Econometrica Supplementary Material

SUPPLEMENT TO "NONPARAMETRIC INSTRUMENTAL VARIABLES ESTIMATION OF A QUANTILE REGRESSION MODEL": MATHEMATICAL APPENDIX

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THIS APPENDIX provides proofs of Theorems 1–3. Theorems 4 and 5 can be proved by following the same steps after conditioning on Z.

A.1. PROOF OF THEOREM 1

The proof is a modification of the proof of Theorem 2 in Bissantz, Hohage, and Munk (2004). By (2.5),

$$\|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W\|^2 + a_n\|\hat{g}\|^2 \leq \|\hat{\mathcal{T}}(g) - q\hat{f}_W\|^2 + a_n\|g\|^2. \tag{A1}$$

In addition, $\mathbf{E}\|\hat{\mathcal{T}}(g) - \mathcal{T}(g)\|^2 = O(\delta_n)$ and $\mathbf{E}\|\hat{f}_W - f_W\|^2 = O(\delta_n)$. Therefore, by Assumption 3,

$$\begin{aligned}\mathbf{E}\|\hat{\mathcal{T}}(g) - q\hat{f}_W\|^2 &\leq 2\mathbf{E}\|\hat{\mathcal{T}}(g) - \mathcal{T}(g)\|^2 + 2q\mathbf{E}\|\hat{f}_W - f_W\|^2 \\ &= O(\delta_n).\end{aligned}$$

Combining this result with (A1) gives

$$\mathbf{E}\|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W\|^2 + a_n\mathbf{E}\|\hat{g}\|^2 \leq C\delta_n + a_n\|g\|^2$$

for some constant $C < \infty$ and all sufficiently large n. Therefore, by Assumption 3,

$$\limsup_{n\to\infty} \mathbf{E}\|\hat{g}\|^2 \leq \|g\|^2.$$

Note, in addition, that $\mathbf{E}\|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W\|^2 \to 0$ as $n \to \infty$. Moreover, Assumptions 1(b) and 2 imply that $\mathcal{T}$ is weakly closed. Consistency now follows from arguments identical to those used to prove Theorem 2 of Bissantz, Hohage, and Munk (2004, p. 1777). *Q.E.D.*

A.2. PROOF OF THEOREM 2

Assumptions 1, 2, and 4–7 hold throughout this section. Let $\langle\cdot,\cdot\rangle$ denote the inner product in $L_2[0, 1]$. Define $\omega_g = (T_g T_g^*)^{-1} T_g g$ and

$$\tilde{g} = g - a_n(T_g^* T_g + a_n I)^{-1} T_g^* \omega_g, \tag{A2}$$

where I is the identity operator. Observe that by (3.4),

$$L\|\omega_g\| < 1. \tag{A3}$$

Let $\hat{r}$ and $\tilde{r}$ be Taylor series remainder terms with the properties that

$$\mathcal{T}(\hat{g}) = qf_W + T_g(\hat{g} - g) + \hat{r} \tag{A4}$$

and

$$\mathcal{T}(\tilde{g}) = qf_W + T_g(\tilde{g} - g) + \tilde{r}, \tag{A5}$$

where $qf_W = \mathcal{T}(g)$. By (3.3), $\|\hat{r}\| \le (L/2)\|\hat{g} - g\|^2$ and $\|\tilde{r}\| \le (L/2)\|\tilde{g} - g\|^2$.

LEMMA A.1: *For any $g \in \mathcal{G}$,*

$$\begin{aligned}
&(1 - L\|\omega_g\|)\|\hat{g} - g\|^2 \\
&\quad \le \|\tilde{g} - g\|^2 + a_n^{-1}\|\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}) + \tilde{r} + qf_W - q\hat{f}_W\|^2 \\
&\qquad + \langle 2T_g(\tilde{g} - g) + a_n\omega_g, \omega_g\rangle + a_n^{-1}\|T_g(\tilde{g} - g)\|^2 \\
&\qquad + 2\langle qf_W - q\hat{f}_W, \omega_g\rangle + 2a_n^{-1}\langle \tilde{r} + qf_W - q\hat{f}_W, T_g(\tilde{g} - g)\rangle \\
&\qquad + 2\langle \hat{\mathcal{T}}(\hat{g}) - \mathcal{T}(\hat{g}), \omega_g\rangle + 2a_n^{-1}\langle \hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}), T_g(\tilde{g} - g)\rangle.
\end{aligned}$$

PROOF: By (A5),

$$\begin{aligned}
&\|\hat{\mathcal{T}}(\tilde{g}) - q\hat{f}_W\|^2 \\
&\quad = \|\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}) + \tilde{r} + qf_W - q\hat{f}_W\|^2 + \|T_g(\tilde{g} - g)\|^2 \\
&\qquad + 2\langle \tilde{r} + qf_W - q\hat{f}_W, T_g(\tilde{g} - g)\rangle + 2\langle \hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}), T_g(\tilde{g} - g)\rangle.
\end{aligned} \tag{A6}$$

Also,

$$\begin{aligned}
\|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W\|^2 = {}& \|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W + a_n\omega_g\|^2 + a_n^2\|\omega_g\|^2 \\
& - 2a_n\langle \hat{\mathcal{T}}(\hat{g}) - \mathcal{T}(\hat{g}), \omega_g\rangle - 2a_n\langle \mathcal{T}(\hat{g}) - qf_W, \omega_g\rangle \\
& - 2a_n\langle qf_W - q\hat{f}_W, \omega_g\rangle - 2a_n^2\|\omega_g\|^2.
\end{aligned} \tag{A7}$$

Moreover,

$$\langle \tilde{g} - g, g\rangle = \langle \tilde{g} - g, T_g^*\omega_g\rangle = \langle T_g(\tilde{g} - g), \omega_g\rangle. \tag{A8}$$

By (A4),

$$\langle \hat{g} - g, g\rangle = \langle \hat{g} - g, T_g^*\omega_g\rangle \tag{A9}$$

$$= \langle T_g(\hat{g} - g), \omega_g \rangle$$
$$= \langle \mathcal{T}(\hat{g}) - qf_W, \omega_g \rangle - \langle \hat{r}, \omega_g \rangle.$$

By (2.5),

$$\|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W\|^2 + a_n\|\hat{g}\|^2 \leq \|\hat{\mathcal{T}}(\tilde{g}) - q\hat{f}_W\|^2 + a_n\|\tilde{g}\|^2. \tag{A10}$$

Rearranging and expanding terms in (A10) gives

$$\begin{aligned} &\|\hat{g} - g\|^2 \\ &\quad \leq a_n^{-1}\big[\|\hat{\mathcal{T}}(\tilde{g}) - q\hat{f}_W\|^2 - \|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W\|^2\big] + \|\tilde{g} - g\|^2 \\ &\quad\quad + 2\langle \tilde{g} - g, g\rangle - 2\langle \hat{g} - g, g\rangle. \end{aligned} \tag{A11}$$

Combining (A11) with (A6)–(A9) gives

$$\begin{aligned} \|\hat{g} - g\|^2 \leq{}& 2\langle \hat{r}, \omega_g\rangle - a_n^{-1}\|\hat{\mathcal{T}}(\hat{g}) - q\hat{f}_W + a_n\omega_g\|^2 + \|\tilde{g} - g\|^2 \\ &+ a_n^{-1}\|\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}) + \tilde{r} + qf_W - q\hat{f}_W\|^2 \\ &+ \langle 2T_g(\tilde{g} - g) + a_n\omega_g, \omega_g\rangle + a_n^{-1}\|T_g(\tilde{g} - g)\|^2 \\ &+ 2\langle qf_W - q\hat{f}_W, \omega_g\rangle + 2a_n^{-1}\langle \tilde{r} + qf_W - q\hat{f}_W, T_g(\tilde{g} - g)\rangle \\ &+ 2\langle \hat{\mathcal{T}}(\hat{g}) - \mathcal{T}(\hat{g}), \omega_g\rangle + 2a_n^{-1}\langle \hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}), T_g(\tilde{g} - g)\rangle. \end{aligned} \tag{A12}$$

The lemma follows by noting that the second term on the right-hand side of (A12) is nonpositive and that $2\langle \hat{r}, \omega_g\rangle \leq L\|\omega_g\|\|\hat{g} - g\|^2$. *Q.E.D.*

LEMMA A.2: *For any* $g \in \mathcal{G}$,

$$\begin{aligned} &a_n^{-1}\|\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}) + \tilde{r} + qf_W - q\hat{f}_W\|^2 \\ &\quad \leq 4a_n^{-1}\left(\|\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g})\|^2 + \frac{L^2}{4}\|\tilde{g} - g\|^4 + \|qf_W - q\hat{f}_W\|^2\right), \end{aligned} \tag{A13}$$

$$\begin{aligned} &\langle 2T_g(\tilde{g} - g) + a_n\omega_g, \omega_g\rangle + a_n^{-1}\|T_g(\tilde{g} - g)\|^2 \\ &\quad = a_n^3\|(T_gT_g^* + a_nI)^{-1}\omega_g\|^2, \end{aligned} \tag{A14}$$

$$\begin{aligned} &\left|2\langle qf_W - q\hat{f}_W, \omega_g\rangle + 2a_n^{-1}\langle \tilde{r} + qf_W - q\hat{f}_W, T_g(\tilde{g} - g)\rangle\right| \\ &\quad \leq L\|\omega_g\|\|\tilde{g} - g\|^2 + 2a_n\|qf_W - q\hat{f}_W\|\|(T_gT_g^* + a_nI)^{-1}\omega_g\| \\ &\quad\quad + La_n\|\tilde{g} - g\|^2\|(T_gT_g^* + a_nI)^{-1}\omega_g\|, \end{aligned} \tag{A15}$$

and

$$\text{(A16)}\quad \left|2\langle\hat{\mathcal{T}}(\hat{g})-\mathcal{T}(\hat{g}),\omega_g\rangle+2a_n^{-1}\langle\hat{\mathcal{T}}(\tilde{g})-\mathcal{T}(\tilde{g}),T_g(\tilde{g}-g)\rangle\right|$$
$$\le 2a_n\|\hat{\mathcal{T}}(\tilde{g})-\mathcal{T}(\tilde{g})\|\|(T_gT_g^*+a_nI)^{-1}\omega_g\|$$
$$+2\|\omega_g\|\big\|[\hat{\mathcal{T}}(\hat{g})-\mathcal{T}(\hat{g})]-[\hat{\mathcal{T}}(g)-\mathcal{T}(g)]\big\|$$
$$+2\|\omega_g\|\big\|[\hat{\mathcal{T}}(\tilde{g})-\mathcal{T}(\tilde{g})]-[\hat{\mathcal{T}}(g)-\mathcal{T}(g)]\big\|.$$

PROOF: Inequality (A13) follows from (A5) and the relation

$$\|A+B+C\|^2\le 4(\|A\|^2+\|C\|^2+\|C\|^2)$$

for any functions A, B, and C.

To show (A14), note that

$$\text{(A17)}\quad a_n(T_gT_g^*+a_nI)^{-1}=I-T_g(T_g^*T_g+a_nI)^{-1}T_g^*.$$

By (A2),

$$\text{(A18)}\quad T(\tilde{g}-g)=-a_nT_g(T_g^*T_g+a_nI)^{-1}T_g^*\omega_g.$$

It follows from (A17) and (A18) that

$$\text{(A19)}\quad a_n\omega_g+T(\tilde{g}-g)=a_n^2(T_gT_g^*+a_nI)^{-1}\omega_g.$$

Taking the squares of the norms of both sides of (A19) and expanding the term on the left-hand side yields

$$\text{(A20)}\quad a_n^2\|\omega_g\|^2+\|T_g(\tilde{g}-g)\|^2+2\langle a_n\omega_g,T_g(\tilde{g}-g)\rangle$$
$$=a_n^4\|(T_gT_g^*+a_nI)^{-1}\omega_g\|^2.$$

Then (A14) follows by dividing both sides of (A20) by a_n.

We now turn to (A15). First note that

$$\text{(A21)}\quad \langle\tilde{r}+qf_W-q\hat{f}_W,T_g(\tilde{g}-g)\rangle$$
$$=\langle\tilde{r}+qf_W-q\hat{f}_W,T_g(\tilde{g}-g)+a_n\omega_g\rangle-\langle\tilde{r}+qf_W-q\hat{f}_W,a_n\omega_g\rangle.$$

It follows from (A19) and (A21) that

$$\text{(A22)}\quad 2\langle qf_W-q\hat{f}_W,\omega_g\rangle+2a_n^{-1}\langle\tilde{r}+qf_W-q\hat{f}_W,T_g(\tilde{g}-g)+a_n\omega_g\rangle$$
$$=2a_n\langle\tilde{r}+qf_W-q\hat{f}_W,(T_gT_g^*+a_nI)^{-1}\omega_g\rangle-2\langle\tilde{r},\omega_g\rangle.$$

Then (A15) follows by applying the Cauchy–Schwarz and triangle inequalities to (A22).

Now we prove (A16). Observe that by (A19) and algebra like that yielding (A21),

$$\begin{aligned}
&2\langle \hat{\mathcal{T}}(\hat{g}) - \mathcal{T}(\hat{g}), \omega_g\rangle + 2a_n^{-1}\langle \hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}), T_g(\tilde{g} - g)\rangle \\
&\quad = 2a_n\langle \hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g}), (T_g T_g^* + a_n I)^{-1}\omega_g\rangle \\
&\qquad + 2\big\langle [\hat{\mathcal{T}}(\hat{g}) - \mathcal{T}(\hat{g})] - [\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g})], \omega_g\big\rangle,
\end{aligned}$$

which yields (A16) by the Cauchy–Schwarz and triangle inequalities. *Q.E.D.*

LEMMA A.3: *The following relations hold uniformly over* $H \in \mathcal{H}$.

(a) $\|\tilde{g} - g\|^2 = O[n^{-(2\beta-1)/(2\beta+\alpha)}]$;

(b) $a_n^{-1}\|\tilde{g} - g\|^4 = O[n^{-(2\beta-1)/(2\beta+\alpha)}]$;

(c) $a_n^3\|(T_g T_g^* + a_n I)^{-1}\omega_g\|^2 = O[n^{-(2\beta-1)/(2\beta+\alpha)}]$;

(d) $a_n\|\tilde{g} - g\|^2\|(T_g T_g^* + a_n I)^{-1}\omega_g\| = O[n^{-(2\beta-1)/(2\beta+\alpha)}]$;

(e) $a_n\|qf_W - q\hat{f}_W\|\|(T_g T_g^* + a_n I)^{-1}\omega_g\| = O_p[n^{-(2\beta-1)/(2\beta+\alpha)}]$;

(f) $a_n\|\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g})\|\|(T_g T_g^* + a_n I)^{-1}\omega_g\| = O_p[n^{-(2\beta-1)/(2\beta+\alpha)}]$;

(g) *there are random variables* $\Delta_n = O_p[n^{-(\beta-1/2)/(2\beta+\alpha)}]$ *and* $\Gamma_n = o_p(1)$ *such that* $\|[\hat{\mathcal{T}}(\hat{g}) - \mathcal{T}(\hat{g})] - [\hat{\mathcal{T}}(g) - \mathcal{T}(g)]\| \le \Delta_n\|\hat{g} - g\| + \Gamma_n\|\hat{g} - g\|^2$;

(h) $\|[\hat{\mathcal{T}}(\tilde{g}) - \mathcal{T}(\tilde{g})] - [\hat{\mathcal{T}}(g) - \mathcal{T}(g)]\| = O_p[n^{-(2\beta-1)/(2\beta+\alpha)}]$.

PROOF: To prove (a), note that by (A2) and $T_g^*\omega_g = g$,

$$\begin{aligned}
\tilde{g}(x) - g(x) &= -a_n\sum_{j=1}^{\infty}\frac{1}{\lambda_j + a_n}\phi_j(x)\langle\phi_j, T_g^*\omega_g\rangle \\
&= -a_n\sum_{j=1}^{\infty}\frac{b_j}{\lambda_j + a_n}\phi_j(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\tilde{g} - g\|^2 &= a_n^2\sum_{j=1}^{\infty}\frac{b_j^2}{(\lambda_j + a_n)^2} \\
&= O[n^{-(2\beta-1)/(2\beta+\alpha)}],
\end{aligned}$$

where the second line follows from arguments identical to those used to prove equation (6.4) of Hall and Horowitz (2005). This proves (a). It follows from (a) that

$$a_n^{-1}\|\tilde{g} - g\|^4 = O[n^{-(2\beta-1)/(2\beta+\alpha)}]$$

whenever $\alpha < 2\beta - 1$, thereby proving (b).

We now turn to (c). Define $\psi_j = T_g\phi_j/\|T_g\phi_j\|$. Use $\omega_g = (T_gT_g^*)^{-1}T_g g$ and the singular value decomposition $T_g^*\psi_j = \lambda_j^{1/2}\phi_j$ to obtain

$$\begin{aligned}
(T_gT_g^* + a_nI)^{-1}\omega_g &= \sum_{j=1}^{\infty} \frac{1}{\lambda_j(\lambda_j + a_n)}\psi_j\langle\psi_j, T_g g\rangle \\
&= \sum_{j=1}^{\infty} \frac{1}{\lambda_j(\lambda_j + a_n)}\psi_j\langle T_g^*\psi_j, g\rangle \\
&= \sum_{j=1}^{\infty} \frac{1}{\lambda_j(\lambda_j + a_n)}\psi_j\langle \lambda_j^{1/2}\phi_j, g\rangle \\
&= \sum_{j=1}^{\infty} \frac{b_j}{\lambda_j^{1/2}(\lambda_j + a_n)}\psi_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(T_gT_g^* + a_nI)\omega_g\|^2 &= \sum_{j=1}^{\infty} \frac{b_j^2}{\lambda_j(\lambda_j + a_n)^2} \\
&= O[a_n^{(2\beta-3\alpha-1)/\alpha}]
\end{aligned} \tag{A23}$$

by arguments like those used to prove equation (6.4) of Hall and Horowitz (2005). Therefore, (c) follows from $a_n = C_a n^{-\alpha/(2\beta+\alpha)}$.

To prove (d), note that by (A23),

$$\begin{aligned}
a_n\|(T_gT_g^* + a_nI)^{-1}\omega_g\| &= O[a_n^{(2\beta-\alpha-1)/(2\alpha)}] \\
&= O[n^{-(2\beta-\alpha-1)/(4\beta+2\alpha)}].
\end{aligned} \tag{A24}$$

Therefore, (d) follows from (A24) and (a), because $\alpha < 2\beta - 1$. Now by Assumptions 2 and 5(b),

$$\|\hat{f}_W - f_W\|^2 = O_p[n^{-(2\beta-1+\alpha)/(2\beta+\alpha)}].$$

Moreover, because $\mathcal{T}$ is a density only with respect to its w argument and can be estimated with the one-dimensional nonparametric rate of convergence, we have

$$\|\hat{\mathcal{T}} - \mathcal{T}\|^2 = O_p[n^{-(2\beta-1+\alpha)/(2\beta+\alpha)}]$$

uniformly over $\mathcal{H}$. Therefore, (e) and (f) follow from (A24).

We now turn to (g). By the mean value theorem,

$$\begin{aligned}&\big\{[\hat{\mathcal{T}}(\hat g)-\mathcal{T}(\hat g)]-[\hat{\mathcal{T}}(g)-\mathcal{T}(g)]\big\}(w)\\&\quad=\int_0^1\{\hat f_{YXW}[\bar g(x),x,w]-f_{YXW}[\bar g(x),x,w]\}[\hat g(x)-g(x)]\,dx,\end{aligned}$$

where $\bar g$ is between $\hat g$ and g. Then by the Cauchy–Schwarz inequality,

$$\begin{aligned}&\big\|[\hat{\mathcal{T}}(\hat g)-\mathcal{T}(\hat g)]-[\hat{\mathcal{T}}(g)-\mathcal{T}(g)]\big\|^2\\&\quad=\int_0^1\bigg(\int_0^1\{\hat f_{YXW}[\bar g(x),x,w]-f_{YXW}[\bar g(x),x,w]\}\\&\qquad\times[\hat g(x)-g(x)]\,dx\bigg)^2dw\\&\quad\le\int_0^1\bigg(\int_0^1\{\hat f_{YXW}[\bar g(x),x,w]-f_{YXW}[\bar g(x),x,w]\}^2\,dx\\&\qquad\times\int_0^1[\hat g(x)-g(x)]^2\,dx\bigg)dw\\&\quad=\int_0^1\int_0^1\{\hat f_{YXW}[\bar g(x),x,w]-f_{YXW}[\bar g(x),x,w]\}^2\,dx\,dw\,\|\hat g-g\|^2.\end{aligned}$$

But

$$\begin{aligned}&\int_0^1\int_0^1\{\hat f_{YXW}[\bar g(x),x,w]-f_{YXW}[\bar g(x),x,w]\}^2\,dx\,dw\\&\quad=O_p[n^{-(2\beta-1)/(2\beta+\alpha)}]+\|\hat g-g\|^2o_p(1)\end{aligned}$$

by Assumptions 2 and 5(b), thereby yielding (g). Finally, (h) can be proved by combining (a) with arguments similar to those used to prove (g). The lemma is now proved because the foregoing arguments hold uniformly over $H\in\mathcal{H}$. *Q.E.D.*

PROOF OF THEOREM 2: The theorem follows by combining the results of Lemmas A.1–A.3 with $L\|\omega_g\|<1$. *Q.E.D.*

A.3. PROOF OF THEOREM 3

It suffices to find a sequence of finite-dimensional models $\{g_n\}\in\mathcal{H}$ for which

$$\liminf_{n\to\infty}\mathbf{P}_H[\|\tilde g_n-g_n\|^2>Dn^{-(2\beta-1)/(2\beta+\alpha)}]>0.$$

To this end, let m denote the integer part of $n^{-1/(2\beta+\alpha)}$ and let f_{XW} denote the density of (X, W). Let $r_w = (2\beta+\alpha-1)/2$. Assume that $f_{XW}(x, w) \leq C$ for all $(x, w) \in [0, 1]^2$ and some constant $C < \infty$. Let

$$Y = g_n(X) + U,$$

where U is independent of (X, W) and $P(U \leq 0) = q$. Let F_U and f_U, respectively, denote the distribution function and density of U. Assume that $f_U(0) > 0$ and that F_U is twice continuously differentiable everywhere with $|F_U''(u)| < M$ for all u and some $M < \infty$. Define the operator Q on $L_2[0, 1]$ by

$$(\Pi g)(x) = \int_0^1 \pi(x, z) g(z)\, dz$$

for any $g \in L_2[0, 1]$, where

$$\pi(x, z) = f_U(0)^2 \int_0^1 f_{XW}(x, w) f_{XW}(z, w)\, dw.$$

Let $\{\lambda_j, \phi_j : j = 1, 2, \ldots\}$ denote the orthonormal eigenvalues and eigenvectors of Π ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots > 0$. Assume that $j^\alpha \lambda_j$ is bounded away from 0 and 1 for all j. Set

$$g_n(x) = \theta \sum_{j=m}^{\infty} j^{-\beta} \phi_j(x)$$

for some finite, constant $\theta > 0$. Then for any $h \in L_2[0, 1]$,

$$(\mathcal{T} h)(w) = \int_0^1 F_U[h(x) - g_n(x)] f_{XW}(x, w)\, dx,$$

and the Fréchet derivative of $\mathcal{T}$ at g_n is

$$(T_{g_n} h)(w) = f_U(0) \int_0^1 f_{XW}(x, w)[h(x) - g_n(x)]\, dx.$$

Assumption 6 is satisfied with $L = MC$ whenever $\theta > 0$ is sufficiently small.

Now let $\hat{\theta}$ be an estimator of θ. Then

$$\hat{g}(x) \equiv \hat{\theta} \sum_{j=m}^{\infty} j^{-\beta} \phi_j(x)$$

is an estimator of $g_n(x)$. Moreover,

$$\|\hat{g} - g_n\|^2 = (\hat{\theta} - \theta)^2 R_n, \tag{A25}$$

where $R_n = \sum_{j=m}^{\infty} j^{-2\beta}$. Note that $n^{(2\beta-1)/(2\beta+\alpha)} R_n$ is bounded away from 0 and 1 as $n \to \infty$. In addition, f_W is estimated by

$$\hat{f}_w(w) = q^{-1}(\mathcal{T}\hat{g})(w).$$

Define $\psi_j = T_g\phi_j/\|T_g\phi_j\|$. Then a Taylor series approximation and singular value expansion give

$$\begin{aligned}
&\hat{f}_W(w) - f_W(w) \\
&\quad = q^{-1}(\hat{\theta}-\theta)\sum_{j=m}^{\infty} j^{-\beta}(T_g\phi_j)(w) + (\hat{\theta}-\theta)^2 O[n^{-(2\beta-1)/(2\beta+\alpha)}] \\
&\quad = q^{-1}(\hat{\theta}-\theta)\sum_{j=m}^{\infty} j^{-\beta}\lambda_j^{1/2}\psi_j(w) + (\hat{\theta}-\theta)^2 O[n^{-(2\beta-1)/(2\beta+\alpha)}].
\end{aligned}$$

Now

$$n^{(2\beta+\alpha-1)/(2\beta+\alpha)}\left\|\sum_{j=m}^{\infty} j^{-\beta}\lambda_j^{1/2}\psi_j\right\|^2$$

is bounded away from 0 and ∞ as $n \to \infty$. Therefore, there is a finite constant $C_\theta > 0$ such that

$$(\hat{\theta}-\theta)^2 \geq C_\theta n^{(2\beta+\alpha-1)/(2\beta+\alpha)}\|\hat{f}_W - f_W\|^2. \tag{A26}$$

Combining (A25) and (A26) shows that there is a finite constant $C_g > 0$ such that

$$n^{(2\beta-1)/(2\beta+\alpha)}\|\hat{g} - g_n\|^2 \geq C_g n^{(2\beta+\alpha-1)/(2\beta+\alpha)}\|\hat{f}_W - f_W\|^2. \tag{A27}$$

The theorem now follows from (A27) and the observation that with $r_w = (2\beta + \alpha - 1)/2$, $O_p[n^{(2\beta+\alpha-1)/(2\beta+\alpha)}]$ is the fastest possible minimax rate of convergence of $\|\hat{f}_W - f_W\|^2$. *Q.E.D.*

A.4. PROOF OF (3.5)

Rewrite (3.4) as

$$L < \left(\sum_{j=1}^{\infty}\frac{b_j^2}{\lambda_j}\right)^{-1}. \tag{A28}$$

By a Taylor series expansion,

$$\mathcal{T}(g_1) - \mathcal{T}(g_2) - T_{g_2}(g_1 - g_2)$$
$$= 0.5 \int_0^1 \frac{\partial f_{YXW}[\bar{g}(x), x, w]}{\partial y} [g_1(x) - g_2(x)]^2 \, dx,$$

where $\bar{g}$ is between g_1 and g_2. Therefore, it follows from (3.3) that

$$L < \sup_{y,x,w} \left| \frac{\partial f_{YXW}(y, x, w)}{\partial y} \right|. \tag{A29}$$

It follows from (A28) and (A29) that (3.5) is a sufficient condition for (3.4). *Q.E.D.*

A.5. AN ADDITIONAL FIGURE FOR MONTE CARLO EXPERIMENTS

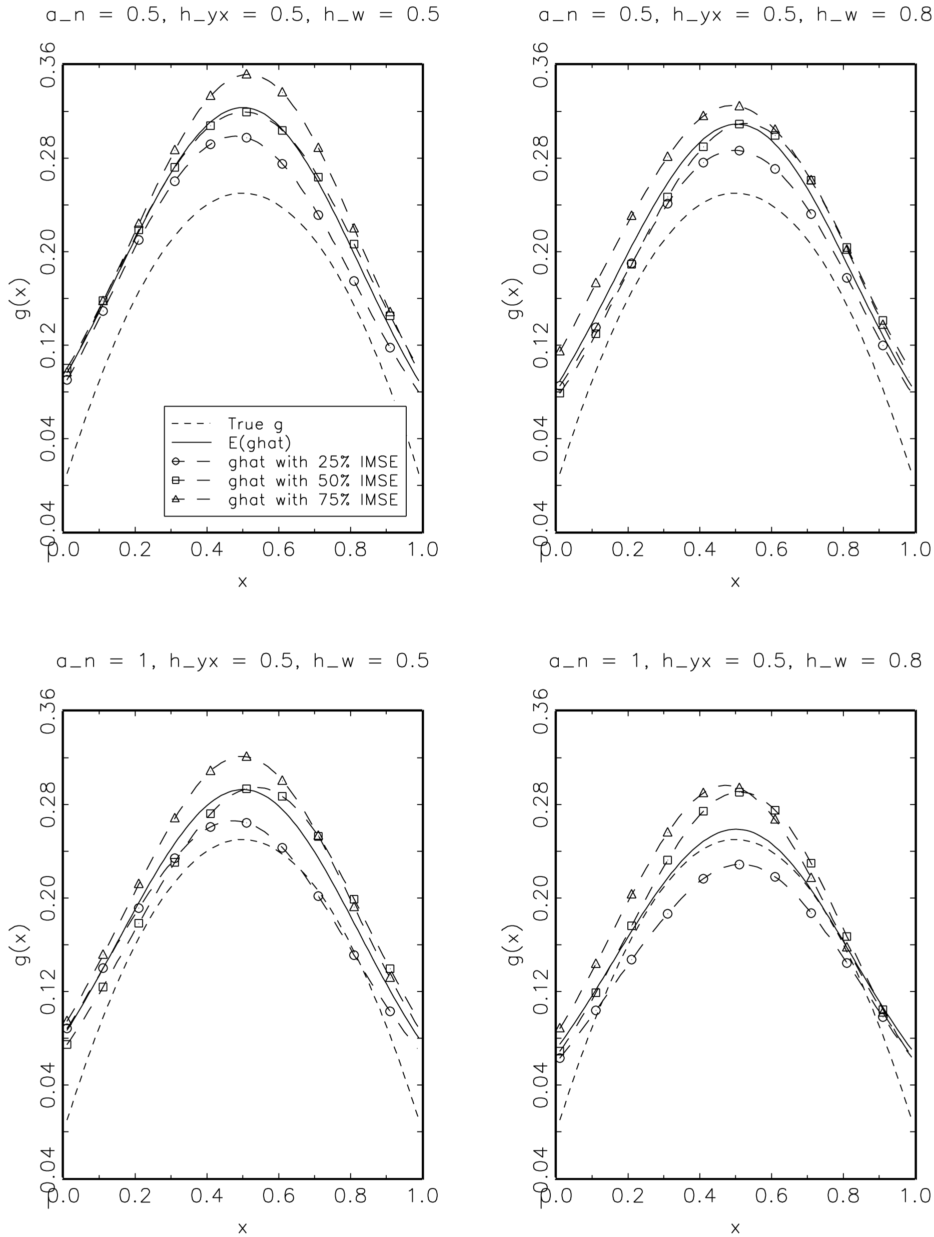

FIGURE A1.—Monte Carlo results for $n = 800$.

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