# SUPPLEMENT TO "SOLVING THE FELDSTEIN–HORIOKA PUZZLE WITH FINANCIAL FRICTIONS"
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### 1. TECHNICAL APPENDIX 1

THIS APPENDIX DEMONSTRATES how the optimal allocation in the bond-enforcement model can be decentralized into a competitive equilibrium with either national default risk or resident default risk. Under national default risk, the government makes the default choice, while under resident default risk, consumers make the default choice. In the literature, Kehoe and Perri (2002) argued that national default risk is more realistic because governments play an important role in collecting and repaying foreign debt. Wright (2006) argued that resident default risk is also important from historical observation. In this appendix, instead of taking a stand on which default risk is more important, we show that our bond-enforcement model can be decentralized in either way.

We first present the general setup of the decentralized models with either national default risk or resident default risk. Then we decentralize the bond-enforcement model through a tax on contract repayments under national default risk. The role of such tax is to make consumers endogenize the effect of private overborrowing on the economy-wide debt limits faced by the government. Last, we show the decentralization through a subsidy on international debt payments under resident default risk. The subsidy on international debt payments lowers the consumer's default incentive and so restores borrowing to the centralized borrowing level in the benchmark case. Given that we have a production economy, the capital stock is also distorted under the decentralized model. We therefore need a tax on capital returns to implement the optimal capital stock in the decentralized models, as in Kehoe and Perri (2002).

### 1.1. *General Decentralization Setup*

In each country, there are three types of agents: a representative firm, a continuum of consumers, and a government. The representative firm uses capital and labor to produce with the Cobb–Douglas production function as specified in the paper. The consumers, with identical preferences and wealth, make decisions over consumption, investment, domestic borrowing, and foreign borrowing. The government collects taxes from and makes lump-sum transfer to the domestic consumers.

Domestic financial markets feature a complete set of assets and domestic contracts have full enforceability. In contrast, international financial markets offer only one noncontingent bond and international contracts have limited

enforceability. The modeling choices are motivated by the empirical observation that international capital flow mainly takes the form of debt,[1] and within a country there are richer forms of borrowing and lending. Furthermore, there is no international court to enforce the international financial contracts, as commonly argued in the literature.

### 1.1.1. *National Default Risk*

In this case, the representative consumer decides how much to borrow domestically and internationally. The government collects the domestic consumers' repayments to international lenders and chooses whether to default by comparing the defaulting welfare with the nondefaulting welfare. The international lenders cannot price-discriminate among borrowing countries and charge the risk-free rate on the loans. With limited enforceability of debt contracts, the international lenders will impose borrowing limits to ensure that all the debt allowed will be repaid with certainty. The government imposes taxes or subsidies on debt repayments (both domestic and international) and capital returns to ensure that aggregate private borrowing meets such sovereign borrowing limits.

More specifically, each consumer maximizes the expected utility, given by

$$(1.1) \qquad U(s_0) = \sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c(a^t)),$$

subject to the sequential budget constraints

$$c(a^t) + k(a^t) + \sum_{a_{t+1}|a^t} q_H(a^t, a_{t+1}) b_H(a^t, a_{t+1}) + b(a^t)$$

$$\leq w(a^t) + (1 - \tau_k(a^t)) r_k(a^t) k(a^{t-1})$$

$$+ (1 + \tau_b(a^t)) [b_H(a^{t-1}, a_t) + R b(a^{t-1})] + T(a^t)$$

and the no-Ponzi constraints

$$b_H(a^t, a_{t+1}) \geq -D, \quad b(a^t) \geq -D,$$

where $b_H(a^t, a_{t+1})$ denotes the domestic Arrow securities, $q_H(a^t, a_{t+1})$ denotes the price of these securities, $b(a^t)$ denotes the foreign assets or bonds, $w(a^t)$ denotes the wage rate, $r_k(a^t)$ denotes the return to capital, $\tau_b(a^t)$ denotes the tax or subsidy on debt repayments, and $\tau_k(a^t)$ denotes the tax on the return to capital. Since there is no aggregate uncertainty in the model, the world interest rate $R$ is constant over time.

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[1]Kraay, Loayza, Servėn, and Ventura (2005) documented that roughly three-quarters of net north–south capital flow takes the form of net lending.

A representative firm chooses capital and labor so that marginal products and marginal costs are equalized for each input, that is,

$$w(a^t) = (1 - \alpha)a_t k(a^{t-1})^\alpha,$$
$$r_k(a^t) = \alpha a_t k(a^{t-1})^{\alpha-1} + 1 - \delta.$$

The government chooses to default if the default welfare $\tilde{V}^{\text{AUT}}$ is higher than the nondefaulting welfare $U(s^t)$, as defined in equation (1.1). The default value is given by[2]

$$\tilde{V}^{\text{AUT}}(a^t, k(a^{t-1}), b_H(a^{t-1}, a_t)) = \max \sum_{m=t}^{\infty} \sum_{a^m} \beta^{m-t} \pi(a^m|a^t) u(c(a^m)),$$

subject to

$$c(a^m) + k(a^m) + \sum_{a_{m+1}|a^m} q_H(a^m, a_{m+1}) b_H(a^m, a_{m+1})$$

$$\leq w(a^m) + r_k(a^m) k(a^{m-1}) + b_H(a^{m-1}, a_m).$$

The assumptions on the international lenders ensure no default in equilibrium because the country cannot borrow the amount at which they have incentive to default. Note that the domestic consumers are homogenous, which implies that $b_H(a^{t-1}, a_t)$ is zero in equilibrium. The borrowing limits that the international lenders impose depend on the shock realization and the accumulated capital stock.

DEFINITION 1.1: A *competitive equilibrium under national default risk* under a sequence of taxes $\{\tau_b(a^t), \tau_k(a^t)\}$ is a set of allocations $\{c(a^t), b_H(a^t, a_{t+1}), b(a^t), k(a^t)\}$, transfers $T(a^t)$, and prices $\{w(a^t), r_k(a^t), q_H(a^t, a_{t+1}), R\}$ such that the following conditions are satisfied:

   (i) Given the prices, taxes, and transfers, each consumer in a country chooses $\{c(a^t), b_H(a^t, a_{t+1}), b(a^t), k(a^t)\}$ to maximize the utility subject to the budget constraints and the no-Ponzi constraints.

   (ii) The representative firm's first-order conditions are satisfied.

   (iii) The no-default condition is satisfied at any $a^t$: $U(a^t) \geq \tilde{V}^{\text{AUT}}(a^t, k(a^{t-1}), b_H(a^{t-1}, a_t))$.

   (iv) The government's budget constraints are satisfied:

$$T(a^t) = \tau_k(a^t) r_k(a^t) k(a^{t-1}) - \tau_b(a^t)[b_H(a^{t-1}, a_t) + Rb(a^{t-1})].$$

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[2]For simplicity we assume that there is no output drop after default, but all the arguments will go through with the output drop.

(v) The domestic Arrow-security markets clear, that is, $b_H(a^t, a_{t+1}) = 0$.

(vi) The international bond markets clear, that is, $\int b(a^t)\,d\mu = 0$, where $\mu$ denotes the invariant distribution of countries in the world.

PROPOSITION 1.1: *There exists a sequence of taxes $\{\tau_b(a^t), \tau_k(a^t)\}$ such that the optimal allocation in the bond-enforcement model can be supported as a competitive equilibrium under national default risk.*

PROOF: The proof is constructive. The key of the proof is to construct a set of taxes under the decentralized model and to show that the necessary and sufficient conditions that characterize the decentralized model coincide with those of the centralized model.

We start by characterizing our benchmark bond-enforcement model. Let $\gamma(a^t)$ be the Lagrangian multiplier of the enforcement constraint in the centralized model, and let $M(a^t)$ be the sum of the Lagrangian multipliers, that is, $M(a^t) = 1 + \gamma(s^0) + \cdots + \gamma(a^t)$. In equilibrium, the first-order conditions of a country are the Euler equations for capital and international borrowing, which are given by

$$(1.2) \qquad u_c(a^t) = \beta \sum_{a^{t+1}} \pi(a^{t+1}|a^t) \left[ \frac{M(a^{t+1})}{M(a^t)} u_c(a^{t+1})(F_k(a^{t+1}) + 1 - \delta) \right.$$

$$\left. - \frac{\gamma(a^{t+1})}{M(a^t)} V_k^{\text{AUT}}(a^{t+1}) \right],$$

$$(1.3) \qquad u_c(a^t) = \beta R \sum_{a^{t+1}} \pi(a^{t+1}|a^t) \left[ \frac{M(a^{t+1})}{M(a^t)} u_c(a^{t+1}) \right],$$

and the transversality conditions for capital and international borrowing, which are given by

$$(1.4) \qquad \lim_{T \to \infty} \beta^T u_c(a^T) M(a^T) k(a^T) = 0,$$

$$(1.5) \qquad \lim_{T \to \infty} \beta^T u_c(a^T) M(a^T) b(a^T) = 0.$$

For the decentralized model with national default risk, the Euler equations for capital and international borrowing are given by

$$(1.6) \qquad u_c(a^t) = \beta \sum_{a^{t+1}} \pi(a^{t+1}|a^t)(1 - \tau_k(a^{t+1})) u_c(a^{t+1}) r_k(a^{t+1}),$$

$$(1.7) \qquad u_c(a^t) = \beta R \sum_{a^{t+1}} \pi(a^{t+1}|a^t)(1 + \tau_b(a^{t+1})) u_c(a^{t+1}).$$

The Euler equation for domestic borrowing is given by

$$(1.8) \qquad u_c(a^t)q_H(a^t, a_{t+1}) = \beta\pi(a^{t+1}|a^t)(1 + \tau_b(a^{t+1}))u_c(a^{t+1}).$$

The transversality conditions for capital, domestic Arrow securities, and international bonds are given by

$$(1.9) \qquad \lim_{T\to\infty} \beta^T u_c(a^T)k(a^T) = 0,$$

$$(1.10) \qquad \lim_{T\to\infty} \sum_{a^T} \beta^T u_c(a^T)q_H(a^T, a_{T+1})b_H(a^T, a_{T+1}) = 0,$$

$$(1.11) \qquad \lim_{T\to\infty} \beta^T u_c(a^T)b(a^T) = 0.$$

To implement the optimal allocation in the bond-enforcement model, we set the tax on borrowings as

$$\tau_b(a^t) = \frac{\gamma(a^t)}{M(a^{t-1})}.$$

Note that $\tau_b(a^t) \geq 0$ since the Langrange multiplier is nonnegative. If the enforcement constraint is slack, that is, $\gamma(a^{t+1}) = 0$, then the tax on the domestic and international repayments is also zero. At the same time, the government also needs to levy taxes on capital returns to implement the optimal capital stock in the bond-enforcement model. The tax on capital returns is set as

$$\tau_k(a^t) = \frac{\tau_b(a^t)}{u_c(a^t)r_k(a^t)}(V_k^{\mathrm{AUT}}(a^t) - u_c(a^t)r_k(a^t)).$$

It is easy to show that $V_k^{\mathrm{AUT}}(a^t) \geq u_c(a^t)r_k(a^t)$, which implies $\tau_k(a^t) \geq 0$.

The transversality conditions (1.4) and (1.5) in the bond-enforcement model imply the transversality conditions (1.9) and (1.11) in the decentralized problem since the multiplier $M(a^T)$ is positive. Also the transversality condition (1.10) holds since in equilibrium $b_H(a^t, a_{t+1}) = 0$ for any $(a^t, a_{t+1})$ by construction. It is also easy to check that the optimal solution in the bond-enforcement model satisfies the budget constraints and the market clearing conditions of the competitive equilibrium. *Q.E.D.*

### 1.1.2. *Resident Default Risk*

With resident default risk, consumers not only borrow and lend domestically and internationally, but also make default choices on their foreign debt. As is assumed in the case with national default, the international lenders cannot discriminate between borrowers by charging different prices. To ensure repayment, they will impose individual borrowing limits which depend on the country's shock realization and the individual accumulated capital stock. The

government chooses the tax (or subsidy) on international debt services $\tau_b$, the capital return tax on the nondefaulters $\tau_k$, and the capital return tax on the defaulters $\tilde{\tau}_k$ to implement the optimal allocation in the bond-enforcement model.

Each consumer maximizes the lifetime expected utility

$$\sum_{t=0}^{\infty}\sum_{a^t}\beta^t\pi(a^t)u(c(a^t))$$

subject to the sequential budget constraints

$$c(a^t) + k(a^t) + \sum q_H(a^t, a_{t+1})b_H(a^t, a_{t+1}) + b(a^t)$$
$$\leq w(a^t) + (1 - \tau_k(a^t))r_k(a^t)k(a^{t-1}) + b_H(a^{t-1}, a_t)$$
$$+ (1 - \tau_b(a^t))Rb(a^{t-1}) + T(a^t),$$

the enforcement constraints

$$\sum_{m=t}^{\infty}\sum_{a^m}\beta^{m-t}\pi(a^m|a^t)u(c(a^m)) \geq \hat{V}^{\text{AUT}}(a^t, k(a^{t-1}), b_H(a^{t-1}, a_t)),$$

and the no-Ponzi constraints

$$b_H(a^t, a_{t+1}) \geq -D, \quad b(a^t) \geq -D.$$

In particular, $\hat{V}^{\text{AUT}}$ denotes the default value, given by

$$\hat{V}^{\text{AUT}}(a^t, k(a^{t-1}), b_H(a^{t+1}, a_t)) = \max \sum_{m=t}^{\infty}\sum_{a^m}\beta^{m-t}\pi(a^m|a^t)u(c(a^m)),$$

subject to

$$c(a^m) + k(a^m) + \sum q_H(a^m, a_{m+1})b_H(a^m, a_{m+1})$$
$$\leq w(a^m) + r_k(a^m)(1 - \tilde{\tau}_k(a^m))k(a^{m-1})$$
$$+ b_H(a^{m-1}, a_m) + \tilde{T}(a^m).$$

Note that $T(a^t)$ and $\tilde{T}(a^t)$ denote the lump-sum transfers to nondefaulters and defaulters, respectively.

The firm's problem is standard and is characterized by the first-order conditions

$$w(a^t) = (1 - \alpha)a_t k(a^{t-1})^{\alpha},$$
$$r_k(a^t) = \alpha a_t k(a^{t-1})^{\alpha-1} + 1 - \delta.$$

DEFINITION 1.2: A *competitive equilibrium under resident default risk* with a sequence of taxes $\{\tau_k(a^t), \tilde{\tau}_k(a^t), \tau_b(a^t)\}$ is a set of allocations $\{c(a^t), b_H(a^t, b_{t+1}), b(a^t), k(a^t)\}$ and prices $\{w(a^t), r_k(a^t), q_H(a^t, a_{t+1}), R\}$ such that the following conditions are satisfied:

(i) Given the taxes, transfers, and prices, each consumer in each country chooses $\{c(a^t), b_H(a^t, a_{t+1}), b(a^t), k(a^t)\}$ to maximize the utility subject to the budget constraints, the enforcement constraints, and the no-Ponzi constraints.

(ii) The firm's optimality conditions are satisfied.

(iii) The government's budget constraints are satisfied, that is,

$$T(a^t) = \tau_k(s^t)r_k(s^t)k(s^{t-1}) + \tau_b(s^t)Rb(s^{t-1}),$$

$$\tilde{T}(a^t) = \tilde{\tau}_k(s^t)r_k(s^t)k(s^{t-1}).$$

(iv) The domestic Arrow-security markets clear, that is, $b_H(a^t, a_{t+1}) = 0$.

(v) The international bonds markets clear, that is, $\int b(a^t)\,d\mu = 0$.

PROPOSITION 1.2: *There exists a sequence of taxes* $\{\tau_k(a^t), \tilde{\tau}_k(a^t), \tau_b(a^t)\}$ *such that the optimal allocation in the bond-enforcement model can be supported as a competitive equilibrium under resident default risk.*

PROOF: The proof is constructive. The characterizations of the bond-enforcement model are the same as the previous case. We here characterize the decentralized problem with resident default. Let $\gamma_H(a^t)$ be the Langrange multiplier of the enforcement constraint. The sum of the Langrange multiplier is defined as

$$M_H(a^t) = 1 + \gamma_H(a^0) + \cdots + \gamma_H(a^t).$$

The Euler equations for capital, international borrowing, and domestic borrowing are given by

$$(1.12) \quad u_c(a^t) = \beta \sum_{a^{t+1}} \pi(a^{t+1}|a^t)\left[\frac{M_H(a^{t+1})}{M_H(a^t)}u_c(a^{t+1})r_k(a^{t+1})(1 - \tau_k(a^{t+1}))\right.$$

$$\left. - \frac{\gamma_H(a^{t+1})}{M_H(a^t)}\hat{V}_k^{\text{AUT}}(a^{t+1})\right],$$

$$(1.13) \quad u_c(a^t) = \beta R \sum_{a^{t+1}} \pi(a^{t+1}|a^t)\left[\frac{M_H(a^{t+1})}{M_H(a^t)}(1 - \tau_b(a^{t+1}))u_c(a^{t+1})\right],$$

$$(1.14) \quad u_c(a^t)q_H(a^t, a_{t+1}) = \beta\left[\frac{M_H(a^{t+1})}{M_H(a^t)}u_c(a^{t+1}) - \frac{\gamma_H(a^{t+1})}{M_H(a^t)}\hat{V}_b^{\text{AUT}}(a^{t+1})\right].$$

We set the subsidy on borrowings $\tau_b$, the Langrange multiple $\gamma_H$, and the tax on capital returns $\tau_k$ and $\tilde{\tau}_k$ as

$$\tau_b(a^t) = \frac{\gamma_H(a^t)}{M_H(a^t)} \frac{\hat{V}_b^{\text{AUT}}(a^t)}{u_c(a^t)},$$

$$(1 - \tau_b(a^t))\left(1 + \frac{\gamma_H(a^t)}{M_H(a^t)}\right) = 1 + \frac{\gamma(a^t)}{M(a^t)},$$

$$\tau_k(a^t) = \tau_b(a^t),$$

$$1 - \tilde{\tau}_k(a^t) = \frac{\gamma(a^{t+1})/M(a^t)}{\gamma(a^{t+1})/M(a^t) + \tau_b(a^t)}(1 - \tau_b(a^t)).$$

Note that by construction $\tau_b(a^t) \geq 0$ since $\hat{V}_b^{\text{AUT}}(a^t) > 0$ and $u_c(a^t) \geq 0$. Therefore, the government subsidizes international borrowing. If the enforcement constraint is slack at period $t$, that is, $\gamma_H(a^t) = 0$, the subsidy is zero. In addition, the tax on defaulter's capital returns is higher than that on non-defaulter's capital returns, which provides incentive for the domestic consumers to repay their foreign debt.

By construction, the Euler conditions of capital and international borrowing are identical in the bond-enforcement model and in the decentralized model. Furthermore, the Euler condition of domestic borrowing implies that the domestic risk-free rate is the same as the world risk-free rate, which makes the consumers indifferent between borrowing domestically and internationally. The transversality conditions of the decentralized model can be shown using similar arguments in the case of national default. It is easy to check that the optimal solution in the bond-enforcement model satisfies the budget constraints and the market clearing conditions of the competitive equilibrium.     *Q.E.D.*

## 2. TECHNICAL APPENDIX 2

This appendix describes two alternative strategies to compute the bond-enforcement model. The first strategy invokes the classic technique proposed by Abreu, Pearce, and Stacchetti (1990) (henceforth APS technique) to restate the original dynamic problem in a recursive formulation. The second strategy replaces the enforcement constraints in the original problem with the borrowing constraints. The classic technique discussed in Stokey, Lucas, and Prescott (1989) (henceforth SLP) can be invoked to show that the transformed problem has a recursive formulation. Both strategies make the dynamic programming technique applicable in the computation. We demonstrate the validity of both computational approaches so that readers can refer to the solution strategy with which that they are more familiar.

To make this appendix self-contained, we start by presenting the dynamic problem in the bond-enforcement model. An allocation $x = \{c(a^t), k(a^t),$

$b(a^t)\}_{t=0}^{\infty}$ specifies a sequence of consumption, capital, and bond holdings. Given the world interest rate $R$ and the initial state $s_0 = (a_0, k_0, b_0)$, a country chooses allocation $x$ to solve the *original problem*,

$$\max_x U(x) = \sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c(a^t)),$$

subject to

$$(2.1) \qquad c(a^t) + k(a^t) + b(a^t) \leq a_t k(a^{t-1})^\alpha + (1 - \delta)k(a^{t-1}) + Rb(a^{t-1}),$$

$$(2.2) \qquad c(a^t), k(a^t) \geq 0, \quad b(a^t) \geq -D,$$

$$(2.3) \qquad U(a^{t+1}, x) \geq V^{\text{AUT}}(a^{t+1}, k(a^t)) \quad \forall a^{t+1},$$

where $U(a^{t+1}, x)$ denotes the continuation utility under allocation $x$ from $a^{t+1}$ onward and $V^{\text{AUT}}$ denotes the autarky utility.

## 2.1. *The APS Approach*

The original problem can be restated in the recursive formulation

$$W(a, k, b) = \max_{c, k', b'} u(c) + \beta \sum_{a'|a} \pi(a'|a) W(a', k', b'),$$

subject to

$$(2.4) \qquad c + k' + b' \leq ak^\alpha + (1 - \delta)k + Rb,$$

$$(2.5) \qquad c, k' \geq 0, \quad b' \geq -D,$$

$$(2.6) \qquad W(a', k', b') \geq V^{\text{AUT}}(a', k') \quad \forall a'.$$

The basic logic of this recursive formulation is similar to that in Abreu, Pearce, and Stacchetti (1990), with one key difference in that our problem is dynamic rather than repeated. Capital and bond holdings are endogenous state variables that alter the set of feasible allocations in the following period. Thus, they not only affect the current utility, but also the future prospects in the continuation of the dynamic problem. Nonetheless, we show that the original problem can be restated in such a recursive formulation following Atkeson (1991). His environment has a complete set of assets and private information, while our environment has incomplete markets and complete information. Despite these differences, the adaptation of his approach is straightforward and is presented in the first subsection.

We also offer a detailed computation algorithm in the second subsection. An unusual feature of this algorithm is that the set of "viable" states that permit a nonempty set of feasible allocations depends on the continuation welfare $W$ in

each iteration. It is sufficient start with a $W_0$ sufficiently high and a set of viable states $S_0$ sufficiently large. We provide a proof to demonstrate the validity of the computation algorithm.

### 2.1.1. *Equivalence Between Two Problems*

We start with some definitions. Define the domain $\mathbf{S} = A \times K \times B$, where $A$, $K$, and $B$ all have finite supports. Define a country's utility possibility correspondence $V$ on domain $\mathbf{S}$ to be, for each initial value of $s \in \mathbf{S}$, the set of payoffs which the country can obtain from allocations that satisfy constraints (2.1), (2.2), and (2.3). That is, for each $s \in \mathbf{S}$,

$$V(s) = \{U(x)|x \text{ satisfies constraints (2.1)–(2.3) and } (a_0, k_0, b_0) = s\}.$$

Let domain $\mathbf{S}$ be such that the correspondence $V$ is nonempty valued. Note that such a domain exists. An example is when $B$ only includes nonnegative numbers, $V^{\mathrm{AUT}}(a, k) \in V(a, k, b)$ for all $(a, k, b)$ with $b \geq 0$.

Let $G$ be any correspondence defined over domain $\mathbf{S}$. Assume that $G(s)$ is *nonempty valued and uniformly bounded* for all $s \in \mathbf{S}$. Let $z = (c, k', b')$ denote a vector of current controls. A function $U^c$ is said to be a *continuation value function* with respect to $G$ if it is a selection from the correspondence $G$, that is, $U^c: \mathbf{S} \to \mathbb{R}$ with $U^c(s) \in G(s)$ for all $s$.

DEFINITION 2.1: The pair $(z, U^c)$ of current controls and a continuation value function with respect to $G$ is *admissible with respect to $G$ at $s$* if it satisfies the conditions

(2.7)     $c + k' + b' \leq ak^\alpha + (1 - \delta)k + Rb,$

(2.8)     $c, k' \geq 0, \quad b' \geq -D,$

(2.9)     $U^c(a', k', b') \geq V^{\mathrm{AUT}}(a', k') \quad$ for any $a'$.

Denote the payoff to the country generated by a pair $(z, U^c)$ by $E(z, U^c)(s)$, where

$$E(z, U^c)(s) = u(c) + \beta \sum_{a'|a} \pi(a'|a) U^c(s').$$

Denote the set of payoffs that can be generated by pairs $(z, U^c)$ admissible with respect to $G$ at $s$ by $B(G)(s)$, where

$$B(G)(s) = \{E(z, U^c)(s)|(z, U^c) \text{ admissible w.r.t. } G \text{ at } s\}.$$

DEFINITION 2.2: The correspondence $G$ is *self-generating* if for all $s \in \mathbf{S}$, $G(s) \subseteq B(G)(s)$.

PROPOSITION 2.1—Self-Generating: *If $G$ is self-generating, then for all $s \in \mathbf{S}$, $B(G)(s) \subseteq V(s)$.*

PROOF: This is a standard APS proof. For each $v \in B(G)(s)$, we construct an allocation $x(v)$ such that $U(x(v)) = v$ and $x(v)$ satisfies constraints (2.1)–(2.3). We proceed in three steps. The first step constructs the allocation $x(v)$. The second step verifies that $U(x(v)) = v$. The third step shows that the allocation $x(v)$ satisfies constraints (2.1)–(2.3).

*Step 1.* Choose any $v \in B(G)(s_0)$ for some $s_0$. There is an admissible pair $(z(s_0), U^c(s_0))$ such that $E(z(s_0), U^c(s_0)) = v$. Let $x(s_0; v) = z(s_0)$ and $w(s^1; v) = U^c(s_0)(s_1)$, where $s^1 = \{s_0, s_1\}$ and $s_1 = (a_1, k_1, b_1)$ for each $a_1$ and $(k_1, b_1)$ specified in $z$. Since the pair is admissible with respect to (w.r.t.) a self-generating $G$, we have $w(s^1; v) \in G(s_1) \subseteq B(G)(s_1)$. Thus, there is an admissible pair $(z(s_1), U^c(s_1))$ such that $E(z(s_1), U^c(s_1)) = w(s^1; v)$. Let $x(s^1; v) = z(s_1)$ and $w(s^2; v) = U^c(s_1)(s_2)$. Repeat this procedure to construct the allocation $x(v) = \{x(s^t; v)\}_{t=0}^{\infty}$.

*Step 2.* We need to show that $U(x(v)) = v$ for any $v \in G(s_0)$ for some $s_0$. From the above construction, we have $E(z(s_0), U^c(s_0)) = v$, that is,

$$(2.10) \quad v = u(c) + \beta \sum_{a^1} \pi(a^1|a^0) U^c(s_0)(s_1).$$

From the original problem, we have

$$(2.11) \quad U(x(v)) = u(c) + \beta \sum_{a^1} \pi(a^1|a^0) U(x(v|s^1)),$$

where $x(v|s^1) = \{x(s^t; v)\}_{t=1}^{\infty}$ denotes the infinite sequence of allocations from history $s^1$ onward. Taking the difference between (2.10) and (2.11), we have

$$v - U(x(v)) = \beta \sum_{a^1} \pi(a^1|a^0) \big[ U^c(s_0)(s_1) - U(x(v|s^1)) \big].$$

This implies

$$v - U(x(v)) \leq \beta \sup_{v_1 \in B(G)(s_1)} \big| v_1 - U(x(v|s^1)) \big|.$$

Since this holds for all $v \in B(G)(s_0)$, we have

$$\sup_{v \in B(G)(s_0)} \big| v - U(x(v)) \big| \leq \beta \sup_{v_1 \in B(G)(s_1)} \big| v_1 - U(x(v|s^1)) \big|.$$

Since $\beta < 1$, and $B(G)$ is uniformly bounded under the uniform bound on $G(s)$ and the bound on $u$, we have $v = U(x(v))$ for any $v \in B(G)(s)$ for some $s$. (Note that $u$ is bounded since there exists a maximum of $k$ given concavity of the production function; the consumption therefore will be bounded. A rigorous proof can be found in SLP.)

*Step 3.* It is easy to see that the constructed $x(v)$ for any $v \in B(G)(s)$ for some $s$ satisfies conditions (2.1)–(2.3). *Q.E.D.*

PROPOSITION 2.2—Factorization: $V(s) = B(V)(s)$ *for all* $s$.

PROOF: We show that the utility possibility correspondence $V$ is self-generating, which, by Proposition 2.1, gives us the result that $V = B(V)$. Let $v \in V(s)$ be a payoff generated by the allocation $x(v)$ which satisfies constraints (2.1)–(2.3). Construct $(z(s), U^c(s))$ as follows: $z(s) = x_0(v)$ and $U^c(s)(s_1) = U(x(v|s^1))$. Thus, $E(z(s), U^c(s)) = v$. Obviously, all the constraints of (2.7)–(2.9) are satisfied.                          Q.E.D.

The optimal welfare in the original problem $W(s)$ is defined as

$$W(s) = \sup_{v \in V(s)} v.$$

From Propositions 2.1 and 2.2, $W(s)$ is characterized by the program

(P)          $$W(s) = \sup_{(z, U^c)} u(c) + \beta \sum_{a'|a} \pi(a'|a) U^c(s)(s'),$$

subject to the constraint that $(z, U^c)$ is admissible with respect to $V$ at $s$.

We next prove that the optimal welfare $W$ exists and $W(s) \in V(s)$ for any $s$ by demonstrating that $V(s)$ is compact in Proposition 2.3. Moreover, if $\hat{U}^c$ solves the (P) program, it must be the case that $\hat{U}^c = W$ if $W$ is continuous. Given that the domain **S** has finite support, we can work directly with the vector $U^d = (U^c(s_1), U^c(s_2), \ldots, U^c(s_N))$, with $N = \#A \times \#K \times \#B$.

LEMMA 2.1: *If $G$ has a compact graph, then $B(G)$ has a compact graph*.

PROOF: First, we prove that $B(G)$ has a bounded graph. Let $G$ be a correspondence with a compact graph. Under constraints (2.1) and (2.2), control variables, including consumption, capital, and bond holding, are bounded from both above and below. The state variables $(a, k, b)$ are bounded since the state space is finite. Therefore, the pair of $(z, U^d)$, admissible with respect to $G$ at some $s$ is contained in a bounded subset of a finite dimensional Euclidean space. In addition, the $E$ operator is continuous, which implies $B(G)$ has a bounded graph.

We now prove that $B(G)$ also has a closed graph. Let $\{v_n, s_n\}_{n=1}^{\infty}$ be a sequence in the graph of $B(G)$ that converges to a point $(v, s)$. By the definition of $B(G)$, there exists a sequence of pairs of controls and a continuation value function $\{z_n, U_n^d\}_{n=1}^{\infty}$, where $(z_n, U_n^d)$ is admissible with respect to $G$ at $s_n$ and has payoff $E(z_n, U_n^d)(s_n) = v_n$. Because the space of admissible controls and continuation value functions is bounded, this sequence of pairs converges to some limit point, denoted by $(z, U^d)$. By the continuity of $E$, we have $E(z, U^d) = v$. The set of feasible controls under constraints (2.7)–(2.9) is closed. Thus, $z$ satisfies constraints (2.7)–(2.9). Moreover, $U^d \in G$ since $G$ has a compact graph. Thus $(v, s)$ is in the graph of $B(G)$.                          Q.E.D.

LEMMA 2.2: *If* $\text{graph}(G_1) \subseteq \text{graph}(G_2)$, *then* $\text{graph}(B(G_1)) \subseteq \text{graph}(B(G_2))$.

PROOF: The constraints that define $B(G_1)$ are contained in those that define $B(G_2)$. *Q.E.D.*

PROPOSITION 2.3: *V has a compact graph.*

PROOF: $V$ has a bounded graph because $u$ is bounded. We need to show that $V$ has a closed graph. Define the correspondence $V_1$ to satisfy $\text{graph}(V_1) = \text{closure}(\text{graph}(V))$. By definition, $\text{graph}(V) \subseteq \text{graph}(V_1)$. By Lemma 2.2, $\text{graph}(B(V)) \subseteq \text{graph}(B(V_1))$. By Propositions 2.1 and 2.2, $\text{graph}(V) = \text{graph}(B(V)) \subseteq \text{graph}(B(V_1))$. By Lemma 2.1, $\text{graph}(B(V_1))$ is closed. Because $\text{graph}(V_1)$ is the smallest closed set containing $\text{graph}(V)$, $\text{graph}(V_1) \subseteq \text{graph}(B(V_1))$, which implies that $V_1$ is self-generating. By Proposition 2.1, $\text{graph}(V_1) \subseteq \text{graph}(V)$. Therefore, $V$ has a closed and thus compact graph. *Q.E.D.*

PROPOSITION 2.4: *V and W are continuous.*

PROOF: We have shown that $V$ has a compact graph, which implies that $V$ is upper hemicontinuous. We need to show now that $V$ is lower hemicontinuous. Let $v \in V(s)$ and $(z, U^d)$ be a pair admissible with respect to $V$ at $s$ with $E(z, U^d)(s) = v$. Take $\varepsilon > 0$. Since the payoff function $E$ is continuous in all its arguments, we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $z_1 = (c_1, b_1', k_1')$ with $|b_1' - b'| < \delta_1$, $|k_1' - k'| < \delta_2$ and $c_1 = ak^\alpha + (1-\delta)k + Rb - b_1' - k_1'$, we have $|E(z_1, U^d)(s) - E(z, U^d)(s)| < \varepsilon/2$. Furthermore, for one such particular $z_1 = (c_1, b_1', k_1')$, we may find a $\delta_3 > 0$ such that for all $s_1$ with $|s_1 - s| < \delta_3$, we have $|E(z_1, U^d)(s_1) - E(z_1, U^d)(s)| < \varepsilon/2$. By the triangle inequality, for all $s_1$ with $|s_1 - s| < \delta_3$, $|E(z_1, U^d)(s_1) - E(z, U^d)(s)| < \varepsilon$. Furthermore, $(z_1, U^d)$ is admissible with respect to $V$ at $s_1$. Therefore, $V$ is lower hemicontinuous. Thus, we prove that $V$ is continuous and so is $W$. *Q.E.D.*

PROPOSITION 2.5: *If the value function W is continuous, then the continuation value function $\hat{U}^c$, which solves the program* (P), *satisfies $\hat{U}^c = W$.*

PROOF: The proof proceeds by contradiction. Let $(z, \hat{U}^c)$ be an admissible pair with $\hat{U}^c(s)(s') < W(s')$ for some $s$ and $s'$. Construct an alternative pair $(\tilde{z}, W)$ for this $s$ as follows. We first set $\tilde{k}' = k'$. We then set $\tilde{b}'$ to solve $\sum_{a'|a} \pi(a'|a)W(\tilde{s}') = \sum_{a'|a} \pi(a'|a)\hat{U}^c(s')$. Such a $\tilde{b}'$ exists since $W$ is continuous. Furthermore, $\tilde{b}' < b'$. Finally, we set $\tilde{c} = c + b' - \tilde{b}'$. This pair $(\tilde{z}, W)$ is admissible with respect to $V$ because it satisfies constraints (2.7)–(2.9) and $W(\tilde{s}') \in V(\tilde{s}')$ for any $\tilde{s}'$. Moreover, $E(\tilde{z}, W)(s) > E(z, \hat{U})(s)$, which contradicts with the assumption that $\hat{U}$ is optimal. Thus, $\hat{U} = W$. *Q.E.D.*

### 2.1.2. *Computation Algorithm*

We now describe the computation algorithm to solve the recursive problem. An unusual feature of this algorithm is that the set of "viable" states that permit a nonempty set of feasible allocations depends on the continuation welfare $W$ in each iteration. It is sufficient to start with a $W_0$ sufficiently high and a set of viable states $S_0$ sufficiently large. Specifically, we set $S_0$ to include all the states under which the set of allocations $(c, k', b')$ that satisfy constraints (2.4) and (2.5) is nonempty. We set $W_0$ on $S_0$ as the optimal welfare in the P-problem under constraints (2.4) and (2.5).

We then construct a sequence of the sets of viable states and the associated optimal welfare functions recursively. For each $n \geq 1$, we construct $S_n$ to include all the states that permit a nonempty set of feasible allocations that satisfy constraints (2.4), (2.5), and $W_{n-1}(a, k, b) \geq V^{\mathrm{AUT}}(a, k)$. For each $(a, k, b) \in S_n$, the corresponding optimal welfare $W_n$ is

$$(2.12) \quad W_n(a, k, b) = TW_{n-1}(a, k, b)$$
$$= \max_{c, k', b'} u(c) + \beta \sum_{a'|a} \pi(a'|a) W_{n-1}(a', k', b')$$

over the set of feasible allocations under $(a, k, b)$.

Given our construction of $S_0$ and $W_0$, we have $S_1 \subseteq S_0$ and $W_1 \leq W_0$ on $S_1$. Clearly, the set of viable states $S_n$ decreases as $W_{n-1}$ decreases because the set of feasible allocations decreases. This in turn leads to a lower welfare function $W_n$. Thus, both sequences of $\{S_n\}$ and $\{W_n\}$ are decreasing. Since the state space has finite supports and is compact, the decreasing sequence $\{S_n\}$ converges to a limit $S$ in finite iterations. That is, there exists an $N$ such that $S_n = S$ for any $n \geq N$. On domain $S$, the decreasing sequence $\{W_n\}_{n+N}^{\infty}$ converges to a limit, denoted by $W$.

We next establish that the limits $S$ and $W$ correspond to the set of viable states and the optimal welfare in the original problem. $W^o$ denotes the optimal welfare in the original problem, and $S^o$ denotes the set of initial states $(a_0, k_0, b_0)$ under which there is a nonempty set of allocations $x$ that satisfies constraints (2.1), (2.2), (2.3), and $U(x) \geq V^{\mathrm{AUT}}(a_0, k_0)$. The following theorem proves that $W = W^o$ and $S = S^o$.

PROPOSITION 2.6: *The limits $W$ and $S$ of the sequences $\{W_n\}$ and $\{S_n\}$ constructed above are the same as $S^o$ and $W^o$ in the original problem.*

PROOF: We start by proving $S \subseteq S^o$. For any $s \in S$ associated with $W(s)$, we can construct an infinite sequence of allocations $x$ using the self-generating property of $W$. This sequence of allocations clearly satisfies constraints (2.1), (2.2), (2.3), and $U(x) = W(s) \geq V^{\mathrm{AUT}}(a, k)$. Thus, we have $s \in S^o$.

We next prove that $S^o \subseteq S$ by induction. For any $s \in S^o$, it is clear that $s \in S_0$ by our construction. We need to show that $s \in S_1$. Equivalently, we need to

show that $W_0(s) \geq V^{\text{AUT}}(a, k)$. Since $s \in S^o$, there exists a sequence of allocations $x$ that satisfy constraints (2.1), (2.2), (2.3), and $U(x) \geq V^{\text{AUT}}(a, k)$. Let $z(s) = x(s_0)$ and $U^c(s)(s_1) = U(x|s_1)$. Thus, $(z, U^c)$ is feasible under the P-problem for computing $W_0$. Thus, $W_0(s) \geq U(x)$, which implies that $W_0(s) \geq V^{\text{AUT}}(a, k)$. We then show that if $s \in S_{n-1}$, then $s \in S_n$ with similar arguments. Therefore, we have $S = S^o$ and then $W = W^o$ also follows. *Q.E.D.*

The computation algorithm is straightforward. For any interest rate $R$, we construct the sequences of $\{S_n\}$ and $\{W_n\}$ as described above and find the limits $S$ and $W$. Under the optimal decision rules associated with $W$, we compute the invariant distribution and the excess demand in the bond markets under a given interest rate $R$. We finally update the interest rate and repeat the above process until the bond markets clear.

## 2.2. *The SLP Approach*

As is well known, models with limited enforcement friction present challenges to standard recursive methods because all future consumption enters today's enforcement constraints. Alvarez and Jermann (2000) responded to this challenge by replacing the enforcement constraints with endogenous borrowing constraints, which depend on only current states to make the problem recursive. They prove the validity of this approach using the first-order conditions. These conditions are both necessary and sufficient for optimality in their pure exchange economy because autarky utilities depend on only exogenous shocks and the feasible set is convex. In a production economy, however, the feasible set is not necessarily convex because autarky utilities are also functions of endogenous capital stocks. Thus, we must prove the equivalence between the enforcement constraints and the endogenous borrowing constraints without invoking the first-order conditions.

We construct a transformed problem in which borrowing constraints are chosen to replace the enforcement constraints in the original problem. The *transformed problem* under a debt limit function $B$ is constructed as

$$(2.13) \quad W(a_0, k_0, b_0; B) = \max_{\{c(a^t), k(a^t), b(a^t)\}} \sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c(a^t)),$$

subject to the budget constraints (2.1) and the borrowing constraints

$$b(a^t) \geq -B(a_t, k(a^t)),$$

where $W$ denotes *market welfare*. The debt limit function $B: A \times K \to F$ specifies the amount that a country can borrow given its current shock $a \in A$ and capital choice $k \in K$, where $K$ has finite support and $F \equiv [0, D]$. The debt limit function is noncontingent in the sense that it is independent of future shocks.

This problem has an obvious recursive structure as shown in SLP and is easy to compute. The key is to find the endogenous debt limit function $B$ and to establish the equivalence between the original problem and the transformed problem.

### 2.2.1. *Finding the Endogenous Debt Limit Function*

Now we must find a debt limit function $B^*(a, k')$ for which any solution to the transformed problem under $B^*$ is a solution to the original problem. Clearly, if such a debt limit function exists, it might not be unique.[3] The $B^*$ we construct has the property that the function allows as much borrowing as possible, while at the same time preventing countries from defaulting.

We define an operator $T : F^{\#A \times \#K} \to F^{\#A \times \#K}$ on the debt limit function:

$$(2.14) \quad TB(a, k') \equiv \min_{a'|\pi(a'|a)>0} \left\{ -\tilde{b}(a') : W(a', k', \tilde{b}(a'); B) = V^{\mathrm{AUT}}(a, k') \right\}$$

for any $(a, k')$.

Given any debt limit function $B$, the operator $T$ produces a new debt limit function $TB$, which, for each $(a, k')$, specifies the maximum amount of debt that can be supported without default under all future contingencies. We first need to show that the operator $T$ in (2.14) is well defined, which is established with the following three lemmas.

LEMMA 2.3: *For fixed* $(a, k, B)$, $W(a, k, b; B)$ *is continuous in* $b$; *for fixed* $(a, k, b)$, $W(a, k, b; B)$ *is continuous in* $B$.

LEMMA 2.4: *For fixed* $(a, k, B)$, $W(a, k, b; B)$ *is strictly increasing in* $b$.

LEMMA 2.5: *Suppose that* $\lim_{b \to -\infty} W(a, k, b; B) = -\infty$ *and* $\lim_{b \to \infty} W(a, k, b; B) = \infty$. *Then there exists a unique* $\tilde{b}$ *such that*

$$W(a, k, \tilde{b}; B) = V^{\mathrm{AUT}}(a, k) \quad \text{for fixed } (a, k, B).$$

Lemma 2.3 follows directly from the maximum theorem; Lemma 2.4 follows from the envelope theorem and the strict concavity of the utility function. These two lemmas establish two properties of the market utility, that is, continuity and monotonicity with respect to bond holdings. These two properties and the intermediate value theorem lead to Lemma 2.5, which asserts that there exists one unique level of debt (a *cutoff*) at which countries are indifferent between the market utility and the autarky utility.

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[3]Specifically, any debt limit function under which the feasible set of the transformed problem is a subset of the feasible set of the original problem and includes the optimal solutions of the original problem is a candidate for the transformed problem.

For $TB$ to be a borrowing limit function, we still need to show that it is nonnegative for any $(a, k')$. For any future realization of a shock, consider the market utility with $(a', k', b')$, where $b'$ equals zero bond holding. Clearly, then, the market utility must be at least as high as the autarky utility because staying in the market allows countries to smooth future consumption. Thus, the cutoff bond must be smaller than or equal to zero.

LEMMA 2.6: *If $B(a, k') \geq 0$ for all $(a, k')$, then $TB(a, k') \geq 0$ for all $(a, k')$.*

For a country with shock $a$ and capital decision $k'$, $TB(a, k')$ specifies the maximum amount of borrowing that a country is willing to repay in all future states. From the definition of $TB$, a corollary follows immediately:

COROLLARY 2.1: *The value $b' \geq -TB(a, k')$ if and only if $W(a', k', b'; B) \geq V^{\text{AUT}}(a', k')$ for any $a'$ following $a$.*

We now show that the operator $T$ is monotone and then use this property to prove the existence of the borrowing limit function $B$.

LEMMA 2.7: *For any two borrowing limit functions $B_1$ and $B_2$, if $B_1(a, k') \leq B_2(a, k')$ for all $(a, k')$, then $TB_1(a, k') \leq TB_2(a, k')$ for all $(a, k')$.*

PROOF: By the definition of the operator $T$, we have that

$$TB_1(a, k') = \min_{a'|\pi(a'|a)>0} \{-\tilde{b}(a'; B_1) :$$
$$W(a', k', \tilde{b}(a'; B_1); B_1) = V^{\text{AUT}}(a', k')\},$$
$$TB_2(a, k') = \min_{a'|\pi(a'|a)>0} \{-\tilde{b}(a'; B_2) :$$
$$W(a', k', \tilde{b}(a'; B_2); B_2) = V^{\text{AUT}}(a', k')\}.$$

By $B_1(a, k') \leq B_2(a, k')$ for all $(a, k')$, we have that $W(a, k, b; B_1) \leq W(a, k, b; B_2)$ for any $(a, k, b)$ since the feasible set under $B_1$ is a subset of that under $B_2$. This means that $\tilde{b}(a'; B_1) \geq \tilde{b}(a'; B_2)$ for all the $a'$ and for any given $(a, k')$. Thus, $TB_1(a, k') \leq TB_2(a, k')$ for all $(a, k')$.  *Q.E.D.*

The endogenous debt limit function $B^*$ can be constructed for any $(a, k')$ as

$$(2.15) \qquad B^*(a, k') = \lim_{n \to \infty} T^n B_0(a, k'),$$

where $B_0(a, k') = D$ for all $(a, k')$. The borrowing limit $D$ is specified in the spirit of the natural borrowing limit introduced by Aiyagari (1994). At each point in time, if a country were at the borrowing limit, this limit should ensure

nonnegative consumption. When the bond holding approaches this limit and consumption approaches zero, the utility level goes to negative infinity. Thus, for any feasible allocation, the short-sale constraint will never bind with the presence of the enforcement constraints.

When we start with the borrowing limit function $B_0(a, k') = D$ for all $(a, k')$, we know that $0 \leq B_1(a, k') = TB_0(a, k') \leq D = B_0(a, k')$ for any $(a, k')$ because the short-sale constraint is not binding in the presence of the enforcement constraints. Using the monotone property of operator $T$ for any given $(a, k')$, we have that

$$0 \leq \cdots \leq T^n B_0(a, k') \leq \cdots \leq T^2 B_0(a, k') \leq B_0(a, k') \leq D \quad \text{for any } n.$$

Since any monotone sequence in the compact set converges to a limit in the compact set, fixing any $(a, k')$, the sequence $\{B_0(a, k'), TB_0(a, k'), T^2 B_0(a, k'), \ldots\}$ will converge to a limit, denoted by $B^*(a, k')$; that is,

$$B^*(a, k') = \lim_{n \to \infty} T^n B_0(a, k') \quad \text{for all } (a, k').$$

### 2.2.2. *Equivalence of Two Problems*

PROPOSITION 2.7: *Let $B^*$ be defined as in* (2.15). *Then an allocation is optimal in the transformed problem if and only if it is optimal in the original problem*.

PROOF: Here we prove the "only if" part of the theorem used in the study. The "if" part of the theorem can be proved similarly. Denote the optimal allocation in the transformed problem by $x^T = \{c^T(a^t), k^T(a^t), b^T(a^t)\}$. We need to show that $x^T = \{c^T(a^t), k^T(a^t), b^T(a^t)\}$ is also optimal in the original problem.

We first need to show that the allocation $x^T = \{c^T(a^t), k^T(a^t), b^T(a^t)\}$ is feasible in the original problem. The resource constraints are obviously satisfied. The short-sale constraints are satisfied by monotonicity of operator $T$. We need to show that the enforcement constraints are satisfied. To do that, we begin with the fact that $B^*(a, k') \leq B_n(a, k') = T^n B_0(a, k')$ for any $(a, k')$ and all $n$. From that, we have that

$$b^T(a^t) \geq -B_n(a_t, k^T(a^t)) \quad \text{for all } n.$$

By the corollary, we have that for any $n$,

$$W(a^{t+1}, k^T(a^t), b^T(a^t); B_{n-1}) \geq V^{\text{AUT}}(a^{t+1}, k^T(a^t))$$

for any $a^{t+1}$ following $a^t$. From the continuity of $W$ on $B$ for fixed $(a, k, b)$, we have that

$$W(a^{t+1}, k^T(a^t), b^T(a^t); B^*) \geq V^{\text{AUT}}(a^{t+1}, k^T(a^t))$$

for any $a^{t+1}$ following $a^t$. Furthermore, by the optimality of $x^T$ in the transformed problem, we have that

$$U(a^{t+1}, x^T) = W(a^{t+1}, k^T(a^t), b^T(a^t); B^*) \quad \text{for any } a^{t+1} \text{ following } a^t.$$

From the above two inequalities, we have that

$$U(a^{t+1}, x^T) \geq V^{\mathrm{AUT}}(a^{t+1}, k^T(a^t)) \quad \text{for any } a^{t+1} \text{ following } a^t.$$

Thus, the enforcement constraints are satisfied at each history node.

Second, we need to show that $x^T$ is optimal in the original problem. We show this by contradiction. Assume there is another allocation $x^o = \{c^o(a^t), k^o(a^t), b^o(a^t)\}$ which is feasible under the original problem (superscript $o$ indicates original) and delivers higher welfare than $x^T$; that is,

$$(2.16) \quad \sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c^o(a^t)) > \sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c^T(a^t)).$$

We establish the contradiction by showing that $x^o$ is feasible in the transformed problem. Obviously, the resource constraints in the transformed problem are satisfied. We need to show that $b^o(a^t) \geq -B^*(a_t, k^o(a^t))$. We do that by induction. Clearly, $b^o(a^t) \geq -B_0(a_t, k^o(a^t))$, where $B_0(a, k') = D$ for any $(a, k')$, by construction of $D$. Thus, $x^o$ is feasible under the problem with the borrowing limit $B_0$ and we have that

$$W(a^{t+1}, k^o(a^t), b^o(a^t); B_0) \geq U(a^{t+1}, x^o) \quad \text{for any } a^{t+1} \text{ following } a^t.$$

By the feasibility of $x^o$ in the original problem, we also have that

$$U(a^{t+1}, x^o) \geq V^{\mathrm{AUT}}(a^{t+1}, k^o(a^t)) \quad \text{for any } a^{t+1} \text{ following } a^t.$$

From the above two inequalities, we conclude that

$$W(a^{t+1}, k^o(a^t), b^o(a^t); B_0) \geq V^{\mathrm{AUT}}(a^{t+1}, k^o(a^t))$$

for any $a^{t+1}$ following $a^t$. Thus, $b^o(a^t) \geq -TB_0(a_t, k^o(a^t))$ from the corollary.

Repeating the above arguments, we have that $b^o(a^t) \geq -T^n B_0(a_t, k^o(a^t))$ for all $n$, which implies that $b^o(a^t) \geq -B^*(a_t, k^o(a^t)) = -\lim_{n \to \infty} T^n B_0(a_t, k^o(a^t))$. Thus, $x^o$ is feasible in the transformed problem.

From the optimality of $x^T$ and the feasibility of $x^o$ in the transformed problem, we have that

$$\sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c^T(a^t)) > \sum_{t=0}^{\infty} \sum_{a^t} \beta^t \pi(a^t) u(c^o(a^t)),$$

which contradicts (2.16). Thus, the allocation $x^T$ is optimal in the original problem.                                                                    *Q.E.D.*

The theorem shows that we can compute the transformed problem instead of the complicated original one. We invoke the classic SLP approach to restate the transformed problem recursively. Thus, we use the dynamic programming technique to solve the equilibrium as follows. We first guess a world interest rate. Given this interest rate, we then look for the endogenous debt limit function. We start with $B_0 = D$ and solve the transformed problem under $B_0$. Given $W(a, k, b; B_0)$, we then update the debt limit function according to equation (2.14). We repeat these steps until the debt limit function converges. We next compute the invariant distribution and the excess demand in the bond markets. We finally update the interest rate and repeat the above process until the bond markets clear.

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