B Search and Rest Unemployment Fernando Alvarez and Robert Shimer Additional Appendixes not for Publication

B.1 Derivation Hamilton-Jacobi-Bellman

This appendix proves that if $v(\omega)$ is given by:

$$v(\omega) = \int_{\underline{\omega}}^{\bar{\omega}} R(\omega')\Pi_{\omega'}(\omega';\omega)d\omega' \tag{65}$$

for an arbitrary continuous function $R(\cdot)$ and where the local time function $\Pi_{\omega'}(\cdot)$ is given as in Stokey (2009) Proposition 10.4:

$$\Pi_{\omega'}(\omega';\omega) = \begin{cases} \dfrac{\left(\zeta_2 e^{\zeta_1\omega+\zeta_2\bar{\omega}} - \zeta_1 e^{\zeta_1\bar{\omega}+\zeta_2\omega}\right)\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega')} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega')}\right)}{(\rho+q+\delta)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \underline{\omega} \le \omega' < \omega \\[2em] \dfrac{\left(\zeta_2 e^{\zeta_1\omega+\zeta_2\underline{\omega}} - \zeta_1 e^{\zeta_1\underline{\omega}+\zeta_2\omega}\right)\left(\zeta_2 e^{\zeta_2(\bar{\omega}-\omega')} - \zeta_1 e^{\zeta_1(\bar{\omega}-\omega')}\right)}{(\rho+q+\delta)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \omega \le \omega' \le \bar{\omega}, \end{cases} \tag{66}$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation $\rho+q+\delta = \mu\zeta + \frac{\sigma^2}{2}\zeta^2$, then

$$(\rho+q+\delta)v(\omega) = R(\omega) + \mu v'(\omega) + \frac{\sigma^2}{2}v''(\omega).$$

Proof. Differentiating v with respect to ω we get

$$\begin{aligned} v'(\omega) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega')\Pi_{\omega'\omega}(\omega';\omega)d\omega' \\ v''(\omega) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega')\Pi_{\omega'\omega\omega}(\omega';\omega)d\omega' + R(\omega)\big(\lim_{\omega'\uparrow\omega}\Pi_{\omega'\omega}(\omega';\omega) - \lim_{\omega'\downarrow\omega}\Pi_{\omega'\omega}(\omega';\omega)\big) \end{aligned}$$

where we use that $\Pi_{\omega'}$ is continuous but $\Pi_{\omega'\omega}$ has a jump at $\omega' = \omega$. Then

$$\begin{aligned} &(\rho+q+\delta)v(\omega) - \mu v'(\omega) - \frac{\sigma^2}{2}v''(\omega) \\ &\quad= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\left((\rho+q+\delta)\Pi_{\omega'}(\omega';\omega) - \mu\Pi_{\omega'\omega}(\omega';\omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega';\omega)\right)d\omega' \\ &\qquad\qquad - \frac{\sigma^2}{2}R(\omega)\left(\lim_{\omega'\uparrow\omega}\Pi_{\omega'\omega}(\omega';\omega) - \lim_{\omega'\downarrow\omega}\Pi_{\omega'\omega}(\omega';\omega)\right). \end{aligned}$$

Using the functional form of $\Pi_{\omega'}$ we have, for $\omega' < \omega$:

$$\Pi_{\omega'}(\omega';\omega) = e^{\zeta_1 \omega}\tilde{h}_1(\omega') - e^{\zeta_2 \omega}\tilde{h}_2(\omega')$$

where

$$\begin{aligned}
\tilde{h}_1(\omega') &= \frac{\zeta_2 e^{\zeta_2 \bar{\omega}}\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega')} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega')}\right)}{(\rho+q+\delta)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} \\
\text{and } \tilde{h}_2(\omega') &= \frac{\zeta_1 e^{\zeta_1 \bar{\omega}}\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega')} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega')}\right)}{(\rho+q+\delta)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)}.
\end{aligned}$$

Thus for all $\omega' < \omega$:

$$\begin{aligned}
&(\rho+q+\delta)\Pi_{\omega'}(\omega';\omega) - \mu\Pi_{\omega'\omega}(\omega';\omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega';\omega) \\
&= \left[(\rho+q+\delta) - \zeta_1\mu - (\zeta_1)^2\frac{\sigma^2}{2}\right]e^{\zeta_1\omega}\tilde{h}_1(\omega') - \left[(\rho+q+\delta) - \zeta_2\mu - (\zeta_2)^2\frac{\sigma^2}{2}\right]e^{\zeta_2\omega}\tilde{h}_2(\omega') = 0
\end{aligned}$$

where the last equality follow from the definition of the roots ζ_i . Hence

$$\int_{\underline{\omega}}^{\omega} R(\omega')\left((\rho+q+\delta)\Pi_{\omega'}(\omega';\omega) - \mu\Pi_{\omega'\omega}(\omega';\omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega';\omega)\right)d\omega' = 0.$$

Using a symmetric calculation for $\omega' > \omega$ we have:

$$\int_{\omega}^{\bar{\omega}} R(\omega')\left((\rho+q+\delta)\Pi_{\omega'}(\omega';\omega) - \mu\Pi_{\omega'\omega}(\omega';\omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega';\omega)\right)d\omega' = 0.$$

Next, differentiating $\Pi_{\omega'}(\omega';\omega)$ when $\omega' < \omega$ and when $\omega' > \omega$ and let $\omega' \to \omega$ from below and from above, tedious—but straightforward—algebra, gives:

$$\lim_{\omega'\uparrow\omega}\Pi_{\omega'\omega}(\omega';\omega) - \lim_{\omega'\downarrow\omega}\Pi_{\omega'\omega}(\omega';\omega) = -\frac{\zeta_1\zeta_2}{\rho+q+\delta}.$$

Then use the expression for the roots: $\zeta_1\zeta_2 = -(\rho+q+\delta)/(\sigma^2/2)$. Putting this together proves the result. $\square$

B.2 Industry Social Planner's Problem

In this section we introduce a dynamic programming problem whose solution gives the equilibrium value for the thresholds $\underline{\omega}, \bar{\omega}$. This problem has the interpretation of a fictitious social planner located in a given industry who maximizes net consumer surplus by deciding how many of the agents currently located in the industry work and how many rest and whether to adjust the number of workers in the industry. The equivalence of the solution to this problem with the equilibrium value of an industry's worker. First, it establishes that our market decentralization is rich enough to attain an efficient equilibrium, despite the presence of search frictions. Second, it gives an alternative argument to establish the uniqueness of the equilibrium values for the thresholds $\underline{\omega}$ and $\bar{\omega}$. Third, it connects our results with the decision theoretic literature analyzing investment and labor demand model with costly reversibility.

The industry planner maximizes the net surplus from the production of the final good in an industry with current log productivity $\tilde{x}$ and l workers, taking as given aggregate consumption C and aggregate output Y. The choices for this planner are to increase the number of workers located in this industry (hire), paying $\bar{v}$ to the households for each or them, or to decrease the number of workers located it the industry (fire), receiving a payment $\underline{v}$ for each. Increases and decreases are non-negative, and the prices associated with them have the dimension of an asset value, as opposed to a rental. We let $M(\tilde{x}, l)$ be the value function of this planner, hence:

$$M(\tilde{x}, l) = \max_{l_h, l_f} \mathbb{E}\left(\int_0^\infty e^{-(\rho+\delta)t}\big(\big(S(\tilde{x}(t), l(t)) + \underline{v}ql(t)\big)dt \; - \bar{v}dl_h(t) + \underline{v}dl_f(t)\big)\Big|\, \tilde{x}(0) = \tilde{x}, l(0) = l\right)$$
$$\text{subject to } dl(t) = -ql(t)dt + dl_h(t) - dl_f(t) \;\text{ and }\; d\tilde{x} = \mu_x dt + \sigma_x dz. \tag{67}$$

The $l_h(t)$ and $l_f(t)$ are increasing processes describing the cumulative amount of "hiring" and "firing" and hence $dl_h(t)$ and $dl_f(t)$ intuitively have the interpretation of hiring and firing during period t. The term $ql(t)dt$ represent the exogenous quits that happens in a period of length dt. The planner discounts at rate $\rho + \delta$, accounting both for the discount rate of households and for the rate at which her industry disappears.

The function $S(\tilde{x}, l)$ denotes the return function of the industry social planner per unit of time and is given by

$$S(\tilde{x}, l) = \max_{E \in [0,l]} u'(C) \int_0^{Ee^{\tilde{x}}} \left(\frac{Y}{y}\right)^{\frac{1}{\theta}} dy + b_r(l - E) + \delta l \underline{v}.$$

The first term is the consumer's surplus associated with the particular good, obtained by the output produced by E workers with log productivity $\tilde{x}$. The second term is value of the workers that the planner chooses to send back to the household, receiving $\underline{v}$ for each. The third term is the value of the "sale" of all the workers if the industry shuts down. Setting $q = \delta = b_r = 0$ our problem is formally equivalent to Bentolila and Bertola's (1990) model of a firm deciding employment subject to a hiring and firing cost and to Abel and Eberly's (1996) model of optimal investment subject to costly irreversibility, i.e. a different buying and selling price for capital.

Using the envelope theorem, we find that the marginal value of an additional worker is:

$$
\begin{aligned}
S_l(\tilde{x}, l) &= \max\left\{u'(C)\left(\frac{Y(e^{\tilde{x}})^{\theta-1}}{l}\right)^{\frac{1}{\theta}}, b_r\right\} + \delta\underline{v} \\
&\equiv s\left(\frac{(\theta-1)\tilde{x} + \log Y - \log l}{\theta} + \log u'(C)\right)
\end{aligned}
\tag{68}
$$

where the function $s(\cdot)$ is given by $s(\omega) = \max\{e^{\omega}, b_r\} + \delta\underline{v}$ and is identical to the expression for the per-period value of a worker in our equilibrium, except that $\delta\underline{v}$ is in place of $(q+\delta)\underline{v}$. This is critical to the equivalence between the two problems.

To prove this equivalence, we write the industry social planner's Hamilton-Jacobi-Bellman equation. For each $\tilde{x}$, there are two thresholds, $\underline{l}(\tilde{x})$ and $\bar{l}(\tilde{x})$ defining the range of inaction. The value function $M(\cdot)$ and thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ solve the Hamilton-Jacobi-Bellman equation if the following two conditions are met:

1. For all $\tilde{x}$, and $l \in (\underline{l}(\tilde{x}),\ \bar{l}(\tilde{x}))$ employment decays exponentially with the quits at rate q and hence the value function M solves

$$
(\rho+\delta)M(\tilde{x}, l) = S(\tilde{x}, l) - qM_l(\tilde{x}, l) + \mu_x M_{\tilde{x}}(\tilde{x}, l) + \frac{\sigma_x^2}{2} M_{\tilde{x}\tilde{x}}(\tilde{x}, l). \tag{69}
$$

2. For all $(\tilde{x}, l)$ outside the interior of the range of inaction,

$$
(\rho+\delta)M(\tilde{x}, l) + qlM_l(\tilde{x}, l) - \mu_x M_{\tilde{x}}(\tilde{x}, l) - \frac{\sigma_x^2}{2} M_{\tilde{x}\tilde{x}}(\tilde{x}, l) \leq S(\tilde{x}, l), \tag{70}
$$

$$
\underline{v} = M_l(\tilde{x}, l)\ \forall l \geq \bar{l}(\tilde{x}), \text{ and } \bar{v} = M_l(\tilde{x}, l)\ \forall l \leq \underline{l}(\tilde{x}) \tag{71}
$$

Equation (71) is also referred to as *smooth pasting*. Since $M(\tilde{x}, \cdot)$ is linear outside the range of inaction, a twice-continuously differentiable solution implies *super-contact*, or that for all $\tilde{x}$:

$$
0 = M_{ll}(\tilde{x}, \bar{l}(\tilde{x})) = M_{ll}(\tilde{x}, \underline{l}(\tilde{x})). \tag{72}
$$

According to Verification Theorem 4.1, Section VIII in Fleming and Soner (1993), a twice-continuously differentiable function $M(\tilde{x}, l)$ satisfying equations (69), (71), and (72) solves the industry social planner's problem.

If M is sufficiently smooth, finding the optimal thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ can be stated as a boundary problem in terms of the function $M_l(\tilde{x}, l)$ and its derivatives. To see this, differentiate both sides of equation (69) with respect to l and replace S_l using equation (68):

$$(\rho+\delta+q)M_l(\tilde{x},l) = s\left(\frac{(\theta-1)\tilde{x}+\log Y-\log l}{\theta}+\log u'(C)\right) \\ -qlM_{ll}(\tilde{x},l)+\mu_x M_{\tilde{x}l}(\tilde{x},l)+\frac{\sigma_x^2}{2}M_{\tilde{x}\tilde{x}l}(\tilde{x},l). \quad (73)$$

If the required partial derivatives exist, any solution to the industry social planner's problem must solve equations (71)–(73). Moreover, there is a clear relationship between the value function $v(\omega)$ in the decentralized problem and the marginal value of a worker M_l in the industry social planner's problem:

Lemma 3. *Assume that $\theta \neq 1$ and that the functions $M_l(\cdot)$ and $v(\cdot)$ satisfy*

$$M_l(\tilde{x},l) = v(\omega), \text{ where } \omega = \frac{\log Y+(\theta-1)\tilde{x}-\log l}{\theta}+\log u'(C) \quad (74)$$

and that thresholds functions $\{\underline{l}(\cdot),\bar{l}(\cdot)\}$ and the thresholds levels $\{\underline{\omega},\bar{\omega}\}$ satisfy

$$\log \bar{l}(\tilde{x}) = \log Y+(\theta-1)\tilde{x}-\theta(\underline{\omega}-\log u'(C)) \quad (75)$$
$$\log \underline{l}(\tilde{x}) = \log Y+(\theta-1)\tilde{x}-\theta(\bar{\omega}-\log u'(C)). \quad (76)$$

Then, $M_l(\cdot)$ and $\{\underline{l}(\cdot),\bar{l}(\cdot)\}$ solve equations (71)–(73) for all $\tilde{x}$ and $l \in [\underline{l}(\tilde{x}),\bar{l}(\tilde{x})]$ if and only if $v(\cdot)$ and $\{\underline{\omega},\bar{\omega}\}$ solve equations (12).

Proof. Differentiate equation (74) with respect to $\tilde{x}$ and l to get

$$M_{l\tilde{x}}(\tilde{x},l) = v'(\omega)\frac{\theta-1}{\theta}\,,\; M_{l\tilde{x}\tilde{x}}(\tilde{x},l) = v''(\omega)\left(\frac{\theta-1}{\theta}\right)^2 \text{ and } M_{ll}(\tilde{x},l) = -v'(\omega)\frac{1}{\theta}.$$

Recall that a solution of equation (12) is equivalent to a solution to equations (39), (40), and $v(\bar{\omega})=\bar{v}$ and $v(\underline{\omega})=\underline{v}$. The equivalence between equation (12) and equations (71)–(73) is immediate, recalling the definitions of μ and σ. □

This lemma has important implications. First, it establishes, not surprisingly, that the equilibrium allocation is Pareto Optimal. Second, since the industry social planner's problem is a maximization problem, the solution is easy to characterize. For instance, since the problem is convex, it has at most one solution and hence the equilibrium value of a worker is uniquely defined, for given $u'(C)$ and Y. The fact that v is increasing is then equivalent to the concavity of $S(\tilde{x},\cdot)$. Finally, notice that Proposition 1 establishes existence and uniqueness of the solution to equation (12) only under mild conditions on $s(\cdot)$, i.e. that it was weakly increasing and bounded below. Proposition 1 can be used to extend the uniqueness and existence results of the literature of costly irreversible investment to a wider class of production functions. Currently the literature uses that the production function is of the form $x^{a_x} l^{a_l}$ for some constants a_x and a_l, with $0 < a_l < 1$, as in Abel and Eberly (1996). Proposition 1 shows that the only assumption required is that the production function be concave in l, and that the marginal productivity of the factor l can be written as a function of the ratio of the quantity of the input l to (a power of) the productivity shock x.

B.3 Heterogeneous Industries

This section extends the directed search model to include heterogeneity in households' human capital. In equilibrium, industries can be divided into different classes. Industries that attract households with high human capital pay high wages, but the stochastic process for their wages is a scaled version of the one for an industry that attracts households with less human capital. Still, all industries have the same process for the log full employment wage ω (measured in utils) and the same rest and search unemployment rates. This justifies our fixed effect treatment of US industry wage data in Section 6.1.

We prove in this section that in the directed search model with logarithmic utility, the values of the thresholds $\underline{\omega}$ and $\bar{\omega}$ are the same across industries, although the level of consumption, and hence the wage in units of goods, is different. We omit a proof of a similar result in the random search model, under the assumption that workers with a particular human capital level contact other workers with the same human capital level at rate α, at which point they may join the workers' industry.

We turn now to a description of the directed search model. Households are indexed by one of K human capital types, denoted by h_k satisfying $0 < h_1$ and $h_k < h_{k+1}$, for $k = 1, 2, ..., K-1$, with $h_K = 1$. For notational convenience, let $h_0 = 0$ and $\Delta h_k \equiv h_k - h_{k-1}$. Let H_k denote the cumulative distribution of households' human capital types, so that there are H_k households with human capital $h_j \leq h_k$, and there are $\Delta H_k \equiv H_k - H_{k-1}$ household with human capital type h_k for $k = 1, ..., K$.

Recall that industries are indexed by j which belong to $[0, 1]$. The meaning of type h_k human capital is that such household can work in any industry labeled $j \in (0, h_k]$. Assume

$$\frac{\Delta H_{k+1}}{\Delta h_{k+1}} < \frac{\Delta H_k}{\Delta h_k}, \tag{77}$$

for $k = 1, ..., K-1$. We then look for an equilibrium where type h_k households work in industries of type $j \in (h_{k-1}, h_k]$. In this equilibrium, we talk of both household and industries of type k. For workers to sort themselves across industries in this way, it must be the case that wages are increasing in industry type, and equation (77) insures that labor supply is in fact decreasing in industry type.

Let L_k denote the fraction of members of type k households who are located in type k industries and $L_{0,k}$ denote the fraction located in newly created industries within the k class. Thus $L_k(\Delta H_k/\Delta h_k)$ is the number of household members per type k industry, either working or in rest unemployment.

Households with different human capital have different consumption, and hence different marginal utility. Letting C_k be the consumption per household for those with human capital k, we have that the log full-employment wage for household of type k follows:

$$\omega_k(t) \equiv \frac{\log Y + (\theta - 1)\log x(t) - \log l(t)}{\theta} + \log u'(C_k) \tag{78}$$

where Y is aggregate output, $x(t)$ is industry productivity, and $l(t)$ is the number of workers in the industry. We characterize an equilibrium where the process for ω_k is identical for all k.

Proposition 7. Assume log utility, $u(C) \equiv \log C$, and that equation (77) holds. Let $(L^*, \underline{\omega}^*, \bar{\omega}^*)$ be the equilibrium values for the model without heterogeneity. Then there is an equilibrium of the model with heterogeneity with $(L_k, \bar{\omega}_k, \underline{\omega}_k) = (L^*, \bar{\omega}^*, \underline{\omega}^*)$ for all k and

$$\frac{C_k}{C_{k'}} = \left(\frac{\Delta h_k \Delta H_{k'}}{\Delta H_k \Delta h_{k'}}\right)^{1/\theta}.$$

Proof. For the processes $\{\omega_k(t)\}$ to be identical across industries, the difference in the log of the marginal utilities must be compensated by a difference in the level of the employment per industry, so that any two industries in classes k and k' created at the same time and with the same history of shocks have employment l_k and $l_{k'}$ satisfying

$$\log l_k(t) - \log l_{k'}(t) = \theta(\log u'(C_k) - \log u'(C_{k'})).$$

Aggregating across shocks and using the logarithmic utility assumption and the conjecture about the nature of equilibrium, the number of workers located in type k industries is

$$\frac{L^* \Delta H_k C_k^\theta}{\Delta h_k} \equiv \beta \tag{79}$$

for all $k = 1, ..., K$ and some constant β.

The distribution f evaluated at the upper bound still satisfies

$$\frac{\sigma^2}{2} f'(\bar{\omega}) - \left(\mu + \frac{\theta\sigma^2}{2}\right) f(\bar{\omega}) = \delta \frac{L_0^*}{L^*}, \tag{80}$$

where L_0^* is the fraction of workers in a new industry, independent of k in the proposed equilibrium. The requirement that the log full-employment wages is $\bar{\omega}$ in new industries implies

$$\frac{L_0^* \Delta H_k C_k^\theta}{\Delta h_k} = Y x_0^{\theta-1} e^{-\theta\bar{\omega}}. \tag{81}$$

From equation (79), the left hand side is $\beta L_0^*/L^*$. Eliminate L_0^*/L^* using equation (80) to get

$$\beta = \phi_1 Y, \text{ where } \phi_1 \equiv \frac{\delta x_0^{\theta-1} e^{-\theta\bar{\omega}}}{\frac{\sigma^2}{2} f'(\bar{\omega}) - \left(\mu + \frac{\theta\sigma^2}{2}\right) f(\bar{\omega})}. \tag{82}$$

In each industry class k we can solve for the productivity consistent with (l, ω, Y, C_k) as:

$$x = \xi(l, \omega, Y, C_k) \equiv \left(\frac{l e^{\theta\omega} C_k^\theta}{Y}\right)^{\frac{1}{\theta-1}}. \tag{83}$$

Then using the production function, output in a industry in such an industry class, with l workers and log full-employment wage ω, is

$$Q(l, \xi(l, \omega, Y, C_k)) = Y^{\frac{-1}{\theta-1}} \left(e^\omega l C_k\right)^{\frac{\theta}{\theta-1}} \min\{1, e^\omega / b_r\}^\theta. \tag{84}$$

Using this notation, we can write the analog of equation (48) as

$$\begin{aligned} Y &= \left(\sum_{k=1}^K \int_{h_{k-1}}^{h_k} Q\big(l(j,t), \xi(l(j,t), \omega(j,t), Y, C_k)\big)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}} \\ &= \left(\sum_{k=1}^K \int_{h_{k-1}}^{h_k} Q\left(\frac{L^* \Delta H_k}{\Delta h_k}, \xi\left(\frac{L^* \Delta H_k}{\Delta h_k}, \omega(j,t), Y, C_k\right)\right)^{\frac{\theta-1}{\theta}} \frac{l(j,t)}{\frac{L^* \Delta H_k}{\Delta h_k}} dj\right)^{\frac{\theta}{\theta-1}}. \end{aligned}$$

The second equation follows because $Q(\cdot, \xi(\cdot, \omega, Y, C_k))^{\frac{\theta-1}{\theta}}$ is linear in l (equation 84). To solve this, we change the variable of integration from the name of the industry j to its log full-employment wage ω and number of workers l. Let $\tilde{f}(\omega, l)$ be the density of the joint invariant distribution of workers in industries (ω, l), as discussed in Appendix A.4. Notice that under our hypothesis this distribution is the same for all k. Then

$$Y = \left(\sum_{k=1}^{K} \Delta h_k \int_{\underline{\omega}}^{\bar{\omega}} \int_0^\infty Q\left(\frac{L^* \Delta H_k}{\Delta h_k}, \xi\left(\frac{L^* \Delta H_k}{\Delta h_k}, \omega, Y, C_k \right) \right)^{\frac{\theta-1}{\theta}} \frac{l}{\frac{L^* \Delta H_k}{\Delta h_k}} \tilde{f}(\omega, l)\, dl\, d\omega \right)^{\frac{\theta}{\theta-1}}.$$

Since $f(\omega) = \int_0^\infty \frac{l \Delta h_k}{L^* \Delta H_k} \tilde{f}(\omega, l)\, dl$, we can solve the inner integral to obtain

$$Y = \left(\sum_{k=1}^{K} \Delta h_k \int_{\underline{\omega}}^{\bar{\omega}} Q\left(\frac{L^* \Delta H_k}{\Delta h_k}, \xi\left(\frac{L^* \Delta H_k}{\Delta h_k}, \omega, Y, C_k \right) \right)^{\frac{\theta-1}{\theta}} f(\omega) d\omega \right)^{\frac{\theta}{\theta-1}},$$

without characterizing the joint density $\tilde{f}$. Using equation (84) and simplifying,

$$Y = L^* \left(\sum_{k=1}^{K} \Delta H_k C_k \right) \left(\int_{\underline{\omega}}^{\bar{\omega}} e^{\omega} \min\{1, e^{\omega}/b_r\}^{\theta-1} f(\omega) d\omega \right). \tag{85}$$

Since total output in the economy is consumed by the households,

$$Y = \sum_{k=1}^{K} \Delta H_k C_k. \tag{86}$$

Then equation (85) implies

$$L^* = \left(\int_{\underline{\omega}}^{\bar{\omega}} e^{\omega} \min\{1, e^{\omega}/b_r\}^{\theta-1} f(\omega) d\omega \right)^{-1}. \tag{87}$$

This defines L^*. Next, substitute for C_k in equation (86) using equation (79):

$$Y^{\theta} = \frac{\beta}{L^*} \left(\sum_{k=1}^{K} \Delta H_k^{\frac{\theta-1}{\theta}} \Delta h_k^{\frac{1}{\theta}} \right)^{\theta}. \tag{88}$$

Eliminate β using equation (82) to get an expression for total output.

$$Y = \left(\frac{\phi_1}{L^*} \right)^{\frac{1}{\theta-1}} \left(\sum_{k=1}^{K} \Delta H_k^{\frac{\theta-1}{\theta}} \Delta h_k^{\frac{1}{\theta}} \right)^{\frac{\theta}{\theta-1}}. \tag{89}$$

This defines Y. Finally, one can go back to equation (82) to determine β and then to equation (79) to pin down C_k, closing the model. Note that assumption (77) implies consumption is increasing in k.

To prove that a type k household prefer to work on industry k to other industries $j = 1, ..., k-1$, we show that wages are increasing in k. This follows because, with logarithmic utility, the actual wage is the product of ω, whose distribution is independent of k, and consumption. $\square$