SUPPLEMENT TO "ESTIMATING DERIVATIVES IN NONSEPARABLE MODELS WITH LIMITED DEPENDENT VARIABLES"
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APPENDIX

This supplement provides proofs of Theorem 3.1 and Lemma 4.1.

Proof of Theorem 3.1: To show (5) it is sufficient to prove the derivative formula (A.1) for ∇_1, the partial derivative with respect to the first element of x, that is,

$$
\text{(A.1)} \quad \nabla_1 \int M(x, u) I_M(x) \, d\mu(u)
$$
$$
= \int \nabla_1 M(x, u) I_M(x) \, d\mu(u) - H(x) \nabla_1 G_H(x) - L(x) \nabla_1 G_L(x).
$$

The left-hand side of (A.1) is written as

$$
\lim_{\varepsilon \to 0} \left[\int M(x + \varepsilon \mathbf{e_1}, u) I_M(x + \varepsilon \mathbf{e_1}) \, d\mu(u) \right.
$$
$$
\left. - \int M(x, u) I_M(x) \, d\mu(u) \right] / \varepsilon
$$
$$
= \lim_{\varepsilon \to 0} \int [M(x + \varepsilon \mathbf{e_1}, u) - M(x, u)] I_M(x + \varepsilon \mathbf{e_1}) \, d\mu(u) / \varepsilon
$$
$$
+ \lim_{\varepsilon \to 0} \int M(x, u) [I_M(x + \varepsilon \mathbf{e_1}) - I_M(x)] \, d\mu(u) / \varepsilon
$$
$$
= T_1 + T_2,
$$

where $\mathbf{e_1} = \pm(1, 0, \ldots, 0)$. Assumptions 2, 4, and 5 imply $\lim_{\varepsilon \to 0} I_M(x + \varepsilon \mathbf{e_1}) = I_M(x)$ a.s. Thus, Assumption 4 and the Lebesgue dominated convergence theorem imply that $T_1 = \int \nabla_1 M(x, u) I_M(x) \, d\mu(u)$. We now consider T_2. By the definition of I_M and Assumption 2,

$$
I_M(x + \varepsilon \mathbf{e_1}) - I_M(x)
$$
$$
= \big[I\{L(x + \varepsilon \mathbf{e_1}) < M(x + \varepsilon \mathbf{e_1}, U)\}
$$
$$
+ I\{M(x + \varepsilon \mathbf{e_1}, U) < H(x + \varepsilon \mathbf{e_1})\} \big]
$$
$$
- \big[I\{L(x) < M(x, U)\} + I\{M(x, U) < H(x)\} \big]
$$

a.s. for all ε sufficiently close to zero. So, T_2 can be written as

$$
\begin{aligned}
T_2 = \lim_{\varepsilon \to 0} \int & M(x, u) \big[I\{L(x + \varepsilon \mathbf{e_1}) < M(x + \varepsilon \mathbf{e_1}, u)\} \\
& - I\{L(x) < M(x, u)\} \big] d\mu(u)/\varepsilon \\
+ \lim_{\varepsilon \to 0} \int & M(x, u) \big[I\{M(x + \varepsilon \mathbf{e_1}, u) < H(x + \varepsilon \mathbf{e_1})\} \\
& - I\{M(x, u) < H(x)\} \big] d\mu(u)/\varepsilon.
\end{aligned}
$$

Since $I\{L(x + \varepsilon \mathbf{e_1}) < M(x + \varepsilon \mathbf{e_1}, u)\} = 1 - I\{M(x + \varepsilon \mathbf{e_1}, u) \leq L(x + \varepsilon \mathbf{e_1})\}$ for all ε sufficiently close to zero, the following lemma completes the proof.
                                                                                                    Q.E.D.

LEMMA A.1: *Under Assumptions 1–5,*

$$
\begin{aligned}
\text{(A.2)} \quad \lim_{\varepsilon \to 0} \int & M(x, u) \big[I\{M(x + \varepsilon \mathbf{e_1}, u) < H(x + \varepsilon \mathbf{e_1})\} \\
& - I\{M(x, u) < H(x)\} \big] d\mu(u)/\varepsilon \\
& = -H(x) \nabla_1 G_H(x).
\end{aligned}
$$

PROOF: It is sufficient to show that both an upper bound and a lower bound of the left-hand side of (A.2) converge to the right-hand side as $\varepsilon \to 0$. The left-hand side of (A.2) equals

$$
\begin{aligned}
\text{(A.3)} \quad \lim_{\varepsilon \to 0} \int & M(x, u) I\{M(x + \varepsilon \mathbf{e_1}, u) < H(x + \varepsilon \mathbf{e_1})\} \\
& \times I\{M(x, u) \geq H(x)\} d\mu(u)/\varepsilon, \\
- \lim_{\varepsilon \to 0} \int & M(x, u) I\{M(x + \varepsilon \mathbf{e_1}, u) \geq H(x + \varepsilon \mathbf{e_1})\} \\
& \times I\{M(x, u) < H(x)\} d\mu(u)/\varepsilon.
\end{aligned}
$$

Since the argument is analogous, we only show the result for an upper bound.

For any small $\varepsilon > 0$ that satisfies the neighborhood condition in Assumption 4, by the mean value theorem there exists $0 < \tilde{\varepsilon} < \varepsilon$ such that $M(x + \varepsilon \mathbf{e_1}, U) = M(x, U) + \nabla M(x + \tilde{\varepsilon} \mathbf{e_1}, U)\varepsilon$ a.s. Thus, by Assumption 4,

$$
\begin{aligned}
M(x + \varepsilon \mathbf{e_1}, U) & \leq M(x, U) + \sup_{0 < \tilde{\varepsilon} < \varepsilon} \nabla M(x + \tilde{\varepsilon} \mathbf{e_1}, U)\varepsilon \\
& \leq M(x, U) + \sup_{x' \in N(x, \varepsilon)} \nabla M(x', U)\varepsilon \\
& \leq M(x, U) + B(U)\varepsilon
\end{aligned}
$$

a.s., where $N(x, \varepsilon)$ is a neighborhood around x with radius ε. Analogously by replacing the supremum with the infimum, we can show that $M(x + \varepsilon e_1, U) \geq M(x, U) - B(U)\varepsilon$ a.s.

By these inequalities, (A.3) can be bounded from above by

$$\lim_{\varepsilon \to 0} \int H(x + \varepsilon \mathbf{e_1}) I\{M(x + \varepsilon \mathbf{e_1}, u) < H(x + \varepsilon \mathbf{e_1})\}$$

$$\times I\{M(x, u) \geq H(x)\} \, d\mu(u)/\varepsilon$$

$$+ \lim_{\varepsilon \to 0} \int B(u) I\{M(x + \varepsilon \mathbf{e_1}, u) < H(x + \varepsilon \mathbf{e_1})\}$$

$$\times I\{M(x, u) \geq H(x)\} \, d\mu(u)$$

$$- \lim_{\varepsilon \to 0} \int H(x + \varepsilon \mathbf{e_1}) I\{M(x + \varepsilon \mathbf{e_1}, u) \geq H(x + \varepsilon \mathbf{e_1})\}$$

$$\times I\{M(x, u) < H(x)\} \, d\mu(u)/\varepsilon$$

$$+ \lim_{\varepsilon \to 0} \int B(u) I\{M(x + \varepsilon \mathbf{e_1}, u) \geq H(x + \varepsilon \mathbf{e_1})\}$$

$$\times I\{M(x, u) < H(x)\} \, d\mu(u).$$

By Assumptions 2, 4, and 5, the Lebesgue dominated convergence theorem implies that the second term and the fourth term converge to zero. The first term and the third term can be rewritten as

$$\lim_{\varepsilon \to 0} H(x + \varepsilon \mathbf{e_1}) \int \left[I\{M(x + \varepsilon \mathbf{e_1}, u) < H(x + \varepsilon \mathbf{e_1})\} \right.$$

$$\left. - I\{M(x, u) < H(x)\} \right] d\mu(u)/\varepsilon,$$

which is the right-hand side of (A.2) under Assumptions 2 and 3. The conclusion is obtained.   *Q.E.D.*

PROOF OF LEMMA 4.1: The basic idea of the proof is as follows. First, independently from x, we pick any strictly increasing distribution functions F_1 and F_2 with continuous densities f_1 and f_2 such that

$$(A.4) \quad \sup_x |H(x) - L(x)| \{F_j^{-1}(p_H) - F_j^{-1}(p_L)\} \max_{\tilde{u}_j \in [F_j^{-1}(p_L), F_j^{-1}(p_H)]} f_j(\tilde{u}_j)$$

$$< 2\varepsilon(p_2 - p_1)$$

for $j = 1, 2$, where p_1, p_2, p_L, and p_H satisfy Assumption 3' and ε satisfies Assumption 2'. Since $\sup_x |H(x) - L(x)|$ is bounded by a constant from Assumption 3', it is possible to choose such F_1 and F_2. We set the joint density of $\tilde{U}$ as $f_{\tilde{U}}(\tilde{u}_1, \tilde{u}_2) = f_1(\tilde{u}_1) f_2(\tilde{u}_2)$. Second, we pick any point x. Third, for the given

x, we show the existence of $(\tilde{M}_0(x), \tilde{M}_1(x), \tilde{M}_2(x))$ satisfying the equivalence $\Psi(x) = E[M(X, U)|X = x, I_M(X) = 1] = E[\tilde{M}(X, \tilde{U})|X = x, I_M(X) = 1]$. Fourth, observe that we can apply this argument for any x with the same F_1 and F_2 above to show the equivalence on $\Psi(x)$ for all x (note: F_1 and F_2 do not depend on x by definition). Finally, showing that $\tilde{M}(x, \tilde{u})$ is differentiable in x (for almost every $\tilde{u}$) implies $E[\nabla M(X, U)|X = x, I_M(X) = 1] = E[\nabla \tilde{M}(X, \tilde{U})|X = x, I_M(X) = 1]$. Hereafter, we show the third and final steps.

The third step. For given x, we want to find $(\tilde{M}_0(x), \tilde{M}_1(x), \tilde{M}_2(x))$ such that $\tilde{M}(x, \tilde{u}) = \tilde{M}_0(x) + \tilde{M}_1(x)\tilde{u}_1 + \tilde{M}_2(x)\tilde{u}_2$, $G_L(x) = \Pr\{\tilde{M}(X, \tilde{U}) \leq L(X)|X = x\}$, $G_H(x) = \Pr\{\tilde{M}(X, \tilde{U}) \geq H(X)|X = x\}$, and $\Psi(x) = E[\tilde{M}(X, \tilde{U})|I_M(X) = 1, X = x]$. For notational convenience, we hereafter drop the arguments x from functions and suppress the tilde, denoting $(\tilde{M}_0(x), \tilde{M}_1(x), \tilde{M}_2(x))$ as (M_0, M_1, M_2) and $(\tilde{u}_1, \tilde{u}_2)$ as (u_1, u_2). Note that

$$(A.5) \quad G_L = \int_{-\infty}^{\infty} f_1(u_1) F_2\left(\frac{L - M_0 - M_1 u_1}{M_2}\right) du_1$$

$$= \int_{-\infty}^{\infty} f_2(u_2) F_1\left(\frac{L - M_0 - M_2 u_2}{M_1}\right) du_2,$$

$$(A.6) \quad 1 - G_H = \int_{-\infty}^{\infty} f_1(u_1) F_2\left(\frac{H - M_0 - M_1 u_1}{M_2}\right) du_1$$

$$= \int_{-\infty}^{\infty} f_2(u_2) F_1\left(\frac{H - M_0 - M_2 u_2}{M_1}\right) du_2,$$

$$(A.7) \quad \Psi G_M = M_0 G_M$$
$$+ M_1 \int_{-\infty}^{\infty} u_1 f_1(u_1)$$
$$\times \left[F_2\left(\frac{H - M_0 - M_1 u_1}{M_2}\right) - F_2\left(\frac{L - M_0 - M_1 u_1}{M_2}\right)\right] du_1$$
$$+ M_2 \int_{-\infty}^{\infty} u_2 f_2(u_2)$$
$$\times \left[F_1\left(\frac{H - M_0 - M_2 u_2}{M_1}\right) - F_1\left(\frac{L - M_0 - M_2 u_2}{M_1}\right)\right] du_2.$$

Reparameterize so that $\lambda = M_1/M_2$. By holding λ constant, we can find $M_0^*(\lambda)$ and $M_2^*(\lambda)$ that solve (A.5) and (A.6) with respect to M_0 and M_2, respectively. Let l_λ and h_λ denote the solutions to $G_L = \int_{-\infty}^{\infty} f_1(u_1) F_2(l_\lambda - \lambda u_1) du_1$ and $1 - G_H = \int_{-\infty}^{\infty} f_1(u_1) F_2(h_\lambda - \lambda u_1) du_1$, respectively. Then by the definitions, $M_0^*(\lambda)$ and $M_2^*(\lambda)$ are written as $M_0^*(\lambda) = \frac{h_\lambda L - l_\lambda H}{h_\lambda - l_\lambda}$ and $M_2^*(\lambda) = \frac{H - L}{h_\lambda - l_\lambda}$.

By substituting these solutions, the right-hand side of the expression for ΨG_M above can be regarded as a function of λ (denote the function by $m(\lambda)$). Thus, for the conclusion, it is sufficient to check the existence of $\lambda^* > 0$ that solves $\Psi G_M = m(\lambda)$. Note that $m(\lambda)$ is continuous in λ because of the continuity of F_1 and F_2. Thus, by the intermediate value theorem and Assumption 2′, the existence of λ^* can be verified by showing

$$(A.8) \qquad \lim_{\lambda \to 0} m(\lambda) < (L + \varepsilon)G_M, \quad \lim_{\lambda \to \infty} m(\lambda) > (H - \varepsilon)G_M$$

for some $\varepsilon > 0$ satisfying Assumption 2′.

We now show the first statement of (A.8). Note that $h_\lambda \to h_0$ and $l_\lambda \to l_0$ as $\lambda \to 0$, where h_0 and l_0 solve $F_2(h_0) = 1 - G_H$ and $F_2(l_0) = G_L$, respectively, and that $m(\lambda) \to LG_M + \frac{H-L}{h_0-l_0} \int_{l_0}^{h_0} (u - l_0)f_2(u)\,du$ as $\lambda \to 0$. Since $G_M > p_2 - p_1$ by Assumption 3′, the requirement (A.4) on F_2 implies the first statement of (A.8). Similarly, since $m(\lambda) \to HG_M - \frac{H-L}{h_\infty-l_\infty} \int_{l_\infty}^{h_\infty} (h_\infty - u)f_1(u)\,du$ as $\lambda \to \infty$ (where h_∞ and l_∞ solve $F_1(h_\infty) = 1 - G_H$ and $F_1(l_\infty) = G_L$, respectively), the requirement (A.4) on F_1 implies the second statement of (A.8). This completes the proof of the third step.

The final step. Since $(\tilde{M}_0(x), \tilde{M}_1(x), \tilde{M}_2(x))$ satisfies (A.5)–(A.7) for all x and Assumptions 2′–4′ guarantee the differentiability of $(\tilde{M}_0(x), \tilde{M}_1(x), \tilde{M}_2(x))$, it follows that $\tilde{M}(x, \tilde{u})$ is differentiable in x for almost every $\tilde{u}$.
Q.E.D.

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