Econometrica Supplementary Material

SUPPLEMENT TO "BOOTSTRAPPING REALIZED VOLATILITY"

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THIS SUPPLEMENT IS ORGANIZED as follows. First, we introduce some notation. Second, we provide auxiliary lemmas (and their proofs) used to derive the cumulant expansions in Appendix A of the paper. Last, we prove Proposition 4.2 and part (c) of Proposition 4.3, which were not included in Appendix B.

NOTATION

Recall that $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} \sigma_u^2\,du < \infty$, and for any $q > 0$, $\overline{\sigma_h^q} \equiv h^{1-q/2} \times \sum_{i=1}^{1/h} (\sigma_i^2)^{q/2} \equiv h^{1-q/2} \sum_{i=1}^{1/h} \sigma_i^q$, where $\sigma_i^q \equiv (\sigma_i^2)^{q/2}$. Note that in general $\overline{\sigma_h^q} \neq \overline{\sigma^q} \equiv \int_0^1 \sigma_u^q\,du$. We let $\sigma_{q,p} \equiv \overline{\sigma^q}/(\overline{\sigma^p})^{q/p}$ for any $q, p > 0$. When $\overline{\sigma^q}$ is replaced with $\overline{\sigma_h^q}$ we write $\sigma_{q,p,h}$. Similarly, $R_{q,p} \equiv R_q/(R_p)^{q/p}$. We let $\mu_q = E|Z|^q$, where $Z \sim N(0, 1)$ and $q > 0$, and note that $\mu_2 = 1$, $\mu_4 = 3$, $\mu_6 = 15$, and $\mu_8 = 105$. Since $\mu_2 = 1$, we can write $\overline{\sigma^2} = \mu_2 \overline{\sigma^2}$, which will be convenient for proving the results for the wild bootstrap (WB).

Write

$$T_h = S_h \left(\frac{\hat{V}}{V_h} \right)^{-1/2} = S_h (1 + \sqrt{h} U_h)^{-1/2},$$

where

$$S_h \equiv \frac{\sqrt{h^{-1}}(R_2 - \mu_2 \overline{\sigma^2})}{\sqrt{V_h}} \quad \text{and} \quad U_h \equiv \frac{\sqrt{h^{-1}}(\hat{V} - V_h)}{V_h},$$

and $V_h = \mathrm{Var}(\sqrt{h^{-1}} R_2) = (\mu_4 - \mu_2^2)\overline{\sigma_h^4}$. The proof of Lemma S.2 below relies heavily on the fact that

$$R_2 - \mu_2 \overline{\sigma^2} = \sum_{i=1}^{1/h} (r_i^2 - \mu_2 \sigma_i^2) \quad \text{and}$$

$$\hat{V} - V_h = \frac{\mu_4 - \mu_2^2}{\mu_4} h^{-1} \sum_{i=1}^{1/h} (r_i^4 - \mu_4 \sigma_i^4),$$

where for any $q > 0$, $|r_i|^q - \mu_q \sigma_i^q$ are (conditionally on σ) independent with zero mean since $r_i = \sigma_i u_i$, where $u_i \sim$ i.i.d. $N(0, 1)$.

Similarly, let $S_h^* \equiv (\sqrt{h^{-1}}(R_2^* - E^*(R_2^*)))/\sqrt{V^*}$, $U_h^* \equiv (\sqrt{h^{-1}}(\hat{V}^* - V^*))/V^*$, where $V^* = \mathrm{Var}^*(h^{-1/2} R_2^*)$ and $\hat{V}^*$ is a consistent estimator of V^*. Then $T_h^* =$

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$S_h^*(1+\sqrt{h}U_h^*)^{-1/2}$. For the independent and identically distributed (i.i.d.) bootstrap, $V^* = R_4 - R_2^2$ and $\hat{V}^* = R_4^* - R_2^{*2}$. For the WB, $V^* = (\mu_4^* - \mu_2^{*2})R_4$ and $\hat{V}^* = ((\mu_4^* - \mu_2^{*2})/\mu_4^*)R_4^*$.

Finally, note that throughout we will use $\sum_{i\neq j\neq\cdots\neq k}$ to denote a sum where all indices differ, for example, $\sum_{i\neq j\neq k} \equiv \sum_{i\neq j, i\neq k, j\neq k}$.

AUXILIARY LEMMAS

LEMMA S.1: *Let q, p, and s be positive even integers. It follows that*

$$\text{(S1)} \qquad \sum_{i\neq j}^{1/h} \sigma_i^q \sigma_j^p = h^{-2+(q+p)/2}\left(\overline{\sigma_h^q}\;\overline{\sigma_h^p} - h\,\overline{\sigma_h^{q+p}}\right),$$

$$\begin{aligned}\text{(S2)} \qquad \sum_{i\neq j\neq l}^{1/h} \sigma_i^q \sigma_j^p \sigma_l^s &= h^{-3+(q+p+s)/2}\left(\overline{\sigma_h^q}\right)\left(\overline{\sigma_h^p}\right)\left(\overline{\sigma_h^s}\right) \\ &\quad - h^{-2+(q+p+s)/2}\left(\overline{\sigma_h^{q+p}}\;\overline{\sigma_h^s} + \overline{\sigma_h^{q+s}}\;\overline{\sigma_h^p} + \overline{\sigma_h^q}\;\overline{\sigma_h^{p+s}}\right) \\ &\quad + 2h^{-1+(q+p+s)/2}\;\overline{\sigma_h^{q+p+s}}.\end{aligned}$$

LEMMA S.2: *Under Assumption* H, *conditionally on σ,*

$$\text{(a1)} \qquad E|r_i|^q = \mu_q \sigma_i^q,$$

$$\text{(a2)} \qquad V_h \equiv \operatorname{Var}\left(h^{-1/2}R_2\right) = (\mu_4 - \mu_2^2)\overline{\sigma_h^4},$$

$$\text{(a3)} \qquad E\left[\left(R_2 - \mu_2\overline{\sigma^2}\right)^3\right] = h^2(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3)\overline{\sigma_h^6},$$

$$\begin{aligned}\text{(a4)} \qquad E\left[\left(R_2 - \mu_2\overline{\sigma^2}\right)^4\right] &= 3h^2(\mu_4 - \mu_2^2)^2\left(\overline{\sigma_h^4}\right)^2 \\ &\quad + h^3(\mu_8 - 4\mu_2\mu_6 + 12\mu_2^2\mu_4 - 6\mu_2^4 - 3\mu_4^2)\overline{\sigma_h^8},\end{aligned}$$

$$\text{(a5)} \qquad E\left[\left(R_2 - \mu_2\overline{\sigma^2}\right)(\hat{V} - V_h)\right] = h\frac{(\mu_4 - \mu_2^2)(\mu_6 - \mu_2\mu_4)}{\mu_4}\;\overline{\sigma_h^6},$$

$$\begin{aligned}\text{(a6)} \qquad &E\left[\left(R_2 - \mu_2\overline{\sigma^2}\right)^2(\hat{V} - V_h)\right] \\ &\quad = h^2\frac{\mu_4 - \mu_2^2}{\mu_4}(\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4)\overline{\sigma_h^8},\end{aligned}$$

$$\begin{aligned}\text{(a7)} \qquad E\left[\left(R_2 - \mu_2\overline{\sigma^2}\right)^3(\hat{V} - V_h)\right] &= 3h^2\frac{(\mu_4 - \mu_2^2)^2(\mu_6 - \mu_2\mu_4)}{\mu_4} \\ &\quad \times \overline{\sigma_h^4\sigma_h^6} + O(h^3), \quad \text{as} \quad h \to 0,\end{aligned}$$

(a8) $$E\big[\big(R_2-\mu_2\overline{\sigma^2}\big)^4(\hat{V}-V_h)\big] = h^3\frac{\mu_4-\mu_2^2}{\mu_4}\begin{bmatrix} 4(\mu_6-3\mu_2\mu_4+2\mu_2^3)(\mu_6-\mu_2\mu_4)\big(\overline{\sigma_h^6}\big)^2 \\ +6(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4)(\mu_4-\mu_2^2)\overline{\sigma_h^4}\;\overline{\sigma_h^8}\end{bmatrix} + O(h^4) \quad as \quad h\to 0,$$

(a9) $$E\big[\big(R_2-\mu_2\overline{\sigma^2}\big)(\hat{V}-V_h)^2\big] = \frac{(\mu_4-\mu_2^2)^2}{\mu_4^2}(\mu_{10}-2\mu_4\mu_6-\mu_2\mu_8+2\mu_2\mu_4^2)h^2\overline{\sigma_h^{10}} = O(h^2) \quad as \quad h\to 0,$$

(a10) $$E\big[\big(R_2-\mu_2\overline{\sigma^2}\big)^2(\hat{V}-V_h)^2\big] = h^2\frac{(\mu_4-\mu_2^2)^2}{\mu_4^2}\Big((\mu_4-\mu_2^2)(\mu_8-\mu_4^2)\overline{\sigma_h^4}\;\overline{\sigma_h^8}+2(\mu_6-\mu_2\mu_4)^2\big(\overline{\sigma_h^6}\big)^2\Big) + O(h^3) \quad as \quad h\to 0,$$

(a11) $$E\big[\big(R_2-\mu_2\overline{\sigma^2}\big)^3(\hat{V}-V_h)^2\big] = O(h^3)+O(h^4) \quad as \quad h\to 0,$$

(a12) $$E\big[\big(R_2-\mu_2\overline{\sigma^2}\big)^4(\hat{V}-V_h)^2\big] = h^3\frac{(\mu_4-\mu_2^2)^2}{\mu_4^2}\begin{bmatrix} 3(\mu_4-\mu_2^2)^2(\mu_8-\mu_4^2)\big(\overline{\sigma_h^4}\big)^2\overline{\sigma_h^8} \\ +12(\mu_4-\mu_2^2)(\mu_6-\mu_2\mu_4)^2\big(\overline{\sigma_h^6}\big)^2\overline{\sigma_h^4}\end{bmatrix} + O(h^4) \quad as \quad h\to 0.$$

LEMMA S.3: *Under Assumption* H, *conditionally on* σ,

$$E(S_h)=0,$$
$$E(S_h^2)=1,$$
$$E(S_h^3)=\sqrt{h}B_1\sigma_{6,4,h},$$
$$E(S_h^4)=3+hB_2\sigma_{8,4,h},$$
$$E(S_hU_h)=A_1\sigma_{6,4,h},$$
$$E(S_h^2U_h)=\sqrt{h}(A_2\sigma_{8,4,h}),$$

and as $h\to 0$,

$$E(S_h^3U_h)=A_3\sigma_{6,4,h}+O(h),$$
$$E(S_h^4U_h)=\sqrt{h}[D_1\sigma_{8,4,h}+D_2\sigma_{6,4,h}^2]+O\big(h^{3/2}\big),$$

$$E(S_h U_h^2) = O(h^{1/2}), \quad E(S_h^3 U_h^2) = O(h^{1/2}),$$
$$E(S_h^2 U_h^2) = [C_1 \sigma_{8,4,h} + C_2 \sigma_{6,4,h}^2] + O(h),$$
$$E(S_h^4 U_h^2) = [E_1 \sigma_{8,4,h} + E_2 \sigma_{6,4,h}^2] + O(h).$$

The constants A_1, A_2, B_1, B_2, *and* C_1 *are as in Theorem* A.1, *and* $A_3 = 3A_1$, $C_2 = 2A_1^2$, $D_1 = 6A_2$, $D_2 = 4A_1B_1$, $E_1 = 3C_1$, *and* $E_2 = 12A_1^2$.

REMARK 1: The WB analogue of Lemma S.2 replaces $\overline{\sigma_h^q}$ with R_q and μ_q with $\mu_q^* = E^*|\eta_i|^q$. The WB analogue of Lemma S.3 replaces $\sigma_{q,p,h}$ with $R_{q,p}$ (and μ_q with $\mu_q^* = E^*|\eta_i|^q$), yielding for example, $E^*(S_h^{*3}) = \sqrt{h}(B_1^* R_{6,4,h})$, where $B_1^* = (\mu_6^* - 3\mu_2^*\mu_4^* + 2\mu_2^{*3})/(\mu_4^* - \mu_2^{*2})^{3/2}$.

Lemma S.7 below is the i.i.d. bootstrap analogue of Lemma S.3. The next results are auxiliary in proving Lemma S.7.

LEMMA S.4: *Let* $r_i^* \sim$ *i.i.d. from* $\{r_i : i = 1, \dots, 1/h\}$. *Under Assumption* H, *conditionally on* σ, *for any* $q > 0$ *and for any* $i = 1, \dots, 1/h$,

(a1) $\quad E^*(|r_i^*|^q) = h^{q/2} R_q \quad \text{and} \quad E^*(R_q^*) = R_q = O_P(1),$

(a2) $\quad E^*[(r_i^{*2} - hR_2)^2] = h^2(R_4 - R_2^2),$

(a3) $\quad E^*[(r_i^{*2} - hR_2)^3] = h^3(R_6 - 3R_4R_2 + 2R_2^3),$

(a4) $\quad E^*[(r_i^{*2} - hR_2)^4] = h^4(R_8 - 4R_6R_2 + 6R_4R_2^2 - 3R_2^4),$

(a5) $\quad E^*[(r_i^{*2} - hR_2)^5] = h^5(R_{10} - 5R_8R_2 + 10R_6R_2^2 - 10R_4R_2^3 + 4R_2^5),$

(a6) $\quad E^*[(r_i^{*2} - hR_2)^6] = h^6(R_{12} - 6R_{10}R_2 + 15R_8R_2^2 - 20R_6R_2^3 + 15R_4R_2^4 - 5R_2^6),$

(a7) $\quad E^*[(r_i^{*2} - hR_2)^q] = O_P(h^q) \quad \text{for any } q \geq 7, \quad \text{as} \quad h \to 0,$

(a8) $\quad E^*[(r_i^{*4} - h^2R_4)^2] = h^4(R_8 - R_4^2),$

(a9) $\quad E^*[(r_i^{*2} - hR_2)(r_i^{*4} - h^2R_4)] = h^3(R_6 - R_4R_2),$

(a10) $\quad E^*[(r_i^{*2} - hR_2)^2(r_i^{*4} - h^2R_4)] = h^4(R_8 - R_4^2 - 2R_6R_2 + 2R_4R_2^2),$

(a11) $\quad E^*[(r_i^{*2} - hR_2)^3(r_i^{*4} - h^2R_4)] = h^5(R_{10} - R_4R_6 - 3R_8R_2 + 3R_4^2R_2 + 3R_6R_2^2 - 3R_4R_2^3),$

(a12) $\quad E^*[(r_i^{*2} - hR_2)^4(r_i^{*4} - h^2R_4)] = h^6\begin{pmatrix} R_{12} - R_4R_8 - 4R_{10}R_2 + 4R_4R_6R_2 + 6R_8R_2^2 \\ - 6R_4^2R_2^2 - 4R_6R_2^3 + 4R_4R_2^4 \end{pmatrix},$

(a13) $\quad E^*\big((r_i^{*2} - hR_2)(r_i^{*4} - h^2R_4)^2\big) = h^5(R_{10} - 2R_4R_6 - R_8R_2 + 2R_4^2R_2),$

(a14) *for any* $q, p > 0$, $E^*[(r_i^{*2} - hR_2)^q (r_i^{*4} - h^2 R_4)^p]$
$$= O_P(h^{q+2p}) \quad as \quad h \to 0.$$

LEMMA S.5: *Let* $r_i^* \sim$ *i.i.d. from* $\{r_i : i = 1, \dots, 1/h\}$. *Under Assumption* H, *conditionally on* σ,

(a1) $V^* \equiv \operatorname{Var}^*\big(h^{-1/2} R_2^*\big) = R_4 - R_2^2,$

(a2) $\hat{V}^* - V^* = R_4^* - R_4 - [(R_2^* - R_2)^2 + 2R_2(R_2^* - R_2)],$

(a3) $E^*[(R_2^* - R_2)^3] = h^2(R_6 - 3R_4R_2 + 2R_2^3),$

(a4) $E^*[(R_2^* - R_2)^4] = h^2[3(R_4 - R_2^2)^2] + h^3(R_8 - 4R_6R_2 + 12R_4R_2^2 - 6R_2^4 - 3R_4^2),$

(a5) $E^*[(R_2^* - R_2)^5] = h^3[10(R_6 - 3R_4R_2 + 2R_2^3)(R_4 - R_2^2)] + O_P(h^4) \quad as \quad h \to 0,$

(a6) $E^*[(R_2^* - R_2)^6] = h^3[15(R_4 - R_2^2)^3] + O_P(h^4) \quad as \quad h \to 0,$

(a7) $E^*[(R_2^* - R_2)^q] = O_P(h^4) \quad for \quad q = 7, 8, \quad as \quad h \to 0,$

(a8) $E^*[(R_2^* - R_2)(R_4^* - R_4)] = h(R_6 - R_4R_2),$

(a9) $E^*[(R_2^* - R_2)^2(R_4^* - R_4)] = h^2(R_8 - R_4^2 - 2R_6R_2 + 2R_4R_2^2),$

(a10) $E^*[(R_2^* - R_2)^3(R_4^* - R_4)] = 3h^2(R_6 - R_4R_2)(R_4 - R_2^2) + O_P(h^3) \quad as \quad h \to 0,$

(a11) $E^*[(R_2^* - R_2)^4(R_4^* - R_4)]$
$$= h^3 \begin{bmatrix} 4(R_6 - 3R_4R_2 + 2R_2^3)(R_6 - R_4R_2) \\ + 6(R_4 - R_2^2)(R_8 - R_4^2 - 2R_6R_2 + 2R_4R_2^2) \end{bmatrix} + O_P(h^4) \quad as \quad h \to 0,$$

(a12) $E^*[(R_2^* - R_2)^5(R_4^* - R_4)] = h^3[15(R_4 - R_2^2)^2(R_6 - R_4R_2)] + O_P(h^4) \quad as \quad h \to 0,$

(a13) $E^*[(R_2^* - R_2)^6(R_4^* - R_4)] = O_P(h^4) \quad as \quad h \to 0,$

(a14) $E^*[(R_2^* - R_2)(R_4^* - R_4)^2] = h^2(R_{10} - 2R_4R_6 - R_8R_2 + 2R_4^2R_2),$

(a15) $E^*[(R_2^* - R_2)^2(R_4^* - R_4)^2]$
$$= h^2[(R_4 - R_2^2)(R_8 - R_4^2) + 2(R_6 - R_4R_2)^2] + O_P(h^3) \quad as \quad h \to 0,$$

(a16) $E^*[(R_2^* - R_2)^3(R_4^* - R_4)^2] = O_P(h^3) \quad as \quad h \to 0,$

$$\text{(a17)}\qquad E^*[(R_2^*-R_2)^4(R_4^*-R_4)^2]=h^3\begin{bmatrix}3(R_4-R_2^2)^2(R_8-R_4^2)\\+12(R_6-R_4R_2)^2(R_4-R_2^2)\end{bmatrix}+O_P(h^4)\quad as\quad h\to 0.$$

LEMMA S.6: *Let $r_i^*\sim$ i.i.d. from $\{r_i: i=1,\dots,1/h\}$. Under Assumption* H, *conditionally on σ, as $h\to 0$,*

$$\text{(a1)}\qquad E^*[(R_2^*-R_2)(\hat{V}^*-V^*)]=h(R_6-3R_4R_2+2R_2^3)+O_P(h^2),$$

$$\text{(a2)}\qquad E^*[(R_2^*-R_2)^2(\hat{V}^*-V^*)]$$
$$=h^2\begin{bmatrix}(R_8-R_4^2-2R_6R_2+2R_4R_2^2)-3(R_4-R_2^2)^2\\-2R_2(R_6-3R_4R_2+2R_2^3)\end{bmatrix}+O_P(h^3),$$

$$\text{(a3)}\qquad E^*[(R_2^*-R_2)^3(\hat{V}^*-V^*)]=h^2[3(R_4-R_2^2)(R_6-3R_4R_2+2R_2^3)]+O_P(h^3),$$

$$\text{(a4)}\qquad E^*[(R_2^*-R_2)^4(\hat{V}^*-V^*)]$$
$$=h^3\begin{bmatrix}4(R_6-3R_4R_2+2R_2^3)(R_6-R_4R_2)\\+6(R_4-R_2^2)(R_8-R_4^2-2R_6R_2+2R_4R_2^2)\end{bmatrix}$$
$$-h^3[15(R_4-R_2^2)^3]$$
$$-h^3[20R_2(R_6-3R_4R_2+2R_2^3)(R_4-R_2^2)]+O_P(h^4),$$

$$\text{(a5)}\qquad E^*[(R_2^*-R_2)(\hat{V}^*-V^*)^2]=O_P(h^2),$$

$$\text{(a6)}\qquad E^*[(R_2^*-R_2)^2(\hat{V}^*-V^*)^2]$$
$$=h^2\begin{bmatrix}(R_4-R_2^2)(R_8-R_4^2)+2(R_6-R_4R_2)^2\\-12(R_6-R_4R_2)(R_4-R_2^2)(R_2)\\+4(R_2)^2[3(R_4-R_2^2)^2]\end{bmatrix}+O_P(h^3),$$

$$\text{(a7)}\qquad E^*[(R_2^*-R_2)^3(\hat{V}^*-V^*)^2]=O_P(h^3),$$

$$\text{(a8)}\qquad E^*[(R_2^*-R_2)^4(\hat{V}^*-V^*)^2]$$
$$=h^3\big[3(R_4-R_2^2)^2(R_8-R_4^2)+12(R_6-R_4R_2)^2(R_4-R_2^2)\big]$$
$$-h^3[60(R_4-R_2^2)^2(R_6-R_4R_2)(R_2)]$$
$$+h^3[60(R_4-R_2^2)^3(R_2)^2]+O_P(h^4).$$

LEMMA S.7: *Let $r_i^*\sim$ i.i.d. from $\{r_i: i=1,\dots,1/h\}$. Under Assumption* H, *conditionally on σ,*

$$E^*(S_h^*)=0,$$

$$E^*(S_h^{*2}) = 1,$$
$$E^*(S_h^{*3}) = \sqrt{h}\tilde{B}_1,$$
$$E^*(S_h^{*4}) = 3 + h\tilde{B}_2,$$

and as $h \to 0$,

$$E^*(S_h^* U_h^*) = \tilde{A}_1 + O_P(h),$$
$$E^*(S_h^{*2} U_h^*) = \sqrt{h}\tilde{A}_2 + O_P\big(h^{3/2}\big),$$
$$E^*(S_h^{*3} U_h^*) = \tilde{A}_3 + O_P(h),$$
$$E^*(S_h^{*4} U_h^*) = \sqrt{h}\tilde{D} + O_P\big(h^{3/2}\big),$$
$$E^*(S_h^* U_h^{*2}) = O_P\big(h^{1/2}\big), \quad E^*(S_h^{*3} U_h^{*2}) = O_P\big(h^{1/2}\big),$$
$$E^*(S_h^{*2} U_h^{*2}) = \tilde{C} + O_P(h), \quad E^*(S_h^{*4} U_h^{*2}) = \tilde{E} + O_P(h).$$

The bootstrap constants $\tilde{A}_1, \tilde{A}_2, \tilde{B}_2, \tilde{C}, \tilde{D}$, *and* $\tilde{E}$ *are as in Theorem* A.2, *and* $\tilde{A}_3$ *and* $\tilde{B}_1$ *are such that* $\tilde{A}_3 = 3\tilde{A}_1$ *and* $\tilde{B}_1 = \tilde{A}_1$.

PROOFS OF LEMMAS S.1–S.7

PROOF OF LEMMA S.1: For (S1), note that

$$\begin{aligned}
\sum_{i \neq j}^{1/h} \sigma_i^q \sigma_j^p &= \left(\sum_{i=1}^{1/h} \sigma_i^q\right)\left(\sum_{j=1}^{1/h} \sigma_j^p\right) - \left(\sum_{i=1}^{1/h} \sigma_i^{q+p}\right) \\
&= h^{-1+q/2}\left(h^{1-q/2}\sum_{i=1}^{1/h} \sigma_i^q\right) h^{-1+p/2}\left(h^{1-p/2}\sum_{j=1}^{1/h} \sigma_j^p\right) \\
&\quad - h^{-1+(q+p)/2}\left(h^{1-(q+p)/2}\sum_{i=1}^{1/h} \sigma_i^{q+p}\right) \\
&= h^{-2+(q+p)/2}\big(\overline{\sigma_h^q}\;\overline{\sigma_h^p} - h\;\overline{\sigma_h^{q+p}}\big).
\end{aligned}$$

For (S2), note that

$$\begin{aligned}
\sum_{i \neq j \neq k} \sigma_i^q \sigma_j^p \sigma_k^s &= \left(\sum_{i=1}^{1/h} \sigma_i^q\right)\left(\sum_{j=1}^{1/h} \sigma_j^p\right)\left(\sum_{k=1}^{1/h} \sigma_k^s\right) - \sum_{i=1}^{1/h} \sigma_i^{q+p+s} \\
&\quad - \sum_{i \neq j} \sigma_i^{q+p}\sigma_j^s - \sum_{i \neq j} \sigma_i^{q+s}\sigma_j^p - \sum_{i \neq j} \sigma_i^q \sigma_j^{p+s},
\end{aligned}$$

and then proceed as for (S1). *Q.E.D.*

PROOF OF LEMMA S.2: (a1) Equality (a1) follows from $r_i = \sigma_i u_i$, where $u_i \sim$ i.i.d. $N(0,1)$.

(a2) Note that $R_2 = \sum_{i=1}^{1/h} r_i^2$, where r_i^2 is (conditional on σ) independent with $\text{Var}(r_i^2) = E(r_i^4) - (E(r_i^2))^2 = \mu_4\sigma_i^4 - (\mu_2\sigma_i^2)^2 = (\mu_4 - \mu_2^2)\sigma_i^4$, with $\sigma_i^4 \equiv (\sigma_i^2)^2$. To prove the remaining results, we use the multinomial formula to compute the coefficients in the expansions. In particular, we have that

$$(a_1 + a_2 + \cdots + a_d)^n = \sum_{\substack{n_1, n_2, \ldots, n_d \geq 0 \\ n_1 + \cdots + n_d = n}} \frac{n!}{n_1! n_2! \cdots n_d!} a_1^{n_1} a_2^{n_2} \cdots a_d^{n_d}.$$

(a3) Write

$$\begin{aligned} I_1 &\equiv E\big[\big(R_2 - \mu_2\overline{\sigma^2}\big)^3\big] \\ &= E\left[\sum_{i=1}^{1/h}\sum_{j=1}^{1/h}\sum_{k=1}^{1/h}(r_i^2 - \mu_2\sigma_i^2)(r_j^2 - \mu_2\sigma_j^2)(r_k^2 - \mu_2\sigma_k^2)\right]. \end{aligned}$$

The only nonzero contribution to I_1 is when $i = j = k$, in which case we get $E[(r_i^2 - \mu_2\sigma_i^2)^3] = (\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3)\sigma_i^6$ and $I_1 = h^2(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3)\overline{\sigma_h^6}$, proving (a3).

(a4) Using the independence and zero mean property of $\{r_i^2 - \mu_2\sigma_i^2\}$, we have that

$$\begin{aligned} &E\big[\big(R_2 - \mu_2\overline{\sigma^2}\big)^4\big] \\ &\quad = \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^4] + 3\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2]E[(r_j^2 - \mu_2\sigma_j^2)^2] \\ &\quad = E[(u_i^2 - \mu_2)^4]\sum_{i=1}^{1/h}\sigma_i^8 + 3\big(E[(u_i^2 - \mu_2)^2]\big)^2\sum_{i\neq j}^{1/h}\sigma_i^4\sigma_j^4 \\ &\quad = (\mu_8 - 3\mu_2^4 + 6\mu_2^2\mu_4 - 4\mu_2\mu_6)h^3\overline{\sigma_h^8} \\ &\qquad + 3(\mu_4 - \mu_2^2)^2\big[h^2\big(\big(\overline{\sigma_h^4}\big)^2 - h\big(\overline{\sigma_h^8}\big)\big)\big] \\ &\quad = 3h^2(\mu_4 - \mu_2^2)^2\big(\overline{\sigma_h^4}\big)^2 \\ &\qquad + h^3(\mu_8 - 4\mu_2\mu_6 + 12\mu_4\mu_2^2 - 6\mu_2^4 - 3\mu_4^2)\big(\overline{\sigma_h^8}\big), \end{aligned}$$

where we have made use of Lemma S.1 and of the results

$$E[(u_i^2 - \mu_2)^4] = E[u_i^8 - 4u_i^2\mu_2^3 + 6u_i^4\mu_2^2 - 4u_i^6\mu_2 + \mu_2^4]$$

$$= \mu_8 - 3\mu_2^4 + 6\mu_2^2\mu_4 - 4\mu_2\mu_6.$$

$$E[(u_i^2 - \mu_2)^2] = E[u_i^4 - 2u_i^2\mu_2 + \mu_2^2] = \mu_4 - \mu_2^2.$$

(a5) We have

$$\begin{aligned}
E\big[\big(R_2 - \mu_2\overline{\sigma^2}\big)(\hat{V} - V_h)\big] &= \frac{\mu_4 - \mu_2^2}{\mu_4} h^{-1} \\
&\quad \times \sum_{i=1}^{1/h} E(r_i^6 - r_i^2\mu_4\sigma_i^4 - \mu_2\sigma_i^2 r_i^4 + \mu_2\mu_4\sigma_i^6) \\
&= \frac{\mu_4 - \mu_2^2}{\mu_4} h^{-1} h^2 (\mu_6 - \mu_2\mu_4)\overline{\sigma_h^6} \\
&= h\frac{(\mu_4 - \mu_2^2)(\mu_6 - \mu_2\mu_4)}{\mu_4}\overline{\sigma_h^6}.
\end{aligned}$$

(a6) We have

$$\begin{aligned}
&E\big(\big(R_2 - \mu_2\overline{\sigma^2}\big)^2(\hat{V} - V_h)\big) \\
&\quad = \frac{\mu_4 - \mu_2^2}{\mu_4} h^{-1} \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2(r_i^4 - \mu_4\sigma_i^4)] \\
&\quad = h^2 \frac{(\mu_4 - \mu_2^2)(\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4)}{\mu_4}\overline{\sigma_h^8}.
\end{aligned}$$

(a7) Write $E((R_2 - \mu_2\overline{\sigma^2})^3(\hat{V} - V_h)) = ((\mu_4 - \mu_2^2)/\mu_4)h^{-1}I_2$, where by the independence and mean zero property of $|r_i|^q - \mu_2\sigma_i^q$,

$$\begin{aligned}
I_2 &= \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^3(r_i^4 - \mu_4\sigma_i^4)] \\
&\qquad + 3\sum_{i\neq j}^{1/h} E\big[(r_i^2 - \mu_2\sigma_i^2)^2(r_j^2 - \mu_2\sigma_j^2)(r_j^4 - \mu_4\sigma_j^4)\big] \\
&= M_1 \sum_{i=1}^{1/h} \sigma_i^{10} + 3E[(u_i^2 - \mu_2)^2]E[(u_j^2 - \mu_2)(u_j^4 - \mu_4)] \sum_{i\neq j}^{1/h} \sigma_i^4\sigma_j^6 \\
&= 3h^3(\mu_4 - \mu_2^2)(\mu_6 - \mu_2\mu_4)\overline{\sigma_h^4}\;\overline{\sigma_h^6} + O(h^4),
\end{aligned}$$

given Lemma S.1, the fact that $\overline{\sigma_h^{10}} = O(1)$ under our assumptions, and where $M_1 = E[(u_i^2 - \mu_2)^3(u_i^4 - \mu_4)]$ is a constant, and $E[(u_i^2 - \mu_2)^2] = \mu_4 - \mu_2^2$ and $E[(u_j^2 - \mu_2)(u_i^4 - \mu_4)] = \mu_6 - \mu_2\mu_4$.

(a8) Write $E((R_2 - \mu_2\overline{\sigma^2})^4(\hat{V} - V_h)) = ((\mu_4 - \mu_2)^2/\mu_4)h^{-1}I_3$, where by the independence and mean zero property of $|r_i|^q - \mu_2\sigma_i^q$, and Lemma S.1,

$$
\begin{aligned}
I_3 &= \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^4(r_i^4 - \mu_4\sigma_i^4)] \\
&\quad + 4\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^3]E[(r_j^2 - \mu_2\sigma_j^2)(r_j^4 - \mu_4\sigma_j^4)] \\
&\quad + 6\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2]E[(r_j^2 - \mu_2\sigma_j^2)^2(r_j^4 - \mu_4\sigma_j^4)] \\
&= M_1 h^5\overline{\sigma_h^{12}} + 4M_2\big(h^4\big(\overline{\sigma_h^6}\big)^2 - h^5\overline{\sigma_h^{12}}\big) + 6M_3\big(h^4\overline{\sigma_h^4}\ \overline{\sigma_h^8} - h^5\overline{\sigma_h^{12}}\big) \\
&= h^4\big[4M_2\big(\overline{\sigma_h^6}\big)^2 + 6M_3\overline{\sigma_h^4}\ \overline{\sigma_h^8}\big] + O(h^5),
\end{aligned}
$$

where $M_1 \equiv E[(u_i^2-\mu_2)^4(u_i^4-\mu_4)]$, $M_2 \equiv E[(u_i^2-\mu_2)^3]E[(u_j^2-\mu_2)(u_j^4-\mu_4)] = (\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3)(\mu_6 - \mu_2\mu_4)$, and $M_3 \equiv E[(u_i^2 - \mu_2)^2]E[(u_i^2 - \mu_2)^2(u_i^4 - \mu_4)] = (\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4)(\mu_4 - \mu_2^2)$, and given the fact that $\overline{\sigma_h^q} = O(1)$ under our assumptions.

(a9) Write $E((R_2 - \mu_2\overline{\sigma^2})(\hat{V} - V_h)^2) = ((\mu_4 - \mu_2^2)^2/\mu_4^2)h^{-2}\sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)(r_i^4 - \mu_4\sigma_i^4)^2] = O(h^2)$.

(a10) Write $E((R_2 - \mu_2\overline{\sigma^2})^2(\hat{V} - V_h)^2) = ((\mu_4 - \mu_2^2)^2/\mu_4^2)h^{-2}I_4$, where by the independence and mean zero property of $|r_i|^q - \mu_2\sigma_i^q$,

$$
\begin{aligned}
I_4 &= \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2(r_i^4 - \mu_4\sigma_i^4)^2] \\
&\quad + \sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2]E[(r_j^4 - \mu_4\sigma_j^4)^2] \\
&\quad + 2\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)(r_i^4 - \mu_4\sigma_i^4)]E[(r_j^2 - \mu_2\sigma_j^2)(r_j^4 - \mu_4\sigma_j^4)] \\
&= D_1 h^5\overline{\sigma_h^{12}} + D_2\big(h^4\overline{\sigma_h^4}\ \overline{\sigma_h^8} - h^5\overline{\sigma_h^{12}}\big) + 2D_3\big(h^4\big(\overline{\sigma_h^6}\big)^2 - h^5\overline{\sigma_h^{12}}\big) \\
&= h^4\big(D_2\overline{\sigma_h^4}\ \overline{\sigma_h^8} + 2D_3\big(\overline{\sigma_h^6}\big)^2\big) + O(h^5),
\end{aligned}
$$

given Lemma S.1, and where $D_1 = E[(u_i^2 - \mu_2)^2(u_i^4 - \mu_4)^2]$, $D_2 = E[(u_i^2 - \mu_2)^2]E[(u_j^4 - \mu_4)^2] = (\mu_4 - \mu_2^2)(\mu_8 - \mu_4^2)$, and $D_3 = [E((u_i^2 - \mu_2)(u_i^4 - \mu_4))]^2 = (\mu_6 - \mu_2\mu_4)^2$.

(a11) Write $E((R_2 - \mu_2\overline{\sigma^2})^3(\hat{V} - V_h)^2) = ((\mu_4 - \mu_2^2)^2/\mu_4^2)h^{-2}I_5$, with

$$\begin{aligned}
I_5 &= \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^3(r_i^4 - \mu_4\sigma_i^4)^2] \\
&\quad + \sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^3]E[(r_j^4 - \mu_4\sigma_j^4)^2] \\
&\quad + 3\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2]E[(r_j^2 - \mu_2\sigma_j^2)(r_j^4 - \mu_4\sigma_j^4)^2] \\
&\quad + 6\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^2(r_i^4 - \mu_4\sigma_i^4)]E[(r_j^2 - \mu_2\sigma_j^2)(r_j^4 - \mu_4\sigma_j^4)] \\
&= K_1 h^6 \overline{\sigma_h^{14}} + K_2\big(h^5\overline{\sigma_h^6}\,\overline{\sigma_h^8} - h^6\overline{\sigma_h^{14}}\big) + 3K_3\big(h^5\overline{\sigma_h^4}\,\overline{\sigma_h^{10}} - h^6\overline{\sigma_h^{14}}\big) \\
&\quad + 6K_4\big(h^5\overline{\sigma_h^8}\,\overline{\sigma_h^6} - h^6\overline{\sigma_h^{14}}\big) \\
&= h^5\big(K_2\overline{\sigma_h^6}\,\overline{\sigma_h^8} + 3K_3\overline{\sigma_h^4}\,\overline{\sigma_h^{10}} + 6K_4\overline{\sigma_h^8}\,\overline{\sigma_h^6}\big) \\
&\quad + h^6(K_1 - K_2 - 3K_3 - 6K_4)\overline{\sigma_h^{14}},
\end{aligned}$$

where we have used the independence and mean zero property of $|r_i|^q - \mu_2\sigma_i^q$, Lemma S.1, and where K_1 through K_4 are constants that depend on μ_q. Since $\overline{\sigma_h^q} = O(1)$, the result follows.

(a12) Write $E((R_2 - \mu_2\overline{\sigma^2})^4(\hat{V} - V_h)^2) = ((\mu_4 - \mu_2^2)^2/\mu_4^2)h^{-2}I_6$, with

$$\begin{aligned}
I_6 &= \sum_{i=1}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^4(r_i^4 - \mu_4\sigma_i^4)^2] \\
&\quad + \sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^4]E[(r_j^4 - \mu_4\sigma_j^4)^2] \\
&\quad + 8\sum_{i\neq j}^{1/h} E[(r_i^2 - \mu_2\sigma_i^2)^3(r_i^4 - \mu_4\sigma_i^4)]E[(r_j^2 - \mu_2\sigma_j^2)(r_j^4 - \mu_4\sigma_j^4)]
\end{aligned}$$

$$
\begin{aligned}
&+6\sum_{i\neq j}^{1/h} E[(r_i^2-\mu_2\sigma_i^2)^2(r_i^4-\mu_4\sigma_i^4)^2]E[(r_j^2-\mu_2\sigma_j^2)^2]\\
&+4\sum_{i\neq j}^{1/h} E[(r_i^2-\mu_2\sigma_i^2)^3]E[(r_j^2-\mu_2\sigma_j^2)(r_j^4-\mu_4\sigma_j^4)^2]\\
&+6\sum_{i\neq j}^{1/h} E[(r_i^2-\mu_2\sigma_i^2)^2(r_i^4-\mu_4\sigma_i^4)]E[(r_j^2-\mu_2\sigma_j^2)^2(r_j^4-\mu_4\sigma_j^4)]\\
&+3\sum_{i\neq j\neq k}^{1/h} E[(r_i^2-\mu_2\sigma_i^2)^2]E[(r_j^2-\mu_2\sigma_j^2)^2]E[(r_k^4-\mu_4\sigma_k^4)^2]\\
&+12\sum_{i\neq j\neq k}^{1/h} E[(r_i^2-\mu_2\sigma_i^2)^2]E[(r_j^2-\mu_2\sigma_j^2)(r_j^4-\mu_4\sigma_j^4)]\\
&\times E[(r_k^2-\mu_2\sigma_k^2)(r_k^4-\mu_4\sigma_k^4)]\\
={}& h^5\big[3J_7\big(\overline{\sigma_h^4}\big)^2\overline{\sigma_h^8}+12J_8\big(\overline{\sigma_h^6}\big)^2\overline{\sigma_h^4}\big]+O(h^6)+O(h^7),
\end{aligned}
$$

given the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$, Lemma S.1, and where $J_7=(E[(u_i^2-\mu_2)^2])^2E[(u_i^4-\mu_4)^2]=(\mu_4-\mu_2^2)^2(\mu_8-\mu_4^2)$ and $J_8=E[(u_i^2-\mu_2)^2](E[(u_i^2-\mu_2)(u_i^4-\mu_4)])^2=(\mu_4-\mu_2^2)(\mu_6-\mu_2\mu_4)^2$. *Q.E.D.*

PROOF OF LEMMA S.3: The first two results are obvious given S_h. The remaining results follow from the definition of S_h and Lemma S.2. *Q.E.D.*

PROOF OF LEMMA S.4: Part (a1) follows from the properties of the i.i.d. bootstrap. The remaining results follow from (a1), given the binomial expansions. Note in particular that since $R_q=O_P(1)$, it follows that $E^*[(r_i^{*2}-hR_2)^q]=O_P(h^q)$. For instance, for (a2), $E^*[(r_i^{*2}-hR_2)^2]=E^*(r_i^{*4}-2r_i^{*2}hR_2+(hR_2)^2)=h^2(R_4-R_2^2)$. The other results follow similarly. *Q.E.D.*

PROOF OF LEMMA S.5: For (a1), since r_i^* are i.i.d. from $\{r_i: i=1,\dots,1/h\}$, it follows that

$$
V^*=h^{-1}\operatorname{Var}^*\left(\sum_{i=1}^{1/h} r_i^{*2}\right)=h^{-1}\sum_{i=1}^{1/h}\operatorname{Var}^*(r_i^{*2})=h^{-2}\operatorname{Var}^*(r_1^{*2}).
$$

But $\operatorname{Var}^*(r_1^{*2})=E^*(r_1^{*4})-(E^*(r_1^{*2}))^2=h^2R_4-(hR_2)^2$. Thus, $V^*=R_4-R_2^2$. Part (a2) follows because $V^*=R_4-R_2^2$ and $\hat{V}^*=R_4^*-R_2^{*2}$. For the remaining of the proof, note that $\sum_{i\neq j}^{1/h} 1 = h^{-2} - h^{-1}$, $\sum_{i\neq j\neq k} 1 = h^{-3} + 2h^{-1} - 3h^{-2}$, and $\sum_{i\neq j\neq k\neq m}^{1/h} 1 = h^{-4} - 6h^{-3} + 11h^{-2} - 6h^{-1}$. In addition, note that

$$R_2^* - R_2 = \sum_{i=1}^{1/h}(r_i^{*2} - hR_2) \quad \text{and} \quad R_4^* - R_4 = h^{-1}\sum_{i=1}^{1/h}(r_i^{*4} - h^2R_4),$$

where for any, $q > 0$, $\{|r_i^*|^q - h^{q/2}R_q\}$ are (conditionally on the sample) i.i.d. with zero mean and $R_q = O_P(1)$. Using this independence property, we evaluate the bootstrap expectations of the sums of products and cross-products of $|r_i^*|^q - h^{q/2}R_q$ by relying on Lemma S.4 to compute the appropriate bootstrap moments of products and cross-products of $|r_i^*|^q - h^{q/2}R_q$. We proceed as in the proof of Lemma S.2 and use the multinomial expansions to compute the number of coefficients in each sum. *Q.E.D.*

PROOF OF LEMMA S.6: Using part (a2) of Lemma S.5, for $q = 1, \dots, 4$, we can write

$$\begin{aligned} E^*[(R_2^* - R_2)^q(\hat{V}^* - V^*)] &= E^*[(R_2^* - R_2)^q(R_4^* - R_4)] - E^*[(R_2^* - R_2)^{2+q}] \\ &\quad - 2(R_2)E^*[(R_2^* - R_2)^{1+q}] \\ &\equiv I_1^q - I_2^q - I_3^q. \end{aligned} \tag{S3}$$

Similarly, for $q = 1, \dots, 4$, note that

$$\begin{aligned} &E^*[(R_2^* - R_2)^q(\hat{V}^* - V^*)^2] \\ &\quad = E^*[(R_2^* - R_2)^q(R_4^* - R_4)^2] - 2E^*[(R_2^* - R_2)^{2+q}(R_4^* - R_4)] \\ &\qquad - 4(R_2)E^*[(R_2^* - R_2)^{1+q}(R_4^* - R_4)] + E^*[(R_2^* - R_2)^{4+q}] \\ &\qquad + 4(R_2)E^*[(R_2^* - R_2)^{3+q}] + 4(R_2)^2E^*[(R_2^* - R_2)^{2+q}]. \end{aligned}$$

For (a1), set $q = 1$ in (S3). We have that

$$\begin{aligned} I_1^1 &= E^*[(R_2^* - R_2)(R_4^* - R_4)] = h(R_6 - R_4R_2), \\ I_2^1 &= E^*[(R_2^* - R_2)^3] = h^2(R_6 - 3R_4R_2 + 2R_2^3), \\ I_3^1 &= 2R_2E^*[(R_2^* - R_2)^2] = 2R_2[h(R_4 - R_2^2)] \end{aligned}$$

by Lemma S.5 (a8), (a3) and (a1), respectively. Thus

$$\begin{aligned} &E^*[(R_2^* - R_2)(\hat{V}^* - V^*)] \\ &\quad = h[(R_6 - R_4R_2) - 2R_2(R_4 - R_2^2)] - h^2(R_6 - 3R_4R_2 + 2R_2^3) \\ &\quad = h(R_6 - 3R_4R_2 + 2R_2^3) + O_P(h^2). \end{aligned}$$

The remaining results follow similarly. *Q.E.D.*

PROOF OF LEMMA S.7: The proof follows the proof of Lemma S.3, given the definition of V^* and given Lemmas S.5 and S.6. *Q.E.D.*

PROOFS OF PROPOSITIONS 4.2 AND 4.3(C) IN SECTION 4

PROOF OF PROPOSITION 4.2: When $g(z) = \log z$, we have that

$$q_{1,\log}(x) = \frac{\sqrt{2}}{3}\frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}}(2x^2+1) + \frac{(\overline{\sigma^4})^{1/2}}{\sqrt{2}}\frac{g''_{\log}(\overline{\sigma^2})}{g'_{\log}(\overline{\sigma^2})}$$
$$= \frac{1}{\sqrt{2}}\frac{(\overline{\sigma^4})^{1/2}}{\overline{\sigma^2}}\left[x^2\left(\frac{4}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2} - 1\right) + \frac{2}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}\right],$$

whereas for $g(z) = z$, $g''(z) = 0$ and

$$q_1(x) = \frac{\sqrt{2}}{3}\frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}}(2x^2+1)$$
$$= \frac{1}{\sqrt{2}}\frac{(\overline{\sigma^4})^{1/2}}{\overline{\sigma^2}}\left[x^2\left(\frac{4}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}\right) + \frac{2}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}\right].$$

Since $(\overline{\sigma^4})^2 \leq \overline{\sigma^6\sigma^2}$ by the Cauchy–Schwarz inequality, it follows that $\frac{1}{3} \leq \frac{4}{3}(\overline{\sigma^6\sigma^2}/(\overline{\sigma^4})^2) - 1 < \frac{4}{3}(\overline{\sigma^6\sigma^2}/(\overline{\sigma^4})^2)$, which implies that $q_{1,\log}(x) < q_1(x)$ for x fixed and nonzero. When $x = 0$, it follows trivially that $q_{1,\log}(0) = q_1(0)$. *Q.E.D.*

PROOF OF PROPOSITION 4.3(C): Define $C = 4\overline{\sigma^6}/(\sqrt{2}(\overline{\sigma^4})^{3/2})$ and $C^* = (15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3)/(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}$, and note that $C > 0$. It suffices to prove that $|C - C^*| \leq |C|$, which in turn is equivalent to proving $0 \leq C^* \leq 2C$. Next we show that $C^* \geq 0$. The Jensen inequality implies that $\overline{\sigma^4} \geq (\overline{\sigma^2})^2$, and since $\overline{\sigma^4} > 0$, it follows that the denominator of C^* is positive. For the numerator of C^*, note that we can write

$$15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2\left(\overline{\sigma^2}\right)^3 \geq 15\overline{\sigma^6} - 9\left(\overline{\sigma^4}\right)^{3/2} + 2\left(\overline{\sigma^2}\right)^3$$
$$\geq 9\left(\overline{\left(\sigma^4\right)^{3/2}} - \left(\overline{\sigma^4}\right)^{3/2}\right) + 6\overline{\sigma^6} + 2\left(\overline{\sigma^2}\right)^3,$$

using $-(\overline{\sigma^2})^2 \geq -\overline{\sigma^4}$. Since the function $\psi(x) = x^{3/2}$ for $x > 0$ is convex, we have that $\overline{(\sigma^4)^{3/2}} - (\overline{\sigma^4})^{3/2} \geq 0$, which implies $15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3 \geq 6\overline{\sigma^6} +$

$2(\overline{\sigma^2})^3 > 0$, proving that the numerator of C^* is also positive. Next we prove $C^*/C \leq 2$. We can write

$$\frac{C^*}{C} = \frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3}{8\overline{\sigma^6}} \frac{2\sqrt{2}(\overline{\sigma^4})^{3/2}}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}} \equiv C_1 \times C_2.$$

We show that $C_1 \leq 2$ and $C_2 \leq 1$. First, note that

$$\begin{aligned} &\frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3}{8\overline{\sigma^6}} \leq 2 \\ &\iff \quad 15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2\left(\overline{\sigma^2}\right)^3 \leq 16\overline{\sigma^6} \\ &\iff \quad 0 \leq \overline{\sigma^6} + 7\overline{\sigma^4}\,\overline{\sigma^2} + 2\overline{\sigma^2}\left(\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right), \end{aligned}$$

which proves the result since $(\overline{\sigma^4} - (\overline{\sigma^2})^2) \geq 0$ and $0 \leq \overline{\sigma^6} + 7\overline{\sigma^4}\,\overline{\sigma^2}$. Finally, we have that

$$\begin{aligned} \frac{2\sqrt{2}(\overline{\sigma^4})^{3/2}}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}} \leq 1 \quad &\iff \quad 8\left(\overline{\sigma^4}\right)^3 \leq \left(3\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)^3 \\ &\iff \quad 8\left(\overline{\sigma^4}\right)^3 \leq \left(2\overline{\sigma^4} + \left(\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)\right)^3, \end{aligned}$$

which holds true since $(\overline{\sigma^4} - (\overline{\sigma^2})^2) \geq 0$. *Q.E.D.*

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