Econometrica Supplementary Material

THE AFFILIATION EFFECT IN FIRST-PRICE AUCTIONS: SUPPLEMENTARY MATERIAL

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In Pinkse and Tan (2005) we show the existence of a new effect called the *affiliation effect*, which can cause equilibrium bids to be decreasing in the number of bidders in first-price auctions with conditionally independent private values. Here we analyze what happens when the number of bidders tends to infinity. We also derive sufficient conditions for the expected winning bid to be increasing in the number of bidders. We further generalize some of the results in Pinkse and Tan (2005) to general affiliated private-values models.

KEYWORDS: Affiliation effect, first-price auctions, affiliated private values.

1. INTRODUCTION AND NOTATION

IN PINKSE AND TAN (2005) (henceforth referred to as PT) we show that, in first-price auctions with conditionally independent private values (CIPV), equilibrium bids can be decreasing in the number of bidders n. In this supplement we show that asymptotically bids *will* be increasing in n. We also provide sufficient conditions for the expected winning bid (i.e., price) to be increasing in n. Moreover, we extend some of the results in PT to general affiliated private-values (APV) models.

Following PT, $(x_1, \dots, x_n)$ is an affiliated and exchangeable random vector of private valuations with support $[\underline{x}, \bar{x}]^n$. From Milgrom and Weber (1982) it follows that the equilibrium bid in any APV model of first-price auctions with a reserve price r is given by

$$B(x,n) = x - \int_r^x \exp\left\{-\int_t^x R(s,n)\,ds\right\} dt, \tag{1}$$

with

$$R(x,n) = \frac{f_{n-1}(x|x)}{F_{n-1}(x|x)},$$

where $F_{n-1}(\cdot|x)$ denotes the conditional distribution function of $\max_{j\neq i} x_j$ given $x_i = x$ and $f_{n-1}(\cdot|x)$ the corresponding density. We refer to R as the reverse hazard.

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In the particular case of the CIPV model—i.e., a class of APV models in which bidders' private valuations, $x_1, \dots, x_n$, are affiliated through a random variable z but in which they are independent conditional on z—the reverse hazard is given by

$$R(x,n) = (n-1)\frac{\int_{\underline{z}}^{\bar{z}} H^{n-2}(x|z)h^2(x|z)g(z)\,dz}{\int_{\underline{z}}^{\bar{z}} H^{n-1}(x|z)h(x|z)g(z)\,dz}, \tag{2}$$

where $H(x|z)$ and $h(x|z)$ are the conditional distribution and density function of x_i given z and G, g are the distribution and density function of z with support $[\underline{z}, \bar{z}]$. We assume that h satisfies the strict MLRP.

In Section 2.1 we show that if the number of bidders becomes large, then R is again increasing in n. Section 2.2 contains the results about price. Finally in Section 3 we establish our results for general APV models.

2. CONDITIONAL INDEPENDENT PRIVATE-VALUES MODEL

2.1. *Asymptotics*

In PT, Lemmas 1 and 2, we show that if and only if R can be decreasing in n then so can B. Using this property, we also show there (Corollary 1) that B can indeed be decreasing in n. However, as the following proposition indicates, in the limit R is everywhere increasing in n and hence so is B.

PROPOSITION 1: *In the CIPV model, let the first partial derivatives of $H(x|z)$ and $h(x|z)$ with respect to z be continuous in both x, z and let $H(x|z)h(x|z)$ be bounded away from zero on $[r, \bar{x}] \times [\underline{z}, \bar{z}]$. Then, for sufficiently large n, $\max_{x\in[r,1]}(R(x,n) - R(x,n+1))$ is negative.*

For the proof see Appendix A.

Please note that Proposition 1 implies that bids are increasing in n if the number of bidders is sufficiently large.

2.2. *Price*

We now offer sets of sufficient conditions for the expected maximum bid (price),

$$P_n = \int_r^{\bar{x}} B(x,n) f_n(x)\,dx,$$

to be monotonically increasing in n, where $f_n(x)$ is the density function of the maximum valuation $X_n = \max_{j=1}^n x_j$. We have been unable to prove that price is always increasing in n, but if a counterexample exists, it must be fairly extreme.

An increase in n generates three effects on price: the *competition and affiliation effects*, which were introduced in Section 3.2 of PT, and a *sampling effect*. The sampling effect arises because the presence of an additional bidder results in an additional draw from the valuation distribution. One additional draw from the valuation distribution causes the highest valuation to be greater with probability $1/(n+1)$ and the same with probability $n/(n+1)$. Since equilibrium bids are increasing in x, the sampling effect exerts an upward influence on dP_n/dn. This is reflected in Lemma 1 below.

Let

$$\Delta_n(x) \equiv \frac{\frac{\partial B}{\partial n}(x,n)}{\frac{\partial B}{\partial x}(x,n)} - \frac{\frac{\partial F_n}{\partial n}(x)}{\frac{\partial F_n}{\partial x}(x)}, \tag{3}$$

where $F_n(x)$ is the distribution function corresponding to $f_n(x)$. In equation (3), the term $-(\partial F_n/\partial n)(x)$ quantifies the sampling effect and is positive. Due to the affiliation effect, $(\partial B/\partial n)(x,n)$ can be negative. As long as the sampling effect is strong enough to offset the affiliation effect, price will be increasing in n.

LEMMA 1: *Let $G_{Xn}(b) = P(B(X_n, n) \le b)$. Then*

$$\forall x:\quad \Delta_n(x) \ge 0 \quad \Longleftrightarrow \quad \forall b:\quad \frac{\partial G_{Xn}}{\partial n}(b) \le 0.$$

PROOF: Let B^{-1} be such that $B^{-1}(B(x,n),n) = x$ for all x, n. Note that $G_{Xn}(b) = F_n(B^{-1}(b,n))$. Hence, using t as shorthand for $B^{-1}(b,n)$,

$$\begin{aligned}\frac{\partial G_{Xn}}{\partial n}(b) &= \frac{\partial F_n}{\partial n}(t) + f_n(t)\frac{\partial B^{-1}}{\partial n}(b,n) \\ &= \frac{\partial F_n}{\partial n}(t) - f_n(t)\frac{\frac{\partial B}{\partial n}(t,n)}{\frac{\partial B}{\partial x}(t,n)} = -f_n(t)\Delta_n(t).\end{aligned}$$

Q.E.D.

Lemma 1 shows that even if bids are decreasing in n over a range of x's, the sampling effect can nevertheless cause the distribution of the winning bid for $n+1$ bidders to first-order stochastically dominate the distribution for n bidders. First-order stochastic dominance is sufficient for an increase in expectation.

For an interesting class of distributions first-order stochastic dominance holds, as Proposition 2 demonstrates.

PROPOSITION 2: *Suppose, for some function λ, $\forall x, z$: $H(x|z) = \exp(\lambda(x)z)$. Then $\forall x$: $\Delta_n(x) \ge 0$.*

PROOF: Let $K_j(x, n) = \int z^j \exp(n\lambda(x)z)$, $j = 0, 1, 2$. Then

$$R(x, n) = (n-1)\lambda'(x)\frac{K_2(x, n)}{K_1(x, n)} = \frac{n-1}{n}\frac{\partial \log K_1}{\partial x}(x, n),$$

such that

$$\begin{aligned}(4)\qquad &\int_t^x \frac{\partial R}{\partial n}(s, n)\,ds \\ &= n^{-2}\int_t^x R(s, n)\,ds + \frac{n-1}{n}\left(\frac{\partial \log K_1}{\partial n}(x, n) - \frac{\partial \log K_1}{\partial n}(t, n)\right) \\ &\geq \frac{n-1}{n}\lambda(x)\frac{K_2(x, n)}{K_1(x, n)}.\end{aligned}$$

Denote the right-hand side in (4) by $A_n(x)$. Note further that

$$R(x, n)\frac{\frac{\partial F_n}{\partial n}(x)}{\frac{\partial F_n}{\partial x}(x)} = (n-1)\lambda'(x)\frac{K_2(x, n)}{K_1(x, n)}\frac{\lambda(x)K_1(x, n)}{n\lambda'(x)K_1(x, n)} = A_n(x).$$

Hence,

$$\begin{aligned}\frac{\partial B}{\partial n}(x, n) &= \int_r^x \int_t^x \frac{\partial R}{\partial n}(s, n)\,ds \exp\left(-\int_t^x R(s, n)\,ds\right)dt \\ &\geq A_n(x)\int_r^x \exp\left(-\int_t^x R(s, n)\,ds\right)dt \\ &\geq \frac{A_n(x)}{R_n(x)}\frac{\partial B}{\partial x}(x, n) = \frac{\partial B}{\partial x}(x, n)\frac{\frac{\partial F_n}{\partial n}(x)}{\frac{\partial F_n}{\partial x}(x)},\end{aligned}$$

such that $\Delta_n(x) \geq 0$. *Q.E.D.*

Examples of distributions that are of the special form used in Proposition 2 are power distributions $H(x, z) = x^z$ ($\lambda(x) = \log x$) and the Wilson distribution of Example 1 in PT.

There are alternative conditions which are sufficient for price to be increasing in n but do not imply first-order stochastic dominance of the distributions of maximum bids. Consider the CIPV model and define

$$V_n(x) = \int p_n(z|\mathcal{W}, x)\frac{h'(x|z)}{h(x|z)}\,dz.$$

Note that

$$R(x, n) = \frac{d \log f_n}{dx}(x) - V_n(x),$$

so that price can be rewritten as

$$
\begin{aligned}
P_n &= \int_r^{\bar{x}} f_n(x) B(x, n)\, dx \\
&= \int_r^{\bar{x}} f_n(x) \left(x - \int_r^{x} \frac{f_n(t)}{f_n(x)} \exp\left\{ \int_t^{x} V_n(s)\, ds \right\} dt \right) dx \\
&= \int_r^{\bar{x}} f_n(x) I_n(x)\, dx,
\end{aligned}
$$

where $I_n(x) = x - \int_x^{\bar{x}} \exp\{\int_x^t V_n(s)\, ds\}\, dt$.

PROPOSITION 3: *In the CIPV model, if* $\forall\, x \in (\underline{x}, \bar{x}]$: $V_n(x) \geq 0$ *and if* $I_n(r) \geq 0$, *then price is increasing in* n.

PROOF: Note that

$$
\begin{aligned}
\frac{dP_n}{dn} = &\int_r^{\bar{x}} \frac{\partial f_n}{\partial n}(x) I_n(x)\, dx \\
&- \int_r^{\bar{x}} f_n(x) \int_x^{\bar{x}} \int_x^{t} \frac{\partial V_n}{\partial n}(s)\, ds \exp\left\{ \int_x^{t} V_n(s)\, ds \right\} dt\, dx.
\end{aligned}
$$

Now if $L(x|z) = -\log H(x|z)$, $\rho(x, z) = h'(x|z)/h(x, z)$, then, by Lemma 3 of PT,

$$
\begin{aligned}
\frac{\partial V_n}{\partial n}(x) &= \int p_n(z, \mathcal{W}, x) L(x, z)\, dz \int p_n(z, \mathcal{W}, x) \rho(x, z)\, dz \\
&\quad - \int p_n(z, \mathcal{W}, x) L(x|z) \rho(x, z)\, dz \\
&\leq 0,
\end{aligned}
$$

since both L and ρ are increasing in z by affiliation. Finally, applying integration by parts yields

$$
\int_r^{\bar{x}} \frac{\partial f_n}{\partial n}(x) I_n(x)\, dx = -\frac{\partial F_n}{\partial n}(r) I_n(r) - \int_r^{1} \frac{\partial F_n}{\partial n}(x) I_n'(x)\, dx,
$$

where $I_n'(x) = 2 + V_n(x)(x - I_n(x)) \geq 0$ by the assumption that $V_n(x) \geq 0$. The claim follows since $I_n(r) \geq 0$ by assumption and since $\partial F_n/\partial n \leq 0$. *Q.E.D.*

Note that the assumption of $V_n(x) \geq 0$ is implied by the restriction that the conditional density $h(x|z)$ is weakly increasing in x, which is satisfied for a large class of distributions. However, $V_n(x) < 0$ whenever h is decreasing in x for all z. Here is an example:

$$H(x|z) = \frac{1 - \exp(-a(z)x)}{1 - \exp(-a(z))}$$

for $x \in [0, 1]$ and $z \in [\underline{z}, \bar{z}]$, where $a(z)$ is positive and decreases with z. Note that $h'(x|z)/h(x|z) = -a(z) < 0$.

If the reserve price is chosen to maximize the seller's expected revenue (or price), then the condition that $I_n(r) \geq 0$ in Proposition 3 is implied. This is stated in the following corollary.

COROLLARY 1: *In the CIPV model, if* $\forall x \in (\underline{x}, \bar{x}]$, $V_n(x) \geq 0$ *and if the seller sets the reserve price optimally, then price is increasing in* n.

PROOF: Note that

$$\frac{dP_n}{dr} = -f_n(r)I_n(r).$$

A necessary condition for an interior optimal reserve price is that $I_n(r) = 0$. The claim follows from Proposition 3. *Q.E.D.*

3. AFFILIATED PRIVATE-VALUES MODEL

We now generalize some of the results in PT to the case of general APV models. We will continue to assume exchangeability and affiliation and moreover impose that the joint valuation distribution of $(x_1, \ldots, x_m)$ does not depend on $n > m$.

3.1. *Affiliation Effect*

The first generalization is related to the affiliation effect. As in PT, we define

$$\forall x, n: \quad R_Q(x, n) = (n - 1)R(x, 2),$$
$$\Delta R(x, n) = R(x, n) - R_Q(x, n).$$

R_Q is naturally increasing in n and ΔR is decreasing in n. In the IPV case ΔR necessarily equals zero and no affiliation effect then exists; an affiliation effect exists if and only if $\Delta R(x, n) < 0$ for some values of x, n. In the APV case ΔR is not everywhere equal to zero; we now show that it cannot be positive since $\Delta R(x, 2) = 0$.

PROPOSITION 4: $\Delta R(x, n)$ *is decreasing in* n.

For the proof see Appendix B.

Proposition 4 is a generalization of Proposition 1 of PT.

3.2. *Bids at $\bar{x}$*

In PT, Proposition 3, we show that equilibrium bids in the CIPV models are necessarily increasing in n at the top of the value distribution, $\bar{x}$. Proposition 5 below establishes that it holds for general APV models, also. The result in Proposition 5 is important because it can be used as the basis of a test between APV and CV models since $B(\bar{x}, n)$ can be decreasing in n in CV models.

PROPOSITION 5: *$B(\bar{x}, n)$ is increasing in n.*

For the proof see Appendix B.

4. SUMMARY

We have established some additional interesting results over and above those contained in PT. We have shown that when the number of bidders continues to increase, then the reverse hazards will eventually once again be uniformly increasing in the number of bidders and hence so will bids. We have also derived some sufficient conditions for price to be increasing in the number of bidders. Finally, we have provided several generalizations of the results contained in PT.

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APPENDIX A: PROOF OF PROPOSITION 1

We show that for sufficiently large n, $\max_{x\in[r,\bar{x}]}((R(x,n)-R(x,n+1))H(x|\underline{z})/h(x|\underline{z}))<0$. Assume without loss of generality that $\underline{x}=\underline{z}=0$ and $\bar{x}=\bar{z}=1$. Let $T_x(z)=H(x|z)/H(x|0)$, $A_x(z)=h(x|z)/h(x|0)$, and $\phi_{nx}(z)=T_x^{n-2}(z)A_x(z)g(z)$. Then, omitting arguments,

$$
\begin{aligned}
(5)\qquad & \max_{x\in[r,1]}\big((R(x,n)-R(x,n+1))H(x|0)/h(x|0)\big)\\
&=\max_{x\in[r,1]}\left((n-1)\frac{\int\phi_{nx}A_x}{\int\phi_{nx}T_x}-n\frac{\int\phi_{nx}A_xT_x}{\int\phi_{nx}T_x^2}\right)\\
&=\max_{x\in[r,1]}\left(n\frac{\int\phi_{nx}A_x\int\phi_{nx}T_x^2-\int\phi_{nx}A_xT_x\int\phi_{nx}T_x}{\int\phi_{nx}T_x\int\phi_{nx}T_x^2}-\frac{\int\phi_{nx}A_x}{\int\phi_{nx}T_x}\right)\\
&\le n\max_{x\in[r,1]}\left(\frac{\int\phi_{nx}A_x\int\phi_{nx}T_x^2-\int\phi_{nx}A_xT_x\int\phi_{nx}T_x}{\int\phi_{nx}T_x\int\phi_{nx}T_x^2}\right)-\min_{x\in[r,1]}\frac{\int\phi_{nx}A_x}{\int\phi_{nx}T_x}.
\end{aligned}
$$

We first show that the second right-hand side term in the inequality in (5) is bounded away from zero, even in the limit. Indeed,

$$\min_{x\in[r,1]}\frac{\int\phi_{nx}A_x}{\int\phi_{nx}T_x}\ge\min_{x\in[r,1]}\frac{\min_{z\in[0,1]}A_x(z)}{\max_{z\in[0,1]}T_x(z)}>0.$$

We now show that the first term on the right-hand side of the inequality in (5) is $o(1)$,[2] which implies that the right-hand side in (5) is negative for sufficiently large n. Our proof will make repeated use of the mean value theorem to establish that the equivalent condition

$$\max_{x\in[r,1]}\left(\frac{\int \phi_{nx}A_x\int \phi_{nx}T_x^2-\int \phi_{nx}A_xT_x\int \phi_{nx}T_x}{\int \phi_{nx}T_x\int \phi_{nx}T_x^2}\right)=o(n^{-1}) \tag{6}$$

is satisfied. By the mean value theorem and $T_x(0)=A_x(0)=1$,

$$T_x(z)=1+T_x'(z^*(z))z,\quad A_x(z)=1+A_x'(\tilde z(z))z, \tag{7}$$

with $z^*(z), \tilde z(z)$ some numbers between 0 and z. Let $\bar T_x(z)=T_x'(z^*(z))$, $\bar A_x(z)=A_x'(\tilde z(z))$. Introduce the following shorthand notation.

$$\begin{aligned} S_{1x}&=\int \phi_{nx},\quad S_{2x}=\int \phi_{nx}\bar A_x z,\\ S_{3x}&=\int \phi_{nx}\bar T_x z,\quad S_{4x}=\int \phi_{nx}\bar T_x^2 z^2,\quad \text{and}\\ S_{5x}&=\int \phi_{nx}\bar T_x\bar A_x z^2. \end{aligned} \tag{8}$$

Observe that each of the integrals in (6) can be expressed in terms of the S-variables defined in (8) using the expansion in (7), e.g.,

$$\int \phi_{nx}A_x=\int \phi_{nx}+\int \phi_{nx}\bar A_x z=S_{1x}+S_{2x}.$$

Using similar expansions for the other integrals in (5), the left-hand side in (6) is

$$\begin{aligned} &\max_{x\in[r,1]}\frac{(S_{1x}+S_{2x})(S_{1x}+2S_{3x}+S_{4x})-(S_{1x}+S_{2x}+S_{3x}+S_{5x})(S_{1x}+S_{3x})}{(S_{1x}+S_{3x})(S_{1x}+2S_{3x}+S_{4x})}\\ &\quad=\max_{x\in[r,1]}\frac{S_{1x}S_{4x}+S_{2x}S_{3x}+S_{2x}S_{4x}-S_{1x}S_{5x}-S_{3x}^2-S_{3x}S_{5x}}{S_{1x}^2+3S_{1x}S_{3x}+S_{1x}S_{4x}+2S_{3x}^2+S_{3x}S_{4x}}.\\ &\quad=\max_{x\in[r,1]}\frac{\frac{S_{4x}}{S_{1x}}+(\frac{S_{2x}}{S_{1x}})(\frac{S_{3x}}{S_{1x}})+(\frac{S_{2x}}{S_{1x}})(\frac{S_{4x}}{S_{1x}})-\frac{S_{5x}}{S_{1x}}-(\frac{S_{3x}}{S_{1x}})^2-(\frac{S_{3x}}{S_{1x}})(\frac{S_{5x}}{S_{1x}})}{1+3\frac{S_{3x}}{S_{1x}}+\frac{S_{4x}}{S_{1x}}+2(\frac{S_{3x}}{S_{1x}})^2+(\frac{S_{3x}}{S_{1x}})(\frac{S_{4x}}{S_{1x}})}. \end{aligned} \tag{9}$$

Let $\psi_n=n^{-\kappa}$ for some $\kappa\in(1/2,1)$. Note that (9) is $O(\psi_n^2)$ if we can show that $\max_{x\in[r,1]}(S_{jx}/S_{1x})=O(\psi_n)$, $j=2,3$, and $\max_{x\in[r,1]}(S_{jx}/S_{1x})=O(\psi_n^2)$, $j=4,5$. We will establish that $S_{4x}/S_{1x}=O(\psi_n^2)$, where the remaining results follow similarly. Let

$$D_{nx}=\{z\in[0,1]:\bar T_x(z)=0\vee z\le \psi_n/|\bar T_x(z)|\},$$

with $\vee=$ "or" and let D_{nx}^c be the complement of D_{nx}. Now, by the definition of D_{nx}, D_{nx}^c, and (7),

$$\forall z\in D_{nx}:\quad \bar T_x^2z^2\le\psi_n^2,\quad\text{and}\quad \forall z\in D_{nx}^c:\quad T_x^{n-2}=(1-\bar T_xz)^{n-2}\le(1-\psi_n)^{n-2}.$$

Hence,

$$S_{4x}=\int \phi_{nx}\bar T_x^2z^2=\int_{D_{nx}}\phi_{nx}\bar T_x^2z^2+\int_{D_{nx}^c}\phi_{nx}\bar T_x^2z^2\le\psi_n^2S_{1x}+\int_{D_{nx}^c}(1-\psi_n)^{n-2}A_x\bar T_x^2z^2g. \tag{10}$$

[2] O and o mean "order of," i.e., $a_n=o(b_n)$ if $\lim_{n\to\infty}a_n/b_n=0$ and $a_n=O(b_n)$ if $\lim_{N\to\infty}\max_{n>N}|a_n/b_n|<\infty$.

Since $(1-\psi_n)^{n^\kappa} = (1-n^{-\kappa})^{n^\kappa} \to 1/e$ as $n \to \infty$ and $A_x \bar{T}_x^2 g$ is uniformly bounded by assumption, the second term on the right-hand side of (10) goes to zero exponentially in n, uniformly in x.

Finally, we show that $\min_{x\in[r,1]} S_{1x}$ decreases to zero at a rate no faster than $1/n$. Let

$$D^*_{nx} = \{z \in [0,1] : \bar{T}_x(z) = 0 \vee z \leq 1/(n|\bar{T}_x(z)|)\}.$$

Then, again by (7),

$$\min_{x\in[r,1]} S_{1x} \geq \min_{x\in[r,1]} \int_{D^*_{nx}} \phi_{nx} \geq \min_{x\in[r,1]} \int_{D^*_{nx}} (1-1/n)^{n-2} A_x g.$$

Let λ denote the Lebesgue measure. Then $\lambda(D^*_{nx}) \geq 1/(n \max_{x\in[r,1],z\in[0,1]} \bar{T}_x(z))$ by assumption. Further, for any $x \in [r,1]$, $\min_{z\in D^*_{nx}} T_x(z) = \min_{z\in D^*_{nx}} (1+\bar{T}_x z) \geq 1 - 1/n$. Hence, $\min_{x\in[r,1]} S_{1x}$ indeed decreases to 0 at rate no faster than $1/n$ and $\max_{x\in[r,1]}(S_{4x}/S_{1x}) = O(\psi_n^2)$.

APPENDIX B: PROOFS OF PROPOSITIONS 4 AND 5

Let P^* be a pseudo-density, e.g.,

$$P^*(X_1 = x_1, X_2 \leq x_2) = \frac{\partial}{\partial x_1} P(X_1 \leq x_1, X_2 \leq x_2),$$

and similarly for other quantities.

LEMMA 2: *If X, Y are affiliated, then, for all $x' \geq x$, $y' \geq y$,*

$$\frac{f(x'|Y \leq y')}{f(x'|Y \leq y)} - \frac{f(x|Y \leq y')}{f(x|Y \leq y)} \geq 0.$$

PROOF: It is sufficient to show that

$$f(x'|Y \leq y') f(x|Y \leq y) - f(x|Y \leq y') f(x'|Y \leq y) \geq 0,$$

or equivalently that

$$P^*(X = x', Y \leq y') P^*(X = x, Y \leq y) - P^*(X = x, Y \leq y') P^*(X = x', Y \leq y) \geq 0.$$

Now,

$$\begin{aligned}
&P^*(X = x', Y \leq y') P^*(X = x, Y \leq y) - P^*(X = x, Y \leq y') P^*(X = x', Y \leq y) \\
&\quad = \big(P^*(X = x', Y \leq y) + P^*(X = x', y < Y \leq y')\big) P^*(X = x, Y \leq y) \\
&\qquad - \big(P^*(X = x, Y \leq y) + P^*(X = x, y < Y \leq y')\big) P^*(X = x', Y \leq y) \\
&\quad = P^*(X = x', y < Y \leq y') P^*(X = x, Y \leq y) \\
&\qquad - P^*(X = x, y < Y \leq y') P^*(X = x', Y \leq y) \\
&\quad = \int_y^{y'} \int_0^y \big(f(x',t') f(x,t) - f(x,t') f(x',t)\big)\, dt\, dt' \geq 0.
\end{aligned}$$

Q.E.D.

PROOF OF PROPOSITION 4: Since

$$\Delta R(x, n+1) - \Delta R(x,n) = n\left(\frac{R(x,n+1)}{n} - \frac{R(x,n)}{n-1}\right) + \frac{R(x,n)}{n-1} - R(x,2),$$

it suffices to show that

$$\forall x, n: \quad R(x, n+1)/n - R(x,n)/(n-1) \leq 0.$$

Let $Y_n^* = \max_{i=3,\dots,n} X_i$. Note that

$$\frac{R(x,n)}{n-1} = \frac{P^*(X_1 = x, X_2 = x, Y_n^* \le x)}{P^*(X_1 = x, Y_n \le x)} \tag{11}$$

$$= \frac{P^*(X_2 = x | X_1 = x, Y_n^* \le x, X_{n+1} \le 1)}{P(X_2 \le x | X_1 = x, Y_n^* \le x, X_{n+1} \le 1)}.$$

Similarly,

$$\frac{R(x,n+1)}{n} = \frac{P^*(X_2 = x | X_1 = x, Y_n^* \le x, X_{n+1} \le x)}{P(X_2 \le x | X_1 = x, Y_n^* \le x, X_{n+1} \le x)}.$$

The result then follows from Lemma 2. *Q.E.D.*

PROOF OF PROPOSITION 5: It suffices to show that

$$\forall x: \quad \int_x^{\bar{x}} R(s,n)\, ds \text{ is increasing in } n.$$

Let $\Psi(x, y; n) = \log P^*(Y_n \le y, X_1 = x)$, where $Y_n = \max_{i=2,\dots,n} X_i$, such that

$$R(x,n) = \frac{\partial \Psi}{\partial y}(x, x; n).$$

Hence,

$$\int_x^{\bar{x}} R(s,n)\, ds = \Psi(\bar{x}, \bar{x}; n) - \Psi(x, x; n) - \int_x^{\bar{x}} \frac{\partial \Psi}{\partial x}(s, s; n)\, ds.$$

Note that $\Psi(\bar{x}, \bar{x}; n)$ is independent of n and $\Psi(x, x; n)$ is decreasing in n. It hence remains to be shown that $\partial\Psi/\partial x$ is decreasing in n. Now

$$\begin{aligned} &\Psi(x, y; n+1) - \Psi(x, y; n) \\ &\quad = \log \frac{P^*(Y_{n+1} \le y, X_1 = x)}{P^*(Y_n \le y, X_1 = x)} \\ &\quad = \log \frac{P^*(Y_n \le y, X_{n+1} \le y, X_1 = x)}{P^*(Y_n \le y, X_{n+1} \le \bar{x}, X_1 = x)} \\ &\quad = \log \frac{P^*(X_1 = x | X_{n+1} \le y, Y_n \le y)}{P^*(X_1 = x | X_{n+1} \le \bar{x}, Y_n \le y)} + \log \frac{P(X_{n+1} \le y, Y_n \le y)}{P(X_{n+1} \le \bar{x}, Y_n \le y)}. \end{aligned} \tag{12}$$

The second right-hand side term in equation (12) does not depend on x. The first right-hand side term is decreasing in x by Lemma 2. *Q.E.D.*

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