statements hold:*

- In any subgame perfect equilibrium, player 1's (discounted average expected) payment, S , is such that $S \in [E[v] - c, E[v]]$.*
- Fix any $\delta > \frac{1}{2}$. Then any $S \in [E[v] - c, E[v]]$ can be supported as an equilibrium payment of player 1.*

S.4.2. Stackelberg Payoff and Strategy

Next we consider the range of insistent cutoff C . Let $D(C)$ be the set of best responses of player 2 to player 1's insistent strategy with cutoff C . We can then denote the *Stackelberg payoff* of player 1 as

$$\sup_C \inf_{d \in D(C)} u_1(C, d),$$

where $u_1(C, d)$ is the negative of player 1's payment, to be consistent with the standard maximin payoff notion in repeated games.

Note that

$$D(C) = \begin{cases} \{C\} & \text{if } C > E[v] - c, \\ \{s : s \geq E[v] - c\} & \text{if } C = E[v] - c, \\ \{s : s > C\} & \text{if } C < E[v] - c. \end{cases}$$

The reason is as follows. If $C > E[v] - c$, then player 2 should demand just C , since a lower demand is clearly dominated by C , while a higher demand is rejected, leading to a payoff of $E[v] - c$. If $C < E[v] - c$, then player 2 is better off making a losing demand. If the cutoff is exactly $E[v] - c$, player 2 is indifferent between a losing demand and an acceptable demand $E[v] - c$.

With $D(C)$ in place, it is clear that

$$\sup_C \inf_{d \in D(C)} u_1(C, d) = -(E[v] - c).$$

However, this sup-inf payoff is not achieved by the insistent strategy $C = E[v] - c$ because

$$\inf_{d \in D(E[v]-c)} u_1(E[v] - c, d) = -E[v].$$

That is, faced with cutoff $E[v] - c$, player 2's best response that minimizes player 1's payoff is to make a losing demand. Note that

$$\inf_{d \in D(E[v]-c+1/n)} u_1\left(E[v] - c + \frac{1}{n}, d\right) = -\left(E[v] - c + \frac{1}{n}\right).$$

Hence, the Stackelberg payoff $\sup_C \inf_{d \in D(C)} u_1(C, d)$ is achieved by a sequence of cutoffs $C_n = E[v] - c + \frac{1}{n}$, but the Stackelberg strategy does not exist.

S.4.3. Payoff Bound

To simplify our analysis, we make the assumption that player 2's demand is bounded above by a constant $\bar{s} > 0$.

LEMMA S.3: *Fix any δ and any equilibrium. There exists $\tilde{p} \in (0, 1)$ such that at any history with $p > \tilde{p}$, player 2 demands C for sure.*

PROOF: Define $\tilde{p} \in (0, 1)$ such that

$$\tilde{p}C + (1 - \tilde{p})(E[v] - c) = \tilde{p}(E[v] - c) + (1 - \tilde{p})\bar{s},$$

where the left-hand side is the payoff that player 2 can secure by demanding C at belief \tilde{p} ; the right-hand side is the highest possible payoff that could be obtained when demanding more than C . By definition, if $p > \tilde{p}$, the demand must be C . *Q.E.D.*

LEMMA S.4: *Fix any δ and any Markov perfect equilibrium. Also, fix any belief p and consider a demand $s > C$. Then if the rational type accepts s with positive probability, he must accept any $s' < s$ for sure.*

PROOF: If s is accepted, the continuation (discounted average expected) payment from accepting s must be at least as good as that from rejecting it.

Since rejected demands are not observable, rejecting any demand results in the same continuation payment. Also, by the Markov assumption, accepting any demand strictly above C leads to revelation and, hence, continuation payment at the next period equal to $E[v]$. Then accepting any $s' \in (C, s)$ must be strictly better than rejecting it since it yields a lower immediate payment.

On the other hand, accepting a demand $s' \leq C$ need not lead to revelation, but the continuation payment at the next period must still be bounded above by $E[v]$. Thus, the same arguments imply that such a demand must also be accepted for sure. *Q.E.D.*

THEOREM S.1: *For any $C \in (E[v] - c, E[v])$, $p_0 \in (0, 1)$, and $\varepsilon > 0$, there exists $\tilde{\delta}$ such that rational player 1's (discounted average expected) payment in any Markov perfect equilibrium is at most $C - \varepsilon$ if $\delta > \tilde{\delta}$.*

REMARK: Notice that if we take $C_n = E[v] - c + \frac{1}{n}$, then this theorem says that when the discount factor is large, the rational type's payment is bounded above by a level arbitrarily close to C_n . If we take $n \rightarrow \infty$, we show that player 1 can obtain his Stackelberg payment $E[v] - c$ (note from the previous section that the Stackelberg strategy in this model is not well defined). Therefore, the reputation gain is proportional to c . However, our analysis in the main paper has shown that with informative outside options, the reputation gain is only $(1 - \delta)c$, which approaches 0 as $\delta \rightarrow 1$. This highlights the difference between our model and [Fudenberg and Levine \(1989\)](#): informative outside options are indeed the source of the low reputation benefit.

PROOF OF THEOREM S.1: Fix a Markov perfect equilibrium and consider a deviation by the rational type to mimicking the insistent type: accepting a demand if and only if it is no larger than C . Now consider player 2 in period t before player 1's type is revealed, that is, $p_t \in (0, 1)$.

If player 2 demands C , then his expected payoff in the given equilibrium is bounded below by

$$p_t C + (1 - p_t)(E[v] - c),$$

where $C > E[v] - c$ is accepted for sure by the insistent type whose probability is at least p_t and $E[v] - c$ is the least player 2 could obtain from the rational type in any equilibrium.

If player 2 demands $s > C$, then player 2's payoff is bounded above by

$$p_t(E[v] - c) + (1 - p_t)[a_s s + (1 - a_s)(E[v] - c)],$$

where a_s denotes the probability with which the rational type accepts s .

Therefore, player 2 demands $s > C$ in the given equilibrium only if

$$\begin{aligned} p_t(E[v] - c) + (1 - p_t)[a_s s + (1 - a_s)(E[v] - c)] \\ \geq p_t C + (1 - p_t)(E[v] - c). \end{aligned}$$

This is equivalent to

$$(S.2) \quad a_s \geq \frac{p_t}{1 - p_t} \frac{C - (E[v] - c)}{s - (E[v] - c)} > 0.$$

Hence, we have just shown that if player 2 demands $s > C$, he must anticipate that s is accepted by the rational type with positive probability.

Next we make the following claim.

Claim: Fix any Markov perfect equilibrium. If the rational type deviates to mimic the insistent type, then the posterior in any period is bounded below by p_0 .

Proof of the Claim: At any t , if an offer $s \leq C$ is accepted by player 1 and observed, the posterior belief cannot go down. If player 2's demand puts a positive probability on $s > C$, then s is rejected for sure by the insistent type, while by the previous argument, s must be accepted with positive probability by the rational type. This, together with Lemma S.4 above, implies that any demand less than s must be accepted for sure by the rational type and, hence, rejection is more likely to come from the insistent type. It follows that rejection cannot reduce reputation.

Thus, it follows that if player 2 ever demands $s > C$ in equilibrium, by (S.2), the acceptance probability a_s must be bounded below such that

$$(S.3) \quad a_s \geq \frac{p_0}{1-p_0} \frac{C - (E[v] - c)}{s - (E[v] - c)} \geq \frac{p_0}{1-p_0} \frac{C - (E[v] - c)}{\bar{s} - (E[v] - c)} =: \kappa.$$

Now let us return to the play conditional on the rational type's deviation. Notice that observable histories are only rejection and acceptance of a demand equal to or lower than C . Moreover, the posterior on the insistent type is bounded below by p_0 . If player 2's demand is always no larger than C , then clearly the rational type achieves an average payment of at most C . Suppose that this is not the case, and consider a period t with posterior p_t in which player 2 offers a demand strictly higher than C with positive probability. From Lemma S.3 above, $p_t \leq \tilde{p}$. After rejection in t , by (S.3), the posterior at $t+1$ is at least

$$p_{t+1} \geq \frac{p_t}{p_t + (1-p_t)(1-\kappa)}.$$

Hence, since $p_t \leq \tilde{p} < 1$, we have

$$p_{t+1} \geq \frac{p_t}{\tilde{p} + (1-\tilde{p})(1-\kappa)}.$$

Now, let K be such that

$$p_0 \left[\frac{1}{\tilde{p} + (1-\tilde{p})(1-\kappa)} \right]^K > 1.$$

Such a K exists and is finite because $\tilde{p} + (1-\tilde{p})(1-\kappa) < 1$. Therefore, there can be at most K periods during which player 2 makes a demand strictly larger than C ; otherwise, rejecting such demands K times would take the belief above \tilde{p} at which we know C must be demanded for sure. Note that K is independent of δ . This proves the payment bound of the rational type. *Q.E.D.*

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