SUPPLEMENT TO "SPECULATION IN STANDARD AUCTIONS WITH RESALE"

(*Econometrica*, Vol. 74, No. 3, May, 2006, 753–769)

BY ROD GARRATT AND THOMAS TRÖGER

We extend the model presented in Garratt and Tröger (2006) to environments with multiple symmetric independent private-value bidders and prove the results stated in Section 5 of that paper. Section 2 deals with first-price and Dutch auctions. Section 3 covers second-price auctions. Section 4 explains that the second-price auction results remain valid for English auctions.

KEYWORDS: Standard auctions, speculation, resale, efficiency.

1. EXTENDING THE MODEL TO MULTIPLE PRIVATE-VALUE BIDDERS

WE CONSIDER ENVIRONMENTS with $n \geq 2$ risk-neutral bidders $i = 1, \ldots, n$, called *regular bidders*, who are interested in consuming a single indivisible private good, and an additional risk-neutral bidder s, called the *speculator*. Let $I = \{1, \ldots, n\}$. Let $\tilde{\theta}_i \in [0, 1]$ denote the use value of bidder $i \in I$ for the good and let $\tilde{\theta}_s = \theta_s = 0$ denote the speculator's use value for the good. For all $i \in I \cup \{s\}$, let $\tilde{\theta}_{-i}^{(1)} = \max_{j \in I \setminus \{i\}} \tilde{\theta}_j$ denote the random variable for the highest use value among bidders other than i. The random variables $\tilde{\theta}_1, \ldots, \tilde{\theta}_n$ are stochastically independent and each $\tilde{\theta}_i$ $(i \in I)$ is distributed according to the same distribution F. We assume that F has a density f that is positive and continuous on $[0, 1]$ and identically 0 elsewhere. Because each bidder privately learns her realized use value before the interaction begins, we say, following Vickrey (1961), that the bidders $1, \ldots, n$ are symmetric with independent private values (SIPV). We make the standard assumption that F has a weakly increasing hazard rate.

ASSUMPTION 1: *The mapping* $\theta \mapsto f(\theta)/(1 - F(\theta))$ *is weakly increasing on* $[0, 1)$.

We consider a two-period interaction. In period 1, the good is offered via a sealed-bid first-price auction or second-price auction without reserve price (our analysis extends to English and Dutch auctions; see below). The highest bidder becomes the new owner of the good. Our results do not depend on the tieing rule, but to simplify some proofs we assume that the speculator loses all ties. The bidder who wins in period 1 either consumes the good in period 1 or offers the good for resale in period 2; if she fails to resell the good, she consumes it in period 2. Period-2 payoffs are discounted according to a common factor $\delta \in (0, 1)$.

Modelling Period 1*—The Auction*

Because our focus is on the impact of the speculator and because the regular bidders are ex ante identical, we will focus on equilibria such that all regular bidders use the same bid function β in period 1. This bid function is assumed to be strictly increasing in the winning range (this simplifies the computation of post-auction beliefs). Moreover, any regular-bidder type who does not expect to ever win the auction does not participate (formally, we assume such a type bids 0, but we will show that with bid 0 she never wins).[1]

ASSUMPTION 2: *In equilibrium, all regular bidders use the same bid function* β *in period* 1. *For all* $\theta, \theta' \in [0, 1]$ *with* $\theta > \theta'$, *we have* $\beta(\theta) > \beta(\theta')$ *if the bid* $\beta(\theta')$ *wins in equilibrium with positive probability and we have* $\beta(\theta') = 0$ *otherwise*.

For later use, let us introduce some notation for random vectors of bids and highest-order statistics. For all $i \in I$, let $\tilde{b}_i = \beta(\tilde{\theta}_i)$ denote the random variable for i's bid. Let $\tilde{b}_s$ denote an independent random variable for the speculator's, possibly randomized, bid. For all $i \in I \cup \{s\}$, let $\tilde{\mathbf{b}}_{-i} = (\tilde{b}_j)_{j \in (I \cup \{s\}) \setminus \{i\}}$ denote the random vector of bids by bidders other than i and let $\tilde{b}^{(1)}_{-i} = \max_{j \in (I \cup \{s\}) \setminus \{i\}} \tilde{b}_j$ denote the random variable for the highest among these bids. For all $i, j \in I \cup \{s\}$, let $\tilde{\mathbf{b}}_{-i-j} = (\tilde{b}_k)_{k \in (I \cup \{s\}) \setminus \{i,j\}}$ denote the random vector of bids by bidders other than i and j, and let $\tilde{b}^{(1)}_{-i-j} = \max_{k \in (I \cup \{s\}) \setminus \{i,j\}} \tilde{b}_k$ denote the random variable for the highest among these bids.

The next assumption extends the bid revelation assumptions from Garratt and Tröger (2006).

ASSUMPTION 3: *After a first-price auction, the winner's bid becomes public*; *the losers' bids remain private*. *After a second-price auction, the losers' bids become public*; *the winner's bid remains private*.

These bid revelation assumptions make the first-price auction setting strategically equivalent to the Dutch auction setting. Concerning the second-price auction, the bid revelation assumptions are such that after the auction the same information is revealed as after an English auction. In Section 4 we show that our second-price auction results remain valid in the English auction setting.[2]

By Assumption 3, all bidders make the same bid observations. It is thus reasonable to assume that any two bidders have identical post-auction beliefs

[1] As for further regularity properties of the bid function β, note that we will construct equilibria in the second-price auction case where β is not continuous. In the first-price auction case, there will be a unique equilibrium, where β is always continuous, but not necessarily differentiable.

[2] In the second-price auction setting, we expect that our equilibria remain valid even if the initial seller does not announce *any* information about the bids. This is because in the equilibria that we will construct, the only dependence of the outcome of the resale market on the auction bids is via the highest losing bid, and in a second-price auction, the auction winner learns the highest losing bid from the price she pays.

about a third bidder, even if an unexpected event occurs so that Bayesian updating is not possible, and to assume that it is commonly believed that all bidders have these beliefs. Moreover, it is reasonable to maintain stochastic independence of the post-auction beliefs across bidders because use values are stochastically independent ex ante. Hence, given any auction outcome, we can assume that post-auction beliefs are summarized by a vector of distributions $\mathbf{\Pi} = (\Pi_j)_{j \in I}$, where Π_j represents the post-auction belief about bidder j's use value. Note that $\mathbf{\Pi}$ includes the beliefs about the resale seller's use value unless the resale seller is the speculator.

The post-auction beliefs that occur in our equilibrium constructions are represented by distributions of the following kind. For any $\underline{\theta}, \overline{\theta} \in [0, 1]$ such that $\underline{\theta} \leq \overline{\theta}$, consider a bidder who infers from the observed bidding behavior in period 1 that bidder i's ($i \in I$) use value belongs to the interval $[\underline{\theta}, \overline{\theta}]$. According to Bayes rule, the resulting posterior distribution function $\hat{F}_{[\underline{\theta}, \overline{\theta}]}$ for bidder i's use value is given by

$$\hat{F}_{[\underline{\theta}, \overline{\theta}]}(\theta_i) = \begin{cases} \dfrac{F(\theta_i) - F(\underline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})}, & \text{if } \theta_i \in [\underline{\theta}, \overline{\theta}), \\ 1, & \text{if } \theta_i \geq \overline{\theta}, \\ 0, & \text{if } \theta_i < \underline{\theta}. \end{cases}$$

Note that $\hat{F}_{[\underline{\theta}, \overline{\theta}]}$ is a point distribution if $\overline{\theta} = \underline{\theta}$. If $\overline{\theta} > \underline{\theta}$, the distribution $\hat{F}_{[\underline{\theta}, \overline{\theta}]}$ has (on its support $[\underline{\theta}, \overline{\theta}]$) the same regularity properties as F: a positive and continuous density and an increasing hazard rate. By Assumption 2, the case $\overline{\theta} > \underline{\theta}$ becomes relevant only if an off-equilibrium bid or bid 0 is observed.

Modelling Period 2—The Resale Market

The period-2 environment where resale takes place is determined by the identity $i \in I \cup \{s\}$ and use value θ_i of the resale seller (i.e., the auction winner), and by the post-auction beliefs $\mathbf{\Pi} = (\Pi_j)_{j \in I}$. Any tuple $(\mathbf{\Pi}, \theta_i, i)$ is called a *resale environment*. A *resale mechanism* is a game form to be played in period 2 with players $I \cup \{s\}$, where an outcome is a probability distribution over who gets the good and a vector of side payments among the players, together with an equilibrium such that the participation constraints are satisfied; if multiple equilibria exist, we assume an equilibrium that is best for the resale seller is played. A resale mechanism is called *optimal* if there exists no other resale mechanism that yields a higher expected payoff for the resale seller. For any resale environment $(\mathbf{\Pi}, \theta_i, i)$, let $\hat{\mathcal{M}}(\mathbf{\Pi}, \theta_i, i)$ denote the resale mechanism used.

ASSUMPTION 4: *For all resale environments* $(\mathbf{\Pi}, \theta_i, i)$, *where every standard auction with an optimal reserve price is an optimal resale mechanism*, *the resale mechanism* $\hat{\mathcal{M}}(\mathbf{\Pi}, \theta_i, i)$ *is a standard auction with an optimal reserve price*.

Observe that, according to this assumption, in any resale environment where not all four standard auctions are optimal, *any* resale mechanism may be used. Hence, Assumption 4 covers two important special cases: the resale seller may always use any standard auction with an optimal reserve price[3] or she may always use an optimal resale mechanism.[4] The first case is attractive because the cost of implementing a standard auction is negligible in many applications. The second case is attractive because it allows the resale seller to be fully rational concerning the choice of a resale mechanism if any mechanism can be costlessly implemented.

The reason that Assumption 4 is sufficient for our results is that in the resale environments that are relevant for our equilibrium constructions, any standard auction with an optimal reserve price is an optimal resale mechanism. Other resale environments become relevant if a bidder deviates from her equilibrium bid in an attempt to win the auction more often and subsequently offer the good for resale, but we will show that even if the deviating bidder appropriates the entire surplus in the resale market, no such deviation is profitable.

Equilibrium Resale Environments

The following two classes of resale environments are relevant for our equilibrium constructions: (i) the resale environments where the resale seller is certain about the maximum use value among all bidders; (ii) the resale environments where the resale seller is the speculator and, according to the post-auction beliefs, the n regular bidders have independent private values, where each bidder's value is distributed according to $\hat{F}_{[0,\hat{\theta}]}$ for some $\hat{\theta} \in (0, 1]$.

In class (i), any standard auction with a reserve price equal to the maximum use value is an optimal resale mechanism because the resale seller appropriates the entire surplus. The equilibrium resale allocation is that the resale seller keeps the good if she has the maximum use value and otherwise sells it to the bidder with the maximum use value. In class (ii), it is well known (cf. Myerson (1981)) that any standard auction with reserve price $\hat{r}(\hat{\theta})$ implicitly defined by $\hat{r}(\hat{\theta}) = (F(\hat{\theta}) - F(\hat{r}(\hat{\theta})))/f(\hat{r}(\hat{\theta}))$ is an optimal resale mechanism (observe that the function $\hat{r}$ is uniquely determined on $(0, 1]$, and is strictly increasing and continuous). The equilibrium resale allocation is that the bidder with the

[3]Strictly speaking, an optimal reserve price may not always exist (because post-auction beliefs may be represented by nonsmooth distributions). In such a resale environment, Assumption 4 allows any resale mechanism to be used. In particular, a standard auction with any reserve price that is arbitrarily close to being optimal may be used.

[4]If an optimal resale mechanism does not exist, the resale seller may use any resale mechanism that is arbitrarily close to being optimal. Observe that, due to independent private values in the resale market, an optimal resale mechanism exists and can by computed using Myerson's (1981) methods as long as the resale seller's use value is commonly known and the beliefs about the other bidders are represented by smooth distributions.

highest value obtains the good, unless her value is below $\hat{r}(\hat{\theta})$, in which case the resale seller keeps the good. In both classes (i) and (ii) of resale environments, we use the notation $\mathcal{S}(\check{r})$ to denote any standard auction with any reserve price $\check{r} \geq 0$.

Throughout the rest of this section, we consider the resale environments of class (ii). We derive auxiliary technical results about the resale mechanism $\mathcal{S}(\hat{r}(\hat{\theta}))$ for all $\hat{\theta} \in (0, 1]$. Denote by $\hat{P}_{\hat{\theta}}(\theta)$ the expected payment of a regular bidder with use value $\theta \in [0, 1]$ in the mechanism $\mathcal{S}(\hat{r}(\hat{\theta}))$ and denote by $\hat{Q}_{\hat{\theta}}(\theta)$ the probability that the bidder obtains the good.[5] Properties of the functions $\hat{P}$ and $\hat{Q}$ are summarized in the following lemma.

LEMMA 1: *Consider any* $\hat{\theta} \in (0, 1]$. *For all* $\theta \in [0, \hat{r}(\hat{\theta}))$, *we have* $\hat{P}_{\hat{\theta}}(\theta) = 0$ *and* $\hat{Q}_{\hat{\theta}}(\theta) = 0$. *For all* $\theta \in [\hat{r}(\hat{\theta}), \hat{\theta}]$,

$$\frac{\hat{P}_{\hat{\theta}}(\theta)}{\hat{Q}_{\hat{\theta}}(\theta)} = \theta - \int_{\hat{r}(\hat{\theta})}^{\theta} \frac{F(\theta')^{n-1}}{F(\theta)^{n-1}}\, d\theta', \tag{1}$$

$$\frac{\hat{P}_{\hat{\theta}}(\theta)}{\hat{Q}_{\hat{\theta}}(\theta)} \text{ is weakly increasing in } \theta \text{ and } \hat{\theta}, \tag{2}$$

$$\hat{Q}_{\hat{\theta}}(\theta) = \frac{F(\theta)^{n-1}}{F(\hat{\theta})^{n-1}}. \tag{3}$$

For all $\theta \in [\hat{\theta}, 1]$,

$$\hat{Q}_{\hat{\theta}}(\theta) = 1, \quad \hat{P}_{\hat{\theta}}(\theta) = \hat{P}_{\hat{\theta}}(\hat{\theta}). \tag{4}$$

PROOF: Formulas (1), (3), and (4) are standard. To see (2), use that $\hat{r}$ is increasing and that the derivative of $\hat{P}_{\hat{\theta}}(\theta)/\hat{Q}_{\hat{\theta}}(\theta)$ with respect to θ is nonnegative. *Q.E.D.*

The resale seller's expected revenue in the mechanism $\mathcal{S}(\hat{r}(\hat{\theta}))$ is denoted $M(\hat{\theta})$; also let $M(0) = 0$. Observe that M is strictly increasing. Other properties of M are summarized in the following lemma.

LEMMA 2: *For all* $\hat{\theta} \in (0, 1]$,

$$M(\hat{\theta}) = \int_{\hat{r}(\hat{\theta})}^{\hat{\theta}} \left(\theta - \frac{F(\hat{\theta}) - F(\theta)}{f(\theta)}\right) \frac{d(F(\theta)^n)}{F(\hat{\theta})^n}, \tag{5}$$

[5]Observe that according to the post-auction beliefs, the probability equals 0 that a bidder type in $(\hat{\theta}, 1]$ participates in the resale market. It is nevertheless important to define the functions $\hat{P}_{\hat{\theta}}(\theta)$ and $\hat{Q}_{\hat{\theta}}(\theta)$ for $\theta > \hat{\theta}$, because participation of a type $\theta > \hat{\theta}$ may arise from a deviation in period 1.

$$M'(\hat{\theta}) = n\frac{f(\hat{\theta})}{F(\hat{\theta})}\left(\hat{\theta} - M(\hat{\theta}) - \int_{\hat{r}(\hat{\theta})}^{\hat{\theta}} \frac{F(\theta)^{n-1}}{F(\hat{\theta})^{n-1}}\, d\theta\right). \tag{6}$$

Moreover,

$$M'(0) < \frac{n-1}{n}. \tag{7}$$

The function M is Lipschitz continuous on $[0,1]$ *and its derivative M' is continuous*.

PROOF: Formula (5) is standard from Myerson (1981), while (6) follows from standard differentiation rules. By differentiability, $F(\theta) = f(0)\theta + o(\theta)$. Using this and $\hat{r}(\hat{\theta})/\hat{\theta} \to 1/2$ as $\hat{\theta} \to 0$, (5) can be used to show

$$M'(0) = \lim_{\hat{\theta}\to 0} \frac{M(\hat{\theta})}{\hat{\theta}} = \frac{n-1}{n+1} + \frac{1}{2^n(n+1)}, \tag{8}$$

which implies (7). Using (6) and (8), it can be confirmed that $\lim_{\hat{\theta}\to 0} M'(\hat{\theta}) = M'(0)$. Therefore, M' is continuous on $[0,1]$. Hence, M' is bounded above on $[0,1]$, which implies Lipschitz continuity of M. *Q.E.D.*

The final result in this section shows that the payment of the highest type of bidder who participates in the resale market is higher than the resale seller's expected revenue.

LEMMA 3: *The following holds*:

$$\forall\, \hat{\theta} \in (0,1], \quad \hat{P}_{\hat{\theta}}(\hat{\theta}) > M(\hat{\theta}). \tag{9}$$

PROOF: Using Lemma 1, we find

$$\begin{aligned} M(\hat{\theta}) &= n\int_{\hat{r}(\hat{\theta})}^{\hat{\theta}} \hat{P}_{\hat{\theta}}(\theta)\frac{dF(\theta)}{F(\hat{\theta})} = n\int_{\hat{r}(\hat{\theta})}^{\hat{\theta}} \frac{\hat{P}_{\hat{\theta}}(\theta)}{\hat{Q}_{\hat{\theta}}(\theta)}\hat{Q}_{\hat{\theta}}(\theta)\frac{dF(\theta)}{F(\hat{\theta})} \\ &\overset{(3)}{=} \int_{\hat{r}(\hat{\theta})}^{\hat{\theta}} \frac{\hat{P}_{\hat{\theta}}(\theta)}{\hat{Q}_{\hat{\theta}}(\theta)}\frac{d(F(\theta)^n)}{F(\hat{\theta})^n} \overset{(2)}{\leq} \int_{\hat{r}(\hat{\theta})}^{\hat{\theta}} \frac{\hat{P}_{\hat{\theta}}(\hat{\theta})}{\hat{Q}_{\hat{\theta}}(\hat{\theta})}\frac{d(F(\theta)^n)}{F(\hat{\theta})^n} \\ &\overset{(4)}{=} \hat{P}_{\hat{\theta}}(\hat{\theta})\left(1 - \frac{F(\hat{r}(\hat{\theta}))^n}{F(\hat{\theta})^n}\right) < \hat{P}_{\hat{\theta}}(\hat{\theta}) \end{aligned}$$

because $\hat{r}(\hat{\theta}) < \hat{\theta}$. *Q.E.D.*

Observe that Lemma 3 would be trivial if $n = 1$ because nobody would pay more than the highest participating type. If $n > 1$, the expected payment of

any given bidder type is collected n times if all resale bidders have this type. The proof instead works with any given type's expected payment *divided by her winning probability*. Because this divided payment is increasing in type and because with positive probability nobody wins, the highest type's divided payment exceeds the seller's revenue. However, the highest type wins for sure, which completes the proof.

2. FIRST-PRICE AND DUTCH AUCTIONS WITH RESALE

In this section we construct and discuss the unique perfect Bayesian equilibrium of a first-price auction with resale. Proposition 1 describes the equilibrium. Proposition 2 identifies parameter constellations for which the speculator plays an active role. Proposition 3 evaluates the impact of a resale opportunity on initial seller revenue.

Let β denote bidder i's ($i \in I$) bid in the first-price auction as a function of her use value. Let H denote the probability distribution for the speculator's bid $\tilde{b}_s$.

To define post-auction beliefs, consider any bidder $j \in I$. Let $i \in I \cup \{s\}$ denote the label of the winner and let $b_i \geq 0$ denote the winner's bid. Then the probability distribution $\Pi_j(\cdot|i, b_i)$ denotes the post-auction belief about j's use value held by bidders other than j.

Let $\mathcal{M}(i, b_i, \theta_i)$ denote the resale mechanism used by the resale seller $i \in I \cup \{s\}$ after a first-price auction when $b_i \geq 0$ denotes i's auction winning bid and i's use value is θ_i. For all $i, j \in I \cup \{s\}$ with $j \neq i$, all $b_i \geq 0$, and all θ_i, θ_j, let $P_j(i, b_i, \theta_i, \theta_j)$ denote the net expected transfer from bidder j of type θ_j to the other bidders (including the transfer to i) in the mechanism $\mathcal{M}(i, b_i, \theta_i)$. Let $Q_j(i, b_i, \theta_i, \theta_j)$ denote the probability that bidder j obtains the good. Let $P(i, b_i, \theta_i)$ denote the expected transfer *to* the resale seller i and let $Q(i, b_i, \theta_i)$ denote the probability that the resale seller keeps the good.

For all $i \in I$, bidder i's expected payoff when she bids $b_i \geq 0$ and has the use value θ_i equals

$$
\begin{aligned}
&u_i(b_i, \theta_i) \\
&\quad = E\Bigg[\big(-b_i + \max\{\theta_i, \delta(\theta_i Q(i, b_i, \theta_i) + P(i, b_i, \theta_i))\}\big)\mathbb{1}_{w(b_i, \tilde{\mathbf{b}}_{-i})=i} \\
&\qquad\qquad + \sum_{j \neq i} \delta\big(\theta_i Q_i(j, \tilde{b}_j, \tilde{\theta}_j, \theta_i) - P_i(j, \tilde{b}_j, \tilde{\theta}_j, \theta_i)\big)\mathbb{1}_{w(b_i, \tilde{\mathbf{b}}_{-i})=j}\Bigg],
\end{aligned}
$$

where w denotes the period-1 winner as a function of the bid profile and where the max term reflects the condition that after winning in period 1, bidder i decides optimally whether to consume the good or to offer it for resale. The speculator's payoff when she bids $b_s \geq 0$ is given by

$$
u_s(b_s) = E\big[(-b_s + \delta P(s, b_s, \theta_s))\mathbb{1}_{w(b_s, \tilde{\mathbf{b}}_{-s})=s}\big].
$$

The equilibrium conditions are that post-auction beliefs about auction losers are determined by Bayes rule whenever possible (10), that the resale mechanism is chosen according to Assumption 4 (11), and that period-1 behavior is optimal (12) and (13).

DEFINITION 1: *A tuple* $(\beta, H, \mathcal{M})$ *is a quasisymmetric regular equilibrium of the first-price auction with resale if there exists a belief system* $(\Pi_j(\cdot|i, b_i))_{j\in I, i\in I\cup\{s\}, b_i\geq 0}$ *such that the following conditions hold*:

$$\forall\, i \in I \cup \{s\}, b_i > 0, j \in I \setminus \{i\},$$
$$\Pi_j(\cdot|i, b_i) = \hat{F}_{\beta^{-1}([0,b_i])} \quad \textit{if} \quad [0, b_i] \cap \beta([0, 1]) \neq \emptyset, \tag{10}$$

$$\forall\, i \in I \cup \{s\}, b_i \geq 0, \theta_i, \quad \mathcal{M}(i, b_i, \theta_i) = \hat{\mathcal{M}}\big((\Pi_j(\cdot|i, b_i))_{j\in I}, \theta_i, i\big), \tag{11}$$

$$\forall\, i \in I, \theta_i, \quad \beta(\theta_i) \in \arg\max_{b_i\geq 0} u_i(b_i, \theta_i), \tag{12}$$

$$\Pr\Big[\tilde{b}_s \in \arg\max_{b_s\geq 0} u_s(b_s)\Big] = 1. \tag{13}$$

This equilibrium concept is in the spirit of perfect Bayesian equilibrium, combined with the symmetry and regularity restrictions formulated in Assumption 2, and with two presentational simplifications. First, we have omitted a condition on the post-auction beliefs about the auction winner because it would play no role for our analysis. Second, condition (10) excludes the case $b_i = 0$; the appropriate Bayesian updating condition for this case would depend on the auction tieing rule (whereas, by Assumption 2, ties occur with probability 0 if $b_i > 0$), but the case $b_i = 0$ is irrelevant for our analysis because it will turn out that in equilibrium only positive bids can win the auction with positive probability.

Additional notation is needed to state the main result, Proposition 1, which describes the equilibrium. For all $\theta \in (0, 1]$ and $b \in \mathbb{R}$, define

$$K(\theta, b) = \begin{cases} N(\theta, b) & \text{if } b > \delta M(\theta), \\ \max\{\delta M'(\theta), N(\theta, b)\} & \text{if } b \leq \delta M(\theta), \end{cases} \tag{14}$$

where

$$N(\theta, b) = (n - 1)\frac{f(\theta)}{F(\theta)}(\theta - b). \tag{15}$$

Observe that in a first-price auction with n regular bidders and *without* a resale opportunity, $N(\theta, b)$ is the slope of the equilibrium bid function at θ when b equals type θ's equilibrium bid. Hence, $N(\theta, b)$ represents the competition among the regular bidders in the absence of resale. The function $K(\theta, b)$

equals $N(\theta, b)$ if $b > \delta M(\theta)$; that is, if b and θ are such that, if resale is possible, a speculator who comes in with a bid of b and wins against the regular-bidder types up to θ makes a loss. If type θ's bid b is such that the speculator breaks even ($b = \delta M(\theta)$), then $K(\theta, b)$ is the larger of $N(\theta, b)$ and the slope $\delta M'(\theta)$ that keeps the speculator at the break-even point. In summary, $K(\theta, b)$ is the smallest slope that respects the competition among the regular bidders without resale and does not allow the speculator to make a profit via resale.

For any $\overline{b} > 0$, consider a strictly increasing Lipschitz continuous function $\phi : [0, \overline{b}] \to [0, 1]$ such that $\phi(b) > b \geq \delta M(\phi(b))$ for all $b > 0$. For all $b \in (0, \overline{b}]$ where the derivative $\phi'(b)$ exists, define

$$L_\phi(b) = \begin{cases} \dfrac{R_\phi(b)}{S_\phi(b)} & \text{if } b = \delta M(\phi(b)), \\ 0 & \text{if } b > \delta M(\phi(b)), \end{cases} \tag{16}$$

where

$$\begin{aligned} R_\phi(b) &= 1 - N(\phi(b), b)\phi'(b), \\ S_\phi(b) &= (1 - \delta)\phi(b) - b + \delta \hat{P}_{\phi(b)}(\phi(b)). \end{aligned} \tag{17}$$

Observe that L_ϕ is well defined because $b = \delta M(\phi(b))$ implies $S_\phi(b) > 0$ by (9). Also observe that $L_\phi(b) = 0$ if b is such that the speculator would make a loss if she did bid b and did win against the types in $[0, \phi(b)]$.

Proposition 1 describes the unique equilibrium.[6] Condition (18) establishes a differential equation for the regular bidders' bid function β. The differential equation, which holds almost everywhere according to the Lebesgue measure, reflects the bidding competition among the regular bidders and a no-profit condition for the speculator (recall the properties of the function K explained above). The speculator randomizes her bid according to (19), in which the distribution H is constructed so that it is optimal for the regular bidders to use the bid function defined by (18). Using (16) one sees that the support of H is confined to the points θ where $\beta(\theta) = \delta M(\theta)$ so that the speculator does not make a loss.[7] The post-auction beliefs (20) of the winning bidder are that

[6]The uniqueness property differs from "essential uniqueness" in Garratt and Tröger (2006), where multiple optimal resale prices cannot be excluded because value distributions F with a nonincreasing hazard rate are allowed; cf. Garratt and Tröger (2006, footnote 2) and the text below Proposition 1 of that paper.

[7]The feature that the regular bidders use a strictly increasing bid function while an additional bidder with no private information randomizes over a subset of the regular bidders' bid range also appears in Martinez (2002). She constructs an equilibrium for first-price auctions without resale in an environment with three or more regular bidders and one additional bidder with a commonly known valuation. The differences between Martinez' equilibrium and ours are parallel to the differences between Vickrey's (1961, Appendix III) equilibrium and our equilibrium in the two-bidder case; cf. Garratt and Tröger (2006, footnote 7).

the highest type among the losing regular bidders is the type that would have resulted in a tie. Hence, if one of the regular bidders wins the auction after making her equilibrium bid, she believes that she has the highest use value in the market and thus consumes the good. Condition (21) states that if the auction winner is the speculator, then her resale mechanism is a standard auction with a reserve price that is optimal given her post-auction beliefs.

PROPOSITION 1: *For any regular-bidder number $n \geq 2$, discount factor $\delta \in (0,1)$, and distribution F that satisfies Assumption* 1, *the first-price auction with resale has a unique quasisymmetric regular equilibrium $(\beta, H, \mathcal{M})$. The equilibrium satisfies conditions* (18), (19), *and* (21) *and is supported by beliefs that satisfy* (20):

$$\beta \text{ is Lipschitz}, \quad \beta(0) = 0, \quad \beta'(\theta) = K(\theta, \beta(\theta)) \quad \text{a.e. } \theta \in (0,1), \tag{18}$$

$$\forall b \in (0, \beta(1)], \quad H(b) = \exp\left(-\int_b^{\beta(1)} L_\phi(b)\, db\right), \tag{19}$$

where ϕ denotes the inverse of β,

$$\forall i, b_i \geq 0, j \notin \{s, i\}, \theta_j, \quad \Pi_j(\theta_j | i, b_i) = \min\left\{\frac{F(\theta_j)}{F(\phi(b_i))}, 1\right\}, \tag{20}$$

where we define $\phi(b_i) = 1$ for all $b_i > \beta(1)$, and

$$\forall b_s > 0, \quad \mathcal{M}(s, b_s, 0) = \mathcal{S}\big(\hat{r}(\phi(b_s))\big). \tag{21}$$

PROOF: We first show uniqueness (Lemmas 4–11), then existence (Lemmas 12–15).

Lemma 4 shows that no regular bidder will bid above her value. Lemma 5 establishes a convenient representation for the speculator's payoff function. Lemma 6 shows that, with positive probability, the speculator makes arbitrarily small bids. Lemma 7 shows that the speculator's equilibrium payoff equals 0. Lemma 8 shows that the speculator's bid distribution has no atoms. Lemma 9 states that the regular bidders' equilibrium bid function is continuous. Lemma 10 states that every regular bidder with a positive use value will bid lower than her use value. Lemma 11 shows that the regular bidders' equilibrium bid function satisfies (18) and is uniquely characterized by (18), and the speculator's bid distribution is given by (19). Conditions (20) and (21) are then straightforward. This completes the equilibrium uniqueness proof.

In Lemmas 12 and 14, we apply techniques from the theory of differential inclusions to show that the differential equation (18) has a solution.[8] In doing this, we also recall in Lemma 13 the differential equation satisfied by the standard no-resale first-price auction equilibrium bid function. Lemma 15 completes the equilibrium existence proof.

[8]We thank Jörg Oechssler for help with this part of the proof.

For any bid function β satisfying Assumption 2 and all $b \in [0, \beta(1)]$, let

$$\phi(b) = \sup\{\theta_i \in [0,1] \mid \beta(\theta_i) < b\}, \tag{22}$$

where $\sup \emptyset = 0$.

LEMMA 4: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale and let ϕ be defined by* (22). *Then $\beta(\theta_i) \leq \theta_i$ for all $\theta_i \in [0,1]$ and $\phi(b) \geq b$ for all $b \in [0, \beta(1)]$.*

PROOF: By Assumption 2, $\beta(0) = 0$. Now suppose that $\beta(\theta_i) > \theta_i$ for some $\theta_i > 0$. In particular, $\beta(\theta_i) > 0$ and thus $\beta(\theta_i)$ wins with positive probability by Assumption 2. Therefore, $H(\beta(\theta_i)) > 0$ and $F(\phi(\beta(\theta_i)))^{n-1} > 0$. A deviation to the bid $b_i = \theta_i$ is profitable because

$$\begin{aligned}
&u_i(\theta_i, \theta_i) - u_i(\beta(\theta_i), \theta_i) \\
&\quad = H(\beta(\theta_i))F\big(\phi(\beta(\theta_i))\big)^{n-1}(\beta(\theta_i) - \theta_i) \\
&\qquad + \delta E\Big[\underbrace{\big(\theta_i \hat{Q}_{\phi(\tilde{b}_s)}(\theta_i) - \hat{P}_{\phi(\tilde{b}_s)}(\theta_i)\big)}_{\geq 0} \mathbb{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}, \theta_i < \tilde{b}_s \leq \beta(\theta_i)}\Big] \\
&\quad > 0.
\end{aligned}$$

To prove the second part of the lemma, fix $b > 0$ and consider any $\theta_i < b$. Then $\beta(\theta_i) \leq \theta_i < b$. Because θ_i is arbitrary, $\phi(b) \geq b$. *Q.E.D.*

LEMMA 5: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale and let ϕ be defined by* (22). *Then*

$$\forall\, b > 0, \text{ if } H(b) > 0, \text{ then } u_s(b) = F(\phi(b))^n\big(\delta M(\phi(b)) - b\big).$$

PROOF: Consider any $b > 0$ such that $H(b) > 0$ and any $\theta' < \phi(b)$. It follows that $\beta(\theta') < b$ (otherwise $\beta(\theta) \geq b$ for all $\theta > \theta'$ by Assumption 2, implying $\phi(b) \leq \theta'$). Hence, the set $\{\theta \in [0,1] \mid \beta(\theta) < b\}$ equals $[0, \phi(b))$ or $[0, \phi(b)]$. *Q.E.D.*

The following lemma shows that the speculator makes, with positive probability, arbitrarily small bids. The proof is similar to that of Garratt and Tröger (2006, Lemma 1). One supposes that the speculator's infimum equilibrium bid $\underline{b} > 0$. Then low regular-bidder types never win the auction. For some of these types it is profitable to deviate to a bid slightly above $\underline{b}$ because, combining (9) with a continuity argument, the deviating bid is smaller than the speculator's resale price.

LEMMA 6: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale. Then $H(b) > 0$ for all $b > 0$.*

PROOF: Let $\underline{b} = \inf\{b \mid H(b) > 0\}$. Suppose that $\underline{b} > 0$. Let $U_s \geq 0$ denote the speculator's equilibrium payoff. First consider the case $H(\underline{b}) = 0$ (i.e., no atom at $\underline{b}$). Then there exists a sequence $(b^m)_{m\in\mathbb{N}}$ such that $b^m \to \underline{b}$ as $m \to \infty$, $u_s(b^m) = U_s$, and $b^m > \underline{b}$ for all m. By Lemma 5,

$$\forall m, \quad U_s = F(\phi(b^m))^n\big(\delta M(\phi(b^m)) - b^m\big). \tag{23}$$

By Lemma 4, $\phi(b^m) > 0$; hence (23) implies $\delta M(\phi(b^m)) - b^m \geq 0$. Defining $\underline{\theta} = \lim_m \phi(b^m)$, it follows that $\delta M(\underline{\theta}) \geq \underline{b}$. This together with (9) implies

$$\exists\, \xi > 0, \quad \hat{P}_{\underline{\theta}}(\underline{\theta}) \geq M(\underline{\theta}) + \xi \geq \frac{\underline{b}}{\delta} + \xi. \tag{24}$$

Continuity of $\hat{P}$ by Lemma 1 together with (24) implies that there exist $b' > \underline{b}$ and $\theta' < \underline{\theta}$ such that

$$\forall\, b \in (\underline{b}, b'), \quad \hat{P}_{\phi(b)}(\theta') \geq \frac{\underline{b}}{\delta} + \frac{\xi}{2}. \tag{25}$$

Now consider any bidder $i \in I$ with type $\theta_i = \theta'$. By construction, her equilibrium bid $b_i = \beta(\theta')$ never wins. Moreover, if some bidder $j \in I \setminus \{i\}$ wins, then $b_j = \beta(\theta_j) > \underline{b} \geq b_i$, implying $\theta_j > \theta'$; i.e., bidder j will not resell to bidder i. Therefore, $Q_i(j, b_j, \theta_j, \theta') = 0$ and $P_i(j, b_j, \theta_j, \theta') = 0$. Therefore,

$$u_i(\beta(\theta'), \theta') = \delta E\big[\big(\theta'\hat{Q}_{\phi(\tilde{b}_s)}(\theta') - \hat{P}_{\phi(\tilde{b}_s)}(\theta')\big)\mathbb{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}}\big].$$

On the other hand, for all $b_i > \underline{b}$,

$$\begin{aligned} u_i(b_i, \theta') &\geq \delta E\big[\big(\theta'\hat{Q}_{\phi(\tilde{b}_s)}(\theta') - \hat{P}_{\phi(\tilde{b}_s)}(\theta')\big)\mathbb{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \tilde{b}_s > b_i}\big] \\ &\quad + (\theta' - b_i)\Pr[\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}, \tilde{b}_s \leq b_i]. \end{aligned}$$

Therefore, for all $b_i \in (\underline{b}, b')$,

$$\begin{aligned} &u_i(b_i, \theta') - u_i(\beta(\theta'), \theta') \\ &\quad \geq -\delta E\big[\big(\theta'\hat{Q}_{\phi(\tilde{b}_s)}(\theta') - \hat{P}_{\phi(\tilde{b}_s)}(\theta')\big)\mathbb{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}, \tilde{b}_s \leq b_i}\big] \\ &\qquad + (\theta' - b_i)\Pr[\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}, \tilde{b}_s \leq b_i] \\ &\quad \geq E\big[\big(\delta\hat{P}_{\phi(\tilde{b}_s)}(\theta') - b_i\big)\mathbb{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}, \tilde{b}_s \leq b_i}\big] \\ &\quad \overset{(25)}{\geq} E\left[\left(\underline{b} + \frac{\delta\xi}{2} - b_i\right)\mathbb{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \tilde{b}_s \leq b_i}\right] \\ &\quad = \left(\underline{b} + \frac{\delta\xi}{2} - b_i\right)\underbrace{\Pr[\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}, \tilde{b}_s \leq b_i]}_{\geq \Pr[\tilde{\theta}^{(1)}_{-i} < \underline{\theta}, \tilde{b}_s \leq b_i] > 0}. \end{aligned}$$

Hence, $u_i(b_i, \theta') > u_i(\beta(\theta'), \theta')$ for all $b_i > \underline{b}$ that are sufficiently close to $\underline{b}$. This contradicts (12).

In the case $H(\underline{b}) > 0$, the proof is similar. One defines $\underline{\theta} = \phi(\underline{b})$ and shows that the deviation $b_i = \underline{b}$ is profitable for some type $\theta' < \underline{\theta}$. *Q.E.D.*

LEMMA 7: *In any quasisymmetric regular equilibrium of the first-price auction with resale, the speculator's payoff equals* 0.

PROOF: Let $U_s \geq 0$ denote the speculator's equilibrium payoff. Suppose that $U_s > 0$. This implies $H(0) = 0$. Hence, by Lemma 6 there exists a sequence $(b^m)_{m\in\mathbb{N}}$ such that $b^m \to 0$ as $m \to \infty$, $u_s(b^m) = U_s$, and $b^m > 0$ for all m. By Lemma 5,

$$\forall m, \quad U_s = F(\phi(b^m))^n\big(\delta M(\phi(b^m)) - b^m\big) \leq M(\phi(b^m)).$$

Hence, $M(\underline{\theta}) \geq U_s > 0$, where $\underline{\theta} \equiv \lim_m \phi(b^m)$. Note that

$$\forall\, \theta < \underline{\theta}, \quad \beta(\theta) = 0. \tag{26}$$

By (9), $\hat{P}_{\underline{\theta}}(\underline{\theta}) > U_s$. As in the proof of Lemma 6, continuity of $\hat{P}$, by Lemma 1, now shows the existence of $b' > 0$ and $\theta' < \underline{\theta}$ such that

$$\forall\, b \in (0, b'), \quad \hat{P}_{\phi(b)}(\theta') \geq \frac{U_s}{2}.$$

Consider a bidder $i \in I$ with type $\theta_i = \theta'$. By (26), her equilibrium bid $\beta(\theta') = 0$ never wins. In the same manner as in the proof of Lemma 6, one obtains a contradiction by showing that a deviation to a small positive bid $b_i > 0$ is profitable. *Q.E.D.*

LEMMA 8: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale. Then H is continuous on* $(0, \infty)$.

PROOF: Suppose that there exists $b' > 0$ where H is not continuous; i.e., $\Pr[\tilde{b}_s = b'] > 0$. Define $\theta^m = \phi(b') - 1/m$ for all m large enough such that $\theta^m > 0$. Let $\overline{b} = \lim_{m\to\infty} \beta(\theta^m)$. We have $\overline{b} = b'$ because otherwise $\Pr[\beta(\tilde{\theta}_i) \in (\overline{b}, b')] = 0$, which would imply $u_s((b' + \overline{b})/2) > u_s(b')$.

By (9) there exists $\xi > 0$ such that $\hat{P}_{\phi(b')}(\phi(b')) \geq M(\phi(b')) + \xi \geq b'/\delta + \xi$, because otherwise $u_s(b') < 0$. By Lemma 1, the function $\hat{P}_{\phi(b')}$ is continuous at $\phi(b')$ and thus

$$\hat{P}_{\phi(b')}(\theta^m) > b'/\delta + \xi/2$$

for all large m. Therefore, for large m,

$$\begin{aligned}
&u_i(b', \theta^m) - u_i(\beta(\theta^m), \theta^m) \\
&\quad \geq H(\beta(\theta^m))F\big(\phi(\beta(\theta^m))\big)^{n-1}(\beta(\theta^m) - b') \\
&\qquad + \Pr\big[\tilde{b}_s \in (\beta(\theta^m), b')\big](-1) \\
&\qquad + \Pr[\tilde{b}_s = b']F(\phi(b'))^{n-1}\big((1-\delta)\theta^m + (\delta\hat{P}_{\phi(b')}(\theta^m) - b')\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\liminf_{m\to\infty} u_1(b', \theta^m) - u_1(\beta(\theta^m), \theta^m) \\
&\quad \geq \Pr[\tilde{b}_s = b']F(\phi(b'))^{n-1}(\delta\hat{P}_{\phi(b')}(\theta^m) - b') > 0;
\end{aligned}$$

i.e., for large m, type θ^m has a profitable deviation. *Q.E.D.*

LEMMA 9: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale and let ϕ be defined by* (22). *Then β is continuous. Moreover, $\phi(\beta(\theta_i)) = \theta_i$ for all $\theta_i \in [0, 1]$. The function ϕ is continuous and strictly increasing.*

PROOF: Standard. *Q.E.D.*

LEMMA 10: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale and let ϕ be defined by* (22). *Then $\beta(\theta_i) < \theta_i$ for all $\theta_i \in (0, 1]$. Moreover, $\phi(b) > b$ for all $b \in (0, \beta(1)]$.*

PROOF: Suppose that $\beta(\theta_i) \geq \theta_i > 0$. Then $\beta(\theta_i) > 0$, implying $H(\beta(\theta_i)) > 0$ by Lemma 6. Now Lemma 4 shows that $\beta(\theta_i) = \theta_i$. Finally, a computation similar to that in the proof of Lemma 4 shows that a deviation to the bid $b_i = \theta_i/2$ is profitable—a contradiction. The claim about ϕ now follows from Lemma 9. *Q.E.D.*

The following lemma establishes the differential equation that characterizes the bid function β and simultaneously determines the bid distribution H. The proof relies on local optimality of every regular-bidder type's bid and on the result from Lemma 7 that the speculator's payoff equals 0. Throughout the proof, we distinguish two kinds of bids $b \in (0, \beta(1)]$: we write $\Xi(b) = 0$ if b has a neighborhood where the speculator bids with probability 0 and we write $\Xi(b) = 1$ otherwise.

We begin by using upward optimality of the bid function β to derive inequality (30) in terms of H and the inverse bid function ϕ. Inequality (30) shows that the payoff increase effect from buying more often in the auction and less often

from the speculator, weighted by the probability mass $H(b_i) - H(b_i')$, is dominated by the payoff decrease effect from bidding higher and winning more often against one of the other regular bidders.

We continue by showing that H is locally Lipschitz continuous on $(0, \beta(1)]$. Local Lipschitz continuity around any b with $\Xi(b) = 0$ is easy because H is locally constant there. If $\Xi(b) = 1$, then we use (9) and the result that the speculator obtains 0 payoff (32) to obtain a positive lower bound (33) for the payoff increase effect in (30). Because the payoff increase effect is weighted with the probability mass $H(b_i) - H(b_i')$, we can conclude that H is locally Lipschitz around b.

The payoff decrease effect in (30) is confirmed in (34) and is then used to show in (35) that the slope of the inverse bid function ϕ is bounded above by the slope that would be relevant in the absence of a resale opportunity.

Next we use downward optimality of the bid function β to derive inequality (36), which is parallel to (30). Inequality (36) includes a payoff increase effect from bidding lower and winning less often against one of the other regular bidders, as confirmed in (37).

Combining the payoff decrease effect (34) with the payoff increase effect (37), we obtain the slope of the inverse bid function ϕ in the neighborhood of points where the speculator does not bid (38). Combining this with the 0-payoff condition for the speculator and with the upper bound (35), we obtain a differential equation (41) for ϕ. Inverting this equation yields the differential equation (18) for β. Using the fact that K is strictly decreasing in its second argument, we show that the solution to (18) is unique.

Finally, we combine inequalities (30) and (36), derived from upward and downward optimality of β, with the fact that H is locally Lipschitz to obtain a differential equation for H; see (43). This equation has the unique solution (19).

LEMMA 11: *Let $(\beta, H, \mathcal{M})$ be a quasisymmetric regular equilibrium of the first-price auction with resale and let ϕ be defined by* (22). *Then β satisfies* (18) *and is uniquely determined by these conditions. The distribution H is given by* (19).

PROOF: For all $i \in I$, $b_i \in (0, \beta(1)]$, and $\theta_i \in [0, 1]$, let

$$\check{v}_i(b_i, \theta_i) = H(b_i)F(\phi(b_i))^{n-1}(\theta_i - b_i) + \delta \int_{b_i}^{\infty} F(\phi(b_s))^{n-1}\left(\theta_i \hat{Q}_{\phi(b_s)}(\theta_i) - \hat{P}_{\phi(b_s)}(\theta_i)\right) dH(b_s) \tag{27}$$

denote bidder i's expected payoff if she consumes the good after winning in period 1. Because bidder i can alternatively offer the good for resale,

$$u_i(b_i, \theta_i) \begin{cases} = \check{v}_i(b_i, \theta_i) & \text{if } \phi(b_i) \le \theta_i, \\ \ge \check{v}_i(b_i, \theta_i) & \text{if } \phi(b_i) > \theta_i. \end{cases} \tag{28}$$

Consider $b_i, b_i' \in (0, \beta(1)]$ with $b_i' < b_i$ and $\theta_i' = \phi(b_i')$. Note that

$$\delta \int_{b_i'}^{b_i} F(\phi(b_s))^{n-1} \underbrace{\hat{Q}_{\phi(b_s)}(\theta_i')}_{\text{decreasing in } b_s} \underbrace{\left(\theta_i' - \frac{\hat{P}_{\phi(b_s)}(\theta_i')}{\hat{Q}_{\phi(b_s)}(\theta_i')}\right)}_{\text{decreasing in } b_s \text{ by (1)}} dH(b_s) \tag{29}$$

$$\leq \delta F(\phi(b_i))^{n-1}\big(\phi(b_i') - \hat{P}_{\phi(b_i')}(\phi(b_i'))\big)(H(b_i) - H(b_i')),$$

because $\hat{Q}_{\phi(b_i')}(\phi(b_i')) = 1$. Hence,

$$\begin{aligned}
0 &\overset{(12)}{\geq} u_i(b_i, \theta_i') - u_i(b_i', \theta_i') \\
&\overset{(28)}{\geq} \check{v}_i(b_i, \theta_i') - \check{v}_i(b_i', \theta_i') \\
&\overset{(27)}{=} \big(H(b_i)F(\phi(b_i))^{n-1} - H(b_i')F(\phi(b_i'))^{n-1}\big)\theta_i' \\
&\quad - \big(H(b_i)F(\phi(b_i))^{n-1}b_i - H(b_i')F(\phi(b_i'))^{n-1}b_i'\big) \\
&\quad - \delta \int_{b_i'}^{b_i} F(\phi(b_s))^{n-1}\big(\hat{Q}_{\phi(b_s)}(\theta_i')\theta_i' - \hat{P}_{\phi(b_s)}(\theta_i')\big)\, dH(b_s) \\
&\overset{(29)}{\geq} -H(b_i')k_1(b_i, b_i')(b_i - b_i') + k_2(b_i, b_i')(H(b_i) - H(b_i')),
\end{aligned} \tag{30}$$

where

$$k_1(b_i, b_i') = F(\phi(b_i))^{n-1} - \frac{F(\phi(b_i))^{n-1} - F(\phi(b_i'))^{n-1}}{b_i - b_i'}(\phi(b_i') - b_i'),$$

$$k_2(b_i, b_i') = F(\phi(b_i))^{n-1}\big(\phi(b_i')(1-\delta) + \delta\hat{P}_{\phi(b_i')}(\phi(b_i')) - b_i\big).$$

Consider any $b \in (0, \beta(1)]$. We write $\Xi(b) = 0$ if

$$\exists\, \nu > 0, \quad \Pr[\tilde{b}_s \in (b - \nu, b + \nu)] = 0,$$

and we write $\Xi(b) = 1$ otherwise. By Lemma 7,

$$\begin{aligned}
&\text{if} \quad \Xi(b) = 1, \\
&\text{then} \quad \exists (c^m), \quad c^m \to b, \quad c^m \neq b, \quad \delta M(\phi(c^m)) = c^m.
\end{aligned} \tag{31}$$

Therefore, by continuity of M and ϕ,

$$\text{if} \quad \Xi(b) = 1, \quad \text{then} \quad M(\phi(b)) = b. \tag{32}$$

Suppose that $\Xi(b) = 0$. Then there exists $\nu > 0$ such that in the neighborhood $N(b) = (b - \nu, b + \nu) \cap (0, \beta(1)]$ of b, the function H is constant. In particular,

H is Lipschitz in $N(b)$. Now suppose that $\Xi(b) = 1$. Using (9) and (32), there exists $\xi > 0$ such that for all b_i, b_i' in some neighborhood of b, if $b_i' < b_i$,

$$\begin{aligned} k_2(b_i, b_i') &\geq F(\phi(b_i))^{n-1}\big(\phi(b_i')(1-\delta) + \delta M(\phi(b_i')) + \delta\xi - b_i\big) \\ &\geq F(\phi(b_i))^{n-1}\frac{\delta\xi}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\forall\, b,\ \Xi(b) = 1, \quad \exists\, \underline{k}(b) > 0 \text{ and a neighborhood } \mathcal{N}(b) \ni b, \text{ such that} \\ &\forall\, b_i, b_i' \in \mathcal{N}(b),\ b_i' < b_i, \quad k_2(b_i, b_i') \geq \underline{k}(b). \end{aligned} \tag{33}$$

Together with (30) and $k_1(b_i, b_i') \leq 1$, (33) implies that H is Lipschitz in $\mathcal{N}(b)$. We therefore conclude that in either case, $\Xi(b) = 0$ or $\Xi(b) = 1$, the function H is Lipschitz in a neighborhood of b, proving that H is locally Lipschitz in $(0, \beta(1)]$. In particular, H is differentiable almost everywhere on $(0, \beta(1)]$.

Next we show that

$$\forall\, b_i \in (0, \beta(1)], b_i' \in (0, \beta(1)], b_i' < b_i, \quad k_1(b_i, b_i') \geq 0. \tag{34}$$

Consider any $b \in (0, \beta(1)]$. If $\Xi(b) = 0$, then $k_1(b_i, b_i') \geq 0$ holds for all b_i, b_i' in a neighborhood of b because $H(b_i) = H(b_i')$. If $\Xi(b) = 1$, then (33) together with (30) implies that $k_1(b_i, b_i') \geq 0$ holds for all b_i, b_i' in a neighborhood of b. This completes the proof of (34).

Using the definition of k_1, (34) implies

$$\forall\, b_i \in (0, \beta(1)), \text{ if } \phi \text{ differentiable at } b_i, \text{ then } \phi'(b_i) \leq \frac{1}{N(\phi(b_i), b_i)}. \tag{35}$$

Note that for all $b_i, b_i' \in (0, \beta(1)]$ with $b_i' < b_i$ and $\theta_i = \phi(b_i)$,

$$\begin{aligned} 0 &\overset{(12)}{\leq} u_i(b_i, \theta_i) - u_i(b_i', \theta_i) \\ &\overset{(28)}{=} \check{v}_i(b_i, \theta_i) - \check{v}_i(b_i', \theta_i) \\ &\overset{(27)}{=} \big(H(b_i)F(\phi(b_i))^{n-1} - H(b_i')F(\phi(b_i'))^{n-1}\big)\theta_i \\ &\quad - \big(H(b_i)F(\phi(b_i))^{n-1}b_i - H(b_i')F(\phi(b_i'))^{n-1}b_i'\big) \\ &\quad - \delta\int_{b_i'}^{b_i} F(\phi(b_s))^{n-1}\big(\underbrace{\hat{Q}_{\phi(b_s)}(\theta_i)}_{=1 \text{ by } (4)}\theta_i - \underbrace{\hat{P}_{\phi(b_s)}(\theta_i)}_{=\hat{P}_{\theta_i}(\theta_i) \text{ by } (4)}\big)\, dH(b_s) \\ &\leq -H(b_i)l_1(b_i, b_i')(b_i - b_i') + l_2(b_i, b_i')(H(b_i) - H(b_i')), \end{aligned} \tag{36}$$

where

$$l_1(b_i, b_i') = F(\phi(b_i))^{n-1} - \frac{F(\phi(b_i))^{n-1} - F(\phi(b_i'))^{n-1}}{b_i - b_i'}(\phi(b_i) - b_i'),$$

$$l_2(b_i, b_i') = F(\phi(b_i'))^{n-1}\big(\phi(b_i)(1-\delta) + \delta \hat{P}_{\phi(b_i)}(\phi(b_i)) - b_i'\big).$$

Consider any $b \in (0, \beta(1)]$ such that $\Xi(b) = 0$. Then (36) implies the existence of a neighborhood $\mathcal{N}(b)$ of b such that

$$\forall b_i \in \mathcal{N}(b), b_i' \in \mathcal{N}(b), b_i' < b_i, \quad l_1(b_i, b_i') \leq 0. \tag{37}$$

Taken together, (34) and (37) imply for all $b_i \in \mathcal{N}(b)$ that $F(\phi)^{n-1}$ is differentiable at b_i and

$$\lim_{b_i' \to b_i} \frac{F(\phi(b_i))^{n-1} - F(\phi(b_i'))^{n-1}}{b_i - b_i'} = \frac{F(\phi(b_i))^{n-1}}{\phi(b_i) - b_i}.$$

Hence, ϕ is differentiable at $b_i \in \mathcal{N}(b)$ and

$$\forall b, \Xi(b) = 0, b_i \in \mathcal{N}(b), \quad \phi'(b_i) = \frac{1}{N(\phi(b_i), b_i)}. \tag{38}$$

Note also that

$$\forall b, \delta M(\phi(b)) = b, \text{ if } \phi \text{ differentiable at } b, \text{ then} \tag{39}$$
$$\delta M'(\phi(b))\phi'(b) = \lim_{b_i \searrow b} \frac{\delta M(\phi(b_i)) - \delta M(\phi(b))}{b_i - b} \leq 1,$$

because $\delta M(\phi(b_i)) \leq b_i$ for all b_i, by Lemma 7. By (31),

$$\forall b, \Xi(b) = 1, \text{ if } \phi \text{ differentiable at } b, \text{ then } \delta M'(\phi(b))\phi'(b) = 1. \tag{40}$$

In summary, for all $b \in (0, \beta(1)]$ such that ϕ is differentiable at b,

$$\phi'(b) = \begin{cases} \dfrac{1}{N(\phi(b), b)} & \text{if } \delta M(\phi(b)) < b, \\ \min\left\{\dfrac{1}{N(\phi(b), b)}, \dfrac{1}{\delta M'(\phi(b))}\right\} & \text{if } \delta M(\phi(b)) = b, \end{cases} \tag{41}$$

by (32), (35), (38), (39), and (40). Next we show that β is Lipschitz continuous. By (34), $F(\phi)^{n-1}$ is locally Lipschitz in $(0, 1]$, implying that ϕ is locally Lipschitz in $(0, 1]$. Hence,

$$\forall c, d \in (0, \beta(1)], \quad \phi(d) - \phi(c) = \int_c^d \phi'(b)\, db, \tag{42}$$

where ϕ' is given almost everywhere by (41).

The mapping $b \mapsto N(\phi(b), b)$ is bounded above on $[0, \beta(1)]$ because $N(\phi(b), b) \leq (n-1)f(\phi(b))\phi(b)/F(\phi(b)) \to n-1$ as $b \to 0$. Moreover, M' is bounded above on $[0, 1]$ by Lemma 2. Therefore, ϕ' is bounded below by a positive number. Hence, β is Lipschitz on $[0, 1]$ by (42).

Because β is the inverse of ϕ, (41) implies (18).

As for uniqueness, let β and γ be two Lipschitz continuous functions that satisfy (18). By definition of K, if $\beta(\theta) > \gamma(\theta)$, then $K(\theta, \beta(\theta)) < K(\theta, \gamma(\theta))$ for all $\theta \in (0, 1)$. Therefore, $\beta \leq \gamma$. The same argument shows $\beta \geq \gamma$.

Next we show that H satisfies

$$h(b) \equiv H'(b) = H(b)L_\phi(b) \quad \text{a.e. } b \in (0, \beta(1)]. \tag{43}$$

Consider any $b \in (0, \beta(1)]$. If $\Xi[b] = 0$ and $b = \delta M(\phi(b))$, then $h(b) = 0$ and, by (38), $R_\phi(b) = 0$. Hence, $h(b) = 0 = H(b)L_\phi(b)$, as was to be shown. If $\Xi[b] = 0$ and $b > \delta M(\phi(b))$, then $h(b) = 0 = H(b)L_\phi(b)$ by definition of L_ϕ. Finally suppose that $\Xi[b] = 1$. The function $\hat{r}$ is continuous. Hence, using Lemma 1, the function $b' \mapsto \hat{P}_{\phi(b')}(\phi(b'))$ is continuous on $(0, 1]$. Therefore,

$$\lim_{b' \nearrow b} k_2(b, b') = S_\phi(b)F(\phi(b)) = \lim_{b'' \searrow b} l_2(b'', b) \quad \forall\, b \in (0, \beta(1)]. \tag{44}$$

Because ϕ is differentiable almost everywhere,

$$\lim_{b' \nearrow b} k_1(b, b') = R_\phi(b)F(\phi(b)) = \lim_{b'' \searrow b} l_1(b'', b) \quad \text{a.e. } b \in (0, \beta(1)]. \tag{45}$$

From (30), (36), (44), and (45) we get (43).

It is straightforward that (19) satisfies (43). It remains to be shown that (43) together with the boundary condition $H(\beta(1)) = 1$ (which holds because the speculator does not bid more than necessary to win for sure) has only one solution. Observe that, for all $\varepsilon > 0$, L is bounded above on $[\varepsilon, \beta(1)]$ due to (9). Therefore, applying the Picard–Lindelöf theorem to the differential equation (43) implies that H is unique on $[\varepsilon, 1]$. As a distribution function, H is right-continuous at 0 and thus uniquely determined at 0 as well. *Q.E.D.*

We now turn to the equilibrium existence proof. Lemma 12 shows the existence of a solution for a class of discontinuous differential equations. These equations are constructed by altering a continuous differential equation so that if the solution function hits the boundary value 0, the solution remains there until the slope becomes positive again.

To prove the existence of a solution, the differential equation is transformed into an upper hemicontinuous differential inclusion that allows a set of slopes (including 0) for the solution function if it reaches the value 0 and allows only

the slope 0 if a negative value is reached. For the differential inclusion, a solution exists by a theorem of Aubin and Cellina (1984). One then shows that the solution, in fact, does not take negative values and solves the original differential equation almost everywhere.

LEMMA 12: *Let* $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ *and consider a bounded and continuous function* $\hat{N} : [\underline{\theta}, \overline{\theta}] \times \mathbb{R} \to \mathbb{R}$. *Define*

$$\hat{K}(\theta, b) = \begin{cases} \hat{N}(\theta, b) & \text{if } b > 0, \\ \max\{0, \hat{N}(\theta, b)\} & \text{if } b \leq 0. \end{cases} \tag{46}$$

Then the initial value problem

$$\hat{\beta}(\underline{\theta}) = 0, \quad \hat{\beta}'(\theta) = \hat{K}(\theta, \hat{\beta}(\theta)) \quad \text{a.e. } \theta \in [\underline{\theta}, \overline{\theta}] \tag{47}$$

has a Lipschitz continuous solution $\hat{\beta}$. *For all* θ *such that* $\hat{\beta}(\theta) > 0$, *the function* $\hat{\beta}$ *is differentiable at* θ.

PROOF: Following Aubin and Cellina (1984, p. 101), define

$$\overline{K}(\theta, b) = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, \hat{K}(B_\varepsilon(\theta, b)),$$

where $B_\varepsilon(\theta, b)$ denotes the ε-ball around (θ, b) according to any norm in $\mathbb{R}^2$ and where $\overline{\text{co}}$ denotes the closed-convex-hull operator. Then $\overline{K}$ is an upper hemicontinuous (or, in Aubin and Cellina's (1984) terminology, upper semicontinuous) correspondence, and its values are closed and convex. Moreover, $\overline{K}$ is globally bounded. Therefore, the differential inclusion problem

$$\hat{\beta}'(\theta) \in \overline{K}(\theta, \hat{\beta}(\theta)) \quad \text{a.e. } \theta \in (\underline{\theta}, \overline{\theta}), \quad \hat{\beta}(\underline{\theta}) = 0, \tag{48}$$

has an absolutely continuous solution $\hat{\beta}$ (see Aubin and Cellina (1984, Theorem 4, p. 101)). Note that, by (48), for a.e. $\theta \in [\underline{\theta}, \overline{\theta}]$, if $\hat{\beta}(\theta) < 0$, then $\hat{\beta}'(\theta) \geq 0$. Hence

$$\forall\, \theta \in [\underline{\theta}, \overline{\theta}], \quad \hat{\beta}(\theta) \geq 0. \tag{49}$$

We will now show that $\hat{\beta}$ satisfies (47). First, consider $\theta \in [\underline{\theta}, \overline{\theta}]$ with $\hat{\beta}(\theta) > 0$, or $\hat{\beta}(\theta) = 0$ and $\hat{N}(\theta, 0) \geq 0$. Then $\overline{K}(\theta, \hat{\beta}(\theta)) = \hat{K}(\theta, \hat{\beta}(\theta))$.

Second, consider $\theta \in [\underline{\theta}, \overline{\theta}]$ with $\hat{\beta}(\theta) = 0$ and $\hat{N}(\theta, 0) < 0$. For a.e. such θ, (48) implies $\hat{\beta}'(\theta) \leq 0$. On the other hand, $\hat{\beta}'(\theta) \geq 0$ by (49). Therefore, $\hat{\beta}'(\theta) = 0 = \hat{K}(\theta, \hat{\beta}(\theta))$, completing the proof of (47).

Because $\hat{\beta}$ is absolutely continuous,

$$(50) \qquad \forall\, \theta \in [\underline{\theta}, \overline{\theta}], \quad \hat{\beta}(\theta) = \int_{\underline{\theta}}^{\theta} \hat{K}(\theta', \hat{\beta}(\theta'))\, d\theta'.$$

This together with the fact that $\hat{K}$ is bounded, implies that $\hat{\beta}$ is Lipschitz continuous. If $\hat{\beta}(\theta) > 0$ for some θ, then $\hat{K}(\theta', \hat{\beta}(\theta')) = \hat{N}(\theta', \hat{\beta}(\theta'))$ for all θ' that are sufficiently close to θ. Continuity of $\hat{N}$ together with (50) then implies that $\hat{\beta}$ is differentiable at θ. *Q.E.D.*

Let β^{I} denote the standard equilibrium bid function of the first-price auction with n regular bidders and no resale opportunity. The following result is well known.

LEMMA 13: *Let N be defined as in* (15). *Then*

$$\beta^{\mathrm{I}}(0) = 0 \text{ and } \forall\, \theta \in [0, 1], \quad \beta^{\mathrm{I}'}(\theta) = N(\theta, \beta^{\mathrm{I}}(\theta)).$$

The next lemma constructs the equilibrium bid function β for the regular bidders. One first shows that bidder types close to 0 make the same bid as in the absence of a resale opportunity. To extend the bid function to the higher types, Lemma 12 is applied to a differential equation for the excess of the regular bidders' bid function β over the fictitious bid function δM that corresponds to a 0 payoff for the speculator.

LEMMA 14: *The initial value problem* (18) *has a solution β on* $[0, 1]$ *with the following properties*:

(51) *β is Lipschitz continuous on* $[0, 1]$,

(52) $\forall\, \theta \in (0, 1], \quad \beta(\theta) < \theta$,

(53) *β is strictly increasing on* $[0, 1]$,

(54) $\forall\, \theta \in [0, 1], \quad \beta(\theta) \geq \delta M(\theta)$,

(55) *$\phi \equiv \beta^{-1}$ is Lipschitz continuous on* $[0, \beta(1)]$.

PROOF: Define $\underline{\theta}$ to be the smallest $\theta \in [0, 1]$ with $\beta^{\mathrm{I}}(\theta) = \delta M(\theta)$ (let $\underline{\theta} = 1$ if no such θ exists). Because $\beta^{\mathrm{I}'}(0) = (n-1)/n > \delta M'(0)$ by (7), we have $\underline{\theta} > 0$. Defining $\beta(\theta) = \beta^{\mathrm{I}}(\theta)$ for $\theta \in [0, \underline{\theta}]$, it follows from Lemma 13 that (18) is satisfied for $\theta \in [0, \underline{\theta}]$ and it follows from the theory of first-price auctions without resale that β has the desired properties (51)–(55) on $[0, \underline{\theta}]$.

Define $\hat{N} : [\underline{\theta}, 1] \times \mathbb{R} \to \mathbb{R}$ by

$$\hat{N}(\theta, b) = \begin{cases} N(\theta, b + \delta M(\theta)) - \delta M'(\theta) & \text{if } b \in [0, \theta - \delta M(\theta)], \\ N(\theta, \theta) - \delta M'(\theta) & \text{if } b > \theta - \delta M(\theta), \\ N(\theta, \delta M(\theta)) - \delta M'(\theta) & \text{if } b < 0, \end{cases}$$

and define $\hat{K}$ as in (46) with $\overline{\theta} = 1$. Then, Lemma 12 implies that there exists a Lipschitz continuous $\hat{\beta}$ such that (47) holds. By definition of $\hat{K}$,

$$\forall\, \theta \in [\underline{\theta}, 1], \quad 0 \leq \hat{\beta}(\theta) \leq \theta - \delta M(\theta).$$

Therefore, $\beta(\theta) \equiv \hat{\beta}(\theta) + \delta M(\theta)$, $\theta \in [\underline{\theta}, 1]$, yields a Lipschitz continuous solution for (18).

To prove (52), suppose that $\beta(\theta) \geq \theta$ for some $\theta > \underline{\theta}$. Let θ' be minimal with that property. Then $\beta(\theta') = \theta' > \delta M(\theta')$. Thus, β is differentiable at θ' by Lemma 12. Also, $\hat{N}(\theta, \hat{\beta}(\theta)) = -\delta M'(\theta)$. Hence, $\beta'(\theta') = 0$. Thus, $\beta(\theta) > \theta$ for some $\theta < \theta'$, a contradiction.

To prove (53), observe that

$$\beta'(\theta) \overset{(18)}{=} K(\theta, \beta(\theta)) \overset{(14)}{\geq} N(\theta, \beta(\theta)) \overset{(52)}{>} 0 \quad \text{a.e. } \theta \geq \underline{\theta}.$$

This also implies that β' is bounded below by a positive number on $[\underline{\theta}, 1]$. Hence, ϕ is Lipschitz on $[\beta(\underline{\theta}), \beta(1)]$; i.e., (55) follows. Inequality (54) is immediate from (14) and (18). *Q.E.D.*

The final lemma toward the proof of Proposition 1 shows equilibrium existence. We begin by showing that (19) yields a well-defined distribution function. Next, although by construction of the differential equation (18) the bid function β satisfies the first-order condition for optimal bidding for every regular-bidder type, it remains to be shown that β is globally optimal. To get this, we show quasiconcavity of the regular bidders' payoff function. When considering upward deviations, we must take care of the possibility that a regular bidder deviates to a higher bid in an attempt to offer the good for resale. No such complication arises in the context of downward deviations because such a deviation does not lead to additional resale offers from competing regular bidders. Optimality of the speculator's bid distribution follows by construction.

LEMMA 15: *The first-price auction with resale has a quasisymmetric regular equilibrium* $(\beta, H, \mathcal{M})$.

PROOF: Define β according to Lemma 14 and define $\phi = \beta^{-1}$. The first step is to show that H, as defined by (19), is a well-defined distribution function. For any given $\varepsilon > 0$, (9) implies that there exists $\xi > 0$ such that

$$\forall\, b \in [\varepsilon, \beta(1)], \tag{56}$$
$$\text{if} \quad b = \delta M(\phi(b)), \quad \text{then} \quad \delta \hat{P}_{\phi(b)}(\phi(b)) \geq b + \delta\xi.$$

By definition of K,

$$\beta'(\phi(b)) \geq N(\phi(b), \beta(\phi(b))) \quad \text{a.e. } b \in (0, \beta(1)],$$

implying

$$R_\phi(b) \overset{(17)}{\geq} 0 \quad \text{a.e. } b \in (0, \beta(1)]. \tag{57}$$

From (56) and (57) it follows that L_ϕ (cf. (16)) is well defined almost everywhere and is bounded above on $[\varepsilon, \beta(1)]$ for any given $\varepsilon > 0$. Hence, H is Lipschitz continuous on $[\varepsilon, \beta(1)]$ for every $\varepsilon > 0$. Moreover, because L_ϕ is nonnegative, H is weakly increasing on $(0, \beta(1)]$ and the limit $H(0) = \lim_{b \searrow 0} H(b)$ exists. Therefore, H is a distribution function.

Let post-auction beliefs and resale mechanisms be defined by (10) and (11). It remains to be shown that (12) and (13) hold.

To show (12), first consider any deviating bid $b_i \in (0, \beta(\theta_i))$ of a bidder $i \neq s$ with type $\theta_i \in (0, 1]$. After winning at b_i, it is optimal for bidder i to consume the good because she believes that the highest use value among bidders other than herself is at most $\phi(b_i) < \theta_i$. Also, if some bidder $j \in I \setminus \{i\}$ wins after making her equilibrium bid, then bidder j consumes the good in period 1. Hence, bidder i's payoff is given by

$$\begin{aligned} u_i(b_i, \theta_i) &= H(b_i) F(\phi(b_i))^{n-1} (\theta_i - b_i) \\ &\quad + \delta \int_{b_i}^{\infty} F(\phi(b_s))^{n-1} \hat{U}_{\phi(b_s)}(\theta_i)\, dH(b_s), \end{aligned}$$

where

$$\hat{U}_{\phi(b_s)}(\theta_i) \equiv \theta_i \hat{Q}_{\phi(b_s)}(\theta_i) - \hat{P}_{\phi(b_s)}(\theta_i).$$

Because ϕ and H are locally Lipschitz continuous on $(0, \beta(1)]$, the mapping $b_i \mapsto u_i(b_i, \theta_i)$ has the same property and is differentiable Lebesgue-a.e. in $(0, 1)$. Hence, for a.e. $b_i \in (0, \beta(\theta_i))$,

$$\begin{aligned} &\frac{\partial u_i}{\partial b_i}(b_i, \theta_i) \\ &\quad = H(b_i) F(\phi(b_i))^{n-2} \\ &\qquad \times \big((n-1) f(\phi(b_i)) \phi'(b_i) (\theta_i - b_i) - F(\phi(b_i)) \big) \\ &\qquad + H'(b_i) F(\phi(b_i))^{n-1} \big(\theta_i - b_i - \delta \hat{U}_{\phi(b_i)}(\theta_i) \big) \\ &\quad \overset{\theta_i \geq \phi(b_i)}{\geq} H(b_i) F(\phi(b_i))^{n-1} \\ &\qquad \times \left((n-1) \frac{f(\phi(b_i))}{F(\phi(b_i))} (\phi(b_i) - b_i) \phi'(b_i) - 1 \right) \\ &\qquad + H'(b_i) F(\phi(b_i))^{n-1} \big(\theta_i - b_i - \delta \hat{U}_{\phi(b_i)}(\theta_i) \big). \end{aligned}$$

By definition of R_ϕ and because $H'(b_i) = L_\phi(b_i)H(b_i)$ by (19),

$$\frac{\frac{\partial u_i}{\partial b_i}(b_i, \theta_i)}{H(b_i)F(\phi(b_i))^{n-1}} \geq -R_\phi(b_i) + L_\phi(b_i)\big(\theta_i - b_i - \delta\hat{U}_{\phi(b_i)}(\theta_i)\big). \tag{58}$$

To show that

$$\frac{\partial u_i}{\partial b_i}(b_i, \theta_i) \geq 0 \quad \text{a.e. } b_i \in (0, \beta(\theta_i)), \tag{59}$$

we distinguish two cases. If $b_i > \delta M(\phi(b_i))$, then $\beta(\phi(b_i)) > \delta M(\phi(b_i))$; hence, $\beta'(\phi(b_i)) = N(\phi(b_i), b_i)$ by (18). Thus, $R_\phi(b_i) = 0$. Moreover, $L_\phi(b_i) = 0$ by definition of L_ϕ. Hence, (59) follows from (58).

If $b_i = \delta M(\phi(b_i))$, then $R_\phi(b_i) = L_\phi(b_i)S_\phi(b_i)$ by definition of L_ϕ. Using this and the definition of $S_\phi(b_i)$, (58) implies

$$\begin{aligned}&\frac{\frac{\partial u_i}{\partial b_i}(b_i, \theta_i)}{H(b_i)F(\phi(b_i))^{n-1}}\\ &\quad\geq L_\phi(b_i)\big(\theta_i - \delta\hat{U}_{\phi(b_i)}(\theta_i) - (1-\delta)\phi(b_i) - \delta\hat{P}_{\phi(b_i)}(\phi(b_i))\big).\end{aligned}$$

Now (59) follows from $\hat{U}_{\phi(b_i)}(\theta_i) = \theta_i - \hat{P}_{\phi(b_i)}(\phi(b_i))$ by (4). From (59) it follows that type θ_i cannot gain from deviating to any bid $b_i \in [0, \beta(\theta_i))$.

Now consider a deviating bid $b_i \in (\beta(\theta_i), \beta(1)]$. Suppose first that bidder i offers the good for resale upon winning. We obtain an upper bound $v_i(b_i, \theta_i)$ for bidder i's payoff by assuming she gets the entire surplus that is available in the resale market,

$$\begin{aligned}&v_i(b_i, \theta_i)\\ &\quad= H(b_i)\left(F(\phi(b_i))^{n-1}(\delta\theta_i - b_i) + \delta\int_{\theta_i}^{\phi(b_i)}(\theta' - \theta_i)\,d(F(\theta')^{n-1})\right)\\ &\qquad+ \delta\int_{b_i}^{\infty}F(\phi(b_s))^{n-1}\hat{U}_{\phi(b_s)}(\theta_i)\,dH(b_s).\end{aligned}$$

For a.e. $b_i \in (\beta(\theta_i), \beta(1))$,

$$\begin{aligned}&\frac{\partial v_i}{\partial b_i}(b_i, \theta_i)\\ &\quad\overset{\theta_i \leq \phi(b_i)}{\leq} H(b_i)F(\phi(b_i))^{n-1}\\ &\qquad\times\left((n-1)\frac{f(\phi(b_i))}{F(\phi(b_i))}(\delta\phi(b_i) - b_i)\phi'(b_i) - 1\right)\\ &\qquad+ H'(b_i)F(\phi(b_i))^{n-1}\end{aligned}$$

$$\times \Bigg(\delta\theta_i - b_i - \delta \hat{U}_{\phi(b_i)}(\theta_i) + \delta \int_{\theta_i}^{\phi(b_i)} \frac{\theta' - \theta_i}{F(\phi(b_i))^{n-1}}\, d(F(\theta')^{n-1}) \Bigg).$$

By definition of R_ϕ and $\delta \le 1$, and because $H'(b_i) = L_\phi(b_i)H(b_i)$ by (19),

$$\begin{aligned} (60)\qquad & \frac{\frac{\partial v_i}{\partial b_i}(b_i, \theta_i)}{H(b_i)F(\phi(b_i))^{n-1}} \\ & \le -R_\phi(b_i) + L_\phi(b_i)\Bigg(\delta\theta_i - b_i - \delta \hat{U}_{\phi(b_i)}(\theta_i) + \delta \int_{\theta_i}^{\phi(b_i)} \frac{\theta' - \theta_i}{F(\phi(b_i))^{n-1}}\, d(F(\theta')^{n-1}) \Bigg). \end{aligned}$$

To show that

$$(61)\qquad \frac{\partial v_i}{\partial b_i}(b_i, \theta_i) \le 0 \quad \text{a.e. } b_i \in (\beta(\theta_i), \beta(1)),$$

we distinguish two cases. If $b_i > \delta M(\phi(b_i))$, then $R_\phi(b_i) = 0$ by (18) and $L_\phi(b_i) = 0$ by definition of L_ϕ. Hence, (61) follows from (60).

If $b_i = \delta M(\phi(b_i))$, then $R_\phi(b_i) = L_\phi(b_i)S_\phi(b_i)$. Using this and the definition of $S_\phi(b_i)$, (60) implies

$$\begin{aligned} (62)\qquad & \frac{\frac{\partial v_i}{\partial b_i}(b_i, \theta_i)}{H(b_i)F(\phi(b_i))^{n-1}} \\ & \le L_\phi(b_i)\Bigg(\delta\theta_i - \delta \hat{U}_{\phi(b_i)}(\theta_i) + \delta \int_{\theta_i}^{\phi(b_i)} \frac{\theta' - \theta_i}{F(\phi(b_i))^{n-1}}\, d(F(\theta')^{n-1}) - (1-\delta)\phi(b_i) - \delta \hat{P}_{\phi(b_i)}(\phi(b_i)) \Bigg). \end{aligned}$$

Using the envelope theorem in integral form and integration by parts, the expression in brackets on the right-hand side of (62) can be simplified:

$$\begin{aligned} (\cdots) &= \delta\theta_i - \phi(b_i) + \delta\big(\hat{U}_{\phi(b_i)}(\phi(b_i)) - \hat{U}_{\phi(b_i)}(\theta_i)\big) + \delta \int_{\theta_i}^{\phi(b_i)} \frac{\theta' - \theta_i}{F(\phi(b_i))^{n-1}}\, d(F(\theta')^{n-1}) \\ &= \delta\theta_i - \phi(b_i) + \delta \int_{\theta_i}^{\phi(b_i)} \frac{F(\theta')^{n-1}}{F(\phi(b_i))^{n-1}}\, d\theta' \end{aligned}$$

$$
\begin{aligned}
&+ \delta \int_{\theta_i}^{\phi(b_i)} \frac{\theta' - \theta_i}{F(\phi(b_i))^{n-1}}\, d(F(\theta')^{n-1}) \\
&= \delta\theta_i - \phi(b_i) + \delta\theta' \frac{F(\theta')^{n-1}}{F(\phi(b_i))^{n-1}}\Bigg|_{\theta_i}^{\phi(b_i)} \\
&\quad - \delta\theta_i \frac{F(\phi(b_i))^{n-1} - F(\theta_i)^{n-1}}{F(\phi(b_i))^{n-1}} \\
&= -(1-\delta)\phi(b_i) \leq 0.
\end{aligned}
$$

Hence, (61) follows from (62).

Finally, consider a deviating bid $b_i \in (\beta(\theta_i), \beta(1)]$ and suppose that bidder i consumes the good upon winning. Let $\check{v}_i(b_i, \theta_i)$ denote the corresponding payoff. Using techniques analogous to those leading to (59) and (61), one shows that

$$
\frac{\partial \check{v}_i}{\partial b_i}(b_i, \theta_i) \leq 0 \quad \text{a.e. } b_i \in (\beta(\theta_i), \beta(1)). \tag{63}
$$

An upper bound for bidder i's payoff with any bid $b_i \in (\beta(\theta_i), \beta(1)]$ is

$$
\overline{u}_i(b_i, \theta_i) \equiv \max\{v_i(b_i, \theta_i), \check{v}_i(b_i, \theta_i)\}.
$$

Because the maximum of two locally Lipschitz continuous functions is locally Lipschitz itself, the mapping $b_i \mapsto \overline{u}_i(b_i, \theta_i)$ can be written as the integral over its derivative. Moreover, $\overline{u}_i(\beta(\theta_i), \theta_i) = u_i(\beta(\theta_i), \theta_i)$. Therefore, (61) and (63) imply that for all $b_i \in (\beta(\theta_i), \beta(1)]$,

$$
\begin{aligned}
u_i(b_i, \theta_i) - u_i(\beta(\theta_i), \theta_i) &\leq \overline{u}_i(b_i, \theta_i) - \overline{u}_i(\beta(\theta_i), \theta_i) \\
&= \int_{\beta(\theta_i)}^{b_i} \frac{\partial \overline{u}_i}{\partial b_i}(b, \theta_i)\, db \leq 0.
\end{aligned}
$$

Hence, no type $\theta_i > 0$ has an incentive to deviate. By continuity of u_i, type $\theta_i = 0$ has no incentive to deviate either. This completes the proof of (12).

To complete the equilibrium existence proof, we have to show (13). From (54) it follows that $b_s \geq \delta M(\phi(b_s))$ and thus $u_s(b_s) \leq 0$ for all $b_s \in (0, \beta(1)]$. It remains to be shown that

$$
\Pr[u_s(\tilde{b}_s) < 0] = 0.
$$

Consider the event $u_s(\tilde{b}_s) < 0$. Then $\beta(\phi(\tilde{b}_s)) > \delta M(\phi(\tilde{b}_s))$ and thus $L_\phi(\tilde{b}_s) = 0$. Using (19), the probability of that event is

$$
\begin{aligned}
\int_{(0,\beta(1)]} \mathbb{1}_{L_\phi(b_s)=0} h(b_s)\, db_s &= \int_{(0,\beta(1)]} \mathbb{1}_{L_\phi(b_s)=0} L_\phi(b_s) H(b_s)\, db_s \\
&= 0. \qquad \textit{Q.E.D.}
\end{aligned}
$$

This completes the proof of Proposition 1. *Q.E.D.*

Observe that Proposition 1 allows the possibility that $H(0) = 1$; i.e., the speculator may not play an active role at all. The following Proposition 2 shows that the answer to the question of whether the speculator plays an active role depends on the distribution F, the discount factor δ, and the number of regular bidders n. For any given n, Proposition 2 determines the smallest discount factor δ_n such that the speculator plays an active role for *some* F. To this end, we show that the type of distribution described in Garratt and Tröger (2006, Section 5) is the most favorable for speculative activity so that we can define

$$\delta_n = \min_{z \in [0,1]} \frac{\eta(z,n)}{\rho(z,n)}. \tag{64}$$

The intuition is as follows. Because the distribution of the types below the resale reserve price is irrelevant for the resale revenue, the most favorable distribution for the speculator is the one with the smallest expectation conditional on the information that the value is below the reserve price. One cannot shift all the probability weight to 0 because that would violate the increasing hazard rate assumption. The best one can do is to shift probability weight to lower values until the hazard rate is constant—this yields the exponential shape. Above the resale reserve price, the best situation for the speculator is for all of the probability weight to be concentrated just above the reserve price so that the bidders obtain no information rent. This yields the type of distribution described in Garratt and Tröger (2006, Section 5).

PROPOSITION 2: *For every regular-bidder number $n \geq 2$, there exists a discount factor $\delta_n < 1$ such that the following statements hold in the quasisymmetric regular equilibrium of a first-price or Dutch auction with resale*:

(i) *For all $\delta > \delta_n$, there exists a distribution F that satisfies Assumption* 1 *such that the speculator plays an active role*.

(ii) *For all $\delta \leq \delta_n$ and all F that satisfy Assumption* 1, *the speculator bids* 0 *and the regular bidders use the same bid function as in the absence of a resale opportunity*.

Moreover, $\delta_n \to 1$ as $n \to \infty$.

PROOF: Observe that, for all (F, n, δ),

$$\text{if} \quad H(0) = 1, \quad \text{then} \quad \beta = \beta^{\mathrm{I}}. \tag{65}$$

To see (65), suppose that $H(0) = 1$. Then (19) implies $L_\phi(b) = 0$ a.e. $b \in [0, \beta(1)]$. Hence, $\beta = \beta^{\mathrm{I}}$ by Lemma 13, (16), (17), and (18).[9]

[9]The result that the bid function $\beta = \beta^{\mathrm{I}}$ yields an equilibrium in the absence of bids by the speculator appears in Haile (1999, Theorem 1).

The argument given in Garratt and Tröger (2006, Section 5) together with (65) proves part (i). To prove the "moreover" part, note that

$$\rho(z,n) \leq 1, \quad \min_{z\in[0,1]} \eta(z,n) \underset{n\to\infty}{\to} 1.$$

To prove part (ii), we need additional notation concerning order statistics. For any distribution function D, let $D^{(k,l)}$ denote the distribution function for the lth highest-order statistic among k independent and identically distributed (i.i.d.) random variables ($k = 1, 2, \ldots;\ l = 1, \ldots, k$) that are distributed according to D.

Fix any $n \geq 2$, $\delta \leq \delta_n$, and F with an increasing hazard rate. Suppose that $\beta = \beta^{\mathrm{I}}$. To complete the proof of (ii), it is sufficient to show that $u_s(b_s) \leq 0$ for all $b_s \geq 0$ (because then H with $H(0) = 1$ is a best-response bid distribution of the speculator and β is a best-response bid function of the regular bidders by (65)).

By definition of β^{I}, if $b_s = \beta^{\mathrm{I}}(1)$,

$$u_s(b_s) = \pi(F,n,\delta) \equiv \delta M(1) - \int_0^1 \theta\, dF^{(n-1,1)}(\theta). \tag{66}$$

If $b_s < \beta^{\mathrm{I}}(1)$ and $\hat{\theta} \equiv (\beta^{\mathrm{I}})^{-1}(b_s)$,

$$u_s(b_s) = F(\hat{\theta})^n \left(\delta M(\hat{\theta}) - \int_0^1 \theta\, d\hat{F}^{(n-1,1)}_{[0,\hat{\theta}]}(\theta) \right) = F(\hat{\theta})^n \pi(\hat{F}_{[0,\hat{\theta}]}, n, \delta).$$

Hence, without loss of generality, it is sufficient to show that

$$\pi(F,n,\delta) \leq 0. \tag{67}$$

Rearranging (5), one finds

$$M(1) = n(1-F(\check{r}))F(\check{r})^{n-1}\check{r} + \int_{\check{r}}^1 \theta\, dF^{(n,2)}(\theta),$$

where $\check{r} \equiv \hat{r}(1)$. Hence, using (66),

$$\pi(F,n,\delta) = \delta\left(n(1-F(\check{r}))F(\check{r})^{n-1}\check{r} + \int_{\check{r}}^1 \theta\, dF^{(n,2)}(\theta) \right) - \int_0^1 \theta\, dF^{(n-1,1)}(\theta). \tag{68}$$

Observe that $\check{r} = 1/\lambda(\check{r})$ by definition of the function $\hat{r}$, where $\lambda(\theta) = f(\theta)/(1 - F(\theta))$ denotes for all $\theta \in [0, 1)$ the hazard function for F. We can write

$$F(\theta) = 1 - \exp\left(-\int_0^\theta \lambda(t)\,dt\right).$$

Let $\mu = (1/\check{r}) \int_0^{\check{r}} \lambda(t)\,dt$. Then

$$\int_0^{\check{r}} (\lambda(t) - \mu)\,dt = 0.$$

Because $\lambda(t) - \mu$ is weakly increasing in t,

$$\forall\, \theta \in [0, \check{r}], \quad \int_0^\theta \lambda(t)\,dt \le \int_0^\theta \mu\,dt = \theta\mu.$$

Therefore,

$$\forall\, \theta \in [0, \check{r}], \quad F(\theta) \le J(\theta) \equiv 1 - e^{-\theta\mu}.$$

Also define $J(\theta) = F(\theta)$ for $\theta \in (\check{r}, 1]$. Then $J(\check{r}) = F(\check{r})$ and F stochastically dominates J. Therefore, using (68),

$$\pi(F, n, \delta) \le \check{\pi} \equiv \delta\left(n(1 - J(\check{r}))J(\check{r})^{n-1}\check{r} + \int_{\check{r}}^1 \theta\,dJ^{(n,2)}(\theta)\right) - \int_0^1 \theta\,dJ^{(n-1,1)}(\theta). \tag{69}$$

For arbitrary i.i.d. random variables with densities, the highest of $n - 1$ dominates the second highest of n in terms of the likelihood ratio. Therefore, the expectation of the highest of $n - 1$ conditional on being greater or equal to $\check{r}$ is greater or equal to the respective conditional expectation of the second highest of n,

$$\frac{1}{1 - J^{(n,2)}(\check{r})} \int_{\check{r}}^1 \theta\,dJ^{(n,2)}(\theta) \le \check{e} \equiv \frac{1}{1 - J^{(n-1,1)}(\check{r})} \int_{\check{r}}^1 \theta\,dJ^{(n-1,1)}(\theta).$$

Therefore, using the definition of $\check{\pi}$ from (69),

$$\check{\pi} \le \delta\big(n(1 - J(\check{r}))J(\check{r})^{n-1}\check{r} + (1 - J^{(n,2)}(\check{r}))\check{e}\big) - \int_0^{\check{r}} \theta\,dJ^{(n-1,1)}(\theta) - (1 - J^{(n-1,1)}(\check{r}))\check{e}.$$

Because $J^{(n-1,1)}(\check{r}) \leq J^{(n,2)}(\check{r})$ and $\check{e} \geq \check{r}$, it follows that

$$
\begin{aligned}
(70) \qquad \check{\pi} &\leq \delta\big(n(1-J(\check{r}))J(\check{r})^{n-1}\check{r} + (1-J^{(n,2)}(\check{r}))\check{r}\big) \\
&\quad - \int_0^{\check{r}} \theta \, dJ^{(n-1,1)}(\theta) - (1-J^{(n-1,1)}(\check{r}))\check{r} \\
&= \delta(1-(1-e^{-z})^n)\check{r} \\
&\quad - \check{r}(n-1)\frac{1}{z}\int_0^z \tau e^{-\tau}(1-e^{-\tau})^{n-2}\,d\tau - (1-(1-e^{-z})^{n-1})\check{r} \\
&= \check{r}(\delta\rho(z,n) - \eta(z,n)),
\end{aligned}
$$

where $z \equiv \mu\check{r} = \mu/\lambda(\check{r}) \leq 1$ and we have made the substitution $\tau = \theta\mu$ in the integral. Because $\delta \leq \delta_n$, (64) implies $\delta\rho(z,n) - \eta(z,n) \leq 0$. Hence, (70) together with (69) shows (67). *Q.E.D.*

Proposition 3 shows that the presence of a resale opportunity never reduces initial seller revenue and strictly increases it if the speculator plays an active role. We provide two proofs. In Proof 1 we compare the differential equation (18) with the differential equation that is relevant in the absence of a resale opportunity (cf. Lemma 13) so as to argue that each regular-bidder type bids at least as much in the presence of a resale opportunity as without one and, moreover, the initial seller collects the speculator's bid if she wins. Proof 2 follows the intuition provided in Garratt and Tröger (2006, Section 5).

PROPOSITION 3: *For any regular-bidder number $n \geq 2$, discount factor $\delta \in (0,1)$, and distribution F that satisfies Assumption* 1, *the presence of a resale opportunity does not reduce initial seller revenue from a first-price or Dutch auction, and strictly increases it if (F, n, δ) is such that the speculator plays an active role.*

PROOF 1: From (18) it follows that

$$
(71) \qquad \beta'(\theta) \geq N(\theta, \beta(\theta)) \quad \text{a.e. } \theta \in [0,1].
$$

By $\beta(0) = 0$, Lemma 13, and (71),

$$
(72) \qquad \forall\, \theta \in [0,1], \quad \beta(\theta) \geq \beta^{\mathrm{I}}(\theta).
$$

The result is immediate from (72). *Q.E.D.*

PROOF 2: Consider bidder $i \in I$ with type $\theta_i \in (0,1]$ and bid $b_i \geq 0$. Her expected payoff if she consumes the good after winning in period 1 is given by $\check{v}_i(b_i, \theta_i)$ as defined in (27). Because $\delta < 1$, it is optimal for bidder i to con-

sume the good upon winning if she makes a bid b_i sufficiently close to $\beta(\theta_i)$.[10] Hence, $u_i(b_i, \theta_i) = \check{v}_i(b_i, \theta_i)$ for all b_i sufficiently close to $\beta(\theta_i)$. The mapping $b_i \mapsto \check{v}_i(b_i, \theta_i)$ is locally Lipschitz continuous on $(0, \beta(1)]$ (because ϕ is Lipschitz continuous on $[0, \beta(1)]$ by (18) and H is locally Lipschitz continuous on $(0, \beta(1)]$ by (19)). Hence, for a.e. $b \in (0, \beta(1))$, the optimality of the bid b for type $\theta_i = \phi(b)$ implies the first-order condition

$$
\begin{aligned}
(73) \qquad 0 &= \frac{\partial u_i}{\partial b}(b, \theta_i)\bigg|_{\theta_i=\phi(b)} = \frac{\partial \check{v}_i}{\partial b}(b, \theta_i)\bigg|_{\theta_i=\phi(b)} \\
&= H'(b)F(\phi(b))^{n-1}\big(\phi(b) - b - \delta\big(\phi(b) - \hat{P}_{\phi(b)}(\phi(b))\big)\big) \\
&\quad + H(b)F(\phi(b))^{n-2} \\
&\quad \times \big((n-1)f(\phi(b))\phi'(b)(\phi(b) - b) - F(\phi(b))\big).
\end{aligned}
$$

If $H'(b) > 0$, then $b \le \delta M(\phi(b))$ because otherwise the speculator would make losses with bid b. Hence, if $H'(b) > 0$, then $b < \delta \hat{P}_{\phi(b)}(\phi(b))$ by (9). Hence, (73) implies

$$
(74) \qquad (n-1)f(\phi(b))\phi'(b)(\phi(b) - b) - F(\phi(b)) \begin{cases} < 0 & \text{if } H'(b) > 0, \\ = 0 & \text{if } H'(b) = 0. \end{cases}
$$

Using the bid function β instead of its inverse ϕ, (74) can be equivalently written as

$$
\beta'(\theta) \begin{cases} > N(\theta, \beta(\theta)) & \text{if } H'(\beta(\theta)) > 0, \\ = N(\theta, \beta(\theta)) & \text{if } H'(\beta(\theta)) = 0, \end{cases} \qquad \text{a.e. } \theta \in [0, 1].
$$

Comparing this with Lemma 13, it is immediate that (72) holds and that $\beta(\theta) > \beta^1(\theta)$ for a positive mass of θ-values if the speculator plays an active role.

Q.E.D.

3. THE SECOND-PRICE AUCTION WITH RESALE

In this section we construct and discuss a continuum of pure-strategy perfect Bayesian equilibria for second-price auctions with resale where the speculator plays an active role. Proposition 4 describes the equilibria. Propositions 5 and 6 evaluate the impact of a resale opportunity on initial seller revenue.

Let β denote bidder i's ($i \in I$) bid in the second-price auction as a function of her use value. Let $\hat{b}_s = \tilde{b}_s$ denote the speculator's bid (we do not allow randomization; hence, $\tilde{b}_s$ is a degenerate random variable).

[10]The proof would be slightly different if $\delta = 1$. It would then be optimal for bidder i to offer the good for resale after winning with a bid b_i arbitrarily close to $\beta(\theta_i)$. However, the additional payoff gain from reselling the good is only a second-order effect because the highest losing type, $\phi(b_i)$, is approximately equal to θ_i.

To define post-auction beliefs, consider any bidder $j \in I$ and let $b_j \geq 0$ denote her bid. Let $i \in I \cup \{s\}$ denote the label of the winner. Then the probability distribution $\Pi_j(\cdot|i, b_j)$ denotes the post-auction belief about j's use value of bidders other than j.

Let $\mathcal{M}(i, \mathbf{b}_{-i}, \theta_i)$ denote the resale mechanism used by the resale seller $i \in I \cup \{s\}$ after a second-price auction when the vector of losing bids is $\mathbf{b}_{-i} \in [0, \infty)^n$ and i's use value is θ_i. For all $i, j \in I \cup s$ with $j \neq i$, all $\mathbf{b}_{-i}$, and all θ_i, θ_j, let $P_j(i, \mathbf{b}_{-i}, \theta_i, \theta_j)$ denote the net expected transfer from bidder j of type θ_j to the other bidders (including the transfer to i) in the mechanism $\mathcal{M}(i, \mathbf{b}_{-i}, \theta_i)$. Let $Q_j(i, \mathbf{b}_{-i}, \theta_i, \theta_j)$ denote the probability that bidder j obtains the good. Let $P(i, \mathbf{b}_{-i}, \theta_i)$ denote the expected transfer *to* the resale seller i and let $Q(i, \mathbf{b}_{-i}, \theta_i)$ denote the probability that the resale seller keeps the good.

For all $i \in I$, bidder i's expected payoff when she bids $b_i \geq 0$ and has the use value θ_i equals

$$\begin{aligned} &u_i(b_i, \theta_i) \\ &\quad = E\Bigg[\big(-\tilde{b}_{-i}^{(1)} + \max\big\{\theta_i, \delta(\theta_i Q(i, \tilde{\mathbf{b}}_{-i}, \theta_i) + P(i, \tilde{\mathbf{b}}_{-i}, \theta_i))\big\}\big)\mathbb{1}_{w(b_i, \tilde{\mathbf{b}}_{-i})=i} \\ &\qquad + \sum_{j \neq i} \delta\big(\theta_i Q_i(j, (b_i, \tilde{\mathbf{b}}_{-j-i}), \tilde{\theta}_j, \theta_i) \\ &\qquad\qquad - P_i(j, (b_i, \tilde{\mathbf{b}}_{-j-i}), \tilde{\theta}_j, \theta_i)\big)\mathbb{1}_{w(b_i, \tilde{\mathbf{b}}_{-i})=j}\Bigg], \end{aligned}$$

where w denotes the period-1 winner as a function of the bid profile and where the max term reflects the condition that after winning in period 1, bidder i decides optimally whether to consume the good or offer it for resale. The speculator's payoff when she bids $b_s \geq 0$ is given by

$$u_s(b_s) = E\big[(-\tilde{b}_{-s}^{(1)} + \delta P(s, \tilde{\mathbf{b}}_{-s}, \theta_s))\mathbb{1}_{w(b_s, \tilde{\mathbf{b}}_{-s})=s}\big].$$

The equilibrium conditions are that post-auction beliefs about auction losers are determined by Bayes rule whenever possible (75), that the resale mechanism is chosen according to Assumption 4 (76), and that period-1 behavior is optimal (77) and (78).

DEFINITION 2: *A tuple* $(\beta, \hat{b}_s, \mathcal{M})$ *is a quasisymmetric regular equilibrium of the second-price auction with resale if there exists a belief system* $(\Pi_j(\cdot| i, b_j))_{j \in I, i \in I \cup \{s\}, b_j \geq 0}$ *such that the following conditions hold*:

$$\begin{aligned} &(75) \qquad \forall\, i \in I \cup \{s\}, j \in I \setminus \{i\}, b_j \geq 0, \\ &\qquad\qquad \Pi_j(\cdot|i, b_j) = \hat{F}_{\beta^{-1}(b_j)} \quad \textit{if} \quad b_j \in \beta([0, 1]), \end{aligned}$$

$$\forall i \in I \cup \{s\}, \mathbf{b}_{-i} \in [0,\infty)^n, \theta_i, \tag{76}$$
$$\mathcal{M}(i, \mathbf{b}_{-i}, \theta_i) = \hat{\mathcal{M}}\big((\Pi_j(\cdot|i, b_j))_{j\in I}, \theta_i, i\big),$$

$$\forall i \neq s, \theta_i, \quad \beta(\theta_i) \in \arg\max_{b_i \geq 0} u_i(b_i, \theta_i), \tag{77}$$

$$\hat{b}_s \in \arg\max_{b_s \geq 0} u_s(b_s). \tag{78}$$

This equilibrium concept is in the spirit of perfect Bayesian equilibrium, combined with the symmetry and regularity restrictions formulated in Assumption 2. Like in the first-price auction case (see the explanation below Definition 1), an equilibrium condition on the post-auction beliefs about the auction winner is omitted because it would play no role for our analysis.

The result below describes our equilibria. The structure of these equilibria is analogous to the structure of the equilibria constructed in Garratt and Tröger (2006, Proposition 2). Condition (81) states that the resale mechanism used by the speculator is a standard auction with an optimal reserve price. Observe that if the speculator wins at a strictly positive price, then she is certain about the maximum use value among all bidders (see (82)); hence the standard auction yields the same outcome as a take-it-or-leave-it offer equal to the auction's reserve price.

PROPOSITION 4: *For any regular-bidder number $n \geq 2$, discount factor $\delta \in (0,1)$, distribution F that satisfies Assumption* 1, *and every $\theta^* \in [0,1]$, the second-price auction with resale has a quasisymmetric regular equilibrium $(\beta, \hat{b}_s, \mathcal{M})$ with the properties*

$$\forall \theta_i, \quad \beta(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \in [0, \theta^*), \\ \theta_i & \text{if } \theta_i \in (\theta^*, 1], \end{cases} \tag{79}$$

$$\hat{b}_s = \theta^* - \delta(\theta^* - \hat{P}_{\theta^*}(\theta^*)), \tag{80}$$

$$\forall \mathbf{b}_{-s}, \quad \mathcal{M}(s, \mathbf{b}_{-s}, 0) = \begin{cases} \mathcal{S}(\hat{r}(\theta^*)) & \text{if } b_{-s}^{(1)} = 0, \\ \mathcal{S}(\theta^*) & \text{if } b_{-s}^{(1)} \in (0, \theta^*), \\ \mathcal{S}(b_{-s}^{(1)}) & \text{if } b_{-s}^{(1)} \in (\theta^*, 1]. \end{cases} \tag{81}$$

It is supported by beliefs that satisfy

$$\forall i, j \notin \{s, i\}, b_j, \theta_j, \quad \Pi_j(\theta_j|i, b_j) = \begin{cases} \hat{F}_{[0,\theta^*]}(\theta_j) & \text{if } b_j = 0, \\ \mathbb{1}_{\theta_j \geq \theta^*} & \text{if } b_j \in (0, \theta^*), \\ \mathbb{1}_{\theta_j \geq b_j} & \text{if } b_j \in (\theta^*, 1]. \end{cases} \tag{82}$$

The proof is a straightforward generalization of the arguments leading to the proof of Proposition 2 in Garratt and Tröger (2006), except for the following complication. With multiple private-value bidders, we have to show that a

regular bidder with type $\theta_i < \theta^*$ will not deviate to a bid in $[\hat{b}_s, \theta^*]$ and subsequently offer the good for resale if she wins. A detailed proof is below; here is a sketch of the argument. Suppose that bidder i makes such a deviation and suppose if she wins, then she appropriates the entire expected surplus that is available in the resale market. Because an increase in θ_i changes the resale surplus only if all bidders other than i have types below θ_i, the marginal change in expected resale surplus due to an increase in θ_i equals the probability that all bidders other than i have types below θ_i. However, this probability equals (or is an upper bound for if θ_i is below the resale reserve price $\hat{r}(\theta^*)$) the probability that bidder i obtains the good if she bids 0 and waits for the speculator's resale mechanism. Hence, using the envelope theorem, the derivative of bidder i's equilibrium payoff with respect to θ_i is at most as great as the derivative of the expected resale surplus obtained from the deviation. Hence, if for any type $\theta_i < \theta^*$, the deviation is profitable, then the deviation is also profitable for type $\theta_i = \theta^*$. However, for type θ^* the deviation is not profitable by construction of $\hat{b}_s$.

PROOF OF PROPOSITION 4: The proof of (75), (76), and (78) is straightforward. The proof of (77) is straightforward, too, except for one step: we have to show that a bidder $i \in I$ with type $\theta_i \leq \theta^*$ cannot profit from deviating to a bid $b_i \in [\hat{b}_s, \theta^*]$ and offering the good for resale if she wins. Observe that bidder i's resale payoff cannot exceed the entire expected surplus that is available in the resale market:

$$u_i(b_i, \theta_i) \leq \overline{u}_i(\theta_i) \equiv \left(-\hat{b}_s + \delta E[\max\{\theta_i, \tilde{\theta}_{-i}^{(1)}\} \mid \tilde{\theta}_{-i}^{(1)} \leq \theta^*]\right) F(\theta^*)^{n-1}. \tag{83}$$

A straightforward computation shows that

$$\frac{\partial \overline{u}_i}{\partial \theta_i}(\theta_i) = \delta F(\theta_i)^{n-1}. \tag{84}$$

Bidder i's equilibrium payoff equals

$$u_i(0, \theta_i) = \delta(\theta_i \hat{Q}_{\theta^*}(\theta_i) - \hat{P}_{\theta^*}(\theta_i)) F(\theta^*)^{n-1}.$$

The envelope theorem implies

$$\frac{\partial u_i}{\partial \theta_i}(0, \theta_i) = \begin{cases} \delta F(\theta_i)^{n-1} & \text{if } \theta_i > \hat{r}(\theta^*), \\ 0 & \text{if } \theta_i < \hat{r}(\theta^*). \end{cases} \tag{85}$$

By (80),

$$u_i(0, \theta^*) \geq \overline{u}_i(\theta^*). \tag{86}$$

Hence, for all $\theta_i' \leq \theta^*$,

$$u_i(0, \theta_i') - \overline{u}_i(\theta_i') \overset{(86)}{\geq} u_i(0, \theta_i') - u_i(0, \theta^*) - (\overline{u}_i(\theta_i') - \overline{u}_i(\theta^*))$$
$$= \int_{\theta_i'}^{\theta^*} \left(\frac{\partial \overline{u}_i}{\partial \theta_i}(\theta_i) - \frac{\partial u_i}{\partial \theta_i}(0, \theta_i) \right) d\theta_i \overset{(84),(85)}{\geq} 0,$$

which completes the proof. *Q.E.D.*

In contrast to the first-price/Dutch auction setting (cf. Proposition 3), the presence of a resale opportunity may either increase or decrease initial seller revenue, depending on which θ^*-equilibrium is played. This comparison assumes that in the absence of a resale opportunity all bidders use the dominant strategy of bidding their use values.

PROPOSITION 5: *For any regular-bidder number $n \geq 2$, discount factor $\delta \in (0, 1)$, and distribution F that satisfies Assumption* 1, *the second-price auction with resale has quasisymmetric regular equilibria such that the initial seller's expected revenue is larger* (*alternatively*, *smaller*) *than the expected revenue that results when all bidders bid their use values*.

PROOF: For all $\theta^* \in [0, 1]$, let $R(\theta^*)$ denote the initial seller's expected revenue in a θ^*-equilibrium. In particular, $\pi(0)$ is the expected revenue of the initial seller when every bidder bids her use value in period 1. We will show that $R(\theta^*) > R(0)$ for all θ^* sufficiently close to 0 and $R(\theta^*) < R(0)$ for all θ^* sufficiently close to 1.

Let $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$ denote the highest and second-highest use values among the regular bidders:

$$R(\theta^*) - R(0) = \Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]\big(\hat{b}_s - E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]\big) + \Pr[\tilde{\theta}^{(1)} < \theta^*](0 - E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(1)} < \theta^*]). \tag{87}$$

We have $\Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}] = nF(\theta^*)^{n-1}(1 - F(\theta^*)) \to 0$ as $\theta^* \to 1$. Thus, $R(\theta^*) < R(0)$ for all θ^* sufficiently close to 1.

Let us now consider the case where θ^* is close to 0. From (80) it follows that $\hat{b}_s \geq \hat{P}_{\theta^*}(\theta^*)$. Hence, (87) implies

$$R(\theta^*) - R(0) \geq \Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]\big(\hat{P}_{\theta^*}(\theta^*) - E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]\big) - \Pr[\tilde{\theta}^{(1)} < \theta^*]E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(1)} < \theta^*]. \tag{88}$$

By Lemma 1 and because $\hat{r}'(0) = 1/2$,

$$\begin{aligned}\frac{\hat{P}_{\theta^*}(\theta^*)}{\theta^*} &= 1 - \frac{1}{\theta^* F(\theta^*)^{n-1}} \int_{\theta^*/2}^{\theta^*} (f(0)\theta + o(\theta))^{n-1}\, d\theta + \frac{o(\theta^*)}{\theta^*} \\ &= 1 - \frac{1}{\theta^* F(\theta^*)^{n-1}} \int_{\theta^*/2}^{\theta^*} \left(f(0)^{n-1}\theta^{n-1} + o(\theta^{n-1})\right) d\theta + \frac{o(\theta^*)}{\theta^*} \\ &= 1 - \frac{f(0)^{n-1}}{\theta^* F(\theta^*)^{n-1}} \left(\frac{\theta^{*n}}{n}\left(1 - \frac{1}{2^n}\right) + o(\theta^{*n})\right) + \frac{o(\theta^*)}{\theta^*}.\end{aligned}$$

Therefore,

$$\lim_{\theta^* \to 0} \frac{\hat{P}_{\theta^*}(\theta^*)}{\theta^*} = 1 - \frac{1}{n}\left(1 - \frac{1}{2^n}\right) = \frac{n-1}{n} + \frac{1}{n2^n}. \tag{89}$$

Similarly,

$$\lim_{\theta^* \to 0} \frac{E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]}{\theta^*} = \frac{n-1}{n} \tag{90}$$

and

$$\lim_{\theta^* \to 0} \frac{E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(1)} < \theta^*]}{\theta^*} = \frac{n-1}{n+1}. \tag{91}$$

Moreover,

$$\frac{\Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]}{\Pr[\tilde{\theta}^{(1)} < \theta^*]} = n\frac{1 - F(\theta^*)}{F(\theta^*)} \xrightarrow{\theta^* \to 0} \infty. \tag{92}$$

Taking (88), (89), (90), (91), and (92) together implies that $R(\theta^*) > R(0)$ if θ^* is small. *Q.E.D.*

The final Proposition 6 assumes that the initial seller is a player whose objective is to maximize expected revenue. We maintain the assumption that the initial seller is restricted to a second-price auction, but assume that she can set an arbitrary reserve price before the auction begins. Proposition 6 shows that the initial seller's revenue in such an extended second-price auction game with resale can be smaller than her revenue in the absence of a resale opportunity (where all bidders have a dominant strategy to bid their use values).

Observe that a special case covered by Assumption 4 is that the resale seller is, like the initial seller, restricted to a second-price auction with reserve price. Hence, Proposition 6 shows that the presence of a resale opportunity can be harmful to the initial seller even if she has access to the same class of sales mechanisms as the resale seller.

PROPOSITION 6: *For any regular-bidder number $n \geq 2$ and distribution F that satisfies Assumption* 1, *let R^* denote the initial seller's revenue in a second-price auction when all bidders bid their use values and she chooses an optimal reserve price. Then, for all sufficiently large n and for all δ sufficiently close to* 1, *the extended second-price auction game with resale, where the initial seller can set a reserve price, has a perfect Bayesian equilibrium such that the initial seller's expected revenue is smaller than R^*.*

PROOF: Let r^* denote the reserve price that is optimal for the initial seller if all agents bid their use values in period 1. We will show that if δ is sufficiently close to 1, then there exists an equilibrium such that the initial seller sets a reserve price $r > r^*$ and her expected revenue is smaller than R^*.

Because $r^* < 1$ is independent of n (Myerson (1981)) and $R^* \to 1$ as $n \to \infty$,

$$r^* < \delta R^*, \tag{93}$$

assuming that n is sufficiently large and δ is sufficiently close to 1.

Observe that the speculator's payoff in the θ^*-equilibrium with $\theta^* = 1$ constructed in Proposition 4 equals δR^*, and her auction bid $\hat{b}_s > R^*$ by (80) and Lemma 3. Hence, the 1-equilibrium remains valid, with a reduced payoff for the speculator, if the initial seller sets any reserve price $r < \delta R^*$.

By (93), there exists $\underline{r}$ such that $r^* < \underline{r} < \delta R^*$. Using $\underline{r}$ we can construct continuation equilibria for the continuation games following any reserve price $r \geq 0$ by the initial seller. If $r < \underline{r}$, let the 1-equilibrium be played in the continuation game. If $r > \underline{r}$, let the θ^*-equilibrium with $\theta^* = 0$ be played in the continuation game. If $r = \underline{r}$, let either the 1-equilibrium or the 0-equilibrium be played, depending on which of these two leads to a higher initial seller revenue.

Given these continuation equilibria, every reserve price $r < \underline{r}$ results in an expected revenue of $r < R^*$ for the initial seller. Any reserve price $r > \underline{r}$ leads to an initial seller revenue below R^* because otherwise we would have $r = r^*$ by definition of r^*. The reserve price $r = \underline{r}$ leads to the initial seller revenue $\underline{r} < R^*$ if the 1-equilibrium is played and leads to an initial seller revenue below R^* if the 0-equilibrium is played (if the revenue R^* were reached, we would have $\underline{r} = r^*$ by definition of r^*). In summary, any reserve price $r \geq 0$ leads to an initial seller revenue below R^*. *Q.E.D.*

4. THE ENGLISH AUCTION WITH RESALE

In this section we explain why the equilibrium outcomes that we have constructed for second-price auctions with resale remain valid if in period 1 an English auction, as modelled by Milgrom and Weber (1982), takes place. Observe that the second-price auction and the English auction are not strategically equivalent because we are considering environments with three or more bidders.

After the English auction is completed, the losers' bids are publicly known, while the winner's bid remains private. This corresponds exactly to the bid revelation that we have assumed in our analysis of the second-price auction with resale. Hence, if the bidders' stopping points in the English auction are identical to their bids in the second-price auction, the bidders' payoffs are the same across the two auction formats. Thus, the only thing we need to argue to show that the equilibrium outcomes of Proposition 4 remain valid for the English auction is that bidding incentives are the same as in a second-price auction. The difference between the second-price auction and the English auction is that the losing bids become public during instead of after the auction, so that bidders can revise their beliefs each time a bidder drops out.

We define bidding in the English auction as follows. Every bidder $i \in I$ with use value $\theta_i \in [0, 1]$ stops bidding at price 0 if $\theta_i < \theta^*$ and is willing to bid up to her use value if $\theta_i > \theta^*$, independently of who stays in and how long. The speculator is willing to bid up to $\hat{b}_s$, also independently of who stays in and how long.

To see that these bidding strategies are optimal, consider first the speculator's bidding incentives, given the regular bidders' strategies. At the bid 0, it is better to stay in than to drop out because the latter means she foregoes her chances of winning and making a resale profit. If some regular bidder stays in beyond 0 as well, Bayesian updating requires the speculator to believe that the use value of this regular bidder is distributed on the support $[\theta^*, 1]$. She expects the regular bidder to stay in up to her use value, which is beyond $\hat{b}_s$. Hence, it is optimal for the speculator to drop out at $\hat{b}_s$. If one regular bidder deviates by stopping at a bid in $(0, \theta^*]$, the speculator switches to the belief that the use value of this regular bidder equals θ^*. Thus, if the speculator wins, then she uses the same resale mechanism as in the second-price auction case.

Now consider the bidding incentives of a bidder $i \in I$ with use value $\theta_i \in [0, 1]$. If $\theta_i < \theta^*$, bidder i has no incentive to stay in beyond the bid 0, for the same reasons as in the second-price auction case. Finally, consider the case $\theta_i > \theta^*$. Bidder i will stay in until the price reaches her use value because she expects the same from all other regular bidders once they stay in at positive bids. When bidder i observes that a competing regular bidder drops out at a bid in $(0, \theta^*]$, she switches to the belief that this regular bidder has the use value θ^*, and it remains optimal for bidder i to stay in until the price reaches her use value.

Dept. of Economics, University of California at Santa Barbara, CA 93106-9210, U.S.A.; garratt@econ.ucsb.edu

and

Dept. of Economics, University of Bonn, Adenauerallee 24-26, 53113 Bonn, Germany; ttroeger@uni-bonn.de.

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