Econometrica Supplementary Material

SUPPLEMENT TO "MONOTONICITY AND IMPLEMENTABILITY"
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S.1. DOMAINS WITH CONVEX CLOSURE

SAKS AND YU (2005) PROVED that if D is convex, then every monotone deterministic allocation rule is implementable. We prove in this supplement the following generalization of their result:

THEOREM S1: *Every domain with a convex closure is a proper monotonicity domain.*

S.1.1. *Preparations*

First we recall the definitions of monotonicity and cyclic monotonicity. An allocation rule f is called *monotone* if

$$\text{(S1)} \qquad \langle f(v) - f(w), v - w\rangle \geq 0 \quad \text{for every } v, w \in D,$$

and f is called *cyclically monotone* if for every $k \geq 2$, for every k vectors in D (not necessarily distinct), $v_1, v_2, \dots, v_k$, the inequality

$$\text{(S2)} \qquad \sum_{i=1}^{k} \langle v_i - v_{i+1}, f(v_i)\rangle \geq 0,$$

holds, where v_{k+1} is defined to be v_1. By taking $k = 2$ in (S2) it can be seen that every cyclically monotone allocation rule is monotone.

Let $f : D \to \bar{Z}(A)$ be monotone and finite-valued, where D is an arbitrary set. Let $y^1, \dots, y^m \in R^A$ be the distinct values of f. That is, for every $v \in D$, there exists $1 \leq j \leq m$ such that $f(v) = y^j$, and every y^j is attained at some valuation. If $m > 1$, for $j \neq k$ define

$$\text{(S3)} \qquad \delta(j, k) = \delta_{D,f}(j, k) = \inf_{v \in D, f(v) = y^j} \langle v, y^j - y^k\rangle.$$

If $w \in D$ satisfies $f(w) = y^k$, then, by monotonicity, $\langle v, y^j - y^k\rangle \geq \langle w, y^j - y^k\rangle$. Therefore, $\delta(j, k) > -\infty$. Furthermore,

$$\delta(j, k) \geq \sup_{v \in D, f(v) = y^k} \langle v, y^j - y^k\rangle = -\delta(k, j).$$

Hence,

$$\text{(S4)} \qquad \delta(j, k) + \delta(k, j) \geq 0 \quad \forall j \neq k.$$

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As (S2) can be written

$$\text{(S5)} \qquad \sum_{i=1}^{k} \langle v_i, f(v_i) - f(v_{i-1}) \rangle \geq 0,$$

where v_0 is defined to be v_k, the following useful lemma has been noted by many authors (see, e.g., Heydenreich et al. (2009) and Saks and Yu (2005)).

LEMMA S2: *Let $f : D \to \bar{Z}(A)$ be finite-valued and monotone.*
(a) *f is cyclically monotone if and only if for every sequence $j_1, j_2, \dots, j_k$, $k \geq 2$, such that $j_s \neq j_{s+1}$ for $1 \leq s < k$ the inequality*

$$\text{(S6)} \qquad \sum_{i=1}^{k} \delta(j_i, j_{i+1}) \geq 0$$

holds, where j_{k+1} is defined to be j_1.
(b) *If, in addition to the monotonicity, $\delta(j,k) + \delta(k,j) = 0$ for every $j \neq k$, then f is cyclically monotone if and only if the inequalities (S6) are satisfied as equalities.*

For every j, let

$$D_j = \{v \in D \mid \langle v, y^j - y^k \rangle \geq \delta(j,k) \ \forall k,\ k \neq j\}.$$

Obviously, $f(v) = y^j$ implies $v \in D_j$. Hence, $D = \bigcup_{j=1}^m D_j$.
The following sufficient condition will be useful.

LEMMA S3: *Let $f : D \to \bar{Z}(A)$ be finite-valued and monotone. If $\bigcap_{j=1}^m D_j \neq \emptyset$, then f is cyclically monotone.*

PROOF: Let $v \in D$ be in the intersection. Hence $\langle v, y^j - y^k \rangle \geq \delta(j,k)$ for all $j \neq k$. We claim that

$$\text{(S7)} \qquad \langle v, y^j - y^k \rangle = \delta(j,k) \quad \text{for all } j \neq k.$$

Indeed, $v \in D_j$ implies $\langle v, y^j - y^k \rangle \geq \delta(j,k)$, and $v \in D_k$ implies $\langle v, y^k - y^j \rangle \geq \delta(k,j)$. Therefore, from (S4) we obtain (S7). By plugging (S7) in (S6) it follows that (S6) is satisfied with equality for every sequence of indices, and hence f is cyclically monotone. *Q.E.D.*

We next show that to prove that a set is a proper monotonicity domain, it suffices to prove that its closure is a proper monotonicity domain. For a domain D, we denote its closure by $\mathrm{cl}(D)$.

LEMMA S4: *If* cl(D) *is a proper monotonicity domain*, *so is* D.

PROOF: Suppose cl(D) is a proper monotonicity domain and let $f:D \to \bar{Z}(A)$ be a finite-valued monotone function on D. Extend f to cl(D) as follows: For every $v \in \mathrm{cl}(D) \setminus D$, there exists a sequence v_n, $n \geq 1$, in D such that $v_n \to v$. For some j, there exists an infinite numbers of indices n such that $f(v_n) = y^j$. Hence for every $v \in \mathrm{cl}(D) \setminus D$, there exists j and a sequence $v_n \in D$ such that $v_n \to v$ and $f(v_n) = y^j$ for every $n \geq 1$. Let $f(v) = y^j$ for such arbitrary j. It is easily verified that the extension of f is monotone on cl(D). Therefore, it is cyclically monotone on cl(D) and, therefore, f is cyclically monotone on D. *Q.E.D.*

We will use a characterization of cyclically monotone functions that can easily be derived from Section 24 in Rockafellar (1970).

THEOREM S5—Rockafellar: *Let* $D \subseteq R^A$ *be a convex and nonempty subset of valuations*, *and let* $f:D \to \bar{Z}(A)$.

(a) f *is cyclically monotone on* D *if and only if there exists a real-valued function* U *on* D *such that*[1]

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle \quad \forall v_1, v_2 \in D. \tag{S8}$$

(b) *If each of the functions* $U_1, U_2 : D \to R$ *satisfies* (S8), *then the functions differ by a constant*. *That is*, *there exists a real number* α *such that*

$$U_1(v) = U_2(v) + \alpha \quad \forall v \in D. \tag{S9}$$

(c) *Suppose that* $U:D \to R$ *satisfies* (S8) *and let* $v_1 \neq v_2 \in D$. *Then the real-valued function*

$$\phi(t) = \big\langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \big\rangle, \tag{S10}$$

defined for every $t \in [0, 1]$, *is nondecreasing and*

$$U(v_2) - U(v_1) = \int_0^1 \phi(t)\, dt, \tag{S11}$$

where the integral is computed in the sense of Riemann.[2]

The main tool in proving Theorem S1 is the following theorem.

[1] $U(v)$ can be interpreted as the utility function of the agent when her valuation is v.

[2] A nondecreasing function is Riemann integrable. It is also Borel measurable and, therefore, its Riemann integral equals its Lebesgue integral.

THEOREM S6: *Let $D = H_1 \cup H_2$ be a closed convex set, where each H_i is closed convex and nonempty. Let $f : D \to \bar{Z}(A)$ be monotone (not necessarily finite-valued). If f is cyclically monotone on every H_i then f is cyclically monotone on D.*

PROOF: Because D and the sets H_i are closed, $H_1 \cap H_2 \neq \emptyset$. Let v^* be a fixed valuation in $H_1 \cap H_2$. By part (a) of Theorem S5, there exists U_1 on H_1 that satisfies (S8) on H_1. By adding a constant, we can choose U_1 such that $U_1(v^*) = 0$. Similarly there exists $U_2 : H_2 \to R$ that satisfies (S8) on H_2 and $U_2(v^*) = 0$. By part (b) of Theorem S5, $U_1 = U_2$ on $H_1 \cap H_2$. Hence we can define a function U on D by $U(v) = U_i(v)$ for $v \in H_i$. To show that f is cyclically monotone on D, it suffices by part (a) to show that (S8) is satisfied by U on D. Let then $v_1 \neq v_2$ in D. Obviously we can consider only the cases $v_1 \in H_1 \setminus H_2$ and $v_2 \in H_2 \setminus H_1$. Because H_1, H_2, and D are closed, and $v_1 \in H_1 \setminus H_2$ and $v_2 \in H_2 \setminus H_1$, the interval (v_1, v_2) intersects $H_1 \cap H_2$, say $w = v_1 + s(v_2 - v_1)$, $0 < s < 1$, is a valuation at the intersection. By applying part (c) of Theorem S5 to v_1 and w in H_1, and by a simple change of variables, we get

$$U(w) - U(v_1) = \int_0^s \big\langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \big\rangle \, dt$$

and, similarly,

$$U(v_2) - U(w) = \int_s^1 \big\langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \big\rangle \, dt.$$

Therefore,

$$U(v_2) - U(v_1) = \int_0^1 \big\langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \big\rangle \, dt.$$

By the monotonicity of f, the integrand is nondecreasing in t and, therefore, the integral is greater than or equal to the value of the integrand at $t = 0$. Hence,

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle. \qquad \textit{Q.E.D.}$$

S.1.2. *Proof of Theorem S1*

We first show that it suffices to prove that every compact convex set is a proper monotonicity domain. Let D be a set such that $\mathrm{cl}(D)$ is convex. By Lemma S4 it suffices to prove that $\mathrm{cl}(D)$ is a proper monotonicity domain.

Assume the result holds for every compact convex set, and assume to the negative that $f : \mathrm{cl}(D) \to \bar{Z}(A)$ is a finite-valued monotone randomized allocation rule, which is not cyclically monotone. Therefore, there exist $v_1, v_2, \ldots, v_k$

in cl(D) that contradict (S2). Let K be the convex hull of these valuations. Then f is finite-valued and monotone on K, and it is not cyclically monotone, contradicting our assumption that the assertion holds for compact convex sets.

We prove the theorem for compact convex sets by a double induction process. The first induction is on the number of distinct values, $m(D, f)$ of f on D. If $m(D, f) = 1$, then obviously f is cyclically monotone. Let $m > 1$, and assume we have already proven that for every compact convex D and for every monotone randomized allocation rule $f : D \to \bar{Z}(A)$ with $m(f, D) < m$, f is cyclically monotone on D. We proceed to prove it for every $m(D, f) = m$.

For every (D, f) with $f(D) = \{y^1, \dots, y^m\}$, let $r(D, f)$ be the maximal number r, $1 \le r \le m$, for which, for every set F of r distinct values in $\{1, \dots, m\}$, the intersection $\bigcap_{j \in F} D_j \neq \emptyset$. We prove our result by induction on $r(D, f)$. Let then $r(D, f) = 1$. Since $m > 1$, there exists $j \neq k$ such that $D_j \cap D_k = \emptyset$. Since D_j and D_k are compact and convex, we can strongly separate them. That is, there exist $0 \neq y \in R^A$ and $\alpha \in R$ such that

$$\langle v, y \rangle < \alpha < \langle w, y \rangle \quad \forall v \in D_j, \forall w \in D_k.$$

Denote $H_1 = \{v \in D \mid \langle v, y \rangle \le \alpha\}$ and $H_2 = \{v \in D \mid \langle v, y \rangle \ge \alpha\}$. On each H_i, the function f takes at most $m - 1$ values and, therefore, by the first induction hypothesis f is cyclically monotone on each H_i. By Theorem S6, f is cyclically monotone on D. Suppose the theorem is proved for $1, \dots, r - 1$, $2 \le r \le m$. We now prove it for $r(D, f) = r$. If $r = m$, the result follows from Lemma S3. If $r < m$, there exists a set of indices of cardinality $r + 1$, which, without loss of generality we take to be $\{1, \dots, r + 1\}$, such that $\bigcap_{j=1}^{r} D_j \neq \emptyset$ and $\bigcap_{j=1}^{r+1} D_j = \emptyset$. The convex compact sets $\bigcap_{j=1}^{r} D_j$ and D_{r+1} must be strongly separated. That is, there exist $0 \neq y \in R^A$ and $\alpha \in R$ such that

$$\langle v, y \rangle < \alpha < \langle w, y \rangle \quad \forall v \in \bigcap_{j=1}^{r} D_j, \forall w \in D_{r+1}.$$

Let $H_1 = \{v \in D \mid \langle v, y \rangle \le \alpha\}$ and $H_2 = \{v \in D \mid \langle v, y \rangle \ge \alpha\}$. On H_1, the function f does not take the value y^{r+1} and, therefore, by our first induction hypothesis, f is cyclically monotone. On H_2, if $m(H_2, f) < m$, then f is implementable on H_2 by the first induction hypothesis. Suppose $m(H_2, f) = m$. Since $H_2 \subseteq D$, $\delta_{H_2, f}(j, k) \ge \delta_{D, f}(j, k)$ for every $j \neq k$. Therefore, for every j, $H_{2_j} \subseteq D_j$, where $H_{2_j} = \{v \in H_2 \mid \langle v, y^j - y^k \rangle \ge \delta_{H_2, f}(j, k)\}$. Hence, $\bigcap_{j=1}^{r} H_{2_j} \subseteq H_2 \cap (\bigcap_{j=1}^{r} D_j) = \emptyset$, implying $r(H_2, f) < r$. Therefore, by our second induction hypothesis, f is cyclically monotone on H_2. Hence f is cyclically monotone on D by Theorem S6. *Q.E.D.*

S.1.3. *A Note on General Monotone Allocation Rules*

The definitions of monotonicity and cyclic monotonicity are not restricted to functions that take only subprobability values. Hence, every function $f : D \to R^A$ that satisfies (S1) ((S2)) is called *monotone* (*cyclically monotone*). Such general functions can be used, for example, in models with divisible goods. It is, therefore, interesting to note that without any change in the proofs, Theorem S1 holds for such functions. Therefore, the following result holds.

THEOREM S7: *Let $D \subseteq R^A$ be a domain with a convex closure. Every finite-valued monotone function $f : D \to R^A$ is cyclically monotone.*

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