Econometrica Supplementary Material

SUPPLEMENT TO "PREEMPTIVE POLICY EXPERIMENTATION": APPENDIX

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THIS APPENDIX PROVES THE MAIN RESULTS OF THE PAPER. The proof of Lemma 1 is omitted since this result is a restatement of Proposition 1 in Callander (2011a). We next present a formal characterization of R's best responses to the various possible outcomes resulting from D's first period policy choice.

LEMMA S1: (a) *If* $p_1 > 0$, *then* R *chooses a policy satisfying*

(i) $p_2^* = 0$ *if* $\psi(p_1) < -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$,

(ii) $p_2^* > p_1$ *such that* $E[\psi(p_2^*)] = \gamma - \frac{\sigma^2}{2\mu}$ *if* $\psi(p_1) \in [-\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma, \gamma - \frac{\sigma^2}{2\mu})$,

(iii) $p_2^* = p_1$ *if* $\psi(p_1) \in [\gamma - \frac{\sigma^2}{2\mu}, \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}]$, *and*

(iv) $p_2^* \in (0, p_1)$ *if* $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$.

(b) *If* $p_1 < 0$, *then* R *chooses a policy satisfying*

(i) $p_2^* > 0$ *such that* $E[\psi(p_2^*)] = \gamma - \frac{\sigma^2}{2\mu}$ *if* $\psi(p_1) < \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$,

(ii) $p_2^* = p_1$ *if* $\psi(p_1) \in [\gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}, \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}]$, *and*

(iii) $p_2^* \in (p_1, 0)$ *if* $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$.

PROOF: (a) Suppose D chooses a policy of the form $p_1 > 0$ and the ultimate outcome satisfies $\psi(p_1) < -\gamma$. Note that if R then chooses a policy of the form $p_2 > p_1$ in the second period, the expected policy outcome is then $\psi(p_1) + \mu(p_2 - p_1)$ and the variance in this policy outcome is $(p_2 - p_1)\sigma^2$. From this it follows that R's expected utility from the second period from choosing a policy $p_2 > p_1$ is $u_R(p_2) = -[\gamma - (\psi(p_1) + \mu(p_2 - p_1))]^2 - (p_2 - p_1)\sigma^2$. Differentiating $u_R(p_2)$ with respect to p_2 indicates that if p_2 is an optimal policy for R, it must be the case that $2\mu[\gamma - \psi(p_1) - \mu(p_2 - p_1)] - \sigma^2 = 0$, which in turn implies that $p_2 - p_1 = -\frac{\sigma^2}{2\mu^2} + \frac{\gamma - \psi(p_1)}{\mu}$. By substituting this into the expression for R's expected utility, it follows that if it is optimal for R to use a policy of the form $p_2 > p_1$ in the second period, it must be the case that R obtains an expected utility $u_R(p_2) = \frac{\sigma^4}{4\mu^2} - \frac{(\gamma - \psi(p_1))\sigma^2}{\mu}$.

By contrast, if R chooses the policy $p_2 = 0$ in the second period, then R obtains an expected utility of $u_R(0) = -4\gamma^2$. From this it follows that R prefers to choose $p_2 = 0$ over a policy of the form $p_2 > p_1$ in the second period if and only if $\frac{\sigma^4}{4\mu^2} - \frac{(\gamma - \psi(p_1))\sigma^2}{\mu} < -4\gamma^2$. Since this holds if and only if $\psi(p_1) <$

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$-\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$, it follows that R chooses the policy $p_2 = 0$ in the second period if and only if $\psi(p_1) < -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$.

Now suppose that $\psi(p_1) \geq -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$, but $\psi(p_1) \leq \gamma$. If it is optimal for R to choose a policy of the form $p_2 > p_1$ in the second period, then it must be the case that the optimal $p_2 > p_1$ from the first paragraph of the proof of this step satisfies $p_2 - p_1 > 0$. From this it follows that if it is optimal for R to choose a policy of the form $p_2 > p_1$ in the second period, then it must be the case that $p_2 - p_1 = -\frac{\sigma^2}{2\mu^2} + \frac{\gamma - \psi(p_1)}{\mu} > 0$, meaning it must be the case that $\psi(p_1) < \gamma - \frac{\sigma^2}{2\mu}$. From this it follows that R chooses a policy of the form $p_2 > p_1$ in the second period if and only if $\psi(p_1) \geq -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$ and $\psi(p_1) < \gamma - \frac{\sigma^2}{2\mu}$. Moreover, in these cases, the optimal policy p_2^* satisfies $E[\psi(p_2^*)] = \gamma - \frac{\sigma^2}{2\mu}$.

Next suppose that $\psi(p_1) > \gamma$ and consider what happens if R chooses a policy of the form $p_2 \in (0, p_1)$ in the second period. Note that if R chooses a policy of the form $p_2 \in (0, p_1)$, then the expected policy outcome is $-\gamma + \frac{p_2}{p_1}(\psi(p_1) + \gamma)$ and the variance in this policy outcome is $\frac{p_2(p_1 - p_2)}{p_1}\sigma^2$. From this it follows that if R chooses a policy of the form $p_2 \in (0, p_1)$, then R obtains an expected utility of $u_R(p_2) = -[2\gamma - \frac{p_2}{p_1}(\psi(p_1) + \gamma)]^2 - \frac{p_2(p_1 - p_2)}{p_1}\sigma^2$ in the second period. Differentiating with respect to p_2 indicates that if $p_2 \in (0, p_1)$ is an optimal policy for R, it must be the case that $2(\psi(p_1) + \gamma)[2\gamma - \frac{p_2}{p_1}(\psi(p_1) + \gamma)] + (2p_2 - p_1)\sigma^2 = 0$. Simple algebra then implies that the optimal p_2 satisfies

$$p_1 - p_2 = \frac{p_1}{2} \frac{2(\psi(p_1) - \gamma)(\psi(p_1) + \gamma) - p_1\sigma^2}{(\psi(p_1) + \gamma)^2 - p_1\sigma^2}.$$

Now if R prefers to choose a policy of the form $p_2 \in (0, p_1)$ over the policy p_1, it must be the case that the derivative $u_R'(p_2) < 0$ when this derivative is evaluated at $p_2 = p_1$. This takes place if and only if $2\frac{\psi(p_1)+\gamma}{p_1}[2\gamma - (\psi(p_1) + \gamma)] + \sigma^2 < 0$. Simple algebra then indicates that this inequality holds if and only if $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$, so from this it follows that R will not choose a policy of the form $p_2 \in (0, p_1)$ unless $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$. Furthermore, R will indeed prefer a policy of the form $p_2 \in (0, p_1)$ over $p_2 = 0$ in these cases for any reasonable choices of $p_1 > 0$ (e.g., if $p_1 < \frac{16\gamma^2}{\sigma^2}$).

Combining all these results indicates that R chooses a policy of the form $p_2 = 0$ satisfying $E[\psi(p_2^*)] = \gamma - \frac{\sigma^2}{2\mu}$ in the second period if $\psi(p_1) < -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$, R chooses a policy of the form $p_2 > p_1$ satisfying $E[\psi(p_2)] = \gamma - \frac{\sigma^2}{2\mu}$ if $\psi(p_1) \in [-\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma, \gamma - \frac{\sigma^2}{2\mu})$, R chooses a policy of the form $p_2 = p_1$

if $\psi(p_1) \in [\frac{\sigma^2}{2\mu}, \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}]$, and R chooses a policy of the form $p_2 \in (0, p_1)$ if $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$.

(b) Now suppose D chooses a policy of the form $p_1 < 0$. Note that if R chooses a policy of the form $p_2 > 0$, then we know from the same reasoning as in Lemma 1 that R will choose a policy satisfying $-\gamma + \mu p_2 = \gamma - \frac{\sigma^2}{2\mu}$, meaning R will choose the policy $p_2 = \frac{2\gamma}{\mu} - \frac{\sigma^2}{2\mu^2}$. Thus, if R chooses a policy of the form $p_2 > 0$, then the expected policy outcome will be $-\gamma + \mu p_2 = \gamma - \frac{\sigma^2}{2\mu}$ and the variance in this policy outcome will be $p_2\sigma^2 = \frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{2\mu^2}$. From this it follows that if R chooses a policy of the form $p_2 > 0$, then R obtains an expected utility from the second period equal to $-(\gamma - (\gamma - \frac{\sigma^2}{2\mu}))^2 - (\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{2\mu^2}) = \frac{\sigma^4}{4\mu^2} - \frac{2\gamma\sigma^2}{\mu}$.

Now if R chooses the policy $p_2 = p_1$, then R obtains a utility from the second period equal to $-(\gamma - \psi(p_1))^2$. From this it follows that R prefers to choose the optimal policy $p_2 > 0$ over the policy $p_2 = p_1$ if and only if $-(\gamma - \psi(p_1))^2 < \frac{\sigma^4}{4\mu^2} - \frac{2\gamma\sigma^2}{\mu}$, which holds if and only if $\psi(p_1) < \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$ (or $\psi(p_1) > \gamma + \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$). Thus, if the final policy outcome satisfies $\psi(p_1) < \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$, then R will choose the policy $p_2 = \frac{2\gamma}{\mu} - \frac{\sigma^2}{2\mu^2}$.

Now suppose that $\psi(p_1) > \gamma$ and consider what happens if R chooses a policy of the form $p_2 \in (p_1, 0)$ in the second period. Note that if R chooses a policy of the form $p_2 \in (p_1, 0)$ in the second period, then the expected policy outcome is $-\gamma + \frac{p_2}{p_1}(\psi(p_1) + \gamma)$ and the variance in this policy outcome is $\frac{p_2(p_2-p_1)}{p_1}\sigma^2$, meaning R obtains an expected utility of $u_R(p_2) = -[2\gamma - \frac{p_2}{p_1}(\psi(p_1) + \gamma)]^2 - \frac{p_2(p_2-p_1)}{p_1}\sigma^2$ in the second period. Differentiating with respect to p_2 then indicates that if $p_2 \in (p_1, 0)$ is an optimal policy for R, it must be the case that $2(\psi(p_1) + \gamma)[2\gamma - \frac{p_2}{p_1}(\psi(p_1) + \gamma)] - (2p_2 - p_1)\sigma^2 = 0$. Simple algebra then implies that the optimal p_2 satisfies

$$p_2 - p_1 = -\frac{p_1}{2}\frac{2(\psi(p_1) - \gamma)(\psi(p_1) + \gamma) + p_1\sigma^2}{(\psi(p_1) + \gamma)^2 + p_1\sigma^2}.$$

Now if R prefers to choose a policy of the form $p_2 \in (p_1, 0)$ over the policy p_1, it must be the case that the derivative $u'_R(p_2)$ is positive when evaluated at $p_2 = p_1$. This holds if and only if $2\frac{\psi(p_1)+\gamma}{p_1}[2\gamma - (\psi(p_1) + \gamma)] - \sigma^2 > 0$, and simple algebra then indicates that this in turn holds if and only if $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$. From this it follows that R will not choose a policy of the form $p_2 \in (p_1, 0)$ unless $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$. And R will indeed prefer the optimal $p_2 \in (p_1, 0)$ over the optimal $p_2 > 0$ in these cases for any reasonable choices of $p_1 < 0$ (e.g., $p_1 > \frac{\sigma^2}{\mu^2} - \frac{8\gamma}{\mu}$ is a sufficient condition).

Combining all these results shows that R chooses a policy of the form $p_2 > 0$ satisfying $E[\psi(p_2)] = \gamma - \frac{\sigma^2}{2\mu}$ if $\psi(p_1) < \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$, R chooses the policy $p_2 = p_1$ if $\psi(p_1) \in [\gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}, \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}]$, and R chooses a policy of the form $p_2 \in (p_1, 0)$ if $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$. *Q.E.D.*

PROOF OF THEOREM 1: The proof proceeds in several steps, which are outlined below.

Step 1. We first derive exact expressions for D's expected utility from choosing policies of the form $p_1 > 0$.

Step 2. Next we derive exact expressions for D's expected utility from choosing policies of the form $p_1 < 0$.

Step 3. We then derive expressions for the difference between D's expected utility from choosing a policy of the form $p_1 > 0$ and D's expected utility from choosing $p_1 = 0$, and we rewrite these expressions in terms of a variable k representing the ratio of γ to $\frac{\sigma^2}{4\mu}$ and a variable β representing the ratio of p_1 to $\frac{\sigma^2(k-1)^4}{48\mu^2}$.

Step 4. We also derive expressions for the difference between D's expected utility from choosing a policy of the form $p_1 < 0$ and D's expected utility from choosing $p_1 = 0$, and we rewrite these expressions in terms of the variable k defined above and a variable α representing the ratio of $|p_1|$ to $\frac{\sigma^2(k-\sqrt{2k-1})^2}{12\mu^2}$.

The key benefit of this transformation is that now the expressions for the expected utility differences in Steps 3 and 4 both break down into terms representing the short-term expected costs from choosing a less favorable policy initially and a term representing the expected benefits from potentially favorably influencing R's policy choice in the second period.

Step 5. We then illustrate that for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu}$, it must be the case that D prefers the optimal policy $p_1 > 0$ over $p_1 = 0$ and also prefers the optimal policy $p_1 < 0$ over $p_1 = 0$. We do this by showing that when k is only slightly greater than 1, the expected benefits from choosing a policy $p_1 \neq 0$ and potentially influencing R's second period policy choice are an order of magnitude greater than the expected costs.

Step 6. We show that for sufficiently large values of γ, it must be the case that D chooses the policy $p_1 = 0$ by showing that when k is sufficiently large, the short-term expected costs to choosing a policy $p_1 \neq 0$ are exponentially larger than the expected benefits from potentially influencing R's policy choice.

Step 7. We then illustrate that if D prefers the optimal value of $p_1 > 0$ over $p_1 = 0$ for some $\gamma > \frac{\sigma^2}{4\mu}$, then D also prefers the optimal value of $p_1 > 0$ over $p_1 = 0$ for all smaller values of γ that are still greater than $\frac{\sigma^2}{4\mu}$. We do this by showing that the ratio of the expected benefits from potentially influencing R's policy choice to the expected costs from choosing a policy $p_1 > 0$ is increasing in k for such values of k.

Step 8. By combining the insights in Steps 5–7, it follows that if D were restricted to only choosing policies of the form $p_1 \geq 0$, he would use a cutoff strategy characterized by a cutoff $\hat{\gamma}$ such that he chooses $p_1 = 0$ if $\gamma \leq \frac{\sigma^2}{4\mu}$ or $\gamma > \hat{\gamma}$, $p_1 > 0$ if $\gamma \in (\frac{\sigma^2}{4\mu}, \hat{\gamma})$, and would be indifferent between the optimal policy $p_1 > 0$ and $p_1 = 0$ if $\gamma = \hat{\gamma}$.

Step 9. We then illustrate that if D prefers the optimal value of $p_1 < 0$ over $p_1 = 0$ for some $\gamma > \frac{\sigma^2}{4\mu}$, then D also prefers the optimal value of $p_1 < 0$ over $p_1 = 0$ for all smaller values of γ that are still greater than $\frac{\sigma^2}{4\mu}$ by a similar proof technique to that in Step 7.

Step 10. By combining the insights in Steps 5, 6, and 9, it follows that if D were restricted to only choosing policies of the form $p_1 \leq 0$, then he would use a cutoff strategy characterized by some other cutoff $\tilde{\gamma}$ such that he chooses $p_1 = 0$ if $\gamma \leq \frac{\sigma^2}{4\mu}$ or $\gamma > \tilde{\gamma}$, $p_1 < 0$ if $\gamma \in (\frac{\sigma^2}{4\mu}, \tilde{\gamma})$, and would be indifferent between the optimal policy $p_1 < 0$ and $p_1 = 0$ if $\gamma = \tilde{\gamma}$.

Step 11. We then illustrate that the optimal $p_1 > 0$ will afford D a greater utility than the optimal $p_1 < 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu}$ by bounding the utility differences in Steps 3 and 4 for values of k close to 1.

Step 12. We illustrate that the smallest value of $\gamma > \frac{\sigma^2}{4\mu}$ for which $p_1 = 0$ is (weakly) preferred to the optimal value of $p_1 > 0$ must be smaller than the smallest value of $\gamma > \frac{\sigma^2}{4\mu}$ for which $p_1 = 0$ is (weakly) preferred to the optimal value of $p_1 < 0$.

Step 13. We then illustrate that for values of γ such that both the optimal policy $p_1 > 0$ and the optimal policy $p_1 < 0$ are preferred to $p_1 = 0$, it must be the case that there is some intermediate value of γ, say γ', such that D prefers the optimal policy $p_1 > 0$ over the optimal policy $p_1 < 0$ for values of $\gamma < \gamma'$, prefers the optimal policy $p_1 < 0$ over the optimal policy $p_1 > 0$ for values of $\gamma > \gamma'$, and is indifferent between the optimal policy $p_1 > 0$ and the optimal policy $p_1 < 0$ for values of $\gamma = \gamma'$.

Step 14. Finally we note why the policy must vary continuously with γ in regions where D always chooses $p_1 > 0$ or where D always chooses $p_1 < 0$.

Step 15. By combining all these intermediate results, we then note that the equilibrium must be of the form given in the proposition.

We now tackle each of these steps in turn.

STEP 1: If D chooses a policy $p_1 > 0$ and R chooses a policy of the form $p_2 > p_1$ in the second period, then we know from the proof of Lemma S1 that R chooses a second period policy satisfying $p_2 - p_1 = -\frac{\sigma^2}{2\mu^2} + \frac{\gamma - \psi(p_1)}{\mu}$, which will lead to an expected policy outcome $\gamma - \frac{\sigma^2}{2\mu}$ with variance $\frac{(\gamma - \psi(p_1))\sigma^2}{\mu} - \frac{\sigma^4}{2\mu^2}$. Thus, when R chooses a policy of the form $p_2 > p_1$ in the second period, D

obtains an expected utility of $u_D = -[-\gamma - (\gamma - \frac{\sigma^2}{2\mu})]^2 + \frac{\sigma^4}{2\mu^2} - \frac{(\gamma-\psi(p_1))\sigma^2}{\mu}$ from the second period, which simplifies to $-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2} + \frac{\psi(p_1)\sigma^2}{\mu}$. From this it follows that if $\psi(p_1) \geq -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$ and $\psi(p_1) < \gamma - \frac{\sigma^2}{2\mu}$, then D obtains an expected utility of $u_D = -4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}$ in the second period since R chooses a policy of the form $p_2 > p_1$ if $\psi(p_1)$ satisfies these two inequalities.

Also note that D obtains a second period expected utility of $u_D = 0$ if $\psi(p_1) < -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma$ since we have seen that R will choose the policy $p_2 = 0$ in this case. And D will obtain a second period expected utility of $u_D = -(-\gamma - \psi(p_1))^2$ if $\psi(p_1) \geq \gamma - \frac{\sigma^2}{2\mu}$ and $\psi(p_1) < \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$ since we have seen that R simply chooses the policy $p_2 = p_1$ in this case.

Now if $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$, then we know from the proof of Lemma S1 that R will choose a policy $p_2 \in (0, p_1)$ satisfying

$$p_1 - p_2 = \frac{p_1}{2}\frac{2(\psi(p_1)-\gamma)(\psi(p_1)+\gamma) - p_1\sigma^2}{(\psi(p_1)+\gamma)^2 - p_1\sigma^2},$$

which in turn implies that

$$p_2 = p_1\frac{4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2}{2((\psi(p_1)+\gamma)^2 - p_1\sigma^2)}.$$

Thus if $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$, then the expected policy outcome will be

$$\begin{aligned} &-\gamma + \frac{p_2}{p_1}\big(\psi(p_1)+\gamma\big) \\ &\quad = -\gamma + \frac{4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2}{2((\psi(p_1)+\gamma)^2 - p_1\sigma^2)}\big(\psi(p_1)+\gamma\big) \end{aligned}$$

and the variance in this policy outcome will be

$$\begin{aligned} &\frac{p_2(p_1-p_2)}{p_1}\sigma^2 \\ &\quad = \frac{[4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2][2(\psi(p_1)-\gamma)(\psi(p_1)+\gamma) - p_1\sigma^2]}{4((\psi(p_1)+\gamma)^2 - p_1\sigma^2)^2} \\ &\qquad \times p_1\sigma^2, \end{aligned}$$

which means that if $\psi(p_1) > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$, then D obtains a second-period expected utility of

$$-\left(\frac{4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2}{2((\psi(p_1)+\gamma)^2 - p_1\sigma^2)}\big(\psi(p_1)+\gamma\big)\right)^2 - \frac{[4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2][2(\psi(p_1)-\gamma)(\psi(p_1)+\gamma) - p_1\sigma^2]}{4((\psi(p_1)+\gamma)^2 - p_1\sigma^2)^2} \times p_1\sigma^2.$$

Now D's expected utility in the first period from choosing a policy $p_1 > 0$ is $E[u_D^1|p_1] = -\mu^2 p_1^2 - p_1\sigma^2$. Putting all this together, we see that if $f(\psi(p_1); p_1)$ denotes the density corresponding to a normal distribution with mean $-\gamma + \mu p_1$ and variance $p_1\sigma^2$ at the point $\psi(p_1)$, then D's total expected utility for the game from choosing a policy $p_1 > 0$ is

$$\begin{aligned}
E[u_D|p_1] = {} & -\mu^2 p_1^2 - p_1\sigma^2 \\
& + \int_{-4\mu\gamma^2/\sigma^2 - \sigma^2/(4\mu)+\gamma}^{\gamma-\sigma^2/(2\mu)} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
& \times f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& + \int_{\gamma-\sigma^2/(2\mu)}^{\sqrt{\gamma^2+p_1\sigma^2/2}} -\big(-\gamma - \psi(p_1)\big)^2 f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& + \int_{\sqrt{\gamma^2+p_1\sigma^2/2}}^{\infty} -\left(\frac{4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2}{2((\psi(p_1)+\gamma)^2 - p_1\sigma^2)}\big(\psi(p_1)+\gamma\big)\right)^2 \\
& \times f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& + \int_{\sqrt{\gamma^2+p_1\sigma^2/2}}^{\infty} -\big[4\gamma\big(\psi(p_1)+\gamma\big) - p_1\sigma^2\big] \\
& \times \big[2\big(\psi(p_1)-\gamma\big)\big(\psi(p_1)+\gamma\big) - p_1\sigma^2\big] \\
& /\Big(4\Big(\big(\psi(p_1)+\gamma\big)^2 - p_1\sigma^2\Big)^2\Big) p_1\sigma^2 f\big(\psi(p_1); p_1\big)\, d\psi(p_1).
\end{aligned}$$

STEP 2: Now suppose D chooses a policy of the form $p_1 < 0$. If R chooses a policy of the form $p_2 > 0$, then we know from the proof of Lemma S1 that R will choose the policy $p_2 = \frac{2\gamma}{\mu} - \frac{\sigma^2}{2\mu^2}$, meaning the expected policy outcome will be $-\gamma + \mu p_2 = \gamma - \frac{\sigma^2}{2\mu}$ and the variance in this policy outcome will be $p_2\sigma^2 = \frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{2\mu^2}$. Since R chooses a policy of the form $p_2 > 0$ if and only if

$\psi(p_1) < \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$, it then follows that if $\psi(p_1) < \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$, then D's expected utility from the second period will be $-(-\gamma - (\gamma - \frac{\sigma^2}{2\mu}))^2 - (\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{2\mu^2}) = -4\gamma^2 + \frac{\sigma^4}{4\mu^2}$.

Also note that D obtains a second period utility of $-(-\gamma - \psi(p_1))^2$ if $\psi(p_1) \geq \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$ and $\psi(p_1) < \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$ because we know from Lemma S1 that R will choose the policy $p_2 = p_1$ if $\psi(p_1) \geq \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$ and $\psi(p_1) \leq \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$.

Now if $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$, then we know from the proof of Lemma S1 that R chooses a policy $p_2 \in (p_1, 0)$ of the form

$$p_2 - p_1 = -\frac{p_1}{2}\frac{2(\psi(p_1) - \gamma)(\psi(p_1) + \gamma) + p_1\sigma^2}{(\psi(p_1) + \gamma)^2 + p_1\sigma^2},$$

which can also be expressed as

$$p_2 = p_1\frac{4\gamma(\psi(p_1) + \gamma) + p_1\sigma^2}{2((\psi(p_1) + \gamma)^2 + p_1\sigma^2)}.$$

Thus if $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$, then the expected policy outcome will be

$$-\gamma + \frac{p_2}{p_1}\big(\psi(p_1) + \gamma\big) = -\gamma + \frac{4\gamma(\psi(p_1) + \gamma) + p_1\sigma^2}{2((\psi(p_1) + \gamma)^2 + p_1\sigma^2)}\big(\psi(p_1) + \gamma\big)$$

and the variance in this policy outcome will be

$$\begin{aligned}&\frac{p_2(p_2 - p_1)}{p_1}\sigma^2\\ &\quad = -\frac{(4\gamma(\psi(p_1) + \gamma) + p_1\sigma^2)(2(\psi(p_1) - \gamma)(\psi(p_1) + \gamma) + p_1\sigma^2)}{4((\psi(p_1) + \gamma)^2 + p_1\sigma^2)^2}\\ &\qquad \times p_1\sigma^2.\end{aligned}$$

Thus if $\psi(p_1) > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$, then D will obtain a second period expected utility of

$$\begin{aligned}&-\left(\frac{4\gamma(\psi(p_1) + \gamma) + p_1\sigma^2}{2((\psi(p_1) + \gamma)^2 + p_1\sigma^2)}\big(\psi(p_1) + \gamma\big)\right)^2\\ &\quad + \frac{(4\gamma(\psi(p_1) + \gamma) + p_1\sigma^2)(2(\psi(p_1) - \gamma)(\psi(p_1) + \gamma) + p_1\sigma^2)}{4((\psi(p_1) + \gamma)^2 + p_1\sigma^2)^2}\\ &\quad \times p_1\sigma^2.\end{aligned}$$

Now D's expected utility in the first period from choosing a policy $p_1 < 0$ is $E[u_D^1|p_1] = -\mu^2 p_1^2 + p_1\sigma^2$. Putting all this together, we see that if $g(\psi(p_1); p_1)$ denotes the density corresponding to a normal distribution with mean $-\gamma + \mu p_1$ and variance $-p_1\sigma^2$ at the point $\psi(p_1)$, then D's total expected utility for the game from choosing a policy $p_1 < 0$ is

$$
\begin{aligned}
E[u_D|p_1] = {} & -\mu^2 p_1^2 + p_1\sigma^2 + \int_{-\infty}^{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}} \left(-4\gamma^2 + \frac{\sigma^4}{4\mu^2}\right) \\
& \times g\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& + \int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\sqrt{\gamma^2-p_1\sigma^2/2}} -\big(-\gamma - \psi(p_1)\big)^2 \\
& \times g\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& + \int_{\sqrt{\gamma^2-p_1\sigma^2/2}}^{\infty} -\left(\frac{4\gamma(\psi(p_1)+\gamma) + p_1\sigma^2}{2((\psi(p_1)+\gamma)^2 + p_1\sigma^2)}\big(\psi(p_1)+\gamma\big)\right)^2 \\
& \times g\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& + \int_{\sqrt{\gamma^2-p_1\sigma^2/2}}^{\infty} \big(4\gamma\big(\psi(p_1)+\gamma\big) + p_1\sigma^2\big)\big(2\big(\psi(p_1)-\gamma\big) \\
& \times \big(\psi(p_1)+\gamma\big) + p_1\sigma^2\big)/\big(4\big(\big(\psi(p_1)+\gamma\big)^2 + p_1\sigma^2\big)^2\big) \\
& \times p_1\sigma^2 g\big(\psi(p_1); p_1\big)\, d\psi(p_1).
\end{aligned}
$$

STEP 3: Note that when $f(\psi(p_1); p_1)$ denotes the density corresponding to a normal distribution with mean $-\gamma + \mu p_1$ and variance $p_1\sigma^2$ at the point $\psi(p_1)$, then by exploiting the facts that $\int_{-\infty}^{\infty} f(\psi(p_1); p_1)\, d\psi(p_1) = 1$ and $\int_{-\infty}^{\infty} \psi(p_1) f(\psi(p_1); p_1)\, d\psi(p_1) = -\gamma + \mu p_1$, we see that

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& \quad = -4\gamma^2 + \frac{\sigma^4}{4\mu^2} + p_1\sigma^2.
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
& \int_{-4\mu\gamma^2/\sigma^2-\sigma^2/(4\mu)+\gamma}^{\gamma-\sigma^2/(2\mu)} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
& \qquad \times f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
& \quad = -4\gamma^2 + \frac{\sigma^4}{4\mu^2} + p_1\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\gamma-\sigma^2/2\mu}^{\infty}\left(-4\gamma^2+\frac{\gamma\sigma^2}{\mu}+\frac{\psi(p_1)\sigma^2}{\mu}+\frac{\sigma^4}{4\mu^2}\right)\\
&\times f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1)\\
&-\int_{-\infty}^{-4\mu\gamma^2/\sigma^2-\sigma^2/4\mu+\gamma}\left(-4\gamma^2+\frac{\gamma\sigma^2}{\mu}+\frac{\psi(p_1)\sigma^2}{\mu}+\frac{\sigma^4}{4\mu^2}\right)\\
&\times f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1).
\end{aligned}
$$

Substituting this into our expression for $E[u_D|p_1]$ from Step 1 then gives

$$
\begin{aligned}
E[u_D|p_1]={}&-4\gamma^2+\frac{\sigma^4}{4\mu^2}-\mu^2p_1^2\\
&-\int_{\gamma-\sigma^2/2\mu}^{\infty}\left(-4\gamma^2+\frac{\gamma\sigma^2}{\mu}+\frac{\psi(p_1)\sigma^2}{\mu}+\frac{\sigma^4}{4\mu^2}\right)\\
&\times f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1)\\
&-\int_{-\infty}^{-4\mu\gamma^2/\sigma^2-\sigma^2/(4\mu)+\gamma}\left(-4\gamma^2+\frac{\gamma\sigma^2}{\mu}+\frac{\psi(p_1)\sigma^2}{\mu}+\frac{\sigma^4}{4\mu^2}\right)\\
&\times f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1)\\
&+\int_{\gamma-\sigma^2/(2\mu)}^{\sqrt{\gamma^2+p_1\sigma^2/2}}-\big(-\gamma-\psi(p_1)\big)^2 f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1)\\
&+\int_{\sqrt{\gamma^2+p_1\sigma^2/2}}^{\infty}-\left(\frac{4\gamma(\psi(p_1)+\gamma)-p_1\sigma^2}{2((\psi(p_1)+\gamma)^2-p_1\sigma^2)}\big(\psi(p_1)+\gamma\big)\right)^2\\
&\times f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1)\\
&+\int_{\sqrt{\gamma^2+p_1\sigma^2/2}}^{\infty}-\big[4\gamma\big(\psi(p_1)+\gamma\big)-p_1\sigma^2\big]\\
&\times\big[2\big(\psi(p_1)-\gamma\big)\big(\psi(p_1)+\gamma\big)-p_1\sigma^2\big]\\
&/\big(4\big(\big(\psi(p_1)+\gamma\big)^2-p_1\sigma^2\big)^2\big)p_1\sigma^2 f\big(\psi(p_1);\,p_1\big)\,d\psi(p_1).
\end{aligned}
$$

Now if D simply chooses the policy $p_1=0$ in the first period, then D obtains an expected payoff of $-4\gamma^2+\frac{\sigma^4}{4\mu^2}$ from the game since this is the payoff D obtains when the first period policy outcome $\psi(p_1)$ satisfies $\psi(p_1)=0$. By combining this with the previous result, we see that the difference between D's expected utility from choosing a policy of the form $p_1>0$ and D's expected utility from choosing the policy $p_1 = 0$ is

$$
\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\mu^2 p_1^2 - \int_{\gamma-\sigma^2/(2\mu)}^{\infty} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
&\qquad \times f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad - \int_{-\infty}^{-4\mu\gamma^2/\sigma^2-\sigma^2/(4\mu)+\gamma} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
&\qquad \times f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad + \int_{\gamma-\sigma^2/(2\mu)}^{\sqrt{\gamma^2+p_1\sigma^2/2}} -\big(-\gamma - \psi(p_1)\big)^2 f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad + \int_{\sqrt{\gamma^2+p_1\sigma^2/2}}^{\infty} -\left(\frac{4\gamma(\psi(p_1)+\gamma) - p_1\sigma^2}{2((\psi(p_1)+\gamma)^2 - p_1\sigma^2)}\big(\psi(p_1)+\gamma\big)\right)^2 \\
&\qquad \times f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad + \int_{\sqrt{\gamma^2+p_1\sigma^2/2}}^{\infty} -\big[4\gamma\big(\psi(p_1)+\gamma\big) - p_1\sigma^2\big] \\
&\qquad \times \big[2\big(\psi(p_1)-\gamma\big)\big(\psi(p_1)+\gamma\big) - p_1\sigma^2\big] \\
&\qquad /\big(4\big(\big(\psi(p_1)+\gamma\big)^2 - p_1\sigma^2\big)^2\big) \\
&\qquad \times p_1\sigma^2 f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1).
\end{aligned}
$$

We can also rewrite this as

$$
\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\mu^2 p_1^2 - \int_{\gamma-\sigma^2/(2\mu)}^{\infty} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
&\qquad \times f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad - \int_{-\infty}^{-4\mu\gamma^2/\sigma^2-\sigma^2/(4\mu)+\gamma} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
&\qquad \times f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad + \int_{\gamma-\sigma^2/(2\mu)}^{\infty} -\big(-\gamma - \psi(p_1)\big)^2 f\big(\psi(p_1);\, p_1\big)\, d\psi(p_1) \\
&\qquad + O\left(\Pr\left(w > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}\right)\right),
\end{aligned}
$$

where $O(\Pr(w > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}))$ denotes a (positive) term that is of an order no greater than the probability that $w > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}$ when w is a random variable drawn from the distribution with density $f(\psi(p_1); p_1)$. We can further simplify this expression by adding the terms representing integrals from $\gamma - \frac{\sigma^2}{2\mu}$ to ∞ to get

$$
\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\int_{-\infty}^{-4\mu\gamma^2/\sigma^2-\sigma^2/(4\mu)+\gamma} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
&\qquad \times f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
&\qquad - \mu^2 p_1^2 \\
&\qquad - \int_{\gamma-\sigma^2/(2\mu)}^{\infty} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2} + \big(-\gamma - \psi(p_1)\big)^2\right) \\
&\qquad \times f\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
&\qquad + O\left(\Pr\left(w > \sqrt{\gamma^2 + \frac{p_1\sigma^2}{2}}\right)\right).
\end{aligned}
$$

Now we wish to rewrite this expressions in terms of other variables. To do this, let k denote a variable defined by $k \equiv \frac{4\mu\gamma}{\sigma^2}$ so that $\gamma = \frac{k\sigma^2}{4\mu}$. Also suppose that we write p_1 in the form $p_1 = \frac{\beta\sigma^2(k-1)^4}{48\mu^2}$ for some positive number β, define z to be equal to $z \equiv \frac{-\gamma+\mu p_1-\psi(p_1)}{\sigma\sqrt{p_1}}$, and write $\psi(p_1)$ as $\psi(p_1) = -\frac{4\mu\gamma^2}{\sigma^2} - \frac{\sigma^2}{4\mu} + \gamma - \varepsilon\sigma\sqrt{p_1}$ for some $\varepsilon > 0$. Note that this formulation implies that $\mu^2 p_1^2 = \frac{\beta^2(k-1)^8\sigma^4}{2304\mu^2}$. After several steps of algebra, one can also show that this formulation implies that $z = \frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{\beta}} + \varepsilon$ and $-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2} = -\frac{\sigma^4}{\mu^2}[\frac{k(k-1)}{2} + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}]$. Finally, the fact that $z \equiv \frac{-\gamma+\mu p_1-\psi(p_1)}{\sigma\sqrt{p_1}}$ implies that the distribution of z is the same as the distribution of a standard normal variable with mean 0 and variance 1. From this it follows that the integral

$$
\begin{aligned}
&-\int_{-\infty}^{-4\mu\gamma^2/\sigma^2-\sigma^2/(4\mu)+\gamma} \left(-4\gamma^2 + \frac{\gamma\sigma^2}{\mu} + \frac{\psi(p_1)\sigma^2}{\mu} + \frac{\sigma^4}{4\mu^2}\right) \\
&\quad \times f\big(\psi(p_1); p_1\big)\, d\psi(p_1)
\end{aligned}
$$

also equals

$$\frac{\sigma^4}{\mu^2}\int_0^\infty \left[\frac{k(k-1)}{2}+\frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right] \\ \times \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\sqrt{3}/\sqrt{\beta}+\varepsilon)^2/2}\, d\varepsilon.$$

Now suppose that $\psi(p_1)=\gamma-\frac{\sigma^2}{2\mu}+\varepsilon\sigma\sqrt{p_1}$. When this holds, several steps of algebra show that $-4\gamma^2+\frac{\gamma\sigma^2}{\mu}+\frac{\psi(p_1)\sigma^2}{\mu}+\frac{\sigma^4}{4\mu^2}+(-\gamma-\psi(p_1))^2=\frac{\varepsilon\sqrt{\beta}(k-1)^2\sigma^4}{4\sqrt{3}\mu^2}(k+\frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}})$. Also note that if $z\equiv\frac{\gamma+\psi(p_1)-\mu p_1}{\sigma\sqrt{p_1}}$, then z has the same distribution as a standard normal variable with mean 0 and variance 1, and $z=\frac{2\sqrt{3}}{\sqrt{\beta}(k-1)}-\frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}}+\varepsilon$. From this it follows that the integral

$$\int_{\gamma-\sigma^2/2\mu}^\infty \left(-4\gamma^2+\frac{\gamma\sigma^2}{\mu}+\frac{\psi(p_1)\sigma^2}{\mu}+\frac{\sigma^4}{4\mu^2}+\big(-\gamma-\psi(p_1)\big)^2\right) \\ \times f\big(\psi(p_1);p_1\big)\,d\psi(p_1)$$

also equals

$$\frac{\sigma^4}{\mu^2}\int_0^\infty \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\left(k+\frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right) \\ \times \frac{1}{\sqrt{2\pi}} e^{-(2\sqrt{3}/(\sqrt{\beta}(k-1))-\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\varepsilon)^2/2}\, d\varepsilon.$$

Also note that if $\psi(p_1)=\sqrt{\gamma^2+\frac{p_1\sigma^2}{2}}=\frac{\sigma^2}{4\mu}\sqrt{k^2+\frac{\beta(k-1)^4}{6}}$ and $z\equiv\frac{\gamma+\psi(p_1)-\mu p_1}{\sigma\sqrt{p_1}}$, then we can rewrite z as $z=\frac{\sqrt{3}k}{\sqrt{\beta}(k-1)^2}+\frac{\sqrt{3k^2+\beta(k-1)^4/2}}{\sqrt{\beta}(k-1)^2}+\frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}}$. Since the distribution of $z=\frac{\gamma+\psi(p_1)}{\sigma\sqrt{p_1}}-\frac{\mu\sqrt{p_1}}{\sigma}$ is the same as the distribution of a standard normal random variable with mean 0 and variance 1, it follows that

$$O\left(\Pr\left(w>\sqrt{\gamma^2+\frac{p_1\sigma^2}{2}}\right)\right) \\ =O\left(\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\left(\varepsilon+\frac{\sqrt{3}k}{\sqrt{\beta}(k-1)^2}+\frac{\sqrt{3k^2+\beta(k-1)^4/2}}{\sqrt{\beta}(k-1)^2} \\ +\frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right)^2/2\right)d\varepsilon\right).$$

Putting all this together, we see that we can express the difference between D's expected utility from choosing a policy $p_1 > 0$ and D's expected utility from choosing the policy $p_1 = 0$ as

$$\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = \frac{\sigma^4}{\mu^2}\int_0^\infty \left[\frac{k(k-1)}{2} + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right] \\
&\qquad \times \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\sqrt{3}/\sqrt{\beta}+\varepsilon)^2/2}\, d\varepsilon \\
&\qquad - \frac{\sigma^4}{\mu^2}\int_0^\infty \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\left(k + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right) \\
&\qquad \times \frac{1}{\sqrt{2\pi}} e^{-(2\sqrt{3}/(\sqrt{\beta}(k-1))-\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\varepsilon)^2/2}\, d\varepsilon - \frac{\sigma^4}{\mu^2}\frac{\beta^2(k-1)^8}{2304} \\
&\qquad + O\biggl(\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\biggl(-\biggl(\varepsilon + \frac{\sqrt{3}k}{\sqrt{\beta}(k-1)^2} \\
&\qquad + \frac{\sqrt{3k^2 + \beta(k-1)^4/2}}{\sqrt{\beta}(k-1)^2} + \frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\biggr)^2 \Big/ 2\biggr) d\varepsilon\biggr).
\end{aligned}$$

STEP 4: Note that when $g(\psi(p_1); p_1)$ denotes the density corresponding to a normal distribution with mean $-\gamma + \mu p_1$ and variance $-p_1\sigma^2$ at the point $\psi(p_1)$ for some $p_1 < 0$, then the integral

$$\begin{aligned}
&\int_{-\infty}^{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}} \left(-4\gamma^2 + \frac{\sigma^4}{4\mu^2}\right) g\bigl(\psi(p_1); p_1\bigr)\, d\psi(p_1) \\
&\quad = -4\gamma^2 + \frac{\sigma^4}{4\mu^2} \\
&\qquad - \int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty} \left(-4\gamma^2 + \frac{\sigma^4}{4\mu^2}\right) g\bigl(\psi(p_1); p_1\bigr)\, d\psi(p_1).
\end{aligned}$$

Substituting this for our expression for $E[u_D|p_1]$ when $p_1 < 0$ then gives

$$\begin{aligned}
E[u_D|p_1] = {}& -\mu^2 p_1^2 + p_1\sigma^2 - 4\gamma^2 + \frac{\sigma^4}{4\mu^2} \\
&- \int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty} \left(-4\gamma^2 + \frac{\sigma^4}{4\mu^2}\right) g\bigl(\psi(p_1); p_1\bigr)\, d\psi(p_1)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\sqrt{\gamma^2-p_1\sigma^2/2}} -\bigl(-\gamma-\psi(p_1)\bigr)^2 \\
&\times g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1) \\
&+\int_{\sqrt{\gamma^2-p_1\sigma^2/2}}^{\infty} -\left(\frac{4\gamma(\psi(p_1)+\gamma)+p_1\sigma^2}{2((\psi(p_1)+\gamma)^2+p_1\sigma^2)}\bigl(\psi(p_1)+\gamma\bigr)\right)^2 \\
&\times g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1) \\
&+\int_{\sqrt{\gamma^2-p_1\sigma^2/2}}^{\infty} \bigl(4\gamma\bigl(\psi(p_1)+\gamma\bigr)+p_1\sigma^2\bigr) \\
&\times \bigl(2\bigl(\psi(p_1)-\gamma\bigr)\bigl(\psi(p_1)+\gamma\bigr)+p_1\sigma^2\bigr) \\
&/\bigl(4\bigl(\bigl(\psi(p_1)+\gamma\bigr)^2+p_1\sigma^2\bigr)^2\bigr)p_1\sigma^2 g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1).
\end{aligned}
$$

If D simply chooses the policy $p_1 = 0$ in the first period, then D obtains an expected payoff of $-4\gamma^2 + \frac{\sigma^4}{4\mu^2}$ from the game since this is the payoff D obtains when the first period policy outcome $\psi(p_1)$ satisfies $\psi(p_1) = 0$. By combining this with the previous result, we see that the difference between D's expected utility from choosing a policy of the form $p_1 < 0$ and D's expected utility from choosing the policy $p_1 = 0$ is

$$
\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\mu^2 p_1^2 + p_1\sigma^2 \\
&\qquad -\int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty} \left(-4\gamma^2+\frac{\sigma^4}{4\mu^2}\right) g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1) \\
&\qquad +\int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\sqrt{\gamma^2-p_1\sigma^2/2}} -\bigl(-\gamma-\psi(p_1)\bigr)^2 g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1) \\
&\qquad +\int_{\sqrt{\gamma^2-p_1\sigma^2/2}}^{\infty} -\left(\frac{4\gamma(\psi(p_1)+\gamma)+p_1\sigma^2}{2((\psi(p_1)+\gamma)^2+p_1\sigma^2)}\bigl(\psi(p_1)+\gamma\bigr)\right)^2 \\
&\qquad \times g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1) \\
&\qquad +\int_{\sqrt{\gamma^2-p_1\sigma^2/2}}^{\infty} \bigl(4\gamma\bigl(\psi(p_1)+\gamma\bigr)+p_1\sigma^2\bigr) \\
&\qquad \times \bigl(2\bigl(\psi(p_1)-\gamma\bigr)\bigl(\psi(p_1)+\gamma\bigr)+p_1\sigma^2\bigr) \\
&\qquad /\bigl(4\bigl(\bigl(\psi(p_1)+\gamma\bigr)^2+p_1\sigma^2\bigr)^2\bigr)p_1\sigma^2 g\bigl(\psi(p_1);p_1\bigr)\,d\psi(p_1).
\end{aligned}
$$

We can rewrite this as

$$
\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\mu^2 p_1^2 + p_1\sigma^2 \\
&\qquad - \int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty} \left(-4\gamma^2 + \frac{\sigma^4}{4\mu^2}\right) g\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
&\qquad + \int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty} -\big(-\gamma - \psi(p_1)\big)^2 g\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
&\qquad + O\left(\Pr\left(y > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}\right)\right),
\end{aligned}
$$

where $O(\Pr(y > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}))$ denotes a (positive) term that is of an order no greater than the probability that $y > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}$ when y is a random variable drawn from the distribution with density $g(\psi(p_1); p_1)$. We can further simplify this expression by adding the terms representing integrals from $\gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}}$ to ∞ to get

$$
\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\mu^2 p_1^2 + p_1\sigma^2 \\
&\qquad - \int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty} \left(-4\gamma^2 + \frac{\sigma^4}{4\mu^2} + \big(-\gamma - \psi(p_1)\big)^2\right) \\
&\qquad \times g\big(\psi(p_1); p_1\big)\, d\psi(p_1) \\
&\qquad + O\left(\Pr\left(y > \sqrt{\gamma^2 - \frac{p_1\sigma^2}{2}}\right)\right).
\end{aligned}
$$

Now we wish to rewrite this expression in terms of other variables. To do this, again let k denote a variable defined by $k \equiv \frac{4\mu\gamma}{\sigma^2}$ so that $\gamma = \frac{k\sigma^2}{4\mu}$. Also suppose that we write p_1 in the form $-p_1 = \frac{\alpha\sigma^2(k-\sqrt{2k-1})^2}{12\mu^2}$ for some positive number α, defined z to be equal to $z \equiv \frac{\gamma-\mu p_1+\psi(p_1)}{\sigma\sqrt{-p_1}}$, and write $\psi(p_1)$ as $\psi(p_1) = \gamma - \sqrt{\frac{2\gamma\sigma^2}{\mu} - \frac{\sigma^4}{4\mu^2}} + \varepsilon\sigma\sqrt{-p_1}$ for some $\varepsilon > 0$. Note that this formulation implies that $\mu^2 p_1^2 = \frac{\alpha^2(k-\sqrt{2k-1})^4\sigma^4}{144\mu^2}$ and $p_1\sigma^2 = -\frac{\alpha\sigma^4(k-\sqrt{2k-1})^2}{12\mu^2}$. Several steps of algebra also show that this formulation implies that $z = \frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{\alpha}} + \varepsilon$ and $-4\gamma^2 + \frac{\sigma^4}{4\mu^2} + (-\gamma - \psi(p_1))^2 = \frac{\sigma^4}{4\mu^2}[-k^2 + 1 + (k - \sqrt{2k-1})^2(1 + \frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}})^2]$. Finally, the fact that $z = \frac{\gamma-\mu p_1+\psi(p_1)}{\sigma\sqrt{-p_1}}$ implies that the distribution of z is the same as the distribution of a standard normal variable with mean 0 and variance 1. From this it follows that the integral

$$-\int_{\gamma-\sqrt{2\gamma\sigma^2/\mu-\sigma^4/(4\mu^2)}}^{\infty}\left(-4\gamma^2+\frac{\sigma^4}{4\mu^2}+\left(-\gamma-\psi(p_1)\right)^2\right)$$
$$\times\, g\big(\psi(p_1);\, p_1\big)\, d\psi(p_1)$$

also equals

$$\frac{\sigma^4}{\mu^2}\int_0^{\infty}\frac{1}{4}\left[k^2-1-(k-\sqrt{2k-1})^2\left(1+\frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}}\right)^2\right]$$
$$\times\,\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3})+\sqrt{3}/\sqrt{\alpha}+\varepsilon)^2/2}\,d\varepsilon.$$

Also note that if $\psi(p_1)=\sqrt{\gamma^2-\frac{p_1\sigma^2}{2}}=\frac{\sigma^2}{4\mu}\sqrt{k^2+\frac{2\alpha(k-\sqrt{2k-1})^2}{3}}$ and $z\equiv\frac{\gamma-\mu p_1+\psi(p_1)}{\sigma\sqrt{-p_1}}$, then we can rewrite z as $z=\frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}}+\frac{\sqrt{3}(k+\sqrt{k^2+2\alpha(k-\sqrt{2k-1})^2/3})}{2\sqrt{\alpha}(k-\sqrt{2k-1})}$. Since the distribution of z is the same as the distribution of a standard normal random variable with mean 0 and variance 1, it follows that

$$O\left(\Pr\left(y>\sqrt{\gamma^2-\frac{p_1\sigma^2}{2}}\right)\right)$$
$$=O\Bigg(\int_0^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Bigg(-\Bigg(\varepsilon+\frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}}$$
$$+\frac{\sqrt{3}(k+\sqrt{k^2+2\alpha(k-\sqrt{2k-1})^2/3})}{2\sqrt{\alpha}(k-\sqrt{2k-1})}\Bigg)^2\Big/2\Bigg)\,d\varepsilon\Bigg).$$

Putting all this together, we see that we can express the difference between D's utility from choosing the policy $p_1<0$ and D's utility from choosing the policy $p_1=0$ as

$$E[u_D|p_1]-E[u_D|p_1=0]$$
$$=-\frac{\sigma^4}{\mu^2}\frac{\alpha^2(k-\sqrt{2k-1})^4}{144}-\frac{\sigma^4}{\mu^2}\frac{\alpha(k-\sqrt{2k-1})^2}{12}$$
$$+\frac{\sigma^4}{\mu^2}\int_0^{\infty}\frac{1}{4}\left[k^2-1-(k-\sqrt{2k-1})^2\left(1+\frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}}\right)^2\right]$$
$$\times\,\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3})+\sqrt{3}/\sqrt{\alpha}+\varepsilon)^2/2}\,d\varepsilon$$

$$+O\bigg(\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\bigg(-\bigg(\varepsilon + \frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}}$$
$$+\frac{\sqrt{3}(k+\sqrt{k^2+2\alpha(k-\sqrt{2k-1})^2/3})}{2\sqrt{\alpha}(k-\sqrt{2k-1})}\bigg)^2\Big/2\bigg)\,d\varepsilon\bigg).$$

STEP 5: To prove that D prefers the optimal policy $p_1 > 0$ over the policy $p_1 = 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu^2}$, it suffices to show that in the limit as k approaches 1 from above, it must be the case that there is some function β that depends only on k, $\beta(k)$, such that the final expression at the end of Step 3 is guaranteed to be positive when $\beta = \beta(k)$. To see that this holds, note that if $\beta = \frac{1}{k-1}$, then the final expression at the end of Step 3 reduces to

$$\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1=0]\\
&= \frac{\sigma^4}{\mu^2}\int_0^\infty \left[\frac{k(k-1)}{2} + \frac{\varepsilon(k-1)^{3/2}}{4\sqrt{3}}\right]\\
&\quad\times \frac{1}{\sqrt{2\pi}} e^{-((k-1)^{3/2}/(4\sqrt{3})+\sqrt{3(k-1)}+\varepsilon)^2/2}\,d\varepsilon\\
&\quad - \frac{\sigma^4}{\mu^2}\int_0^\infty \frac{\varepsilon(k-1)^{3/2}}{4\sqrt{3}}\left(k+\frac{\varepsilon(k-1)^{3/2}}{4\sqrt{3}}\right)\\
&\quad\times \frac{1}{\sqrt{2\pi}} e^{-(2\sqrt{3}/\sqrt{k-1}-(k-1)^{3/2}/(4\sqrt{3})+\varepsilon)^2/2}\,d\varepsilon - \frac{\sigma^4}{\mu^2}\frac{(k-1)^6}{2304}\\
&\quad + O\bigg(\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\bigg(-\bigg(\varepsilon+\frac{\sqrt{3}k}{(k-1)^{3/2}}+\frac{\sqrt{3k^2+(k-1)^3/2}}{(k-1)^{3/2}}\\
&\quad + \frac{(k-1)^{3/2}}{4\sqrt{3}}\bigg)^2\Big/2\bigg)\,d\varepsilon\bigg).
\end{aligned}$$

Now in the limit as k approaches 1 from above, the first integral in this expression equals a term that is $\Omega(k-1)$ since $\frac{k(k-1)}{2} + \frac{\varepsilon(k-1)^{3/2}}{4\sqrt{3}} \geq \frac{k-1}{2}$ for all $k \geq 1$ and $\varepsilon \geq 0$, and $\lim_{k\to 1}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-((k-1)^{3/2}/(4\sqrt{3})+\sqrt{3(k-1)}+\varepsilon)^2/2}\,d\varepsilon = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2}\,d\varepsilon = \frac{1}{2}$. Further note that this first integral is always positive.

Also note that in the limit as k approaches 1 from above, the second integral in this expression equals a term that is $O(e^{-(6/(k-1))})$ and the third integral in this expression equals a term that is $O(e^{-(6/(k-1)^3)})$. These results hold because $e^{-(2\sqrt{3}/\sqrt{k-1}-(k-1)^{3/2}/(4\sqrt{3})+\varepsilon)^2/2}$ approaches $e^{-(2\sqrt{3}/\sqrt{k-1}+\varepsilon)^2/2}$, which satisfies $e^{-(2\sqrt{3}/\sqrt{k-1}+\varepsilon)^2/2} \leq e^{-((2\sqrt{3}/\sqrt{k-1})^2+\varepsilon^2)/2} = e^{-6/(k-1)}e^{-\varepsilon^2/2}$ for all $\varepsilon \geq 0$, and

$e^{-(\varepsilon+\sqrt{3}k/(k-1)^{3/2}+\sqrt{3k^2+(k-1)^3/2}/(k-1)^{3/2}+(k-1)^{3/2}/(4\sqrt{3}))^2/2}$ approaches $e^{-(\varepsilon+2\sqrt{3}/(k-1)^{3/2})^2/2}$, which satisfies $e^{-(\varepsilon+2\sqrt{3}/(k-1)^{3/2})^2/2} \leq e^{-((2\sqrt{3}/(k-1)^{3/2})^2+\varepsilon^2)/2} = e^{-6/(k-1)^3}e^{-\varepsilon^2/2}$ for all $\varepsilon > 0$.

Finally note that it is clearly the case that the $\frac{\sigma^4}{\mu^2}\frac{(k-1)^6}{2304}$ is $O((k-1)^6)$ in the limit as k approaches 1 from above.

Thus in the limit as k approaches 1 from above, the first term in the above equation is $\Omega(k-1)$ and is positive, and all other terms are $o(k-1)$. From this it follows that the above equation is strictly positive for values of k sufficiently close to 1 and D prefers the optimal policy $p_1 > 0$ over the policy $p_1 = 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu^2}$.

To prove that D prefers the optimal policy $p_1 < 0$ over the policy $p_1 = 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu^2}$, it suffices to show that in the limit as k approaches 1 from above, it must be the case that there is some function α that depends only on k, $\alpha(k)$, such that the final expression at the end of Step 4 is guaranteed to be positive when $\alpha = \alpha(k)$. To see that this holds, note that if $\alpha = \frac{1}{(k-\sqrt{2k-1})}$, then the final expression at the end of Step 4 reduces to

$$\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&\quad = -\frac{\sigma^4}{\mu^2}\frac{(k-\sqrt{2k-1})^2}{144} - \frac{\sigma^4}{\mu^2}\frac{(k-\sqrt{2k-1})}{12} \\
&\qquad + \frac{\sigma^4}{\mu^2}\int_0^\infty \frac{1}{4}\Bigg[k^2 - 1 - (k-\sqrt{2k-1})^2 \\
&\qquad \times \left(1 + \frac{\varepsilon}{\sqrt{3(k-\sqrt{2k-1})}}\right)^2\Bigg]\frac{1}{\sqrt{2\pi}}e^{-(7\sqrt{3(k-\sqrt{2k-1})}/6+\varepsilon)^2/2}\,d\varepsilon \\
&\qquad + O\Bigg(\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\Bigg(-\Bigg(\varepsilon + \frac{\sqrt{k-\sqrt{2k-1}}}{2\sqrt{3}} \\
&\qquad + \frac{\sqrt{3}(k+\sqrt{k^2+2(k-\sqrt{2k-1})^{3/2}/3})}{2\sqrt{k-\sqrt{2k-1}}}\Bigg)^2\Big/2\Bigg)\,d\varepsilon\Bigg).
\end{aligned}$$

Now in the limit as k approaches 1 from above, $\sqrt{2k-1} = \sqrt{k^2-(k-1)^2} = \sqrt{k^2} - h(k) = k - h(k)$, where $h(k)$ is a nonnegative function that satisfies $h(k) = \Theta(k-1)^2$. From this it follows that $k - \sqrt{2k-1} = \Theta((k-1)^2)$ for values of k that are slightly greater than 1. Thus the $-\frac{\sigma^4}{\mu^2}\frac{(k-\sqrt{2k-1})^2}{144}$ term in the above equation is $\Theta((k-1)^4)$ and the $-\frac{\sigma^4}{\mu^2}\frac{(k-\sqrt{2k-1})}{12}$ term is $\Theta((k-1)^2)$.

Now note that $k^2 - 1 - (k - \sqrt{2k-1})^2 = 2k(\sqrt{2k-1} - 1) = 2k(k - 1 - h(k)) = \Theta(k-1)$ in the limit as k approaches 1 from above. This in turn implies that $k^2 - 1 - (k - \sqrt{2k-1})^2(1 + \frac{\varepsilon}{\sqrt{3(k-\sqrt{2k-1})}})^2 = k^2 - 1 - (k - \sqrt{2k-1})^2(1 + \frac{2\varepsilon}{\sqrt{3(k-\sqrt{2k-1})}} + \frac{\varepsilon^2}{3(k-\sqrt{2k-1})}) = \Theta(k-1)$ for all values of $\varepsilon = o(\frac{1}{(k-\sqrt{2k-1})^{1/4}})$.

But in the limit as k approaches 1 from above, the first integral in the above expression approaches $\int_0^\infty \frac{1}{4}[2k(\sqrt{2k-1} - 1) - \frac{2\varepsilon(k-\sqrt{2k-1})^{3/2}}{\sqrt{3}} - \frac{\varepsilon^2(k-\sqrt{2k-1})}{3}]\frac{1}{\sqrt{2\pi}}e^{-\varepsilon^2/2}\,d\varepsilon$. Since $2k(\sqrt{2k-1} - 1) - \frac{2\varepsilon(k-\sqrt{2k-1})^{3/2}}{\sqrt{3}} - \frac{\varepsilon^2(k-\sqrt{2k-1})}{3} = \Theta(k-1)$ for all values of $\varepsilon = o(\frac{1}{(k-\sqrt{2k-1})^{1/4}})$, it then follows that this integral must be $\Theta(k-1)$ for values of k that are slightly greater than 1.

Finally, note that it must be the case that the last integral in the above expression equals a term that is $O(e^{-3/(2(k-1)^2)})$ in the limit as k approaches 1 for the following reason:

$$\left(\varepsilon + \frac{\sqrt{k-\sqrt{2k-1}}}{2\sqrt{3}} + \frac{\sqrt{3}\left(k + \sqrt{k^2 + \frac{2(k-\sqrt{2k-1})^{3/2}}{3}}\right)}{2\sqrt{k-\sqrt{2k-1}}}\right)^2$$

$$\geq \varepsilon^2 + \left(\frac{\sqrt{3}\left(k + \sqrt{k^2 + \frac{2(k-\sqrt{2k-1})^{3/2}}{3}}\right)}{2\sqrt{k-\sqrt{2k-1}}}\right)^2$$

$$\geq \varepsilon^2 + \left(\frac{\sqrt{3}}{\sqrt{k-\sqrt{2k-1}}}\right)^2 = \varepsilon^2 + \frac{3}{k-\sqrt{2k-1}};$$

hence, it must be the case that the integral is $O(e^{-3/(2(k-\sqrt{2k-1}))}) = O(e^{-3/(2(k-1)^2)})$.

Thus in the limit as k approaches 1 from above, the second integral in the above equation is $\Theta(k-1)$ and all other terms are $o(k-1)$. From this it follows that the above equation is strictly positive for values of k sufficiently close to 1 and D prefers the optimal policy $p_1 < 0$ over the policy $p_1 = 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu^2}$.

STEP 6: Note that for any fixed value of $k > 1$, the minimum possible value of $\frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{\beta}}$ that is achieved when $\beta > 0$ is achieved when the derivative of this expression with respect to $\sqrt{\beta}$ is 0, which occurs when $\frac{(k-1)^2}{4\sqrt{3}} - \frac{\sqrt{3}}{\beta} = 0$ or when $\beta = \frac{12}{(k-1)^2}$ and $\sqrt{\beta} = \frac{2\sqrt{3}}{k-1}$. Thus the minimum possible value of $\frac{\sqrt{\beta}(k-1)^2}{4\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{\beta}}$ when $\beta > 0$ is $k-1$. From this it follows that for all values of $\beta > 0$, it is necessarily the case that $e^{-(\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\sqrt{3}/\sqrt{\beta}+\varepsilon)^2/2} \le e^{-((k-1)+\varepsilon)^2/2} \le e^{-(k-1)^2/2}e^{-\varepsilon^2/2}$. Thus, the first integral in the final equation of Step 3 is necessarily no greater than

$$\frac{\sigma^4}{\mu^2}\int_0^\infty \left[\frac{k(k-1)}{2} + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right]\frac{1}{\sqrt{2\pi}}e^{-(k-1)^2/2}e^{-\varepsilon^2/2}\,d\varepsilon.$$

Since this term is $O(k^2e^{-(k-1)^2/2})$ in the limit as k goes to infinity, it follows that the first integral in the final equation of Step 3 is also $O(k^2e^{-(k-1)^2/2})$ in the limit as k goes to infinity.

Now the $-\frac{\sigma^4}{\mu^2}\frac{\beta^2(k-1)^8}{2304}$ term that appears in the final equation of Step 3 is $\Theta(\beta^2(k-1)^8)$ and is negative. And the total of the two last integrals that appear in the final equation of Step 3 is necessarily negative for all k. Since a term that is $\Theta(\beta^2(k-1)^8)$ is necessarily greater in magnitude than a term that is $O(k^2e^{-(k-1)^2/2})$ in the limit as k goes to infinity as long as β does not become arbitrarily small in the limit as k goes to infinity, it thus follows that the final equation in Step 3 is necessarily negative for sufficiently large k as long as this expression is also negative for values of $\beta > 0$ that are arbitrarily close to 0.

But for any fixed $k > 1$, the derivative of the final equation in Step 3 with respect to β is necessarily negative when β is arbitrarily close to 0 because the derivative of each of the integrals with respect to β is an exponentially small term, whereas the derivative of the $-\frac{\sigma^4}{\mu^2}\frac{\beta^2(k-1)^8}{2304}$ term with respect to β is only a polynomially small term. Since the derivative of this last term with respect to β is negative, it then follows that the derivative of the final equation in Step 3 with respect to β is necessarily negative when β is arbitrarily close to 0.

But from this it follows that for sufficiently large values of k, the expression in the final equation in Step 3 is necessarily negative for all values of $\beta > 0$. From this it follows that D prefers to choose the policy $p_1 = 0$ over any policy $p_1 > 0$ when γ is sufficiently large.

Next note that for any fixed value of $k > 1$, the minimum possible value of $\frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{\alpha}}$ that is achieved when $\alpha > 0$ is achieved when the derivative of this expression with respect to $\sqrt{\alpha}$ is 0, which occurs when $\frac{k-\sqrt{2k-1}}{2\sqrt{3}} - \frac{\sqrt{3}}{\alpha} = 0$ or when $\alpha = \frac{6}{k-\sqrt{2k-1}}$ and $\sqrt{\alpha} = \frac{\sqrt{6}}{\sqrt{k-\sqrt{2k-1}}}$. Thus, the minimum possible value of $\frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{\alpha}}$ when $\alpha > 0$ is $\sqrt{2(k-\sqrt{2k-1})}$. From this it follows that for all values of $\alpha > 0$, it is necessarily the case that $e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3})+\sqrt{3}/\sqrt{\alpha}+\varepsilon)^2/2} \le e^{-(\sqrt{2(k-\sqrt{2k-1})}+\varepsilon)^2/2} \le e^{-(k-\sqrt{2k-1})/2}e^{-\varepsilon^2/2}$. Thus, the sum of the two integrals in the final equation of Step 4 is necessarily no greater than

$$\frac{\sigma^4}{\mu^2}\int_0^\infty \frac{1}{4}k^2\frac{1}{\sqrt{2\pi}}e^{-(k-\sqrt{2k-1})/2}e^{-\varepsilon^2/2}\,d\varepsilon.$$

Since this term is $O(k^2e^{(k-\sqrt{2k-1})/2})$ in the limit as k goes to infinity, it follows that the sum of the integrals in the final equation of Step 4 is also $O(k^2e^{(k-\sqrt{2k-1})/2})$ in the limit as k goes to infinity.

Now the $-\frac{\sigma^4}{\mu^2}\frac{\alpha^2(k-\sqrt{2k-1})^4}{144}$ and $-\frac{\sigma^4}{\mu^2}\frac{\alpha(k-\sqrt{2k-1})^2}{12}$ terms that appear in the final equation of Step 4 are both negative, and are also $\Theta(\alpha^2(k-\sqrt{2k-1})^4)$ and $\Theta(\alpha(k-\sqrt{2k-1})^2)$, respectively. Since a term that is $\Theta(\alpha(k-\sqrt{2k-1})^2)$ is necessarily greater in magnitude than a term that is $O(k^2e^{(k-\sqrt{2k-1})/2})$ in the limit as k goes to infinity as long as α does not become arbitrarily small in the limit as k goes to infinity, it thus follows that the final equation in Step 4 is necessarily negative for sufficiently large k as long as this expression is also negative for values of $\alpha > 0$ that are arbitrarily close to 0.

But for any fixed $k > 1$, the derivative of the final equation in Step 4 with respect to α is necessarily negative when α is arbitrarily close to 0 because the derivative of each of the integrals with respect to α is an exponentially small term, whereas the derivative of the $-\frac{\sigma^4}{\mu^2}\frac{\alpha^2(k-\sqrt{2k-1})^4}{144}$ term with respect to α is only a polynomially small term and the derivative of the $-\frac{\sigma^4}{\mu^2}\frac{\alpha(k-\sqrt{2k-1})^2}{12}$ term with respect to α is a constant. Since the derivative of this last term with respect to α is negative, it then follows that the derivative of the final equation in Step 4 with respect to α is necessarily negative when α is arbitrarily close to 0.

But from this it follows that for sufficiently large values of k, the expression in the final equation in Step 4 is necessarily negative for all values of $\alpha > 0$. From this it follows that D prefers to choose the policy $p_1 = 0$ over any policy $p_1 < 0$ when γ is sufficiently large.

STEP 7: In this step, we wish to show that if D prefers the optimal value of $p_1 > 0$ over $p_1 = 0$ for some $\gamma > \frac{\sigma^2}{4\mu}$, then it must be the case that D prefers the optimal value of $p_1 > 0$ over $p_1 = 0$ for all smaller values of γ that are still greater than $\frac{\sigma^2}{4\mu}$. To prove this, it suffices to show that if, for a particular value of $k > 1$, the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation in Step 3 from choosing some value of $\beta > 0$ is strictly positive, then for all smaller values of k that are still greater than 1, the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation in Step 3 from choosing this β is also strictly positive.

To see this, let k denote some particular constant that is greater than 1 such that D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation of Step 3 from choosing some value of $\beta > 0$ is strictly positive. Consider some $k' \in (1, k)$ and compare D's utility from choosing the same value of β under this k' with D's utility from choosing this β under k.

Note that changing the value of k to some $k' \in (1, k)$ changes the value of the first integral in the final equation of Step 3 by a factor $\eta > \frac{(k'-1)^2}{(k-1)^2}$. The value of the term $e^{-(\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\sqrt{3}/\sqrt{\beta}+\varepsilon)^2/2}$ in this integral becomes greater when k is replaced by some $k' \in (1, k)$. And the value of the term $\frac{k(k-1)}{2} + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}$ in this integral changes by a factor no less than $\frac{(k'-1)^2}{(k-1)^2}$ when k is replaced by $k' \in (1, k)$. Thus, changing the value of k to k' changes the value of the first integral in the final equation of Step 3 by a factor η that is larger than $\frac{(k'-1)^2}{(k-1)^2}$.

Next note that changing the value of k to some $k' \in (1, k)$ changes the sum of the remaining terms (which has negative value) by a factor $\nu < \frac{(k'-1)^2}{(k-1)^2}$. The value of the term $-\frac{\sigma^4}{\mu^2}\frac{\beta^2(k-1)^8}{2304}$ changes by a factor of $\frac{(k'-1)^8}{(k-1)^8} < \frac{(k'-1)^2}{(k-1)^2}$ when k is replaced by $k' \in (1, k)$. The value of the term $e^{-(2\sqrt{3}/(\sqrt{\beta}(k-1))-\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\varepsilon)^2/2}$ that appears in the second integral in the final equation of Step 3 becomes smaller when k is replaced by some $k' \in (1, k)$. And the value of the term $\frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}(k + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}})$ that appears in the second integral in the final equation of Step 3 changes by a factor that is strictly less than $\frac{(k'-1)^2}{(k-1)^2}$ when k is replaced by some $k' \in (1, k)$ since the $\frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}$ term changes by a factor of exactly $\frac{(k'-1)^2}{(k-1)^2}$ and the $(k + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}})$ term becomes smaller when k is replaced by some $k' \in (1, k)$. From this it follows that changing the value of k to some $k' \in (1, k)$ changes the remaining terms (which have negative sum) by a factor ν that is strictly less than $\frac{(k'-1)^2}{(k-1)^2}$.

By combining the results in the previous two paragraphs, we see that changing the value of k to some $k' \in (1, k)$ changes the value of the positive terms in Step 3 by a factor that is larger than $\frac{(k'-1)^2}{(k-1)^2}$ and changes the value of the negative terms in Step 3 by a factor that is strictly less than $\frac{(k'-1)^2}{(k-1)^2}$. Thus if D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation of Step 3 from choosing some value of $\beta > 0$ is strictly positive for some particular $k > 1$, then D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation of Step 3 from choosing this β is also strictly positive for some $k' \in (1, k)$. From this it follows that if D prefers the optimal value of $p_1 > 0$ over $p_1 = 0$ for some $\gamma > \frac{\sigma^2}{4\mu}$, then it must be the case that D prefers the optimal value of $p_1 > 0$ over $p_1 = 0$ for all smaller values of γ that are still greater than $\frac{\sigma^2}{4\mu}$.

STEP 8: From the results in Steps 5–7, it follows that if D were restricted to only choosing policies of the form $p_1 \geq 0$, then when $\gamma > \frac{\sigma^2}{4\mu}$, D would use a cutoff strategy characterized by a cutoff $\hat{\gamma}$ such that D would choose the policy $p_1 = 0$ if $\gamma > \hat{\gamma}$ and choose a policy $p_1 > 0$ if $\gamma \in (\frac{\sigma^2}{4\mu}, \hat{\gamma})$. Since D's maximal utility from choosing some particular $p_1 > 0$ must vary continuously with γ, it then follows that D would be indifferent between the optimal policy $p_1 > 0$ and $p_1 = 0$ if $\gamma = \hat{\gamma}$. And we have seen that D chooses the policy $p_1 = 0$ when $\gamma \leq \frac{\sigma^2}{4\mu}$. Thus, if D were restricted to only choosing policies of the form $p_1 \geq 0$, then D would use a cutoff strategy characterized by a cutoff $\hat{\gamma}$ such that D would choose the policy $p_1 = 0$ if $\gamma \leq \frac{\sigma^2}{4\mu}$ or $\gamma > \hat{\gamma}$, choose a policy $p_1 > 0$ if $\gamma \in (\frac{\sigma^2}{4\mu}, \hat{\gamma})$, and would be indifferent between the optimal policy $p_1 > 0$ and $p_1 = 0$ if $\gamma = \hat{\gamma}$.

STEP 9: In this step, we wish to show that if D prefers the optimal value of $p_1 < 0$ over $p_1 = 0$ for some $\gamma > \frac{\sigma^2}{4\mu}$, then it must be the case that D prefers the optimal value of $p_1 < 0$ over $p_1 = 0$ for all smaller values of γ that are still greater than $\frac{\sigma^2}{4\mu}$. To prove this, it suffices to show that if, for a particular value of $k > 1$, the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation in Step 4 from choosing some value of $\alpha > 0$ is strictly positive, then for all smaller values of k that are still greater than 1, the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation in Step 4 from choosing this α is also strictly positive.

Note that D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation in Step 4 from choosing some value of $\alpha > 0$ is strictly positive if and only if this utility difference divided by $(k - \sqrt{2k-1})^2$ is also strictly positive. Thus, to prove the claim given in the previous paragraph, it suffices to show that if the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ that is given in the final equation in Step 4 divided by $(k - \sqrt{2k-1})^2$ is strictly positive for some $k = k^* > 1$, then this expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ divided by $(k - \sqrt{2k-1})^2$ is decreasing in k for all $k \in (1, k^*)$.

The expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ divided by $(k - \sqrt{2k-1})^2$ is equal to

$$
\begin{aligned}
&\frac{E[u_D|p_1] - E[u_D|p_1 = 0]}{(k - \sqrt{2k-1})^2} \\
&\quad = -\frac{\sigma^4}{\mu^2}\frac{\alpha^2(k - \sqrt{2k-1})^2}{144} - \frac{\sigma^4}{\mu^2}\frac{\alpha}{12} \\
&\qquad + \frac{\sigma^4}{\mu^2}\int_0^\infty \frac{1}{4}\left[\frac{k^2-1}{(k-\sqrt{2k-1})^2} - \left(1 + \frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}}\right)^2\right] \\
&\qquad \times \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3}) + \sqrt{3}/\sqrt{\alpha} + \varepsilon)^2/2}\, d\varepsilon
\end{aligned}
$$

$$+O\biggl(\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\biggl(-\biggl(\varepsilon+\frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}}$$
$$+\frac{\sqrt{3}(k+\sqrt{k^2+2\alpha(k-\sqrt{2k-1})^2/3})}{2\sqrt{\alpha}(k-\sqrt{2k-1})}\biggr)^2\Big/2\biggr)\,d\varepsilon\biggr).$$

Now the $-\frac{\sigma^4}{\mu^2}\frac{\alpha^2(k-\sqrt{2k-1})^2}{144}$ term in this expression is strictly decreasing in k since $\frac{d}{dk}[k-\sqrt{2k-1}]=1-\frac{1}{\sqrt{2k-1}}>0$ for all $k>1$. And the $\frac{\sigma^4}{\mu^2}\frac{\alpha}{12}$ term in this expression is independent of k. Thus, to prove that the above expression is decreasing in k for values of k where this expression is positive, it suffices to show that the first integral in this expression is decreasing in k for all $k>1$.

To prove this, we first illustrate that the expression $\frac{k^2-1}{(k-\sqrt{2k-1})^2}$ is decreasing in k for all $k>1$. To see this, note that $\frac{d}{dk}[\frac{k^2-1}{(k-\sqrt{2k-1})^2}]=(2k(k-\sqrt{2k-1})^2-2(k^2-1)(k-\sqrt{2k-1})(1-\frac{1}{\sqrt{2k-1}}))/(k-\sqrt{2k-1})^4$. Thus, $\frac{d}{dk}[\frac{k^2-1}{(k-\sqrt{2k-1})^2}]<0$ if and only if $2k(k-\sqrt{2k-1})^2-2(k^2-1)(k-\sqrt{2k-1})(1-\frac{1}{\sqrt{2k-1}})<0$. But this last expression holds if and only if $k(k-\sqrt{2k-1})-(k^2-1)(1-\frac{1}{\sqrt{2k-1}})<0$, which holds if and only if $\frac{k^2-1}{\sqrt{2k-1}}<k\sqrt{2k-1}-1\Leftrightarrow k^2-1<2k^2-k-\sqrt{2k-1}\Leftrightarrow 0<(k-1)^2+k-\sqrt{2k-1}$, which holds for all $k>1$ since $(k-1)^2>0$ for all $k>1$ and the fact that $k-\sqrt{2k-1}=0$ when $k=1$ and $\frac{d}{dk}[k-\sqrt{2k-1}]>0$ for all $k>1$ implies that $k-\sqrt{2k-1}>0$ when $k>1$. From this it follows that $\frac{k^2-1}{(k-\sqrt{2k-1})^2}$ is decreasing in k for all $k>1$.

Thus when k becomes larger, it is necessarily the case that there is a larger range of values of $\varepsilon>0$ for which the $\frac{1}{4}[\frac{k^2-1}{(k-\sqrt{2k-1})^2}-(1+\frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}})^2]\frac{1}{\sqrt{2\pi}}$ expression in the above integral is negative. And the values of $\varepsilon>0$ for which this expression is negative are relatively more heavily weighted when weights are taken according to $e^{-(\sqrt{\alpha}/(2\sqrt{3})+\sqrt{3}/(\sqrt{\alpha}(k-\sqrt{2k-1}))+\varepsilon/(k-\sqrt{2k-1}))^2/2}$. Furthermore, in the smaller range of values of $\varepsilon>0$ for which the $\frac{1}{4}[\frac{k^2-1}{(k-\sqrt{2k-1})^2}-(1+\frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}})^2]\frac{1}{\sqrt{2\pi}}$ expression in the above integral is positive, it is necessarily the case that this expression becomes smaller as k becomes larger since we have seen that $\frac{\sqrt{k^2-1}}{(k-\sqrt{2k-1})^2}$ is decreasing in k for all $k>1$. By combining all these facts, it follows that the integral

$$\int_0^\infty \frac{1}{4}\biggl[\frac{k^2-1}{(k-\sqrt{2k-1})^2}-\biggl(1+\frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}}\biggr)^2\biggr]$$
$$\times\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{\alpha}/(2\sqrt{3})+\sqrt{3}/(\sqrt{\alpha}(k-\sqrt{2k-1}))+\varepsilon/(k-\sqrt{2k-1}))^2/2}\,d\varepsilon$$

is decreasing in k for values of $k > 1$ when restricting attention to the values of k for which this integral is positive.

Now the first integral that appeared in the expression for $\frac{E[u_D|p_1]-E[u_D|p_1=0]}{(k-\sqrt{2k-1})^2}$ differs from the integral in the above expression only by a factor of $(k - \sqrt{2k-1})^2$ in the exponential. And as k becomes larger, the value of $(k - \sqrt{2k-1})^2$ becomes larger and the entire exponential becomes smaller. From this it follows that since the above expression is decreasing in k for values of $k > 1$ where this expression is positive, it is also the case that the first integral in the expression for $\frac{E[u_D|p_1]-E[u_D|p_1=0]}{(k-\sqrt{2k-1})^2}$ is decreasing in k for values of $k > 1$ where this expression is positive.

Putting all this together shows that the expression for $\frac{E[u_D|p_1]-E[u_D|p_1=0]}{(k-\sqrt{2k-1})^2}$ that was derived earlier is necessarily decreasing in k for values of $k > 1$ where this expression is positive. From this it follows that if D prefers the optimal value of $p_1 < 0$ over $p_1 = 0$ for some $\gamma > \frac{\sigma^2}{4\mu}$, then it must be the case that D prefers the optimal value of $p_1 < 0$ over $p_1 = 0$ for all smaller values of γ that are still greater than $\frac{\sigma^2}{4\mu}$.

STEP 10: From the results in Steps 5, 6, and 9, it follows that if D were restricted to only choosing policies of the form $p_1 \leq 0$, then when $\gamma > \frac{\sigma^2}{4\mu}$, D would use a cutoff strategy characterized by a cutoff $\tilde{\gamma}$ such that D would choose the policy $p_1 = 0$ if $\gamma > \tilde{\gamma}$ and choose a policy $p_1 < 0$ if $\gamma \in (\frac{\sigma^2}{4\mu}, \tilde{\gamma})$. Since D's maximal utility from choosing some particular $p_1 < 0$ must vary continuously with γ, it then follows that D would be indifferent between the optimal policy $p_1 < 0$ and $p_1 = 0$ if $\gamma = \tilde{\gamma}$. And we have seen that D chooses the policy $p_1 = 0$ when $\gamma \leq \frac{\sigma^2}{4\mu}$. Thus, if D were restricted to only choosing policies of the form $p_1 \leq 0$, then D would use a cutoff strategy characterized by a cutoff $\tilde{\gamma}$ such that D would choose the policy $p_1 = 0$ if $\gamma \leq \frac{\sigma^2}{4\mu}$ or $\gamma > \tilde{\gamma}$, choose a policy $p_1 < 0$ if $\gamma \in (\frac{\sigma^2}{4\mu}, \tilde{\gamma})$, and would be indifferent between the optimal policy $p_1 < 0$ and $p_1 = 0$ if $\gamma = \tilde{\gamma}$.

STEP 11: In this step we wish to show that the optimal $p_1 > 0$ will afford D a greater utility than the optimal $p_1 < 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu}$. To prove this, it suffices to show that for values of k that are slightly greater than 1, the maximal utility that D can obtain by choosing a policy $p_1 > 0$ is necessarily greater than the maximal utility D can obtain by choosing a policy $p_1 < 0$.

To prove this, first note that if k is only slightly greater than 1, then D can ensure that the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ given in the final equation in Step 3 is at least $\frac{\sigma^4}{\mu^2}\frac{k(k-1)}{4}$ by choosing some value of $\beta > 0$. Specifically, suppose that D chooses a value of $\beta = \beta(k)$, where

$\beta(k) \equiv \frac{16}{(k-1)^{3/2}}$ is a function that depends only on k. In this case, the final expression at the end of Step 3 reduces to

$$\begin{aligned}
&E[u_D|p_1] - E[u_D|p_1 = 0] \\
&= \frac{\sigma^4}{\mu^2}\int_0^\infty \left[\frac{k(k-1)}{2} + \frac{\varepsilon(k-1)^{5/4}}{\sqrt{3}}\right] \\
&\quad\times \frac{1}{\sqrt{2\pi}} e^{-((k-1)^{5/4}/\sqrt{3}+\sqrt{3}(k-1)^{3/4}/4+\varepsilon)^2/2}\,d\varepsilon \\
&\quad - \frac{\sigma^4}{\mu^2}\int_0^\infty \frac{\varepsilon(k-1)^{5/4}}{\sqrt{3}}\left(k + \frac{\varepsilon(k-1)^{5/4}}{\sqrt{3}}\right) \\
&\quad\times \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{3}/(2(k-1)^{1/4})-(k-1)^{5/4}/\sqrt{3}+\varepsilon)^2/2}\,d\varepsilon - \frac{\sigma^4}{\mu^2}\frac{(k-1)^5}{9} \\
&\quad + O\Bigg(\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\Bigg(-\Bigg(\varepsilon + \frac{\sqrt{3}k}{4(k-1)^{5/4}} + \frac{\sqrt{3k^2+16(k-1)^{5/2}}}{4(k-1)^{5/4}} \\
&\quad + \frac{(k-1)^{5/4}}{\sqrt{3}}\Bigg)^2\Big/2\Bigg)\,d\varepsilon\Bigg).
\end{aligned}$$

Now note that the integral

$$\begin{aligned}
&\int_0^\infty \left[\frac{k(k-1)}{2} + \frac{\varepsilon(k-1)^{5/4}}{\sqrt{3}}\right]\frac{1}{\sqrt{2\pi}} e^{-((k-1)^{5/4}/\sqrt{3}+\sqrt{3}(k-1)^{3/4}/4+\varepsilon)^2/2}\,d\varepsilon \\
&\quad = (k-1)\int_0^\infty \left[\frac{k}{2} + \frac{\varepsilon(k-1)^{1/4}}{\sqrt{3}}\right] \\
&\quad\quad\times \frac{1}{\sqrt{2\pi}} e^{-((k-1)^{5/4}/\sqrt{3}+\sqrt{3}(k-1)^{3/4}/4+\varepsilon)^2/2}\,d\varepsilon.
\end{aligned}$$

Further note that $\frac{k}{2} + \frac{\varepsilon(k-1)^{1/4}}{\sqrt{3}}$ exceeds $\frac{k}{2}$ by an amount $\Theta((k-1)^{1/4})$ and $e^{-((k-1)^{5/4}/\sqrt{3}+\sqrt{3}(k-1)^{3/4}/4+\varepsilon)^2/2}$ differs from $e^{-\varepsilon^2/2}$ by an amount $O((k-1)^{3/4})$. From this it follows that the integral $\int_0^\infty[\frac{k}{2} + \frac{\varepsilon(k-1)^{1/4}}{\sqrt{3}}]\frac{1}{\sqrt{2\pi}}\exp(-(\frac{(k-1)^{5/4}}{\sqrt{3}} + \frac{\sqrt{3}(k-1)^{3/4}}{4} + \varepsilon)^2/2)\,d\varepsilon = \int_0^\infty \frac{k}{2}\frac{1}{\sqrt{2\pi}}e^{-\varepsilon^2/2}\,d\varepsilon + h(k)$, where $h(k)$ is a strictly positive function satisfying $h(k) = \Theta((k-1)^{1/4})$ for values of k close to 1. Since $\int_0^\infty \frac{k}{2}\frac{1}{\sqrt{2\pi}}e^{-\varepsilon^2/2}\,d\varepsilon = \frac{k}{4}$, it then follows that $\int_0^\infty[\frac{k}{2} + \frac{\varepsilon(k-1)^{1/4}}{\sqrt{3}}]\frac{1}{\sqrt{2\pi}}\exp(-(\frac{(k-1)^{5/4}}{\sqrt{3}} + \frac{\sqrt{3}(k-1)^{3/4}}{4} + \varepsilon)^2/2)\,d\varepsilon = \frac{k}{4} + h(k)$, where $h(k)$ is a strictly positive function satisfying $h(k) = \Theta((k-1)^{1/4})$ for values of k close to 1. From this it follows that $\int_0^\infty[\frac{k(k-1)}{2} + \frac{\varepsilon(k-1)^{5/4}}{\sqrt{3}}]\frac{1}{\sqrt{2\pi}}e^{-((k-1)^{5/4}/\sqrt{3}+\sqrt{3}(k-1)^{3/4}/4+\varepsilon)^2/2}\,d\varepsilon = \frac{k(k-1)}{4} + \tilde{h}(k)$, where

$\tilde{h}(k)$ is a strictly positive function satisfying $\tilde{h}(k) = \Theta((k-1)^{5/4})$ for values of k close to 1.

Now for values of k close to 1, it is also the case that the second and third integrals in the expression given for $E[u_D|p_1] - E[u_D|p_1 = 0]$ when D chooses a value of $\beta = \frac{16}{(k-1)^{3/2}}$ are $O(e^{-3/(8\sqrt{k-1})})$ and $O(e^{-3/(8(k-1)^{5/2})})$, respectively. To see this, note that $e^{-(\sqrt{3}/(2(k-1)^{1/4})-(k-1)^{5/4}/\sqrt{3}+\varepsilon)^2/2} \leq e^{-(\sqrt{3}/(2(k-1)^{1/4})-(k-1)^{5/4}/\sqrt{3})^2/2} e^{-\varepsilon^2/2}$ and $e^{-(\sqrt{3}/(2(k-1)^{1/4})-(k-1)^{5/4}/\sqrt{3})^2/2} = O(e^{-3/(8\sqrt{k-1})})$ for values of k close to 1, so the second integral is $O(e^{-3/(8\sqrt{k-1})})$ for values of k close to 1. And $\exp(-(\varepsilon + \frac{\sqrt{3}k}{4(k-1)^{5/4}} + \frac{\sqrt{3k^2+(k-1)^{5/2}}}{4(k-1)^{5/4}} + \frac{(k-1)^{5/4}}{\sqrt{3}})^2/2) \leq e^{-\varepsilon^2/2} \exp(-(\frac{\sqrt{3}k}{4(k-1)^{5/4}} + \frac{\sqrt{3k^2+(k-1)^{5/2}}}{4(k-1)^{5/4}} + \frac{(k-1)^{5/4}}{\sqrt{3}})^2/2) \leq e^{-\varepsilon^2/2} \exp(-(\frac{\sqrt{3}}{2(k-1)^{5/4}} + \frac{(k-1)^{5/4}}{\sqrt{3}})^2/2)$ and $\exp(-(\frac{\sqrt{3}}{2(k-1)^{5/4}} + \frac{(k-1)^{5/4}}{\sqrt{3}})^2/2)$ is $O(e^{-3/(8(k-1)^{5/2})})$ for values of k close to 1, so the third integral is $O(e^{-3/(8(k-1)^{5/2})})$ for values of k close to 1. Also note that $\frac{\sigma^4}{\mu^2}\frac{(k-1)^5}{9}$ is clearly $\Theta((k-1)^5)$ for values of k close to 1.

By combining all the results in the previous two paragraphs, it follows that the expression given for the utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ when D chooses a value of $\beta = \frac{16}{(k-1)^{3/2}}$ is $\frac{\sigma^4}{\mu^2}[\frac{k(k-1)}{4} + \tilde{h}(k) + o((k-1)^{5/4})]$, where $\tilde{h}(k)$ is a strictly positive function satisfying $\tilde{h}(k) = \Theta((k-1)^{5/4})$ for values of k close to 1. From this it follows that if k is only slightly greater than 1, then D can ensure that the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ given in the final equation in Step 3 is at least $\frac{k(k-1)}{4}$ by choosing some value of $\beta > 0$.

Now note that the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ given in the final equation in Step 4 is necessarily less than $\frac{\sigma^4}{\mu^2}\frac{k(k-1)}{4}$ for any value of $\alpha > 0$. To see this, first note that $\sqrt{2k-1} = \sqrt{k^2 - (k-1)^2} = k - \hat{h}(k)$, where $\hat{h}(k)$ is a nonnegative function that satisfies $\hat{h}(k) = \Theta(k-1)^2$. From this it follows that $k^2 - 1 - (k - \sqrt{2k-1})^2 = 2k(\sqrt{2k-1} - 1) = 2k(k-1) - \hat{h}(k)$, where $\hat{h}(k)$ is a nonnegative function that satisfies $\hat{h}(k) = \Theta(k-1)^2$.

From this it follows that $k^2 - 1 - (k - \sqrt{2k-1})^2 < 2k(k-1)$ for values of k close to 1. Thus, $\frac{1}{4}[k^2 - 1 - (k - \sqrt{2k-1})^2] < \frac{k(k-1)}{2}$ and $\frac{1}{4}[k^2 - 1 - (k - \sqrt{2k-1})^2(1 + \frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}})^2] < \frac{k(k-1)}{2}$ for all $\alpha > 0$ and $\varepsilon > 0$ as well. Further note that $e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3})+\sqrt{3}/\sqrt{\alpha}+\varepsilon)^2/2} \leq e^{-\varepsilon^2/2}$ for all $\alpha > 0$. By combining these results, it follows that

$$\int_0^\infty \frac{1}{4}\left[k^2 - 1 - (k - \sqrt{2k-1})^2\left(1 + \frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}}\right)^2\right] \times \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3})+\sqrt{3}/\sqrt{\alpha}+\varepsilon)^2/2}\, d\varepsilon < \int_0^\infty \frac{k(k-1)}{2}\frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2}\, d\varepsilon = \frac{k(k-1)}{4}.$$

Now since $\sqrt{2k-1} = k - \hat{h}(k)$ for some nonnegative function $\hat{h}(k)$ satisfying $\hat{h}(k) = \Theta(k-1)^2$, it follows that $k - \sqrt{2k-1} = \hat{h}(k)$ for some function $\hat{h}(k)$ satisfying $\hat{h}(k) = \Theta(k-1)^2$, and the terms $-\frac{\sigma^4}{\mu^2}\frac{\alpha^2(k-\sqrt{2k-1})^4}{144}$ and $\frac{\sigma^4}{\mu^2}\frac{\alpha(k-\sqrt{2k-1})^2}{12}$ in the final equation in Step 4 are negative terms that are $\Theta(\alpha^2(k-1)^8)$ and $\Theta(\alpha(k-1)^4)$, respectively. Furthermore, the second integral in the final equation in Step 4 is $O(e^{-3/(2\alpha(k-1)^4)})$ since $\exp(-(\varepsilon + \frac{\sqrt{\alpha}(k-\sqrt{2k-1})}{2\sqrt{3}} + \frac{\sqrt{3}(k+\sqrt{k^2+2\alpha(k-\sqrt{2k-1})^2/3})}{2\sqrt{\alpha}(k-\sqrt{2k-1})})^2/2) \le e^{-\varepsilon^2/2}e^{-(\sqrt{3}/(\sqrt{\alpha}(k-\sqrt{2k-1})))^2/2} = e^{-\varepsilon^2/2} \times e^{-3/(2\alpha(k-\sqrt{2k-1})^2)} = O(e^{-3/(2\alpha(k-1)^4)})$.

Combining all these results shows that the final equation in Step 4 equals $\frac{\sigma^4}{\mu^2}[\frac{k(k-1)}{4} - \overline{h}(k)]$ for some nonnegative function $\overline{h}(k)$ that satisfies $\overline{h}(k) = \omega(\alpha(k-1)^4)$. From this it follows that regardless of the value of α that D chooses, the expression for D's utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ given in the final equation in Step 4 is necessarily less than $\frac{\sigma^4}{\mu^2}\frac{k(k-1)}{4}$ for values of k close to 1.

By combining the results for the cases where D chooses a policy $p_1 > 0$ with the results for the cases where D chooses a policy $p_1 < 0$, we see that D can ensure that he obtains an expected utility $E[u_D|p_1]$ satisfying $E[u_D|p_1] - E[u_D|p_1 = 0] > \frac{\sigma^4}{\mu^2}\frac{k(k-1)}{4}$ if D chooses a policy $p_1 > 0$ for values of k close to 1, but D cannot obtain an expected utility $E[u_D|p_1]$ satisfying $E[u_D|p_1] - E[u_D|p_1 = 0] > \frac{\sigma^4}{\mu^2}\frac{k(k-1)}{4}$ if D chooses a policy $p_1 < 0$ for values of k close to 1. From this it follows that the optimal $p_1 > 0$ will afford D a greater utility than the optimal $p_1 < 0$ for values of k that are only slightly greater than 1, and the optimal $p_1 > 0$ will afford D a greater utility than the optimal $p_1 < 0$ for values of γ that are only slightly greater than $\frac{\sigma^2}{4\mu}$. Moreover, similar reasoning indicates that D will prefer the optimal $p_1 > 0$ over the optimal $p_1 < 0$ for all values of $k \in (1, 1.1)$, so the optimal $p_1 > 0$ will afford D a greater utility than the optimal $p_1 < 0$ for values of $\gamma \in (\frac{\sigma^2}{4\mu}, 1.1\frac{\sigma^2}{4\mu})$.

STEP 12: Note that if we differentiate the expression for D's expected utility difference $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 3 with respect to β, then we find that (if we temporarily neglect the final integral, which will turn out to be insignificant in magnitude compared to the other terms)

$$\frac{\partial}{\partial\beta}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$$
$$= \frac{\sigma^4}{\mu^2}\int_0^\infty \frac{1}{\sqrt{2\pi}} h(\varepsilon, \beta, k) e^{-(\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\sqrt{3}/\sqrt{\beta}+\varepsilon)^2/2}\, d\varepsilon$$

$$
\begin{aligned}
&-\frac{\sigma^4}{\mu^2}\int_0^\infty \frac{1}{\sqrt{2\pi}} y(\varepsilon,\beta,k) e^{-(2\sqrt{3}/(\sqrt{\beta}(k-1))-\sqrt{\beta}(k-1)^2/(4\sqrt{3})+\varepsilon)^2/2}\,d\varepsilon\\
&-\frac{\sigma^4}{\mu^2}\frac{\beta(k-1)^8}{1152},
\end{aligned}
$$

where $h(\varepsilon,\beta,k)$ satisfies

$$
\begin{aligned}
h(\varepsilon,\beta,k) = {}& \frac{\varepsilon(k-1)^2}{8\sqrt{3\beta}} + \left(\frac{k(k-1)}{2} + \frac{\varepsilon\sqrt{\beta}(k-1)^2}{4\sqrt{3}}\right)\\
&\times\left(\frac{3}{2\beta^2} + \frac{\sqrt{3}\varepsilon}{2\beta^{3/2}} - \frac{(k-1)^2\varepsilon}{8\sqrt{3\beta}} - \frac{(k-1)^4}{96}\right)
\end{aligned}
$$

and $y(\varepsilon,\beta,k)$ satisfies

$$
\begin{aligned}
&y(\varepsilon,\beta,k)\\
&\quad= \frac{\varepsilon(k-1)^2}{4\sqrt{3}}\\
&\qquad\times\left[\frac{k}{2\sqrt{\beta}} + \frac{\varepsilon(k-1)^2}{4\sqrt{3}} + \left(k\sqrt{\beta} + \frac{\varepsilon\beta(k-1)^2}{4\sqrt{3}}\right)\right.\\
&\qquad\times\left.\left(\frac{6}{\beta^2(k-1)^2} - \frac{(k-1)^4}{96} + \frac{\sqrt{3}\varepsilon}{\beta^{3/2}(k-1)} + \frac{(k-1)^2\varepsilon}{8\sqrt{3\beta}}\right)\right].
\end{aligned}
$$

Also note that if we differentiate the expression for D's expected utility difference $E[u_D|p_1] - E[u_D|p_1=0]$ in the final equation in Step 4 with respect to α, then we find that (if we temporarily neglect the final integral, which will turn out to be insignificant in magnitude compared to the other terms)

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha}\{E[u_D|p_1] - E[u_D|p_1=0]\}\\
&\quad= -\frac{\sigma^4}{\mu^2}\frac{\alpha(k-\sqrt{2k-1})^4}{72} - \frac{\sigma^4}{\mu^2}\frac{(k-\sqrt{2k-1})^2}{12}\\
&\qquad+\frac{\sigma^4}{\mu^2}\int_0^\infty \frac{1}{4\sqrt{2\pi}} w(\varepsilon,\alpha,k) e^{-(\sqrt{\alpha}(k-\sqrt{2k-1})/(2\sqrt{3})+\sqrt{3}/\sqrt{\alpha}+\varepsilon)^2/2}\,d\varepsilon,
\end{aligned}
$$

where $w(\varepsilon,\alpha,k)$ satisfies

$$
\begin{aligned}
&w(\varepsilon,\alpha,k)\\
&\quad= -\left(\frac{\varepsilon}{\sqrt{3\alpha}} + \frac{\varepsilon^2}{3}\right)(k-\sqrt{2k-1})^2
\end{aligned}
$$

$$+\left[k^2-1-(k-\sqrt{2k-1})^2\left(1+\frac{\varepsilon\sqrt{\alpha}}{\sqrt{3}}\right)^2\right]$$
$$\times\left(\frac{3}{2\alpha^2}+\frac{\sqrt{3}\varepsilon}{2\alpha^{3/2}}-\frac{\varepsilon(k-\sqrt{2k-1})^2}{4\sqrt{3\alpha}}-\frac{(k-\sqrt{2k-1})^2}{24}\right).$$

Now for the expression $\frac{\partial}{\partial\beta}\{E[u_D|p_1]-E[u_D|p_1=0]\}$, it is necessarily the case that this partial derivative is either never positive for any values of $\beta>0$ or the partial derivative is initially negative for small values of $\beta>0$, then becomes positive for larger values of $\beta>0$, and then becomes negative once again for even larger values of $\beta>0$. From this it follows that if D prefers to choose some value of $p_1>0$ over $p_1=0$, then it is necessarily the case that the optimal $p_1>0$ for D corresponds to the larger of the two values of $\beta>0$ for which the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1]-E[u_D|p_1=0]\}$ is equal to 0. Similarly, if D prefers to choose some value of $p_1<0$ over $p_1=0$, then it is necessarily the case that the optimal $p_1<0$ for D corresponds to the larger of the two values of $\alpha>0$ for which the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1]-E[u_D|p_1=0]\}$ is equal to 0.

Thus to determine whether D prefers the optimal value of $p_1>0$ over $p_1=0$ for some particular value of $\gamma=\frac{k\sigma^2}{4\mu}$, it suffices to evaluate the expression for $E[u_D|p_1]-E[u_D|p_1=0]$ in the final equation in Step 3 at the larger of the two values of β where the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1]-E[u_D|p_1=0]\}$ is equal to 0. If the expression $E[u_D|p_1]-E[u_D|p_1=0]$ is greater (less) than 0 for this particular value of β, then D prefers the optimal value of $p_1>0$ over $p_1=0$ ($p_1=0$ to the optimal value of $p_1>0$) for this value of γ.

Similarly, to determine whether D prefers the optimal value of $p_1<0$ over $p_1=0$ for some particular value of $\gamma=\frac{k\sigma^2}{4\mu}$, it suffices to evaluate the expression for $E[u_D|p_1]-E[u_D|p_1=0]$ in the final equation in Step 4 at the larger of the two values of α where the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1]-E[u_D|p_1=0]\}$ is equal to 0. If the expression $E[u_D|p_1]-E[u_D|p_1=0]$ is greater (less) than 0 for this particular value of α, then D prefers the optimal value of $p_1>0$ over $p_1=0$ (prefers $p_1=0$ over the optimal value of $p_1>0$) for this value of γ.

Thus the smallest value of $\gamma>\frac{\sigma^2}{4\mu}$ for which $p_1=0$ is (weakly) preferred to the optimal value of $p_1>0$ is smaller than the smallest value of $\gamma>\frac{\sigma^2}{4\mu}$ for which $p_1=0$ is (weakly) preferred to the optimal value of $p_1<0$ if and only if the smallest value of k for which the expression for $E[u_D|p_1]-E[u_D|p_1=0]$ in the final equation in Step 3 is equal to 0 at the larger of the two values of β where the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1]-E[u_D|p_1=0]\}$ is equal to 0 is smaller than the smallest value of k for which the expression for $E[u_D|p_1]-$

$E[u_D|p_1 = 0]$ in the final equation in Step 4 is equal to 0 at the larger of the two values of α where the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0.

Computationally, it follows that the smallest value of k for which the expression for $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 3 is equal to 0 at the larger of the two values of β where the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0 is between 2.592 and 2.593. By contrast, the smallest value of k for which the expression for $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 4 is equal to 0 at the larger of the two values of α where the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0 is necessarily greater than 2.6. When $k = 2.6$, one can verify computationally that the larger of the two values of α where the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0 is $\alpha = 2.6987$, and $E[u_D|p_1] - E[u_D|p_1 = 0]$ is equal to $0.046845\frac{\sigma^4}{\mu^2}$ at these values of k and α. Moreover, the magnitudes of the final integrals in the final equations in Steps 3 and 4 are insignificant for values of k in this range compared to the magnitudes of the other terms and can be safely neglected. From this, it follows that the smallest value of $\gamma > \frac{\sigma^2}{4\mu}$ for which $p_1 = 0$ is (weakly) preferred to the optimal value of $p_1 > 0$ is smaller than the smallest value of $\gamma > \frac{\sigma^2}{4\mu}$ for which $p_1 = 0$ is (weakly) preferred to the optimal value of $p_1 < 0$.

STEP 13: To calculate D's utility difference between choosing the optimal policy $p_1 > 0$ and choosing the optimal policy $p_1 < 0$ for values of $\gamma = \frac{k\sigma^2}{4\mu}$ where both the optimal policy $p_1 > 0$ and the optimal policy $p_1 < 0$ are preferred to $p_1 = 0$, it suffices to calculate the difference between the value of $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 3 at the larger of the two values of β where the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0 and the value of $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 4 at the larger of the two values of α where the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0.

Let $V(k)$ denote the value of this utility difference for any particular k. We already know from earlier analysis that $V(k)$ is necessarily positive for values of k that are only slightly greater than 1 (and, in particular, that $V(k)$ is positive for values of $k \in (1, 1.1)$). We also know from the analysis in the previous step that D prefers to choose $p_1 = 0$ over the optimal value of $p_1 > 0$ for values of $k \geq 2.593$. Thus, to determine the sign of $V(k)$ for values of k where D prefers both the optimal policy $p_1 > 0$ and the optimal policy $p_1 < 0$ to $p_1 = 0$, we can restrict attention to values of $k \in [1.1, 2.593]$.

Computationally, one can calculate the value of $V(k)$ for any $k \in [1.1, 2.593]$ by calculating the difference between the value of $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 3 at the larger of the two values of β where the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0 and the value of $E[u_D|p_1] - E[u_D|p_1 = 0]$ in the final equation in Step 4 at the larger of the two values of α where the partial derivative $\frac{\partial}{\partial\alpha}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0. By doing this for all values of $k \in [1.1, 2.593]$ that are integral multiples of 0.001, it is apparent that $V(k)$ is positive and increasing in k for values of $k \leq 1.381$, that $V(k)$ is decreasing in k for values of $k \in (1.381, 2.469)$, and that $V(k)$ is significantly lower than 0 for values of $k \geq 2.469$. Moreover, the function $V(k)$ only crosses 0 at one value of $k \in (1.381, 2.469)$ (when $k \approx 1.709$), and the slope of $V(k)$ is significantly negative and steady for a range of values around $k \approx 1.709$, precluding any possibility that the sign of $V(k)$ flips several times for values of k near 1.709.

Thus, $V(k) > 0$ for values of $k < 1.709$, $V(k) < 0$ for values of $k > 1.709$, and the only value of $k \in (1, 2.593)$ for which $V(k) = 0$ is $k \approx 1.709$. From this, it follows that for values of γ such that both the optimal policy $p_1 > 0$ and the optimal policy $p_1 < 0$ are preferred to $p_1 = 0$, it must be the case that there is some intermediate value of γ, say γ', such that D prefers the optimal policy $p_1 > 0$ over the optimal policy $p_1 < 0$ for values of $\gamma < \gamma'$, prefers the optimal policy $p_1 < 0$ over the optimal policy $p_1 > 0$ for values of $\gamma > \gamma'$, and is indifferent between the optimal policy $p_1 > 0$ and the optimal policy $p_1 < 0$ for values of $\gamma = \gamma'$.

STEP 14: To see that the optimal policy must vary continuously with γ for ranges of γ where D always chooses a policy $p_1 > 0$, recall that for ranges of $\gamma = \frac{k\sigma^2}{4\mu}$ where D always chooses a policy $p_1 > 0$, D will necessarily choose the policy $p_1 = \frac{\beta\sigma^2(k-1)^4}{48\mu^2}$, where β is the larger of the two values of β where the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ is equal to 0. Let $\beta(k)$ denote this value of β. Since the partial derivative $\frac{\partial}{\partial\beta}\{E[u_D|p_1] - E[u_D|p_1 = 0]\}$ varies continuously with k and β, it follows that $\beta(k)$ must vary continuously with k as well. This in turn implies that the optimal policy $p_1 > 0$ necessarily varies continuously with k (and γ) for ranges of $\gamma = \frac{k\sigma^2}{4\mu}$ where D always chooses a policy $p_1 > 0$.

A similar argument shows that the optimal policy for D must vary continuously with γ for ranges of γ where D always chooses a policy $p_1 < 0$. Thus the optimal policy p_1 for D must vary continuously with γ in regions where D always chooses $p_1 > 0$ or where D always chooses $p_1 < 0$.

STEP 15: By combining all the intermediate results in the previous steps, it follows that the equilibrium is necessarily of the form given in the proposition. We have seen that D chooses the policy $p_1 = 0$ both when γ is sufficiently large and when $\gamma \leq \frac{\sigma^2}{4\mu}$. We also know from the results in Steps 8, 12, and 13 that there is some $\hat{\gamma} > \frac{\sigma^2}{4\mu}$ such that D prefers the optimal policy $p_1 > 0$ over both the policy $p_1 = 0$ and the optimal policy $p_1 < 0$ for all $\gamma \in (\frac{\sigma^2}{4\mu}, \hat{\gamma})$. We then know from the results in Steps 10, 12, and 13 that there is some $\tilde{\gamma} > \hat{\gamma}$ such that D prefers the optimal policy $p_1 < 0$ over both the policy $p_1 = 0$ and the optimal policy $p_1 > 0$ for all $\gamma \in (\hat{\gamma}, \tilde{\gamma})$, and that D then prefers the policy $p_1 = 0$ over all policies $p_1 \neq 0$ for $\gamma > \tilde{\gamma}$. Finally, we know from the result in Step 13 that D is indifferent between the optimal $p_1 > 0$ and the optimal $p_1 < 0$ when $\gamma = \hat{\gamma}$, and from the result in Step 10, we know that D is indifferent between $p_1 = 0$ and the optimal $p_1 < 0$ when $\gamma = \tilde{\gamma}$.

The results in the above paragraphs thus indicate that in equilibrium, D indeed follows a strategy characterized by two cutoffs, γ^* and γ^{**} satisfying $\frac{\sigma^2}{4\mu} < \gamma^* < \gamma^{**}$ such that the following properties are satisfied:

(i) D chooses the policy $p_1 = 0$ if $\gamma \leq \frac{\sigma^2}{4\mu}$ or $\gamma > \gamma^{**}$.

(ii) D chooses a policy $p_1 > 0$ that varies continuously with γ for values of $\gamma \in (\frac{\sigma^2}{4\mu}, \gamma^*)$.

(iii) D chooses a policy $p_1 < 0$ that varies continuously with γ for values of $\gamma \in (\gamma^*, \gamma^{**})$.

(iv) D either chooses a policy $p_1 > 0$ or a policy $p_1 < 0$ when $\gamma = \gamma^*$.

(v) D either chooses a policy $p_1 < 0$ or the policy $p_1 = 0$ when $\gamma = \gamma^{**}$.

Q.E.D.

For the next result that we prove in this appendix, we denote per-period utility by lower-case u_D^t, where the subscript is the player and the superscript is the period, and include arguments only when confusion would otherwise occur. We say that a policy dominates another for a player if it delivers lower variance and a better expected outcome.

PROOF OF THEOREM 2: By Lemma 1, $p_1 = 0$ induces $p_2 = 0$ for $\gamma < \frac{\sigma^2}{4\mu}$. As this gives $U_D = 0$ and $U_D < 0$ for any $p_1 \neq 0$, the result holds.

Hereafter suppose $\gamma > \frac{\sigma^2}{4\mu}$ and consider four cases.

(i) $p_1 = 0$. R experiments as in Lemma 1. Define $z = 2\gamma - \frac{\sigma^2}{2\mu}$. Then $p_2 = \frac{z}{\mu}$ and $EU_D = 0 - z^2 - \frac{z}{\mu}\sigma^2$.

(ii) $p_1 < 0$. This gives $Eu_D^1 < 0$ and, by the dictates of the No Stuck game, R chooses $p_2 = \frac{z}{\mu}$. Thus, $p_1 < 0$ is dominated by $p_1 = 0$.

(iii) $p_1 > \frac{z}{\mu}$. This is dominated by $p_1 = 0$, as D obtains a higher expected utility by choosing $p_1 = 0$ than he would by choosing $p_1 > \frac{z}{\mu}$ even if D always obtained his ideal policy in the second period after choosing $p_1 > \frac{z}{\mu}$.

(iv) $p_1 \in (0, \frac{z}{\mu})$. Suppose $E\psi(p_1) = -\gamma + \lambda$ such that $Eu_D^1 = -\lambda^2 - \frac{\lambda}{\mu}\sigma^2$. The strategy of proof is to show that $Eu_D^2 < -z^2 - \frac{z-\lambda}{\mu}\sigma^2$, which implies $EU_D < (-\lambda^2 - \frac{\lambda}{\mu}\sigma^2) + (-z^2 - \frac{z-\lambda}{\mu}\sigma^2) = -\lambda^2 - z^2 - \frac{z}{\mu}\sigma^2$. As this is strictly less than D's total utility from $p_1 = 0$, $p_1 = 0$ is optimal and the result follows.

The possible outcomes of experiment p_1 are distributed normally around $-\gamma + \lambda$, and Eu_D^2 is the integral over these outcomes factoring in R's optimal

p_2. To circumvent the need for this calculation, we bound the value of this integral by considering matched pairs of outcomes that are arrayed symmetrically around the expected value; that is, are of the form $-\gamma + \lambda \pm \omega$ for all $\omega \geq 0$. Four cases need to be considered:

Case 0: $\omega = 0$. R experiments according to Lemma S1 and $Eu_D^2 = -z^2 - \frac{z-\lambda}{\mu}\sigma^2$.

Case I: $\omega \in (0, z-\lambda)$. By the dictates of the No Stuck game, $p_2 > p_1$ for both $\pm\omega$ and R experiments according to Lemma S1. D's average second period utility across this pair is

$$\begin{aligned}\text{Average}\, Eu_D^2 &= \frac{1}{2}\left(-z^2 - \frac{z-\lambda-\omega}{\mu}\sigma^2\right) + \frac{1}{2}\left(-z^2 - \frac{z-\lambda+\omega}{\mu}\sigma^2\right)\\ &= -z^2 - \frac{1}{2}\left(-\frac{z-\lambda-\omega}{\mu} - \frac{z-\lambda+\omega}{\mu}\right)\sigma^2\\ &= -z^2 - \frac{z-\lambda}{\mu}\sigma^2.\end{aligned}$$

Case II: $\omega \in [z-\lambda, \tilde{\alpha}]$, where $\tilde{\alpha}$ is defined such that $p_2 = p_1$ if and only if $\psi(p_1) \in [\gamma - \frac{\sigma^2}{2\mu}, \gamma + \tilde{\alpha}]$. (Callander (2011) shows that behavior obeys such a cut point.) Following $\psi(p_1) = -\gamma + \lambda - \omega$, R experiments as in Lemma S1 and the variance of $\psi(p_2)$ is $\frac{z-\lambda+\omega}{\mu}\sigma^2 > 2\frac{z-\lambda}{\mu}\sigma^2$, which implies that the average variance across this pair exceeds $\frac{z-\lambda}{\mu}\sigma^2$. As the average expected outcome is more distant than z from D's ideal, the average Eu_D^2 is strictly less than $-z^2 - \frac{z-\lambda}{\mu}\sigma^2$.

Case III: $\omega > \tilde{\alpha}$, such that $p_2 \in (0, p_1)$ and R experiments on the bridge formed by outcome $-\gamma + \lambda + \omega$. This case proceeds analogously to Case II.

As the average $Eu_D^2 \leq -z^2 - \frac{z-\lambda}{\mu}\sigma^2$ in all cases, the integral over all possible outcomes, by the symmetry of the normal distribution, is less than this value, and the required property on Eu_D^2 is established. *Q.E.D.*

PROOF OF THEOREM 3: From the proof of Lemma S1, for $p < 0$,

$$\begin{aligned}Eu_D^1(p) &= -\left(\gamma - \frac{\sigma^2}{2\mu} + \mu p + \gamma\right)^2 + p\sigma^2,\\ \frac{dEu_D^1(p)}{dp} &= -2\left(\gamma - \frac{\sigma^2}{2\mu} + \mu p + \gamma\right)\mu + \sigma^2.\end{aligned}$$

With a one-period horizon, the first order condition is $(\gamma - \frac{\sigma^2}{2\mu} + \mu p^*) = -\gamma + \frac{\sigma^2}{2\mu}$, where the left-hand side is $E\psi(p^*)$ and the right-hand side is $\frac{\sigma^2}{2\mu}$ above D's ideal.

D's second period utility is a truncated draw from the outcome of his first period experiment. If $\psi(p_1) \leq \gamma - \frac{\sigma^2}{2\mu}$ it is ignored by R and, by Lemma S1,

$p_2 = 0$ and the outcome is $\gamma - \frac{\sigma^2}{2\mu}$. On the other hand, if $\psi(p_1) > \gamma - \frac{\sigma^2}{2\mu}$, then R uses the experiment and $p_2 \in [p_1, 0)$ with $E\psi(p_2) > \gamma - \frac{\sigma^2}{2\mu}$ (and possibly positive variance).

D's total utility is then

$$EU_D = -\left(\gamma - \frac{\sigma^2}{2\mu} + \mu p + \gamma\right)^2 + p\sigma^2 + \int_{-\infty}^{\mu p} f\left(x; 0, |p|\sigma^2\right)\Phi(x)\, dx - F\left(\mu p; 0, |p|\sigma^2\right)\left(\gamma - \frac{\sigma^2}{2\mu} + \gamma\right)^2,$$

where $f(\cdot)$ is the Normal probability density function of mean 0 and variance $|p|\sigma^2$, and $\Phi(x)$ measures D's utility from R's second period choice following $\psi(p) = \gamma - \frac{\sigma^2}{2\mu} + \mu p - x$, such that $\Phi(x) < -(\gamma - \frac{\sigma^2}{2\mu} + \gamma)^2$ for all $x < \mu p$. As the integral has strictly positive mass for $p < 0$ and $\frac{dEu_D^1(p)}{dp} = 0$ for $p = 0$ at $\gamma = \frac{\sigma^2}{2\mu}$, the first part of the theorem holds. As $\gamma \to \infty$, $F(\mu p; 0, |p|\sigma^2) \to 1$ for the one-period optimal policy choice (given in Lemma S1), and Eu_D^1 dominates EU_D, establishing the second part of the theorem. *Q.E.D.*

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