SUPPLEMENT TO "SELECTING APPLICANTS"

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APPENDIX A: COMMON KNOWLEDGE WITHOUT ALIGNMENT

THIS SECTION SHOWS HOW TO APPLY now-standard "ironing" logic (see, e.g., Myerson (1981)) to solve the optimal contracting problem under common knowledge of agent type when the upper bound acceptance rule is not monotonic, and therefore the monotonicity constraint (5) may be binding.

First, let us rewrite the function describing the principal's utility for an applicant. Previously, I defined the expected quality in (T, U_A)-space through $U_P(t, u_A) = \mathbb{E}[Q | T = t, U_A = u_A]$. Now define a similar function, l, which tells us the expected quality at a given test result and a *quantile* (rather than a realization) of U_A. Specifically, at each test result t, there is a continuous conditional distribution of U_A which can be rewritten in terms of its quantiles (i.e., by going from a CDF to an inverse CDF): U_A increases in quantile at each t, with quantile 0 at the infimum of the support of $U_A | T = t$ and quantile 1 at the supremum. For $t \in \mathcal{T}$ and $x \in [0, 1]$, let $l(t, x)$ be equal to $U_P(t, u_A)$ for u_A at the xth quantile of the distribution of $U_A | T = t$. Higher x gives higher u_A; alignment up to distinguishability is equivalent to the statement that $l(t, x)$ is weakly increasing in x for every t.

When alignment up to distinguishability fails, there exist test results t for which $l(t, \cdot)$ is not weakly increasing. At these test results, define an *ironed* version of the function l as follows. First, integrate l over quantiles to get $L(t, x) \equiv \int_0^x l(t, x') \, dx'$. Now "iron" L, separately at each test result t, by defining $\overline{L}(t, \cdot)$ to be the convex hull of $L(t, \cdot)$, that is, the highest convex function that is weakly below $L(t, \cdot)$. Finally, let the ironed l be defined as $\overline{l}(t, x) \equiv \frac{\partial \overline{L}}{\partial x}(t, x)$. The function $\overline{l}$ is defined for almost every $x \in [0, 1]$ by convexity of $\overline{L}(t, x)$ in x, and furthermore, $\overline{l}(t, x)$ is weakly increasing in x at every t. At any t for which $l(t, \cdot)$ is weakly increasing, it holds that $\overline{l}(t, x) = l(t, x)$ for all x.[36]

To restate, the principal's utility for an applicant with test result t and agent utility quantile x is $l(t, x)$. The ironed principal utility is $\overline{l}(t, x)$. Loosely speaking, we now proceed as if we were solving for UBAR as in Section 3.1, after replacing true principal utilities for each applicant with ironed—and therefore aligned—utilities.

More formally, let us now write an acceptance rule as $\chi : \mathcal{T} \times [0, 1] \to [0, 1]$, mapping test result and quantile (t, x) into an acceptance probability. As before, implementable acceptance rules must be monotonic—weakly increasing in x—and lead to a total mass of k acceptances. A new ironing constraint also states that any applicants with the same test result and the same ironed principal utility must be given the same acceptance rate:

$$\text{For any } x, x', t \text{ with } \overline{l}(t, x) = \overline{l}(t, x'), \text{ it must hold that } \chi(t, x) = \chi(t, x'). \qquad (31)$$

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[36]If $l(t, x)$ is increasing in x, then $L(t, x)$ is convex in x with $\frac{\partial}{\partial x} L(t, x) = l(t, x)$. Convexity of $L(t, x)$ in x implies that $L(t, x) = \overline{L}(t, x)$ for every x, and therefore that $\overline{l}(t, x) = \frac{\partial}{\partial x} \overline{L}(t, x) = \frac{\partial}{\partial x} L(t, x) = l(t, x)$.

Continuing the ironing procedure, the optimizing acceptance rule χ is constructed as follows: accept k applicants so as to maximize the average ironed principal utility for those accepted, $\frac{1}{k}\mathbb{E}[\chi(T, X) \cdot \bar{l}(T, X)]$, for X uniformly drawn on $[0, 1]$. The value of this problem is unaffected by the ironing constraint (31), so that constraint can be costlessly imposed. The constructed acceptance rule will satisfy monotonicity because $\bar{l}(t, x)$ is weakly increasing in x even if $l(t, x)$ is not.

This acceptance rule amounts to first finding the cutoff ironed principal utility level l^c that will lead to accepting k applicants. The acceptance rule then accepts applicants (t, x) with $\bar{l}(t, x) > l^c$ and rejects those with $\bar{l}(t, x) < l^c$. One can choose arbitrary acceptance probabilities in $[0,1]$ when $\bar{l}(t, x) = l^c$ as long as the total share of applicants accepted is k, and as long as we satisfy the ironing constraint at each t.

One way of satisfying this ironing constraint is to choose a single acceptance probability in $[0, 1]$ for all applicants (t, x) with $\bar{l}(t, x) = l^c$, where the probability is set so that a total of k applicants are accepted. This does indeed give an optimal (implementable) acceptance rule. It involves randomization at any test result t for which there is an interval of x over which $\bar{l}(t, x) = l^c$.

Alternatively, we can satisfy the ironing constraint by choosing acceptance probabilities for those applicants with $\bar{l}(t, x) = l^c$ that are constant in x (as above) but may vary in t. For instance, it is always possible to order the possibly multidimensional test results in $\mathcal{T}$ in such a manner that the acceptance probability for applicants with $\bar{l}(t, x) = l^c$ is set at 1 for test results t below a threshold t^*; 0 for test results above t^*; and some intermediate level in $[0,1]$ for the single threshold test result t^*.

In this alternative way of satisfying the ironing constraint, there is at most a single test result for which an interior acceptance rate is ever used. That is to say, it is always possible to find an optimal contract which is either deterministic, or in which there are stochastic acceptances at just a single test result. When test results are continuously distributed, of course, behavior at any single test result can be disregarded. So with continuously distributed test scores, there exists a deterministic optimal contract.

APPENDIX B: FURTHER ANALYSIS OF THE NORMAL SPECIFICATION UNDER COMMON KNOWLEDGE

Consider the normal specification under common knowledge of the agent's type, as in Section 3.3. The optimal contract was summarized in Proposition 3. Comparative statics on the steepness of the acceptance rate function for this contract were given in Proposition 4, with steeper contracts indicating less agent discretion.

B.1. *Additional Comparative Statics*

In this subsection, I explore comparative statics on the steepness of the optimal contract with respect to the three parameters k, σ_T^2, and σ_Q^2 that were omitted from Proposition 4. One new qualification to bear in mind is that, as we increase σ_Q^2 or σ_T^2, the unconditional variance of the test scores, $\sigma_Q^2 + \sigma_T^2$, increases as well. The coefficient that tells us the relative impact of a one standard deviation increase in test scores, rather than the abso-

lute impact of one unit increase, is $\gamma_T^* \sqrt{\sigma_Q^2 + \sigma_T^2}$.[37] Proposition 7 correspondingly includes comparative statics on this renormalized coefficient when relevant (parts 2 and 3).

PROPOSITION 7: *In the contract of Proposition 3 part 1, the contracting parameter γ_T^* given by* (12) *has the following comparative statics and limits*:
1. γ_T^* *is independent of k.*
2. γ_T^* *and* $\gamma_T^* \sqrt{\sigma_Q^2 + \sigma_T^2}$ *decrease in σ_T^2, with* $\lim_{\sigma_T^2 \to 0} \gamma_T^* = \lim_{\sigma_T^2 \to 0} \gamma_T^* \sqrt{\sigma_Q^2 + \sigma_T^2} = \infty$ *and*
   $\lim_{\sigma_T^2 \to \infty} \gamma_T^* = \lim_{\sigma_T^2 \to \infty} \gamma_T^* \cdot \sqrt{\sigma_Q^2 + \sigma_T^2} = 0.$
3. γ_T^* *and* $\gamma_T^* \sqrt{\sigma_Q^2 + \sigma_T^2}$ *may increase or decrease in σ_Q^2, with* $\lim_{\sigma_Q^2 \to 0} \gamma_T^* = \lim_{\sigma_Q^2 \to 0} \gamma_T^* \times$
   $\sqrt{\sigma_Q^2 + \sigma_T^2} = \infty,$ $\lim_{\sigma_Q^2 \to \infty} \gamma_T^* \in (0, \infty),$ *and* $\lim_{\sigma_Q^2 \to \infty} \gamma_T^* \sqrt{\sigma_Q^2 + \sigma_T^2} = \infty.$

Part 1 reiterates that the steepness of the contract does not depend on the number of people to be hired. Hiring fewer or more applicants just translates the acceptance rate function left or right.

Part 2 finds that as the test becomes less informative, the contract gets flatter: it places less weight on the test results, measured in absolute or relative terms. As the test becomes uninformative, the contract sets a constant acceptance rate across all scores. As the test becomes fully informative, we approach the infinitely steep No Discretion contract in which hiring is entirely based on test results.[38]

Part 3, included for completeness, establishes that the steepness of the contract can vary nonmonotonically with the variance of quality in the population.

B.2. *The Full Discretion Contract*

An agent who has full discretion to select k applicants will choose those with U_A above some fixed level—in Figure 1, above a horizontal line. We can solve explicitly for this Full Discretion acceptance rate under the normal specification. Working through the algebra of Section 3.3, but replacing the UBAR acceptance cutoff line $u_A^c(T)$ with a constant in T, under Full Discretion the agent chooses an acceptance rate of $\Phi(\gamma_T^{\text{FD}} T - \gamma_0)$ for

$$\gamma_T^{\text{FD}} = \frac{\sigma_Q^2}{(\sigma_Q^2 + \sigma_T^2)\sqrt{\sigma_B^2 + \eta}}, \tag{32}$$

with η as defined in (11). Putting together (12) and (32), $\gamma_T^* = \gamma_T^{\text{FD}} + \frac{\sigma_Q^2 \sigma_B^2}{\eta(\sigma_Q^2 + \sigma_T^2)\sqrt{\sigma_B^2 + \eta}}$, and hence $0 < \gamma_T^{\text{FD}} < \gamma_T^*$. The agent with Full Discretion accepts a greater share of applicants at higher test scores ($0 < \gamma_T^{\text{FD}}$) because she places some weight on quality. But, as discussed in Section 3.3, the Full Discretion outcome is flatter than the principal's optimal contract under knowledge of the agent's type ($\gamma_T^{\text{FD}} < \gamma_T^*$).

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[37]$\Phi(\gamma_T^* T - \gamma_0)$ can be rewritten as $\Phi(\gamma_T^* \sqrt{\sigma_Q^2 + \sigma_T^2} \cdot \frac{T}{\sqrt{\sigma_Q^2 + \sigma_T^2}} - \gamma_0)$, where $\frac{T}{\sqrt{\sigma_Q^2 + \sigma_T^2}} \sim \mathcal{N}(0, 1)$.

[38]Taken together, Proposition 4 part 2 and Proposition 7 part 2 confirm in this environment the so-called "ally principle" and "uncertainty principle" of delegation, reviewed in Huber and Shipan (2006). A principal should grant more discretion to an agent when the agent's preferences are more aligned with his own, and when he has more uncertainty about what actions to take.

We can replicate the comparative statics of Propositions 4 and 7 for the Full Discretion outcome rather than the optimal contract, where $\gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2}$ is the coefficient on the z-score of the test result.

PROPOSITION 8: *Under the normal specification, the Full Discretion steepness parameter γ_T^{FD} from (32) has the following comparative statics and limits*:
1. γ_T^{FD} *increases in* σ_S^2, *with* $0 < \lim_{\sigma_S^2 \to 0} \gamma_T^{\mathrm{FD}} < \lim_{\sigma_S^2 \to 0} \gamma_T^* $ *and* $\lim_{\sigma_S^2 \to \infty} \gamma_T^{\mathrm{FD}} \in (0, \infty)$.
2. γ_T^{FD} *decreases in* σ_B^2, *with* $\lim_{\sigma_B^2 \to 0} \gamma_T^{\mathrm{FD}} = \lim_{\sigma_B^2 \to 0} \gamma_T^*$ *and* $\lim_{\sigma_B^2 \to \infty} \gamma_T^{\mathrm{FD}} = 0$.
3. γ_T^{FD} *is independent of* k.
4. γ_T^{FD} *and* $\gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2}$ *decrease in* σ_T^2, *with* $\lim_{\sigma_T^2 \to 0} \gamma_T^{\mathrm{FD}} \in (0, \infty)$, $\lim_{\sigma_T^2 \to 0} \gamma_T^{\mathrm{FD}} \times \sqrt{\sigma_Q^2 + \sigma_T^2} \in (0, \infty)$, *and* $\lim_{\sigma_T^2 \to \infty} \gamma_T^{\mathrm{FD}} = \lim_{\sigma_T^2 \to \infty} \gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2} = 0$.
5. γ_T^{FD} *and* $\gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2}$ *increase in* σ_Q^2, *with* $\lim_{\sigma_Q^2 \to 0} \gamma_T^{\mathrm{FD}} = \lim_{\sigma_Q^2 \to 0} \gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2} = 0$, $0 < \lim_{\sigma_Q^2 \to \infty} \gamma_T^{\mathrm{FD}} < \lim_{\sigma_Q^2 \to \infty} \gamma_T^*$, *and* $\lim_{\sigma_Q^2 \to \infty} \gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2} = \infty$.

There are a few main comparisons to make with the comparative statics on γ_T^* in Propositions 4 and 7. The first (parts 1 and 2) is that the sign of the comparative static on γ_T^{FD} is the same as that on γ_T^* with respect to the agent's information σ_S^2, but the signs are reversed for the comparative static on bias σ_B^2. As discussed in the main text, the principal and agent agree that a more informed agent should have a flatter acceptance rate function. But when the agent is more biased, she prefers flatter acceptance rates, while the principal prefers steeper ones.

Part 2 also confirms that as the agent's bias disappears, the agent's preferred outcome goes to that of the principal's optimal contract: $\gamma_T^{\mathrm{FD}} \to \gamma_T^*$. Without bias, the incentives of the two parties are perfectly aligned.

Finally, part 5 now finds a clean comparative static on σ_Q^2, whereas its sign under the optimal contract (Proposition 7 part 3) was ambiguous. The Full Discretion acceptance rate gets steeper with respect to test scores (in both absolute and relative terms) when the variance of population quality increases. When this variance goes to zero, the agent's preferences are entirely driven by bias, and so the Full Discretion outcome becomes flat even as the principal-optimal contract becomes infinitely steep.

APPENDIX C: APPROXIMATED MECHANISMS IN A FINITE ECONOMY

In this section, I explore how one might implement finite approximations of the optimal contract when the agent's type is commonly known. The body of the paper develops two characterizations of the continuum optimal contract, through the acceptance rate function of Proposition 1 and the minimum average score of Proposition 2, which are summarized for the normal specification in Proposition 3 parts 1 and 2. Here, I separately explore finite approximations of these two contract forms. This exercise is intended to illustrate how one might put these contract forms into practice, while also giving insight into how the two contract forms compare.

C.1. *A Finite Example*

Example Primitives. In the continuum model of the paper, there is a mass 1 of applicants, of which k will be accepted. The aggregate distributions of applicant characteristics

Q, T, S, and B are given by F_Q, $F_{T|Q}$, $F_{S|T,Q}$, and $F_{B|T,S}$. For this section, I instead suppose that there is a finite number N of applicants with characteristics drawn i.i.d. from these distributions, from which kN will be accepted.

Specifically, let the distributions follow the normal specification, with parameters set to $\sigma_Q^2 = 1$, $\sigma_T^2 = 4$, $\sigma_S^2 = 1$, and $\sigma_B^2 = 1$. The agent will accept a share $k = 1/3$ of the applicants.[39] I will consider finite economies with $N = 12$, 24, 48, or 96 applicants, out of which 4, 8, 16, or 32 will be accepted.

Overview of the Mechanisms. In the context of this example, I will go through what I view as the most natural finite approximations of the two contract forms of Proposition 3, with technical details in Appendix C.2. Of course, the exact implementations I consider are certainly not the only possible ways of approximating the continuum contracts for a finite economy. It is doubtless the case that some further tuning could improve payoffs.

To approximate the acceptance rate function of Proposition 3 part 1, I use a *binned acceptance rate* implementation: applicants are divided into bins based on their test scores, and then the agent chooses a specified number of applicants from each bin. I consider bins that put together a uniform number of applicants M for every possible bin size M that is a factor of N. For instance, at $N = 24$, I consider $M = 1, 2, 3, 4, 8, 12, 24$. (The numbers of applicants N have been chosen as multiples of 12 to allow for many possible bin sizes.) The top M scores are binned together, then the next M, and so forth. The manager then selects some predetermined number of applicants from each bin. At the extremes, $M = 1$ corresponds to No Discretion, in which applicants are selected based only on their test scores; and $M = N$ corresponds to Full Discretion, in which the manager can select any kN applicants. After calculating the principal's expected payoffs for every possible M at some fixed N, we can say that the value M yielding the highest payoff is the preferred bin size.

To approximate the average score contract of Proposition 3 part 2, I use a *minimum average score* implementation: the manager can select any kN applicants whose average test score is sufficiently high. I consider two possible floors for the average score. The first is the "naive floor" that is set in advance at the level that is optimal for the continuum contract. Among other concerns with this naive floor, it may be the case that realized test scores were lower than expected and no set of kN applicants have average scores at or above this level. I correct the naive implementation in an ad hoc manner by supposing that if the floor is not achievable, then the manager must select the applicants with the top kN test scores. The second floor I consider is a "responsive floor" that adjusts the floor up or down when the applicant pool has high or low realized test scores; see Appendix C.2.5 below for the adjustment formula. For instance, under the responsive floor, if all test scores go up by some increment, then the floor itself shifts up by the same amount. Moreover, the applicants with the top kN test scores are guaranteed to have an average score above the floor. I consider the responsive floor to be the preferred floor in every case, but I include analysis of the naive floor for comparison.

Numerical Results and Takeaways. I numerically simulated the principal's expected payoffs for each of the contract implementations and values of N discussed above. Table 1 summarizes the results. There are three main takeaways that I draw from this table.

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[39]The numbers were chosen in part to guarantee that in the continuum economy, the optimal contract does considerably better than contracts which give the agent either No Discretion or Full Discretion. See numerical details in Appendix C.2.1.

TABLE 1

PRINCIPAL PAYOFFS IN FINITE ECONOMIES FOR DIFFERENT CONTRACT IMPLEMENTATIONS[a]

	$N = 12$ Accept 4	$N = 24$ Accept 8	$N = 48$ Accept 16	$N = 96$ Accept 32	Continuum Accept 1/3
Benchmarks					
(*Principal Payoff*)					
No Discretion	0.4556	0.4712	0.4794	0.4836	0.4878
Full Discretion	0.4537	0.4693	0.4775	0.4817	0.4859
UBAR upper bound	0.5516	0.5706	0.5805	0.5856	0.5907
Binned Acc Rate					
(*No Disc to UBAR %*)					
Bin size $M = 1$	0%	0%	0%	0%	
2	40	36	34	33	
3	42	48	52	51	
4	**59**	56	62	60	
6	53	68	72	73	
8		**71**	76	77	
12	−2	57	**80**	83	
16			77	84	
24		−2	63	**85**	
32				81	
48			−2	68	
96				−2	
Min Avg Score					
(*No Disc to UBAR %*)					
Naive floor	32	43	62	77	
Responsive floor	**72**	**82**	**93**	**96**	

[a]Payoffs for the binned acceptance rate and minimum average score contracts are reported as a percentage of the way from the No Discretion to the UBAR payoff for the corresponding value of N, rounded to the nearest percent. For each N, I have bolded the best payoff within each contract type. Binned acceptance rate payoffs for bins of size M with $1 < M < N$ and all minimum average score payoffs are calculated from simulations with 100,000 draws each; the standard error of each such payoff is between 0.5 and 1.6 percentage points.

First, we see that for both the binned acceptance rate and the minimum average score contracts, payoffs of the preferred implementation improve as we increase the number of applicants N. Moreover, as we would expect, with larger N the payoffs seem to be approaching those from the "large numbers" continuum model.

Putting numbers to those points, for each of these finite economies I first derive the principal payoffs from the No Discretion and Full Discretion contracts, which can both easily be implemented (by mechanically selecting the applicants with the top test scores, or by letting the agent choose her favorite applicants). It turns out that under the given parameters, the No Discretion contract does slightly better. Next, I derive the payoffs from the outcomes of the upper bound acceptance rule, which is a theoretical upper bound on the performance of an optimal contract that is exactly achieved in the continuum limit (see Appendix C.2.2). Finally, I simulate the performance of the various binned acceptance rate and minimum average score implementations. These simulations find that when accepting 4 out of 12 applicants, a binned acceptance rate contract already achieves 59% of the benefit of moving from No Discretion to the Upper Bound; a minimum average score contract does even better, achieving 72% of the benefit. Accepting 32 out of 96 applicants, a binned acceptance rate contract achieves 85% while a minimum average score contract

achieves 96%. It is not shown in the table, but a minimum average score contract achieves about 99% of this theoretical upper bound when accepting 100 out of 300 applicants.[40]

In other words, the analysis from the continuum model translates well to a finite model of reasonable size. Without solving for exactly optimal contracts in the finite model, I can confirm that these straightforward translations of continuum contracts into finite ones deliver a high share of any possible payoff gains from optimal discretion.

Second, we see that the responsive average score implementation gives higher payoffs than any of the binned acceptance rate implementations. This numerical result does not prove that there would not have been a different finite approximation of "binned acceptance rates" that did even better, of course. But the observation is consistent with the informal argument of Section 5.1 that, in a finite economy, we might expect to prefer some version of a minimum average score contract. Binned acceptance rates impose a number of constraints on the agent while a minimum average score contract links all of the constraints into a single inequality, which can mitigate sampling variation.

Finally, fixing the number of applicants N, we see evidence of the tradeoff over bin size versus number of bins in the binned acceptance rate implementation. As we move from many small bins to fewer large bins by increasing M, the principal payoff tends to increase and then decrease.

C.2. Implementation Details

C.2.1. Continuum Benchmark

For this running example, the (continuum) optimal contract can be expressed in two equivalent ways. First, as an acceptance rate function, the manager accepts a share $\Phi(\gamma_T^* t - \gamma_0^*)$ of applicants with test score t, where from (12) we have $\gamma_T^* = \sqrt{\frac{549}{1280}} \simeq 0.6549$ and we can numerically calculate that $\gamma_0^* \simeq 0.7638$ given $k = 1/3$. Second, as an average test score restriction, the manager accepts her favorite k applicants subject to their average test score being at or above $\kappa^* \simeq 2.0143$, which is the average test score of accepted applicants when using the above acceptance rate function. (The floor will be binding.)

Applicants have a mean quality of 0 with a standard deviation of 1. Hence, if applicants were accepted completely randomly, the principal payoff (the expected quality of hired applicants) would be 0. On the other hand, if quality were perfectly observable and the firm could accept the 1/3 of applicants with the highest quality, the principal payoff would work out to 1.0908.

The optimal contract, which implements the upper bound acceptance rule, leads to a principal payoff of 0.5907. By comparison, under the No Discretion contract in which the applicants with the highest 1/3 of test scores were hired automatically, the principal's payoff would be 0.4878. Under the Full Discretion contract in which the manager selected her favorite 1/3 of applicants, the principal's payoff would be 0.4859.

C.2.2. Finite Economy Benchmarks

Under the No Discretion outcome, the applicants with the top kN out of N test scores are accepted. Under the Full Discretion outcome, the applicants with the top kN out

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[40]Simulating 500,000 draws of $N = 300$ applicants, I found that the principal's payoff from the responsive minimum average score was 98.9% of the way from the No Discretion benchmark to the UBAR upper bound, estimated with a standard error of 0.1%.

of N realizations of $U_A = \mathbb{E}[Q|T, S] + B$ are accepted. Under the Upper Bound Acceptance Rule (UBAR) outcome, the applicants with the top kN out of N realizations of $U_P(T, U_A) = \mathbb{E}[Q|T, U_A]$ are accepted. Note that UBAR is not necessarily implementable by any incentive-compatible contract in the finite economy, but it provides an upper bound for what any incentive-compatible contract can achieve.

As we vary N, the No Discretion, Full Discretion, and UBAR benchmark payoffs all scale linearly with each other as some constants times the expectation of the top kN draws out of N from a standard normal distribution. This expectation term increases in N, asymptoting to the expectation of an appropriately truncated normal.[41]

C.2.3. *Further Notation for the Finite Economy*

Denote the realized test score of applicant $i \in \{1, \ldots, N\}$ by t^i, where without loss of generality we label scores so that $t^1 \leq t^2 \leq \cdots \leq t^N$.

I now define a term $\alpha^{i,N}$, which is a finite approximation of the continuum contract's acceptance rate of the applicant with the ith lowest test score out of N. Recalling that the distribution of test scores in the population is $\mathcal{N}(0, \sigma_Q^2 + \sigma_T^2)$, the "$i$th lowest test score out of N" essentially corresponds to test scores ranging from the $\frac{i-1}{N}$ to $\frac{i}{N}$ quantiles of this distribution. This range of quantiles corresponds to test scores in the interval $[x^{i-1,N}, x^{i,N}]$ for

$$x^{i,N} \equiv \sqrt{(\sigma_Q^2 + \sigma_T^2)} \Phi^{-1}\left(\frac{i}{N}\right).$$

In the continuum economy, the optimal acceptance share at test score t is $\Phi(\gamma_T^* t - \gamma_0^*)$. Integrating the acceptance share over this range of quantiles, the finite approximation of the acceptance rate of the ith lowest scoring applicant out of N, $\alpha^{i,N}$, is given by

$$\alpha^{i,N} \equiv \frac{N}{\sqrt{(\sigma_Q^2 + \sigma_T^2)}} \int_{x^{i-1,N}}^{x^{i,N}} \Phi(\gamma_T^* t - \gamma_0^*) \phi\left(\frac{t}{\sqrt{(\sigma_Q^2 + \sigma_T^2)}}\right) dt.$$

C.2.4. *Approximate Implementation #1: Binned Acceptance Rates*

Here we approximate the continuum contract through one that imposes a specified acceptance rate on some binned sets of applicants. Fixing the number of applicants N, we will have one parameter to optimize over, the bin size M.

For any bin size M that is a factor of the number of applicants N, now create N/M bins of size M each. Given the notation that applicants are labeled in order of lowest to highest test scores, the first bin $\beta_1^{M,N} = \{1, \ldots, M\}$ consists of the M lowest-scoring applicants, the jth bin are applicants $\beta_j^{M,N} = \{(j-1)M + 1, \ldots, jM\}$, and so on up to the highest-scoring applicant bin $\beta_{N/M}^{M,N} = \{N - M + 1, \ldots, N\}$.

Fixing M and N, we will determine the number of applicants to accept in bins $\beta_1^{M,N}, \ldots, \beta_{N/M}^{M,N}$ as follows. At the jth such bin $\beta_j^{M,N}$, recall that the finite approximation of the acceptance rate of applicants in this bin is given by the real number $\sum_{i \in \beta_j^{M,N}} \alpha^{i,N}$. We will simply round these values to integers to find the number of applicants to accept

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[41]It holds for any distribution that the expectation of the top kN draws out of N is increasing in N. For instance, with $k = 1/2$, the top 1 draw out of 2 from a $U[0, 1]$ distribution has an expectation of $2/3$, while for large N, the expectation of the average of the top $N/2$ draws out of N approaches $3/4$.

at bin $\beta_j^{M,N}$. We do the least rounding possible consistent with making sure that when we add up across all bins j, we accept a total of kN applicants.

For instance, suppose we will accept 4 out of $N = 12$ applicants using six bins that are each of size $M = 2$. Adding up the $\alpha^{i,N}$ values, we get—prior to rounding—that we should accept 0.0084 from the first bin, 0.0853 from the second bin, 0.2905 from the third bin, 0.6556 from the fourth bin, 1.1816 from the fifth bin, and 1.7785 from the top bin. Rounding these values, we accept 0 of the applicants from the bottom three bins (the bottom six test scores); 1 of the applicants with the next two higher test scores; 1 of the applicants with the next two higher test scores; and, finally, both of the applicants with the top two test scores. Similarly, when $N = 12$ and $M = 3$, adding up the $\alpha^{i,N}$ values implies that we should accept 0.0344 from the bottom bin, then 0.3498 from the next bin, then 1.1721 from the next one, and finally 2.4437 of the top bin. Rounding as little as possible to get to a total of four accepted applicants, we accept 0 of the applicants with the bottom three test scores, 0 of the next three, 1 of the next three, and finally all 3 of the top three scoring applicants.

From Table 1, the best bin size at $N = 12$ is $M = 4$. For $N = 12$ and $M = 4$, the number of applicants accepted from each of the three bins—from lowest scoring to highest—is 0, 1, 3. The best bin size at $N = 24$ is $M = 8$, in which case the number of applicants accepted from each of the three bins is 0, 2, 6. The best bin size at $N = 48$ is $M = 12$, in which case the number of applicants accepted from each of the four bins is 0, 1, 5, 10. Finally, the best bin size at $N = 96$ is $M = 24$, in which case the number of applicants accepted from the four bins is 0, 3, 9, and 20.

C.2.5. *Approximate Implementation #2: Minimum Average Scores*

Here we approximate the continuum contract through one that lets the agent accept any applicants she wants subject to a constraint on the minimum average test score of those that she hires. I will introduce two possible test score floors. The naive floor is set in advance, while the responsive floor depends on the realized distribution of test scores.

Let the *naive floor* be equal to the predetermined value of $\kappa^* = 2.0143$. Unfortunately, sometimes the top kN applicants have test scores that actually average less than κ^*; in simulations, this happens about 38% of the time with $N = 12$ and 9% of the time with $N = 96$. When this is the case, we simply use the ad hoc correction that the agent must default to the No Discretion rule of accepting the applicants with the top kN test scores.

To motivate the construction of a responsive floor, recall that the naive floor is determined by solving for the average test score of accepted applicants in the continuum limit. This can be thought of as taking a weighted mean over the theoretical distribution of population test scores, with weights given by the acceptance shares. We will create a new responsive floor that is based on a similar weighted average of the realized rather than theoretical distribution of test scores. We will use the weights that we have already solved for above, the $\alpha^{i,N}$ values that give us the finite approximation of the theoretical acceptance rate for the applicant with the ith lowest test score out of N.

In particular, let the *responsive floor* given N applicants with ordered realized test scores $\{t^1, \ldots, t^N\}$ be equal to $\sum_i \frac{\alpha^{i,N}}{kN} t^i$. Since the weights $\frac{\alpha^{i,N}}{kN}$ are all in $[0, 1]$ and add up to 1, we know that the responsive floor is always less than the average of the top kN test scores. Hence, it is always possible to find at least one combination of kN applicants whose average test scores are above the responsive floor.

APPENDIX D: ADDITIONAL ANALYSIS OF SYSTEMATIC BIASES

D.1. *Two-Factor Model Under Common Knowledge of Agent's Type*

Consider the two-factor model of Section 2.4.2 under common knowledge of the agent's type (consisting of F_S and λ). Recalling that $\mathbb{E}[Q_1|T, S] = \mathbb{E}[Q_1|T]$, it holds that $U_A = \mathbb{E}[Q_1|T] + \lambda \mathbb{E}[Q_2|T, S]$. Rearranging,

$$\mathbb{E}[Q_2|T, S] = \frac{U_A - \mathbb{E}[Q_1|T]}{\lambda}.$$

The principal's expected utility $U_P(T, U_A)$ is therefore

$$U_P(T, U_A) = \mathbb{E}[Q_1 + Q_2|T, U_A] = \mathbb{E}[Q_1|T] + \frac{U_A - \mathbb{E}[Q_1|T]}{\lambda} = \frac{(\lambda - 1)}{\lambda}\mathbb{E}[Q_1|T] + \frac{U_A}{\lambda}.$$

Given the assumption that $\lambda > 0$, the coefficient $\frac{1}{\lambda}$ on U_A is positive. Hence, utilities are aligned up to distinguishability.

The sign of the coefficient $\frac{\lambda-1}{\lambda}$ on $\mathbb{E}[Q_1|T]$ in the expression for U_P depends on whether the agent's bias term λ is above or below 1.[42] For the advocate with $\lambda > 1$, there is a positive coefficient on $\mathbb{E}[Q_1|T]$. That means that if the agent is indifferent between an applicant with a low test result and an applicant with a high test result (i.e., indicating high $\mathbb{E}[Q_1|T]$), the principal prefers the one with the high test result. For the cynic with $\lambda < 1$, however, the reverse holds.

One implementation of the upper bound acceptance rule is to specify the acceptance rate $\alpha(T)$ at $\alpha^{\text{UBAR}}(T)$. This acceptance rate is determined only by the joint distribution of $\mathbb{E}[Q_1|T]$ and $\mathbb{E}[Q_2|T, S]$; the principal selects the k applicants with the highest $\mathbb{E}[Q_1|T] + \mathbb{E}[Q_2|T, S]$ as if T and S were both observable. This contract implements the principal's first-best outcomes, thanks to the assumed absence of idiosyncratic biases. Moreover, while the acceptance rate function α^{UBAR} depends on the agent's information structure, it does not depend on the bias term λ. Restating this point, the same acceptance rate function would be optimal for a principal with *any* beliefs on λ, even if λ is not commonly known.

We can also implement UBAR in the alternative manner in which we fix a minimum average score. For the advocate with $\lambda > 1$, we can set the score function as $C(T) = \mathbb{E}[Q_1|T]$. For the cynic with $\lambda < 1$, set the score function as $C(T) = -\mathbb{E}[Q_1|T]$. The contract then specifies that $\mathbb{E}[C(T)|\text{Hired} = 1] \geq \kappa$, for some κ. The different signs of $C(T)$ based on the sign of $(\lambda - 1)$ indicate that advocates prefer to push $\mathbb{E}[Q_1|T]$ to be lower than what the principal wants, whereas cynics prefer $\mathbb{E}[Q_1|T]$ to be higher.

D.2. *Utility Weight on a Public Signal*

One source of systematic bias which is not captured by the two-factor model is that the principal or agent may care directly about the realization of an applicant's hard information. Think about two specific applications.

First, there may be a third party organization (e.g., US News) that rates colleges based on the public hard information of the applicants who matriculate. The college cares about

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[42]Supposing that test results are real numbers normalized so that higher t yields higher $\mathbb{E}[Q_1|T = t]$, a positive coefficient on $\mathbb{E}[Q_1|T]$ corresponds to downward-sloping principal indifference curves in (T, U_A)-space, just as in the normal specification. A negative coefficient yields upward-sloping indifference curves.

its ratings in addition to the "true quality" of its students. So the school is willing to admit a slightly worse applicant who looks better on paper—a worse essay paired with a better SAT score. The admissions officer does not care about ratings, though, and just wants to maximize true student quality.

Second, one or both of the principal and agent may be "prejudiced" or may support "affirmative action" based on an observable characteristic such as race, included as one component of the vector T. This induces a bias—misaligned objectives—if the preferences over the observable characteristic are not perfectly shared by both parties.

To model this, let an applicant's "true quality" be denoted by Q_1. We have distribution $Q_1 \sim F_{Q_1}$ of true quality, with corresponding signal distributions $T \sim F_{T|Q_1}$ and $S \sim F_{S|Q_1,T}$. The expected value of true quality given all information is $\mathbb{E}[Q_1|S, T]$.

Then there is an addition to the principal utility, $Q_{2P}(T)$, and an addition to agent utility, $Q_{2A}(T)$, where $Q_{2P}(\cdot)$ and $Q_{2A}(\cdot)$ are arbitrary functions of the realization of hard information. Utilities for the two players are as follows:

$$P: \quad Q_1 + Q_{2P}(T) = Q,$$
$$A: \quad Q_1 + Q_{2A}(T) = Q + B, \quad \text{for } B = Q_{2A}(T) - Q_{2P}(T).$$

We see that this form of systematic bias shows up as a relationship between the bias realization B and the hard information.

Utilities are aligned up to distinguishability: at any test result, applicants more preferred by the agent are more preferred by the principal. Formally, given T and $U_A = \mathbb{E}[Q_1|S, T] + Q_{2A}(T)$, we can rearrange to get $\mathbb{E}[Q_1|S, T] = U_A - Q_{2A}(T)$. The induced principal utility $U_P(T, U_A)$ is

$$U_P(T, U_A) = \mathbb{E}[Q_1|S, T] + Q_{2P}(T) = U_A - Q_{2A}(T) + Q_{2P}(T),$$

which is increasing in U_A. Hence, we can apply the results of Section 3 to solve for the optimal acceptance rate function. Just as with the two-factor model analyzed in Section D.1, this acceptance rate does not depend on the agent's bias function $Q_{2A}(\cdot)$. Likewise, due to the assumed lack of idiosyncratic bias, the optimal contract implements the first-best payoff for the principal.

APPENDIX E: COMBINING IDIOSYNCRATIC AND SYSTEMATIC BIASES

This section puts together the idiosyncratic biases of the normal specification with the systematic biases of the two-factor model into a *combined model*. I show that the qualitative results for the normal specification in Sections 3.3 and 4 extend to the combined model.

E.1. *Setup of the Combined Model*

As in the two-factor model, quality Q in the combined model can be decomposed into two quality factors Q_1 and Q_2, for which $\mathbb{E}[Q_1|T, S] = \mathbb{E}[Q_1|T]$: the test result reveals everything relevant that can be inferred about Q_1. The private signal S then gives additional information about quality factor Q_2. Adding normally distributed idiosyncratic biases, assume that the principal's utility for hiring an applicant is $Q = Q_1 + Q_2$ and the agent's utility is $Q_1 + \lambda Q_2 + \epsilon_B$, for $\epsilon_B \sim \mathcal{N}(0, \sigma_B^2)$ independent of T, S, and with λ and σ_B^2 in

$\mathbb{R}_{++}$. All together, then, conditional on signals T and S, the principal and agent utilities of hiring an applicant are given by

$$\text{Principal:} \quad \mathbb{E}[Q|T, S] = \mathbb{E}[Q_1 + Q_2|T, S] = \mathbb{E}[Q_1|T] + \mathbb{E}[Q_2|T, S],$$

$$\text{Agent:} \quad U_A \equiv \mathbb{E}[Q_1 + \lambda Q_2|T, S] + \epsilon_B = \mathbb{E}[Q_1|T] + \lambda \mathbb{E}[Q_2|T, S] + \epsilon_B.$$

Let us now add additional distributional assumptions on the two signals. In particular, rather than specifying the conditional distributions $F_{T|Q}$ and $F_{S|Q,T}$, I will write out the expectations of Q_1 and Q_2 given the signals in linear reduced forms. Let the signal realization spaces $\mathcal{T}$ and $\mathcal{S}$ both be equal to $\mathbb{R}$ and assume that

$$\mathbb{E}[Q_1|T] = T,$$

$$\mathbb{E}[Q_2|T, S] = rT + S,$$

for some $r \in \mathbb{R}$. Finally, assume that the distribution of S conditional on T (but unconditional on the quality factors) is given by $S|T \sim \mathcal{N}(0, l \cdot \sigma_2^2)$ for $l \in (0, 1)$ and $\sigma_2 \in \mathbb{R}_{++}$. Note that, while I do not commit to the details of the updating model that would get us these posteriors, it would be straightforward to "microfound" these reduced form assumptions through appropriate joint-normal priors on the two quality factors and normally distributed signals.[43]

We have introduced five relevant parameters: r, σ_2^2, l, λ, and σ_B^2. Two of these, λ and σ_B^2, are familiar as the systematic bias term of the two-factor model and the idiosyncratic bias term of the normal specification. The interpretation of the other three parameters is as follows. First, conditional on the observation of $T = t$, the distribution of Q_2 has mean rt and variance of σ_2^2. A value $r > 0$ indicates a positive correlation of the two quality factors, and $r < 0$ a negative correlation. The parameter l corresponds to the level of the agent's information on Q_2: a more informative private signal means higher l. An agent who perfectly observed the realization of Q_2 would have $l \to 1$, and one who received no private information would have $l \to 0$.

Putting these assumptions together, we can rewrite the utilities:

$$\text{Principal:} \quad \mathbb{E}[Q_1|T] + \mathbb{E}[Q_2|T, S] = (1 + r)T + S,$$

$$\text{Agent:} \quad U_A = \mathbb{E}[Q_1|T] + \lambda \mathbb{E}[Q_2|T, S] + \epsilon_B = (1 + \lambda r)T + \lambda S + \epsilon_B.$$

In the notation $Q = Q_1 + Q_2$, we have that the agent maximizes the expectation of $Q + B$ for $B = (\lambda - 1)(rT + S) + \epsilon_B$.

The agent's type θ consists of three parameters: $l \in (0, 1)$ for information (replacing, but analogous to, σ_S^2 in the normal specification), $\sigma_B^2 \in (0, \infty)$ for idiosyncratic bias, and

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[43]Here is one collection of primitives that would give rise to these reduced form distributional assumptions. Take Q_1 and Q_2 to be joint normally distributed, and have T perfectly reveal Q_1—it has a degenerate distribution at $T = Q_1$. (I have not specified the distribution of T outside of this footnote, but under this assumption the empirical distribution would be normal.) The mean of Q_2, unconditional on other signals, will be linear in T with slope depending on the variances and covariance of Q_1 and Q_2. The agent then receives a private signal equal to Q_2 plus some normally distributed noise (where a higher variance of noise corresponds to less information, and so lower l); normalize the signal S to be the resulting deviation of the posterior belief from the mean. The agent's posterior expectation on Q_2 is normally distributed about the mean with a variance somewhere between 0 (no information) and σ_2^2 (full information).

$\lambda \in (0, \infty)$ for systematic bias.[44] In line with Assumption 1, I take all other parameters to be commonly known.

It holds that $\mathbb{E}[Q|T = t, S]$ and U_A are jointly normally distributed. The marginal distribution of $U_A|T$ is given by

$$U_A|T = t \sim \mathcal{N}\big(\mu_{U_A}(t), \sigma_{U_A}^2\big), \quad \text{for} \tag{33}$$

$$\mu_{U_A}(t) = t(1 + \lambda r), \tag{34}$$

$$\sigma_{U_A}^2 = \lambda^2 l \sigma_2^2 + \sigma_B^2. \tag{35}$$

The marginal distribution of $\mathbb{E}[Q|T = t, S]$ is normal with mean $t(1 + r)$ and variance $l\sigma_2^2$. The covariance of $\mathbb{E}[Q|T = t, S]$ with $U_A|T = t$ is $\lambda l \sigma_2^2$. Hence, we can calculate $U_P(t, u_A) = \mathbb{E}[Q|T = t, U_A = u_A] = \mathbb{E}[\mathbb{E}[Q|T = t, S]|T = t, U_A = u_A]$ as

$$U_P(t, u_A) = \beta_T t + \beta_{U_A} u_A, \quad \text{for} \tag{36}$$

$$\beta_T = 1 - \frac{\lambda l \sigma_2^2 - r \sigma_B^2}{\lambda^2 l \sigma_2^2 + \sigma_B^2}, \tag{37}$$

$$\beta_{U_A} = \frac{\lambda l \sigma_2^2}{\lambda^2 l \sigma_2^2 + \sigma_B^2}. \tag{38}$$

The coefficient β_{U_A} on agent utility is positive, implying that utilities are aligned up to distinguishability. Larger idiosyncratic biases σ_B^2 reduce this coefficient, making beliefs on quality less responsive to agent utilities, but do not affect the sign. Let us look next at the coefficient β_T on test scores. Without idiosyncratic shocks—that is, plugging in $\sigma_B^2 = 0$—β_T reduces to $\frac{\lambda - 1}{\lambda}$ as in the two-factor model (Appendix D.1). Adding idiosyncratic shocks through σ_B^2 pulls the coefficient β_T towards $1 + r$. Putting the effects on β_{U_A} and β_T together, we see that increasing the idiosyncratic preference shocks (larger σ_B^2) takes the principal's belief, at any given test score $T = t$ and utility realization $U_A = u_A$, in the direction of $(1 + r)t$—the estimate of quality given $T = t$, and unconditional on U_A. In the case where there is weakly positive correlation of the two factors ($r \geq 0$), larger idiosyncratic shocks monotonically increase the coefficient β_T. In particular, with $r \geq 0$, the sign of β_T is always positive for advocates ($\lambda > 1$). The sign of β_T can be negative for cynical agents ($\lambda < 1$) but it switches to positive for sufficiently large idiosyncratic biases σ_B^2.

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[44]The combined model here embeds the normal specification after a notational adjustment. Call S and T the signals in the normal specification, and S' and T' the signals in the combined model. The normal specification maps into the combined model if we take $\lambda = 1$, along with

$$S' = \left(S - T \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}\right) \cdot \frac{\sigma_T^2 \sigma_Q^2}{\sigma_S^2 \sigma_T^2 + \sigma_Q^2 \sigma_S^2 + \sigma_Q^2 \sigma_T^2}, \qquad T' = T \cdot \frac{\sigma_S^2 \sigma_Q^2}{\sigma_S^2 \sigma_T^2 + \sigma_Q^2 \sigma_S^2 + \sigma_Q^2 \sigma_T^2},$$

$$\sigma_2^2 = \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2}, \qquad l = \frac{\sigma_Q^2 \sigma_T^2}{\sigma_S^2 \sigma_T^2 + \sigma_Q^2 \sigma_S^2 + \sigma_Q^2 \sigma_T^2}, \qquad r = \frac{\sigma_Q^2 \sigma_T^2}{\sigma_S^2 (\sigma_Q^2 + \sigma_T^2)}.$$

From (36), we see that the distribution of $U_P(t, U_A)$—that is, expected quality conditional on $T = t$ across realizations of U_A—has mean and variance of

$$\mu_{U_P}(t) = t(1+r),$$

$$\sigma^2_{U_P} = \beta^2_{U_A}\sigma^2_{U_A} = \frac{\left(\lambda l\sigma^2_2\right)^2}{\lambda^2 l\sigma^2_2 + \sigma^2_B}.$$

In the normal specification, the principal and agent had equal mean utilities conditional on test score T. But now the coefficients on T differ if the quality factors are correlated ($r \neq 0$) and the agent has a systematic bias ($\lambda \neq 1$). The coefficient on T for the principal is $1 + r$, compared to the agent's $1 + \lambda r$. When there is positive correlation between the two quality factors ($r > 0$), the mean as a function of test score will have a steeper slope for an advocate agent than for the principal, and a flatter slope for a cynic.

E.2. *Contracting With Known Agent Type*

Under common knowledge of agent type, we can solve for the optimal policy exactly as in the normal specification and two-factor model. The upper bound acceptance rule sets a cutoff utility u^c_P and accepts all applicants with $U_P \geq u^c_P$. This is implemented by a normal CDF acceptance rate, $\alpha(T) = \Phi(\gamma_T T - \gamma_0)$, at some appropriate steepness $\gamma_T = \gamma_T^{\text{comb}}$. We can solve for this optimal coefficient from the equations for the distribution of $U_P(t, U_A)$ as

$$\gamma_T^{\text{comb}} = \frac{1+r}{\sigma_{U_P}} = \frac{(1+r)\sqrt{\lambda^2 l\sigma^2_2 + \sigma^2_B}}{\lambda l\sigma^2_2}. \tag{39}$$

The acceptance rate is increasing in the test score (positive γ_T^{comb}) even if β_T is negative, as long as $r \geq -1$. This holds because higher quality applicants tend to have higher test scores, so the principal wants to accept more of them. If there is a sufficiently strong negative correlation between quality factors that $r < -1$, then higher quality applicants tend to have lower test scores (they are lower on the first quality factor), and γ_T^{comb} is negative.[45]

As with the normal specification analyzed in Section 3.3, the linearity of $U_P(T, U_A)$ in both T and U_A, the policy could also be implemented by giving a (binding) floor the average test score of hired workers at some level κ^{comb}, for which I do not provide a formula. Gathering together these observations, we have the following:

PROPOSITION 9: *Under the combined model with common knowledge of the agent's type, the optimal contract can be implemented in either of the following ways. The agent is allowed to hire any set of k applicants, subject to*:
1. *an acceptance rate function of $\alpha(t) = \Phi(\gamma_T^{\text{comb}} T - \gamma_0)$; or,*
2. *an average test score of accepted applicants, $\mathbb{E}[T|\text{Hired} = 1]$, at or above some value κ^{comb}. In this case, the agent will choose applicants so that $\mathbb{E}[T|\text{Hired} = 1] = \kappa^{\text{comb}}$.*

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[45]While we might expect T to be normally distributed, as it would be under the normal prior/normal signal microfoundation of footnote 43, this result did not impose any assumptions on the distribution of T. The distribution of T affects which γ_0 will set the aggregate share of acceptances to k, but does not affect the coefficient γ_T on test scores in the acceptance rate function.

E.3. *Contracting With Unknown Agent Type*

With uncertainty over the agent's type $\theta = (l, \lambda, \sigma_B^2)$, we can replicate much of the analysis of Section 4 in solving for the optimal policy. Say that θ follows distribution function G. Going forward, I write μ_{U_A}, σ_{U_A}, and σ_{U_P} as functions of θ. For this analysis, assume that the unconditional distribution of test scores T is normally distributed, with mean normalized to 0 and variance of Var_T, as motivated in footnote 43.

As in the main text, define Z as an agent utility z-score for a given applicant, and τ and ζ as the average test score and average z-score for a pool of accepted applicants:

$$Z \equiv \frac{U_A - \mu_{U_A}(T; \theta)}{\sigma_{U_A}(\theta)},$$

$$\tau \equiv \mathbb{E}[T|\text{Hired} = 1],$$

$$\zeta \equiv \mathbb{E}[Z|\text{Hired} = 1].$$

The outcome space in terms of (τ, ζ) is exactly as in Lemma 3, with $R_Z = R(k) = \frac{1}{k}\phi(\Phi^{-1}(1-k))$ and $R_T = \sqrt{\sigma_Q^2 + \sigma_T^2 R(k)}$. As before, let $\bar{\tau}(\zeta) \equiv R_T \cdot \sqrt{1 - \frac{\zeta^2}{R_Z^2}}$ be the maximum possible τ for a given $\zeta \in [-R_Z, R_Z]$.

When the agent is of type θ, hiring an applicant with test score T and utility z-score Z gives expected utilities to the agent and principal of

$$U_A = \mu_{U_A}(T; \theta) + \sigma_{U_A}(\theta)Z = (1 + \lambda r)T + \sigma_{U_A}(\theta)Z,$$

$$U_P(T, U_A) = \mu_{U_P}(T) + \sigma_{U_P}(\theta)Z = (1 + r)T + \sigma_{U_P}(\theta)Z.$$

In terms of τ and ζ, agent and principal payoffs for hiring a pool of applicants are

$$\text{A:} \quad V_A = \sigma_{U_A}(\theta)\zeta + (1 + \lambda r)\tau,$$

$$\text{P:} \quad V_P = \sigma_{U_P}(\theta)\zeta + (1 + r)\tau.$$

We see that the agent's behavior depends only on the ratio of $\sigma_{U_A}(\theta)$ to $(1 + \lambda r)$; her problem is equivalent to maximizing $\frac{\sigma_{U_A}(\theta)}{1+\lambda r}\zeta + \tau$, or to maximizing $\sigma_{U_A}(\theta)\frac{1+r}{1+\lambda r}\zeta + (1+r)\tau$. Define ρ as this coefficient on ζ:

$$\rho(\theta) \equiv \sigma_{U_A}(\theta)\frac{1+r}{1+\lambda r} = \sqrt{\lambda^2 l \sigma_2^2 + \sigma_B^2}\frac{1+r}{1+\lambda r}.$$

The coefficient $\rho(\theta)$ is a one-dimensional sufficient statistic for the agent's preferences. For any $\tilde{\rho}$, all agent types θ with $\rho(\theta) = \tilde{\rho}$ act identically. Let the distribution of $\rho(\theta)$ induced by $\theta \sim G$ be given by the cdf H.

Because the principal can never distinguish agents with the same $\rho(\theta)$, it is convenient to define the principal's average value of σ_{U_P} across all agent types with $\rho(\theta) = \tilde{\rho}$ as $\hat{\sigma}_{U_P}(\tilde{\rho})$:

$$\hat{\sigma}_{U_P}(\tilde{\rho}) \equiv \mathbb{E}_{\theta \sim G}\big[\sigma_{U_P}(\theta)|\rho(\theta) = \tilde{\rho}\big].$$

Now rewrite the principal and agent maximization problems as

$$\text{Agent:}\quad \max\bigl(\rho(\theta)\cdot\zeta + (1+r)\bar{\tau}(\zeta)\bigr) - \delta, \tag{40}$$

$$\text{Principal:}\quad \max \mathbb{E}_{\rho(\theta)\sim H}\bigl[\bigl(\hat{\sigma}_{U_P}\bigl(\rho(\theta)\bigr)\cdot\zeta + (1+r)\bar{\tau}(\zeta)\bigr) - \delta\bigr] \tag{41}$$

$$\text{for}\ \delta \equiv (1+r)\bigl(\bar{\tau}(\zeta) - \tau\bigr). \tag{42}$$

Once again, δ represents "money burning" due to taking τ below its maximum possible value. The contract induces a menu of (ζ,δ) from which the agent may select, given her observation of $\rho(\theta)$.

We can now give the analog of Proposition 5.

PROPOSITION 10: *In the combined model, let the distribution H have continuous pdf h over its support, with the support a bounded interval in $\mathbb{R}_+$, and let $\hat{\sigma}_{U_P}(\cdot)$ be continuous over the support. If $H(\tilde{\rho}) + (\tilde{\rho} - \hat{\sigma}_{U_P}(\tilde{\rho}))h(\tilde{\rho})$ is nondecreasing in $\tilde{\rho}$, then the optimal contract takes the same form as in Proposition 5.*

This result embeds Proposition 5—up to some changes of notation—when there is no systematic bias, that is, $\lambda = 1$. In that case, the projection of the agent's type $\rho(\theta) = \sigma_{U_A}(\theta)\frac{1+r}{1+\lambda r}$ is exactly just $\sigma_{U_A}(\theta)$. But we also now have a generalization of the conditions under which the simple contract forms from the body of the paper remain optimal even when agents have a commonly known systematic bias $\lambda \neq 1$, or when there is a distribution of the systematic bias λ across agents.

APPENDIX F: INFERENCE FROM PERFORMANCE DATA

Consider the normal specification, and take σ_Q^2 and σ_T^2 to be commonly known while the agent's type, (σ_S^2, σ_B^2), is not known. (Below, I address how one might also infer σ_Q^2 and σ_T^2 if those were unknown.) I proceed here in a prior-free manner and thus do not specify the principal's prior beliefs over the agent's type.

Let there be two periods over which the agent's type is persistent. In the first period, the principal gives the agent a Full Discretion contract in which she chooses k applicants. For each applicant that is hired, the principal observes the public test result T and also the quality Q—the realized performance. Then, in the second period, the principal uses the first-period data to choose a contract that will select another k applicants.

Assume that the agent selects applicants myopically, that is, her behavior in the first period maximizes her first-period payoff. That is, she has no dynamic consideration for how her behavior affects the contract she will be offered in the future.

In the second period, the principal has access to the acceptance rate as a function of test results, plus the entire distribution of realized qualities for the accepted applicants at each score. I will find that these data are sufficient for the principal to perfectly infer the agent's type, and therefore to set the optimal contract in the second period given the knowledge of her type. Indeed, the principal only needs to look at two moments of the data. Let τ_1 be the average test score of the applicants accepted in the first period, and let ξ_1 be the average realized quality. The principal can calculate the optimal second-period contract from τ_1 and ξ_1. As in Proposition 3 part 2, the contract can be summed up as a requirement that the average test score of accepted applicants in the second period, τ_2, must equal some level κ^*.

LEMMA 5: *Given τ_1 and ξ_1, the principal's period-2 optimal contract allows the agent to accept any k applicants with average test score τ_2 equal to κ^*, with*

$$\kappa^* = \frac{R(k)\sigma_Q^2\sqrt{\sigma_Q^2 + \sigma_T^2}}{\sqrt{\sigma_Q^2 + \frac{\left(\xi_1(\sigma_Q^2 + \sigma_T^2) - \sigma_Q^2\tau_1\right)^2}{(\sigma_Q^2 + \sigma_T^2)\left((\sigma_Q^2 + \sigma_T^2)R(k)^2 - \tau_1^2\right)}}} \tag{43}$$

and $R(\cdot)$ given by (22). Moreover, over the domain of possible ξ_1 and τ_1, it holds that κ^* decreases in ξ_1 and increases in τ_1.

That is, for any fixed average test score in the Full Discretion first period, better ex post performance of the hired applicants ξ_1 leads to a lower required average test score (a flatter contract, one closer to the agent's preferred outcome) in the second period. On the other hand, an agent who picks a higher average test score τ_1 (steeper contract) in the first period is required to pick a higher average test score (steeper contract) in the second period.

Inferring σ_Q^2 and σ_T^2.

What if the principal fundamentals σ_Q^2 and σ_T^2 will be the same from period 1 to period 2, but the values are not known in advance? In fact, these two parameters can also be inferred from the period-1 Full Discretion data. Their imputed values can then be plugged into the formulas above.

To see this, first define Var_T as the empirical variance of the test score distribution across all applicants. This empirical variance is directly observable in period 1. Under the predictions of the model, Var_T will be equal to $\sigma_Q^2 + \sigma_T^2$.

Next, let $\bar{q}_1(t)$ indicate the average realized period-1 quality of accepted applicants at test score t. Suppose that, under full discretion, a share $\alpha(t)$ of applicants are accepted at this test score. Then the model predicts that

$$\bar{q}_1(t) = \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}t + \sigma_{U_P}(\theta)R\big(\alpha(t)\big) = \frac{\sigma_Q^2}{\mathrm{Var}_T}t + \sigma_{U_P}(\theta)R\big(\alpha(t)\big).$$

The value of $\sigma_{U_P}(\theta)$ can be inferred from performance data, with the formula given in (55) in the proof of Lemma 5. Plugging in that formula, and replacing all occurrences of $\sigma_Q^2 + \sigma_T^2$ with Var_T, we get

$$\bar{q}_1(t) = \frac{\sigma_Q^2}{\mathrm{Var}_T}t + \left(\frac{\xi_1\,\mathrm{Var}_T - \sigma_Q^2\tau_1}{\sqrt{\mathrm{Var}_T\left(\mathrm{Var}_T\,R(k)^2 - \tau_1^2\right)}}\right)R\big(\alpha(t)\big).$$

Solving this equation for σ_Q^2 gives a separate estimate of σ_Q^2 at each test score t:

$$\sigma_Q^2 = \frac{\bar{q}_1(t)\sqrt{\mathrm{Var}_T\left(R(k)^2\,\mathrm{Var}_T - \tau_1^2\right)} - R\big(\alpha(t)\big)\,\mathrm{Var}_T\,\xi_1}{t\sqrt{\left(R(k)^2 - \tau_1^2\right)/\mathrm{Var}_T} - R\big(\alpha(t)\big)\tau_1}. \tag{44}$$

Of course, under the theoretical model, the estimate should be identical at every t. With actual performance data, one would presumably want to take an average or a weighted average of these estimates across all of the test scores. At any rate, given an estimate of σ_Q^2 from (44), we have $\sigma_T^2 = \mathrm{Var}_T - \sigma_Q^2$.

APPENDIX G: PROOFS

G.1. *Proofs for Sections 2 and 3*

PROOF OF LEMMA 1: Consider two independent random variables X and Y, for which Y has a log-concave distribution. I seek to show that $\mathbb{E}[X|X + Y = z]$ is weakly increasing in z. Conditioning on $T = t$ and interpreting X as $\mathbb{E}[Q|S, T]$, Y as B, and z as u_A will then yield the desired conclusion that $U_P(t, u_A)$ is increasing in u_A. Specifically, these substitutions give us $X + Y = \mathbb{E}[Q|S, T] + B = U_A$, and $\mathbb{E}[X|X + Y = z, T = t] = \mathbb{E}[\mathbb{E}[Q|S, T]|U_A = u_A, T = t] = \mathbb{E}[Q|U_A = u_A, T = t] = U_P(t, u_A)$, where the second-to-last equality holds by the law of iterated expectations.

To show that $\mathbb{E}[X|X + Y = z]$ is weakly increasing in z (for any prior over X), it suffices to show that the distribution of $X + Y|X = x$ satisfies the monotone likelihood ratio property in x, and thus that higher realizations of $X + Y$ indicate higher posteriors on X. By independence of X and Y, it holds that $X + Y|X = x$ follows the distribution of $Y + x$. In other words, it suffices to show that $Y + x$ has monotone likelihood ratio in x. Indeed, log-concavity of Y implies that $Y + x$ has monotone likelihood ratio in x; see, for example, Marshall and Olkin (2007, Example 2.A.15).

To give some intuition for that final step, indicate the pdf of Y by f_Y and the pdf of $Y + x$ by f_{Y+x}, where $f_{Y+x}(z) = f_Y(z - x)$. The random variable $Y + x$ has monotone likelihood ratio in x if (ignoring zeroes in the denominator) it holds that, for all $\overline{z} > \underline{z}$ and $\overline{x} > \underline{x}$,

$$\frac{f_{Y+\overline{x}}(\overline{z})}{f_{Y+\underline{x}}(\overline{z})} \geq \frac{f_{Y+\overline{x}}(\underline{z})}{f_{Y+\underline{x}}(\underline{z})}, \quad \text{that is,} \quad \frac{f_Y(\overline{z} - \overline{x})}{f_Y(\overline{z} - \underline{x})} \geq \frac{f_Y(\underline{z} - \overline{x})}{f_Y(\underline{z} - \underline{x})}.$$

And log-concavity of Y is equivalent to $f_Y(\overline{z} - \overline{x}) f_Y(\underline{z} - \underline{x}) \geq f(\overline{z} - \underline{x}) f(\underline{z} - \overline{x})$ for all $\overline{z} > \underline{z}$, $\overline{x} > \underline{x}$ (Marshall and Olkin, 2007, Proposition 21.B.8), yielding the expression above.
$$Q.E.D.$$

PROOF OF PROPOSITION 1: Follows from arguments in the text. Under this contract, at each $T = t$ the agent monotonically selects the applicants with $U_A|T = t$ in the top $\alpha^{\mathrm{UBAR}}(t)$ share of the distribution. By the assumed monotonicity of UBAR, these are the same applicants selected by UBAR.
$$Q.E.D.$$

PROOF OF PROPOSITION 2: Given a monotonic upper bound acceptance rule χ^{UBAR}, take some agent utility cutoff function $u_A^c(t)$ consistent with χ^{UBAR} and take a corresponding score function $C(t) = a_0 - a_1 u_A^c(t)$ for $a_1 > 0$. Rearranging, $u_A^c(t) = \frac{a_0}{a_1} - \frac{C(t)}{a_1}$. Assume that the expectation of $C(T)$ exists and is equal to κ, which implies that $C(\cdot)$ and $u_A^c(\cdot)$ are almost everywhere finite-valued.

The agent chooses an acceptance rule χ, a map from test results and agent utilities to acceptance probabilities. Her problem is to choose χ to maximize her objective

$$\frac{1}{k} \mathbb{E}\big[\chi(T, U_A) \cdot U_A\big]$$

subject to the two constraints of accepting k applicants and of setting the average score of hired applicants to at least κ:

$$\mathbb{E}\big[\chi(T, U_A)\big] = k, \tag{45}$$

$$\frac{1}{k}\mathbb{E}\big[\chi(T, U_A) \cdot C(T)\big] \geq \kappa. \tag{46}$$

I claim that this problem is solved by $\chi = \chi^{\text{UBAR}}$. Note that $\chi = \chi^{\text{UBAR}}$ satisfies constraint (45), and by construction satisfies constraint (46) with equality.

Define the Lagrangian function $\mathcal{L}$ (with constraint (45) scaled by $1/k$) as

$$\mathcal{L}(\chi; \lambda_0, \lambda_1)$$
$$\equiv \frac{1}{k}\mathbb{E}\big[\chi(T, U_A) \cdot U_A\big] - \lambda_0\left(\frac{1}{k}\mathbb{E}\big[\chi(T, U_A)\big] - 1\right) + \lambda_1\left(\frac{1}{k}\mathbb{E}\big[\chi(T, U_A) \cdot C(T)\big] - \kappa\right)$$
$$= \mathbb{E}\left[\frac{1}{k}\chi(T, U_A) \cdot \big(U_A - \lambda_0 + \lambda_1\big(a_0 - a_1 u_A^c(T)\big)\big) + \lambda_0 - \lambda_1\kappa\right].$$

By standard Lagrangian logic, in order to show that the problem is solved by $\chi = \chi^{\text{UBAR}}$, it suffices to find $\lambda_0^* \in \mathbb{R}$ and $\lambda_1^* \in \mathbb{R}_+$ such that

$$\max_{\chi} \mathcal{L}\big(\chi; \lambda_0^*, \lambda_1^*\big) = \mathcal{L}\big(\chi^{\text{UBAR}}; \lambda_0^*, \lambda_1^*\big).^{46}$$

I claim that this holds for $\lambda_0^* = a_0/a_1$ and $\lambda_1^* = 1/a_1$, where $\lambda_1^* > 0$ because $a_1 > 0$.

To show $\max_{\chi} \mathcal{L}(\chi; a_0/a_1, 1/a_1) = \mathcal{L}(\chi^{\text{UBAR}}; a_0/a_1, 1/a_1)$, observe that

$$\mathcal{L}\left(\chi; \frac{a_0}{a_1}, \frac{1}{a_1}\right) = \mathbb{E}\left[\frac{1}{k}\chi(T, U_A) \cdot \left(U_A - \frac{a_0}{a_1} + \frac{1}{a_1}\big(a_0 - a_1 u_A^c(T)\big)\right) + \frac{a_0}{a_1} - \frac{1}{a_1}\kappa\right]$$
$$= \mathbb{E}\left[\frac{1}{k}\chi(T, U_A) \cdot \big(U_A - u_A^c(T)\big) + \frac{a_0}{a_1} - \frac{\kappa}{a_1}\right].$$

We see that this expression is indeed pointwise maximized over acceptance rules χ by $\chi = \chi^{\text{UBAR}}$, since $\chi^{\text{UBAR}}(T, U_A)$ is equal to the minimum possible value of 0 when $U_A - u_A^c(T) < 0$ and the maximum possible value of 1 when $U_A - u_A^c(T) > 0$.   *Q.E.D.*

Before I address the proofs of the formally stated results of Section 3.3, let me work out the derivations of formulas (8)–(11) in the text of that section. These will follow from standard updating rules of normal distributions. First, take a multivariate normal random vector X that can be decomposed as $X = (X_1, X_2)$ with mean (μ_1, μ_2) and covariance matrix $\left[\begin{smallmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{smallmatrix}\right]$. The conditional distribution of X_1 given $X_2 = x_2$ is given by

$$X_1 | X_2 = x_2 \sim \mathcal{N}\big(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\big). \tag{47}$$

---

[46]To see why this condition is sufficient, suppose for the sake of contradiction that χ' satisfies (45) and (46) and yields a higher agent objective than χ^{UBAR}. Then $\mathcal{L}(\chi'; \lambda_0^*, \lambda_1^*) > \mathcal{L}(\chi^{\text{UBAR}}; \lambda_0^*, \lambda_1^*)$.

LEMMA 6: *The variables (Q, T, U_A) are joint normally distributed, with means of 0 and covariance matrix of*

$$
\begin{bmatrix}
\sigma_Q^2 & \sigma_Q^2 & \dfrac{\sigma_Q^4(\sigma_T^2 + \sigma_S^2)}{\sigma_Q^2\sigma_T^2 + \sigma_Q^2\sigma_S^2 + \sigma_T^2\sigma_S^2} \\[2.5ex]
\sigma_Q^2 & \sigma_Q^2 + \sigma_T^2 & \sigma_Q^2 \\[2.5ex]
\dfrac{\sigma_Q^4(\sigma_T^2 + \sigma_S^2)}{\sigma_Q^2\sigma_T^2 + \sigma_Q^2\sigma_S^2 + \sigma_T^2\sigma_S^2} & \sigma_Q^2 & \dfrac{\sigma_Q^4(\sigma_T^2 + \sigma_S^2)}{\sigma_Q^2\sigma_T^2 + \sigma_Q^2\sigma_S^2 + \sigma_T^2\sigma_S^2} + \sigma_B^2
\end{bmatrix}.
$$

Given Lemma 6, we can apply (47) to calculate $U_P(T, U_A)$, defined as the expectation of Q conditional on T and U_A, by taking Q as X_1 and (T, U_A) as X_2. Working out the algebra yields Equations (8)–(11).

PROOF OF PROPOSITION 3:   1. Utilities are aligned up to distinguishability, and so we can apply Proposition 1 to find one implementation of the optimal contract. To show the desired result, then, it suffices to show that for any fixed principal utility cutoff u_P^c, as we vary t, the share of applicants with $U_P(t, U_A) \geq u_P^c$ takes the form $\Phi(\gamma_T^* t - \gamma_0)$ for γ_T^* as in (12) and for some γ_0.

The first step is to calculate the conditional distribution of U_A given T. Applying Lemma 6 and (47), taking T as X_1 and U_A as X_2, we find that

$$
U_A | T \sim \mathcal{N}\big(\mu_{U_A}(T), \sigma_{U_A}^2\big), \quad \text{for} \tag{13}
$$

$$
\mu_{U_A}(t) = \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} t, \tag{14}
$$

$$
\sigma_{U_A}^2 = \eta + \sigma_B^2. \tag{15}
$$

(These equations appear in the body of the paper as well, in Section 4.2.)

Restating (13), for any t,

$$
\frac{U_A - \mu_{U_A}(t)}{\sigma_{U_A}} \Big| T = t \sim \mathcal{N}(0, 1). \tag{48}
$$

For any t and any u_P^c, we can now calculate the acceptance rate under UBAR. An applicant with $T = t$ is accepted under UBAR if

$$
\beta_T t + \beta_{U_A} U_A \geq u_P^c
$$

$$
\iff \quad \frac{U_A - \mu_{U_A}(t)}{\sigma_{U_A}} \geq \frac{\dfrac{u_P^c - \beta_T t}{\beta_{U_A}} - \mu_{U_A}(t)}{\sigma_{U_A}}.
$$

Conditional on $T = t$, the LHS of the last line is distributed according to a standard normal. Plugging in $\mu_{U_A}(t)$ from (14) on the RHS and collecting terms, the acceptance condition can be rewritten as

$$
\frac{U_A - \mu_{U_A}(t)}{\sigma_{U_A}} \geq \gamma_0 - \gamma_T^* t,
$$

for $\gamma_0 = \frac{u_P^c}{\beta_{U_A}\sigma_{U_A}}$ and

$$\gamma_T^* = \frac{\beta_T}{\beta_{U_A}\sigma_{U_A}} + \frac{\sigma_Q^2}{(\sigma_Q^2 + \sigma_T^2)\sigma_{U_A}}.$$

So the share of applicants with $U_P(t, U_A) \geq u_P^c$ at test score $T = t$ is $1 - \Phi(\gamma_0 - \gamma_T^* t) = \Phi(\gamma_T^* t - \gamma_0)$.

  As stated in the text, I will not explicitly calculate the optimal value of γ_0 as a function of primitives, as u_P^c is itself a function of k. But plugging (9), (10), and (15) into the above expression for γ_T^* and simplifying yields the expression (12) for γ_T^*.

2. From Proposition 2, it suffices to derive the formula for a cutoff indifference curve, $u_A^c(t)$, and set $C(t)$ as any negative affine transformation. Solving for $u_A^c(t)$ as the solution to $U_P(t, u_A^c(t)) = u_P^c$ for a given u_P^c:

$$U_P(t, u_A^c(t)) = u_P^c$$

$$\stackrel{\text{Eq. (8)}}{\Longrightarrow} \quad \beta_T t + \beta_{U_A} u_A^c(t) = u_P^c$$

$$\Longrightarrow \quad u_A^c(t) = \frac{u_P^c - \beta_T t}{\beta_{U_A}}.$$

$C(t) = t$ is a negative affine transformation of $u_A^c(t)$, a linear function of t with a negative slope. *Q.E.D.*

PROOF OF PROPOSITION 4: Restating (11) and (12),

$$\gamma_T^* = \frac{\sigma_Q^2\sqrt{\eta + \sigma_B^2}}{\eta(\sigma_Q^2 + \sigma_T^2)}$$

$$\text{for } \eta = \frac{\sigma_Q^4\sigma_T^4}{(\sigma_Q^2 + \sigma_T^2)(\sigma_Q^2\sigma_T^2 + \sigma_Q^2\sigma_S^2 + \sigma_T^2\sigma_S^2)}.$$

1. The parameter σ_S^2 appears in γ_T^* only through η. Routine differentiation shows that $\frac{\partial \gamma_T^*}{\partial \eta} < 0$ and $\frac{d\eta}{d\sigma_S^2} < 0$, and so by the chain rule $\frac{d\gamma_T^*}{d\sigma_S^2} > 0$.

  Taking limits,

$$\lim_{\sigma_S^2 \to 0} \gamma_T^* = \frac{1}{\sigma_T^2}\sqrt{\frac{\sigma_T^2\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} + \sigma_B^2}, \quad \text{because } \lim_{\sigma_S^2 \to 0} \eta = \frac{\sigma_T^2\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2},$$

$$\lim_{\sigma_S^2 \to \infty} \gamma_T^* = \infty, \quad \text{because } \lim_{\sigma_S^2 \to \infty} \eta = 0.$$

2. The value η remains constant as we vary σ_B^2. Taking the derivative of γ_T^* with respect to σ_B^2 gives

$$\frac{\sigma_Q^2}{2\eta(\sigma_Q^2 + \sigma_T^2)\sqrt{\eta + \sigma_B^2}} > 0.$$

Taking limits,

$$\lim_{\sigma_B^2 \to 0} \gamma_T^* = \frac{\sigma_Q^2}{\left(\sigma_Q^2 + \sigma_T^2\right)\sqrt{\eta}},$$

$$\lim_{\sigma_B^2 \to \infty} \gamma_T^* = \infty. \hspace{4cm} Q.E.D.$$

G.2. *Proofs for Section 4*

PROOF OF LEMMA 2: Rewriting (15), (18), and (11),

$$\sigma_{U_A}(\theta) = \sqrt{\eta + \sigma_B^2},$$

$$\sigma_{U_P}(\theta) = \frac{\eta}{\sqrt{\eta + \sigma_B^2}}$$

$$\text{for } \eta = \frac{\sigma_Q^4 \sigma_T^4}{\left(\sigma_Q^2 + \sigma_T^2\right)\left(\sigma_T^2 \sigma_Q^2 + \sigma_S^2 \sigma_T^2 + \sigma_S^2 \sigma_Q^2\right)}.$$

1. Observe that η decreases in σ_S^2. Fixing σ_B^2, both σ_{U_A} and σ_{U_P} increase in η.
2. The term η is constant in σ_B^2. Fixing σ_S^2, σ_{U_A} increases in σ_B^2 while σ_{U_P} decreases in σ_B^2.
3. Fixing $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A} > 0$, the range of possible σ_{U_P} is an open interval in $\mathbb{R}_+$. One can achieve the minimum of this interval by taking $\sigma_B \to \tilde{\sigma}_{U_A}$ and $\sigma_S \to \infty$, implying $\sigma_{U_P} \to 0$. It remains to show that the supremum of $\sigma_{U_P}(\theta)$ given $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}$ is $\min\{\tilde{\sigma}_{U_A}, \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2} \frac{1}{\tilde{\sigma}_{U_A}}\}$, or equivalently that this supremum is $\tilde{\sigma}_{U_A}$ for $\tilde{\sigma}_{U_A} \leq \frac{\sigma_Q \sigma_T}{\sqrt{\sigma_Q^2 + \sigma_T^2}}$ and is $\frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2} \frac{1}{\tilde{\sigma}_{U_A}}$ for $\tilde{\sigma}_{U_A} > \frac{\sigma_Q \sigma_T}{\sqrt{\sigma_Q^2 + \sigma_T^2}}$.

   The term η is independent of σ_B^2, and decreases from $\frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2}$ to zero as σ_S^2 increases from zero to infinity. From parts 1 and 2, we maximize $\sigma_{U_P}(\theta)$ given $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}$ over $\theta = (\sigma_S^2, \sigma_B^2)$ by finding the mixture of the lowest σ_S^2 (most information) and the lowest σ_B^2 (least bias) consistent with $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}$.

   For $\tilde{\sigma}_{U_A} \leq \frac{\sigma_Q \sigma_T}{\sqrt{\sigma_Q^2 + \sigma_T^2}}$, we achieve a supremum for σ_{U_P} of $\tilde{\sigma}_{U_A}$ by taking $\sigma_B^2 \to 0$ and setting σ_S^2 so that $\eta = \tilde{\sigma}_{U_A}^2$, implying $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}$.

   For $\tilde{\sigma}_{U_A} > \frac{\sigma_Q \sigma_T}{\sqrt{\sigma_Q^2 + \sigma_T^2}}$, take $\sigma_S^2 \to 0$, which implies $\eta \to \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2}$; and set $\sigma_B^2 = \tilde{\sigma}_{U_A}^2 - \eta$ to get $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}$. Plugging $\eta \to \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2}$ into the identity $\sigma_{U_P}(\theta) = \eta/\sigma_{U_A}(\theta) = \eta/\tilde{\sigma}_{U_A}$ then gives $\sigma_{U_P} \to \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2} \frac{1}{\tilde{\sigma}_{U_A}}$. $\hspace{2cm} Q.E.D.$

PROOF OF LEMMA 3: **Step 1.** Fixing k, let us start by deriving the "upper-right frontier" of (τ, ζ), the pairs that maximize $p\tau + (1-p)\zeta$ for some $p \in [0, 1]$. One maximizes $p\tau + (1-p)\zeta$ by selecting the k applicants with the highest values of $pT + (1-p)Z$. (For $p \in (0, 1)$, these are exactly the applicants above a downward sloping line in

(T, U_A)-space—accepting such applicants induces a normal CDF acceptance rate.) In the population, T and Z are independently normally distributed with means of 0, and respective variances of $\sigma_Q^2 + \sigma_T^2$ and 1. Therefore, $pT + (1-p)Z$ has mean 0 and variance σ_{comb}^2, for $\sigma_{\text{comb}} \equiv \sqrt{p^2(\sigma_Q^2 + \sigma_T^2) + (1-p)^2}$. The applicants with the k highest values of $pT + (1-p)Z$ are those with $\frac{pT+(1-p)Z}{\sigma_{\text{comb}}} \geq r^*$, for r^* satisfying $\Phi(r^*) = 1 - k$. I seek to calculate the expected value of T and of Z conditional on $\frac{pT+(1-p)Z}{\sigma_{\text{comb}}} \geq r^*$.

We have the following joint normal distribution among the three random variables T, Z, and $\frac{pT+(1-p)Z}{\sigma_{\text{comb}}}$:

$$
\begin{bmatrix} T \\ Z \\ \dfrac{pT + (1-p)Z}{\sigma_{\text{comb}}} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_Q^2 + \sigma_T^2 & 0 & \dfrac{p}{\sigma_{\text{comb}}}(\sigma_Q^2 + \sigma_T^2) \\ 0 & 1 & \dfrac{1-p}{\sigma_{\text{comb}}} \\ \dfrac{p}{\sigma_{\text{comb}}}(\sigma_Q^2 + \sigma_T^2) & \dfrac{1-p}{\sigma_{\text{comb}}} & 1 \end{bmatrix} \right).
$$

As in expression (47) of Appendix G.1, we can calculate conditional means of T and Z conditional on any realization $\frac{pT+(1-p)Z}{\sigma_{\text{comb}}} = r$:

$$
\mathbb{E}\left[T \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} = r \right] = \frac{p(\sigma_Q^2 + \sigma_T^2)}{\sigma_{\text{comb}}} r,
$$

$$
\mathbb{E}\left[Z \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} = r \right] = \frac{1-p}{\sigma_{\text{comb}}} r.
$$

This holds for every realization r. Therefore, for every r,

$$
\mathbb{E}\left[T \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \geq r \right] = \frac{p(\sigma_Q^2 + \sigma_T^2)}{\sigma_{\text{comb}}} \mathbb{E}\left[\frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \geq r \right],
$$

$$
\mathbb{E}\left[Z \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \geq r \right] = \frac{1-p}{\sigma_{\text{comb}}} \mathbb{E}\left[\frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \geq r \right].
$$

And given that $\frac{pT+(1-p)Z}{\sigma_{\text{comb}}}$ follows a standard normal, the truncated mean $\mathbb{E}[\frac{pT+(1-p)Z}{\sigma_{\text{comb}}} \mid \frac{pT+(1-p)Z}{\sigma_{\text{comb}}} \geq r]$ is equal to $\frac{\phi(r)}{1-\Phi(r)}$. Evaluating the above expressions at $r = r^*$:

$$
\tau = \mathbb{E}\left[T \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \geq r^* \right] = \frac{p(\sigma_Q^2 + \sigma_T^2)}{\sigma_{\text{comb}}} \frac{\phi(r^*)}{1 - \Phi(r^*)}
$$

$$
= \frac{p(\sigma_Q^2 + \sigma_T^2)}{\sqrt{p^2(\sigma_Q^2 + \sigma_T^2) + (1-p)^2}} R(k),
$$

$$
\zeta = \mathbb{E}\left[Z \,\middle|\, \frac{pT + (1-p)Z}{\sigma_{\text{comb}}} \geq r^* \right] = \frac{1-p}{\sigma_{\text{comb}}} \frac{\phi(r^*)}{1 - \Phi(r^*)} = \frac{1-p}{\sqrt{p^2(\sigma_Q^2 + \sigma_T^2) + (1-p)^2}} R(k).
$$

As p goes from 0 to 1, τ goes from 0 to $R_T = \sqrt{\sigma_Q^2 + \sigma_T^2} R(k)$ and ζ goes from $R_Z = R(k)$ to 0. For any $p \in [0, 1]$,

$$
\frac{\tau^2}{R_T^2} + \frac{\zeta^2}{R_Z^2} = \frac{1}{\sigma_Q^2 + \sigma_T^2} \frac{p^2 (\sigma_Q^2 + \sigma_T^2)^2}{p^2 (\sigma_Q^2 + \sigma_T^2) + (1-p)^2} + \frac{(1-p)^2}{p^2 (\sigma_Q^2 + \sigma_T^2) + (1-p)^2}
$$

$$
= \frac{p^2 (\sigma_Q^2 + \sigma_T^2) + (1-p)^2}{p^2 (\sigma_Q^2 + \sigma_T^2) + (1-p)^2} = 1.
$$

So we see that varying $p \in [0, 1]$ traces out the upper-right boundary of the ellipse $\overline{W}$.

Step 2. We can proceed similarly to show that we trace out the entire boundary of the ellipse as we maximize the four combinations of $\pm p\tau \pm (1-p)\zeta$ for $p \in [0, 1]$. In other words, every (τ, ζ) that is a boundary point of $\overline{W}$ is achieved by some set of k applicants. Moreover, no set of k applicants achieves a pair (τ, ζ) that is outside the boundaries of this ellipse, the interior of which is convex—otherwise this point would yield a higher value of an appropriately signed $\pm p\tau \pm (1-p)\zeta$ than any value on the boundary.

Step 3. Finally, the set of achievable (τ, ζ) across applicant pools is convex: choosing a convex combination of applicants from the two pools yields the same convex combination of the average test score and average z-score (τ, ζ). So all points in the interior of $\overline{W}$ are achievable as well.   *Q.E.D.*

PROOF OF PROPOSITION 5: I will show this result as an application of the one-dimensional delegation results in Amador, Bagwell, and Frankel (2018), an extension of Amador and Bagwell (2013). Specifically, Lemma 2 of Amador, Bagwell, and Frankel (2018) provides sufficiency conditions for the optimality of an action ceiling. As a delegation problem in the framework of those two papers, let the "state" $\sigma_{U_A}(\theta)$ be distributed according to H, let $\zeta \in [0, R_Z]$ be the contractible "action," and let the level of joint "money burning" be $\delta \in \mathbb{R}_+$.[47] Those papers take a contract to be an arbitrary set of allowed actions ζ and an arbitrary function from allowed actions to nonnegative money burning; in my problem, money burning is restricted, bounded at $\delta \leq 2\bar{\tau}(\zeta)$ under action ζ. However, under the conditions of Lemma 2 in Amador, Bagwell, and Frankel (2018), money burning will be identically zero in the optimal delegation contract; any upper bound on money burning will therefore not be binding.[48]

Following the notation of those other papers, the agent's payoff over the state and action, prior to money burning, can be written as

$$
\sigma_{U_A}(\theta) \cdot \zeta + b(\zeta) \quad \text{for } b(\zeta) \equiv \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} R_T \cdot \sqrt{1 - \frac{\zeta^2}{R_Z^2}}. \tag{49}
$$

---

[47]The notation of Amador and Bagwell (2013) and Amador, Bagwell, and Frankel (2018) has state γ distributed according to cdf F, with pdf f; action π; and money burning t. They used $\pi_f(\gamma)$ for the function describing the agent's ideal action, what I will call $\zeta_f(\sigma_{U_A}(\theta))$; and the latter paper used π^* for the principal's ex ante optimal action, what I will call ζ^*. I follow these papers in using $b(\cdot)$ as the component of the agent's payoff function that depends on the action, despite my previous use of b as the bias realization for a given applicant; there should be no confusion between the two distinct terms.

[48]A delegation contract takes a delegation set of allowed actions and a money burning function as direct objects of choice, whereas these are induced objects—expectations over a selected applicant pool—in the problem of the current paper. Hence, even if a delegation contract satisfies the necessary condition of $\delta \leq 2\bar{\tau}(\zeta)$ at each ζ, I still need to show how to find a contract in my setting that implements this delegation outcome.

The principal's payoff prior to money burning can be written as $w(\sigma_{U_A}(\theta), \zeta)$, with

$$w(\tilde{\sigma}_{U_A}, \zeta) \equiv \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) \cdot \zeta + b(\zeta). \tag{50}$$

Money burning of δ reduces both payoffs by that same amount. These payoffs are not just of the general form considered in Amador and Bagwell (2013), but of the form in Equation (6) of Amador, Bagwell, and Frankel (2018): $w(\sigma_{U_A}(\theta), \zeta) = A[b(\zeta) + B(\sigma_{U_A}(\theta)) + C(\sigma_{U_A}(\theta))\zeta]$ for $A = 1$, $B(\sigma_{U_A}(\theta)) = 0$, and $C(\sigma_{U_A}(\theta)) = \hat{\sigma}_{U_P}(\sigma_{U_A}(\theta))$.

Denote the agent's interim optimal action at state $\sigma_{U_A}(\theta)$—her "flexible" action—as $\zeta_f(\sigma_{U_A}(\theta))$. Taking the first-order condition of (49),

$$\zeta_f(\sigma_{U_A}(\theta)) = \frac{\sigma_{U_A}(\theta)R_Z}{\sqrt{\sigma_{U_A}(\theta)^2 + \dfrac{\sigma_Q^4}{(\sigma_Q^2 + \sigma_T^2)^2}\dfrac{R_T^2}{R_Z^2}}}. \tag{51}$$

Denote the principal's ex ante optimal action by ζ^*, the arg max over ζ of the expectation of (50):

$$\zeta^* = \arg \max_{\zeta \in [0, R_Z]} \mathbb{E}_{\sigma_{U_A}(\theta) \sim H}\big[\hat{\sigma}_{U_P}(\sigma_{U_A}(\theta))\big]\zeta + b(\zeta). \tag{52}$$

Because $\hat{\sigma}_{U_P}(\sigma_{U_A}(\theta)) \in (0, \sigma_{U_A}(\theta))$, and because the proposition assumes that $\sigma_{U_A}(\theta)$ has bounded support, it holds that $\mathbb{E}_{\sigma_{U_A}(\theta) \sim H}[\hat{\sigma}_{U_P}(\sigma_{U_A}(\theta))]$ is finite and strictly positive. Moreover, the derivative of $b(\zeta)$ is 0 as $\zeta \to 0$ and minus infinity as $\zeta \to R_Z$. Therefore, ζ^* is interior, contained in $(0, R_Z)$; we have now verified Assumption 2 of Amador, Bagwell, and Frankel (2018).

We can now verify the regularity conditions of Assumption 1 of Amador, Bagwell, and Frankel (2018). Going through the list, (i) w is continuous; (ii) $w(\tilde{\sigma}_{U_A}, \cdot)$ is concave and twice differentiable for every $\tilde{\sigma}_{U_A}$; (iii) $b(\cdot)$ is strictly concave and twice differentiable; (iv) $\zeta_f(\cdot)$ is twice differentiable and strictly increasing; and (v) the function w_ζ, the derivative of w with respect to ζ, is continuous.

Next, let us evaluate w_ζ at the agent's ideal point from (51). Putting together $w_\zeta(\tilde{\sigma}_{U_A}, \zeta) = \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) + b'(\zeta)$ with the fact that the agent's ideal point $\zeta_f(\sigma_{U_A}(\theta))$ is derived from the first-order condition $b'(\zeta_f(\tilde{\sigma}_{U_A})) = -\tilde{\sigma}_{U_A}$, it holds that

$$w_\zeta(\tilde{\sigma}_{U_A}, \zeta_f(\tilde{\sigma}_{U_A})) = \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) - \tilde{\sigma}_{U_A}. \tag{53}$$

From (53) combined with $\hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) \in (0, \tilde{\sigma}_{U_A})$, we see that $w_\zeta(\tilde{\sigma}_{U_A}, \zeta_f(\tilde{\sigma}_{U_A}))$ is strictly negative at each $\tilde{\sigma}_{U_A} > 0$, and is equal to zero in the limit as $\tilde{\sigma}_{U_A} \to 0$ (if this limit is in the support of H); in economic terms, the agent's bias is always towards higher ζ. Therefore, $w_\zeta(\tilde{\sigma}_{U_A}, \zeta_f(\tilde{\sigma}_{U_A}))$ satisfies the sign restrictions of Lemma 2 of Amador, Bagwell, and Frankel (2018) for $\tilde{\sigma}_{U_A}$ at the lower and upper bounds of the support.

To apply that Lemma 2, it remains only to check condition (Gc1) of Amador, Bagwell, and Frankel (2018). The parameter κ appearing in (Gc1) is equal to 1 because—as mentioned above—the payoffs are of the form in Equation (6) of that paper, with $A = 1$. Plugging in $\kappa = 1$ and $w_\zeta(\tilde{\sigma}_{U_A}, \zeta_f(\tilde{\sigma}_{U_A})) = \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) - \tilde{\sigma}_{U_A}$, condition (Gc1) states that $H(\tilde{\sigma}_{U_A}) + (\tilde{\sigma}_{U_A} - \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}))h(\tilde{\sigma}_{U_A})$ is nondecreasing over $\tilde{\sigma}_{U_A}$ in the support of H. This condition is exactly what is assumed in the statement of the proposition.

We have now confirmed all of the hypotheses of Lemma 2 of that paper. We can therefore conclude that in the delegation problem with money burning allowed, the optimal

delegation set is of the form of a ceiling on ζ—possibly everywhere binding, implying a one-point delegation set of $\zeta = \zeta^*$—with money burning δ identically equal to 0.

Finally, as discussed in the body of the paper and illustrated in Figure 5, a ceiling on ζ and no money burning corresponds to an applicant selection contract that takes the form of a floor on the average test score τ. Furthermore, as in Section 3.3, each average test score that the agent may choose is equivalent to a normal CDF acceptance rate function, with higher test scores mapping to steeper normal CDFs. *Q.E.D.*

PROOF OF PROPOSITION 6: Follows immediately from Proposition 11, below. *Q.E.D.*

PROPOSITION 11: *Under the normal specification, suppose that* $\sup_{\tilde{\sigma}_{U_A} \in \mathrm{Supp}\, H} \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) \leq$ $\inf_{\tilde{\sigma}_{U_A} \in \mathrm{Supp}\, H} \tilde{\sigma}_{U_A}$. *Then the optimal contract takes the same form as in Proposition 5. Under either implementation of the optimal contract, the agent will choose applicants so that the floor constraint* ($\gamma_T \geq \Gamma$ *or* $\tau \geq \kappa$) *binds with equality.*

PROOF OF PROPOSITION 11: **Step 1.** Take $\tilde{\sigma}_{U_A}^l < \tilde{\sigma}_{U_A}^h$, and take points (τ^l, ζ^l) and (τ^h, ζ^h) in $\overline{W}$. Suppose an agent with $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}^l$ weakly prefers (τ^l, ζ^l) to (τ^h, ζ^h), and an agent with $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}^h$ weakly prefers (τ^h, ζ^h) to (τ^l, ζ^l). I claim that if $\hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) \leq \tilde{\sigma}_{U_A}^l$, then, conditional on $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}$, the principal weakly prefers (τ^l, ζ^l) to (τ^h, ζ^h).

This claim follows as a straightforward single-crossing argument from (26) and (27). From the two preference orderings, it must be that $\tau^l \geq \tau^h$ and $\zeta^l \leq \zeta^h$. Now, writing out the agent's choice given $\sigma_{U_A}(\theta) = \tilde{\sigma}_{U_A}^l$, it holds that

$$\frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} \cdot \tau^l + \tilde{\sigma}_{U_A}^l \cdot \zeta^l \geq \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} \cdot \tau^h + \tilde{\sigma}_{U_A}^l \cdot \zeta^h.$$

Because $\zeta^l \leq \zeta^h$, the same inequality holds when $\tilde{\sigma}_{U_A}^l$ is replaced by $\hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) \leq \tilde{\sigma}_{U_A}^l$.

Step 2. By the claim in Step 1, under any contract, the principal prefers the (τ, ζ) pair chosen by the agent with $\sigma_{U_A}(\theta)$ equal to the minimum of the support of H to that chosen by any other agent type. So the contract is weakly improved by one which requires the agent to always choose that value (τ, ζ). This new contract, in turn, can be improved by one that specifies that the agent always chooses the principal's ex ante preferred (τ, ζ): the value on the payoff frontier which maximizes

$$\frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} \cdot \tau + \mathbb{E}_{\tilde{\sigma}_{U_A} \sim H}\big[\hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A})\big] \cdot \zeta.$$

This can be implemented by setting an appropriately chosen binding floor on the average test score, or a binding floor on the steepness of a normal CDF acceptance rate function. *Q.E.D.*

PROOF OF LEMMA 4: It is sufficient to confirm that $\hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A})$ is differentiable over the support when the bias is commonly known (and is therefore continuous), and that $\hat{\sigma}'_{U_P}(\tilde{\sigma}_{U_A}) \leq 2$. In that case, Lemma 7 below implies the result: for $\hat{\sigma}_{U_P}$ differentiable, condition (iii) of Lemma 7 amounts to $\hat{\sigma}'_{U_P}(\tilde{\sigma}_{U_A}) \leq 2$.

From (15) and (18),

$$\hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}) = \frac{\tilde{\sigma}_{U_A}^2 - \sigma_B^2}{\tilde{\sigma}_{U_A}} = \tilde{\sigma}_{U_A} - \frac{\sigma_B^2}{\tilde{\sigma}_{U_A}}.$$

Taking the derivative, $\hat{\sigma}'_{U_P}(\tilde{\sigma}_{U_A}) = 1 + \frac{\sigma_B^2}{\tilde{\sigma}_{U_A}^2}$. To show that $\hat{\sigma}'_{U_P}(\tilde{\sigma}_{U_A}) \le 2$, it suffices to show that $\tilde{\sigma}_{U_A}^2 > \sigma_B^2$; and this follows directly from (15), which states that, for any agent type $\theta = (\sigma_S^2, \sigma_B^2)$, it holds that $\sigma_{U_A}^2(\theta) = \eta + \sigma_B^2$, with $\eta > 0$. *Q.E.D.*

LEMMA 7: *Suppose that (i) the distribution H has pdf h, (ii) $h(\tilde{\sigma}_{U_A})$ is nondecreasing in $\tilde{\sigma}_{U_A}$ over the support of the distribution, and (iii) $(2\tilde{\sigma}_{U_A} - \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}))$ is nondecreasing in $\tilde{\sigma}_{U_A}$. Then $H(\tilde{\sigma}_{U_A}) + (\tilde{\sigma}_{U_A} - \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A}))h(\tilde{\sigma}_{U_A})$ is nondecreasing in $\tilde{\sigma}_{U_A}$.*

PROOF OF LEMMA 7: Let $\Delta(\tilde{\sigma}_{U_A}) \equiv \tilde{\sigma}_{U_A} - \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A})$. It holds that $\Delta(\tilde{\sigma}_{U_A}) > 0$. Assumption (iii), that $2\tilde{\sigma}_{U_A} - \hat{\sigma}_{U_P}(\tilde{\sigma}_{U_A})$ is nondecreasing, can be equivalently stated as $\Delta(\bar{\sigma}) + \bar{\sigma} \ge \Delta(\underline{\sigma}) + \underline{\sigma}$ for any $\bar{\sigma} > \underline{\sigma}$ in the support of $\tilde{\sigma}_{U_A}$. In other words, (iii) is equivalent to (iii'):

$$\Delta(\bar{\sigma}) - \Delta(\underline{\sigma}) \ge \underline{\sigma} - \bar{\sigma} \quad \text{for any } \bar{\sigma} > \underline{\sigma}. \tag{iii'}$$

I seek to prove that for any $\bar{\sigma} > \underline{\sigma}$,

$$\big(H(\bar{\sigma}) + \Delta(\bar{\sigma})h(\bar{\sigma})\big) - \big(H(\underline{\sigma}) + \Delta(\underline{\sigma})h(\underline{\sigma})\big) \ge 0.$$

Rewriting the LHS,

$$\big(H(\bar{\sigma}) + \Delta(\bar{\sigma})h(\bar{\sigma})\big) - \big(H(\underline{\sigma}) + \Delta(\underline{\sigma})h(\underline{\sigma})\big)$$
$$= H(\bar{\sigma}) - H(\underline{\sigma}) + \Delta(\bar{\sigma})h(\bar{\sigma}) - \Delta(\underline{\sigma})h(\underline{\sigma}) + \big[\Delta(\bar{\sigma})h(\underline{\sigma}) - \Delta(\bar{\sigma})h(\underline{\sigma})\big]$$
$$= \big[\Delta(\bar{\sigma})\big(h(\bar{\sigma}) - h(\underline{\sigma})\big)\big] + \big[H(\bar{\sigma}) - H(\underline{\sigma})\big] + \big[h(\underline{\sigma})(\Delta(\bar{\sigma}) - \Delta(\underline{\sigma}))\big]$$
$$\ge \big[\Delta(\bar{\sigma})\big(h(\bar{\sigma}) - h(\underline{\sigma})\big)\big] + \big[h(\underline{\sigma})(\bar{\sigma} - \underline{\sigma})\big] + \big[h(\underline{\sigma})\big(\Delta(\bar{\sigma}) - \Delta(\underline{\sigma})\big)\big] \quad \text{by (ii)}$$
$$\ge \big[\Delta(\bar{\sigma})\big(h(\bar{\sigma}) - h(\underline{\sigma})\big)\big] + \big[h(\underline{\sigma})(\bar{\sigma} - \underline{\sigma})\big] + \big[h(\underline{\sigma})(\underline{\sigma} - \bar{\sigma})\big] \quad \text{by (iii')}$$
$$= \Delta(\bar{\sigma})\big(h(\bar{\sigma}) - h(\underline{\sigma})\big) \ge 0 \quad \text{by (ii).} \qquad\qquad \textit{Q.E.D.}$$

G.3. *Additional Appendix Proofs*

PROOF OF PROPOSITION 7: Restating (11) and (12),

$$\begin{cases} \gamma_T^* = \dfrac{\sigma_Q^2\sqrt{\eta + \sigma_B^2}}{\eta(\sigma_Q^2 + \sigma_T^2)}, \\[2em] \gamma_T^*\sqrt{\sigma_Q^2 + \sigma_T^2} = \dfrac{\sigma_Q^2\sqrt{\eta + \sigma_B^2}}{\eta\sqrt{\sigma_Q^2 + \sigma_T^2}}, \end{cases}$$

$$\text{for } \eta = \frac{\sigma_Q^4\sigma_T^4}{(\sigma_Q^2 + \sigma_T^2)(\sigma_Q^2\sigma_T^2 + \sigma_Q^2\sigma_S^2 + \sigma_T^2\sigma_S^2)}.$$

1. The parameter k does not appear in the formula for γ_T^*.

2. Taking $\gamma_T^* = \frac{\sigma_Q^2\sqrt{\eta+\sigma_B^2}}{\eta(\sigma_Q^2+\sigma_T^2)}$ as a function of σ_Q^2, σ_B^2, σ_T^2, and η, we can write $\frac{d\gamma_T^*}{d\sigma_T^2}$ as $\frac{\partial\gamma_T^*}{\partial\sigma_T^2} + \frac{\partial\gamma_T^*}{\partial\eta}\frac{d\eta}{d\sigma_T^2}$. It is easy to confirm by routine differentiation that $\frac{\partial\gamma_T^*}{\partial\sigma_T^2} < 0$, $\frac{\partial\gamma_T^*}{\partial\eta} < 0$, and $\frac{d\eta}{d\sigma_T^2} > 0$. Therefore, $\frac{d\gamma_T^*}{d\sigma_T^2} < 0$.

   Taking $\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2} = \frac{\sigma_Q^2\sqrt{\eta+\sigma_B^2}}{\eta\sqrt{\sigma_Q^2+\sigma_T^2}}$ as a function of σ_Q^2, σ_B^2, σ_T^2, and η, we can write $\frac{d(\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2})}{d\sigma_T^2}$ as $\frac{\partial(\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2})}{\partial\sigma_T^2} + \frac{\partial(\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2})}{\partial\eta}\frac{d\eta}{d\sigma_T^2}$. Once again, $\frac{\partial(\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2})}{\partial\sigma_T^2} < 0$, $\frac{\partial(\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2})}{\partial\eta} < 0$, and $\frac{\partial\eta}{\partial\sigma_T^2} > 0$. Therefore, $\frac{d(\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2})}{d\sigma_T^2} < 0$.

   Taking limits,

$$\lim_{\sigma_T^2\to 0}\gamma_T^* = \lim_{\sigma_T^2\to 0}\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2} = \infty, \quad \text{because } \lim_{\sigma_T^2\to 0}\eta = 0,$$

$$\lim_{\sigma_T^2\to\infty}\gamma_T^* = \lim_{\sigma_T^2\to\infty}\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2} = 0, \quad \text{because } \lim_{\sigma_T^2\to\infty}\eta = \frac{\sigma_Q^4}{\sigma_Q^2+\sigma_S^2}.$$

3. Numerical examples (not shown) verify that, depending on parameters, both γ_T^* and $\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2}$ can either be locally increasing or decreasing in σ_Q^2.

   It is easy to verify that $\lim_{\sigma_Q^2\to 0}\frac{\sigma_T^2\sigma_Q^2}{\sigma_B}\gamma_T^* \to 1$. Therefore, $\lim_{\sigma_Q^2\to 0}\gamma_T^* = \lim_{\sigma_Q^2\to 0}\gamma_T^* \times \sqrt{\sigma_Q^2+\sigma_T^2} = \infty$. Taking $\sigma_Q^2\to\infty$, we get $\lim_{\sigma_Q^2\to\infty}\eta = \frac{\sigma_T^4}{\sigma_T^2+\sigma_S^2}$ and so

$$\lim_{\sigma_Q^2\to\infty}\gamma_T^* = \frac{(\sigma_T^2+\sigma_S^2)\sqrt{\frac{\sigma_T^4}{\sigma_T^2+\sigma_S^2}+\sigma_B^2}}{\sigma_T^4},$$

$$\lim_{\sigma_Q^2\to\infty}\gamma_T^*\sqrt{\sigma_Q^2+\sigma_T^2} = \infty. \qquad\qquad Q.E.D.$$

PROOF OF PROPOSITION 8: Restating (11) and (32),

$$\begin{cases} \gamma_T^{\text{FD}} = \dfrac{\sigma_Q^2}{(\sigma_Q^2+\sigma_T^2)\sqrt{\sigma_B^2+\eta}}, \\[2ex] \gamma_T^{\text{FD}}\sqrt{\sigma_Q^2+\sigma_T^2} = \dfrac{\sigma_Q^2}{\sqrt{\sigma_Q^2+\sigma_T^2}\sqrt{\sigma_B^2+\eta}}, \end{cases}$$

$$\text{for } \eta = \frac{\sigma_Q^4\sigma_T^4}{(\sigma_Q^2+\sigma_T^2)(\sigma_Q^2\sigma_T^2+\sigma_Q^2\sigma_S^2+\sigma_T^2\sigma_S^2)}.$$

1. The parameter σ_S^2 appears only in γ_T^{FD} through η. Routine differentiation shows that $\frac{d\gamma_T^{\text{FD}}}{d\eta} < 0$ and $\frac{d\eta}{d\sigma_S^2} < 0$, and so by the chain rule, $\frac{d\gamma_T^{\text{FD}}}{d\sigma_S^2} > 0$.

Taking limits,

$$\lim_{\sigma_S^2 \to 0} \gamma_T^{\text{FD}} = \frac{\sigma_Q^2}{\left(\sigma_Q^2 + \sigma_T^2\right)\sqrt{\dfrac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2} + \sigma_B^2}}, \quad \text{because} \ \lim_{\sigma_S^2 \to 0} \eta = \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2},$$

$$\lim_{\sigma_S^2 \to \infty} \gamma_T^{\text{FD}} = \frac{\sigma_Q^2}{\sigma_B\left(\sigma_Q^2 + \sigma_T^2\right)}, \quad \text{because} \ \lim_{\sigma_S^2 \to \infty} \eta = 0.$$

From the proof of Proposition 4, $\lim_{\sigma_S^2 \to 0} \gamma_T^* = \frac{1}{\sigma_T^2}\sqrt{\frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2} + \sigma_B^2}$. On the other hand, $\lim_{\sigma_S^2 \to 0} \gamma_T^{\text{FD}}$ can be written as

$$\lim_{\sigma_S^2 \to 0} \gamma_T^{\text{FD}} = \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2\left(\sigma_B^2 + \dfrac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2}\right)}\sqrt{\frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2} + \sigma_B^2}.$$

Observing that $\dfrac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2(\sigma_B^2 + \frac{\sigma_Q^2 \sigma_T^2}{\sigma_Q^2 + \sigma_T^2})} = \dfrac{1}{\sigma_T^2 + \frac{\sigma_B^2}{\sigma_Q^2}(\sigma_Q^2 + \sigma_T^2)} < \dfrac{1}{\sigma_T^2}$, we see that $\gamma_T^{\text{FD}} < \gamma_T^*$.

2. The value η remains constant as we vary σ_B^2. Taking the derivative of γ_T^{FD} with respect to σ_B^2 gives

$$-\frac{\sigma_Q^2}{2\left(\sigma_Q^2 + \sigma_T^2\right)\left(\eta + \sigma_B^2\right)^{\frac{3}{2}}} < 0.$$

Taking limits,

$$\lim_{\sigma_B^2 \to 0} \gamma_{\text{FD}}^* = \frac{\sigma_Q^2}{\left(\sigma_Q^2 + \sigma_T^2\right)\sqrt{\eta}},$$

$$\lim_{\sigma_B^2 \to \infty} \gamma_T^{\text{FD}} = 0.$$

From the proof of Proposition 4, we see that $\lim_{\sigma_B^2 \to 0} \gamma_{\text{FD}}^* = \lim_{\sigma_B^2 \to 0} \gamma_T^*$.

3. The parameter k does not appear in the formula for γ_T^{FD}.

4. Taking $\gamma_T^{\text{FD}} = \frac{\sigma_Q^2}{(\sigma_Q^2 + \sigma_T^2)\sqrt{\sigma_B^2 + \eta}}$ as a function of σ_Q^2, σ_B^2, σ_T^2, and η, we can write $\frac{d\gamma_T^{\text{FD}}}{d\sigma_T^2}$ as $\frac{\partial \gamma_T^{\text{FD}}}{\partial \sigma_T^2} + \frac{\partial \gamma_T^{\text{FD}}}{\partial \eta}\frac{d\eta}{d\sigma_T^2}$. It is easy to confirm that $\frac{\partial \gamma_T^{\text{FD}}}{\partial \sigma_T^2} < 0$, $\frac{\partial \gamma_T^{\text{FD}}}{\partial \eta} < 0$, and $\frac{d\eta}{d\sigma_T^2} > 0$. Therefore, $\frac{d\gamma_T^{\text{FD}}}{d\sigma_T^2} < 0$.

Taking $\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2} = \frac{\sigma_Q^2}{\sqrt{\sigma_Q^2 + \sigma_T^2}\sqrt{\sigma_B^2 + \eta}}$ as a function of σ_Q^2, σ_B^2, σ_T^2, and η, we can write $\frac{d(\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2})}{d\sigma_T^2}$ as $\frac{\partial(\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2})}{\partial \sigma_T^2} + \frac{\partial(\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2})}{\partial \eta}\frac{d\eta}{d\sigma_T^2}$. Once again, $\frac{\partial(\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2})}{\partial \sigma_T^2} < 0$, $\frac{\partial(\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2})}{\partial \eta} < 0$, and $\frac{\partial \eta}{\partial \sigma_T^2} > 0$. Therefore, $\frac{d(\gamma_T^{\text{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2})}{d\sigma_T^2} < 0$.

Taking limits,

$$\lim_{\sigma_T^2 \to 0} \gamma_T^{\mathrm{FD}} = \frac{1}{\sigma_B}, \quad \text{because } \lim_{\sigma_T^2 \to 0} \eta = 0,$$

$$\lim_{\sigma_T^2 \to \infty} \gamma_T^{\mathrm{FD}} = 0, \quad \text{because } \lim_{\sigma_T^2 \to \infty} \eta = \frac{\sigma_Q^4}{\sigma_Q^2 + \sigma_S^2},$$

$$\lim_{\sigma_T^2 \to 0} \gamma_T^{\mathrm{FD}} \sqrt{\sigma_Q^2 + \sigma_T^2} = \frac{\sigma_Q}{\sigma_B}, \quad \text{because } \lim_{\sigma_T^2 \to 0} \eta = 0,$$

$$\lim_{\sigma_T^2 \to \infty} \gamma_T^{\mathrm{FD}} \sqrt{\sigma_Q^2 + \sigma_T^2} = 0, \quad \text{because } \lim_{\sigma_T^2 \to \infty} \eta = \frac{\sigma_Q^4}{\sigma_Q^2 + \sigma_S^2}.$$

5. Taking $\gamma_T^{\mathrm{FD}} = \frac{\sigma_Q^2}{(\sigma_Q^2 + \sigma_T^2)\sqrt{\sigma_B^2 + \eta}}$ as a function of σ_Q^2, σ_B^2, σ_T^2, and η, we can write $\frac{d\gamma_T^{\mathrm{FD}}}{d\sigma_Q^2}$ as

$$
\begin{aligned}
\frac{d\gamma_T^{\mathrm{FD}}}{d\sigma_Q^2} &= \frac{\partial \gamma_Q^{\mathrm{FD}}}{\partial \sigma_Q^2} + \frac{\partial \gamma_T^{\mathrm{FD}}}{\partial \eta} \cdot \frac{d\eta}{d\sigma_Q^2} \\[2mm]
&= \frac{\sigma_T^2}{\left(\sigma_Q^2 + \sigma_T^2\right)^2 \sqrt{\sigma_B^2 + \eta}} \\[2mm]
&\quad - \frac{\sigma_Q^2}{2\left(\sigma_Q^2 + \sigma_T^2\right)\left(\sigma_B^2 + \eta\right)^{\frac{3}{2}}} \cdot \frac{\sigma_Q^2 \sigma_T^6\left(2\sigma_Q^2\sigma_S^2 + \sigma_Q^2\sigma_T^2 + 2\sigma_S^2\sigma_T^2\right)}{\left(\sigma_Q^2 + \sigma_T^2\right)^2\left(\sigma_Q^2\sigma_S^2 + \sigma_Q^2\sigma_T^2 + \sigma_S^2\sigma_T^2\right)} \\[2mm]
&= \frac{\sigma_T^2}{\left(\sigma_Q^2 + \sigma_T^2\right)^2 \sqrt{\sigma_B^2 + \eta}} \\[2mm]
&\quad - \frac{\sigma_Q^2}{2\left(\sigma_Q^2 + \sigma_T^2\right)\left(\sigma_B^2 + \eta\right)^{\frac{3}{2}}} \cdot \frac{\eta \sigma_T^2\left(2\sigma_Q^2\sigma_S^2 + \sigma_Q^2\sigma_T^2 + 2\sigma_S^2\sigma_T^2\right)}{\sigma_Q^2\left(\sigma_Q^2 + \sigma_T^2\right)\left(\sigma_Q^2\sigma_S^2 + \sigma_Q^2\sigma_T^2 + \sigma_S^2\sigma_T^2\right)} \\[2mm]
&= \frac{\sigma_T^2}{\left(\sigma_Q^2\sigma_T^2\right)^2\left(\sigma_B^2 + \eta\right)^{\frac{3}{2}}}\left(\sigma_B^2 + \eta - \eta \cdot \frac{2\sigma_Q^2\sigma_S^2 + \sigma_Q^2\sigma_T^2 + 2\sigma_S^2\sigma_T^2}{2\left(\sigma_Q^2\sigma_S^2 + \sigma_Q^2\sigma_T^2 + \sigma_S^2\sigma_T^2\right)}\right) > 0.
\end{aligned}
$$

And without doing more algebra, if γ_T^{FD} is positive and increasing in σ_Q^2, and if $\sqrt{\sigma_Q^2 + \sigma_T^2}$ is positive and increasing in σ_Q^2, then clearly their product $\gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2}$ is increasing in σ_Q^2.

Taking limits, as $\sigma_Q^2 \to 0$ it is easy to see from the above formulas that $\lim_{\sigma_Q^2 \to 0} \gamma_T^{\mathrm{FD}} = \lim_{\sigma_Q^2 \to 0} \gamma_T^{\mathrm{FD}}\sqrt{\sigma_Q^2 + \sigma_T^2} = 0$. As $\sigma_Q^2 \to \infty$, we have $\eta \to \frac{\sigma_T^4}{\sigma_T^2 + \sigma_S^2}$ and

so

$$\lim_{\sigma_Q^2 \to \infty} \gamma_T^{\mathrm{FD}} = \frac{1}{\sqrt{\sigma_B^2 + \dfrac{\sigma_T^4}{\sigma_T^2 + \sigma_S^2}}},$$

$$\lim_{\sigma_Q^2 \to \infty} \gamma_T^{\mathrm{FD}} \sqrt{\sigma_Q^2 + \sigma_T^2} = \infty.$$

We can rewrite $\lim_{\sigma_Q^2 \to \infty} \gamma_T^{\mathrm{FD}}$ as $\dfrac{(\sigma_Q^2 + \sigma_T^2)\sqrt{\sigma_B^2 + \frac{\sigma_T^4}{\sigma_T^2 + \sigma_S^2}}}{\sigma_T^4 + \sigma_B^2(\sigma_T^2 + \sigma_S^2)}$, which is less than $\lim_{\sigma_Q^2 \to \infty} \gamma_T^* = $

$\dfrac{(\sigma_T^2 + \sigma_S^2)\sqrt{\sigma_B^2 + \frac{\sigma_T^4}{\sigma_T^2 + \sigma_S^2}}}{\sigma_T^4}$ from the proof of Proposition 4 part 3. *Q.E.D.*

PROOF OF PROPOSITION 9: Follows from arguments in the text. *Q.E.D.*

PROOF OF PROPOSITION 10: Given the notation that has been introduced, all of the arguments follow exactly as in Section 4 and Proposition 5. *Q.E.D.*

PROOF OF LEMMA 5: 1. The optimal period-2 contract under common knowledge of the agent's type sets the average test score to some value κ^*. I first show how to derive κ^* in terms of the period-1 outcome.

As a preliminary step, recall that in the notation of Section 4.2, conditional on any agent type, the payoffs from any set of accepted applicants are determined by the average test score τ and the average z-score ζ. Since the agent accepts k applicants, and applicant test scores have an unconditional distribution that is normal with mean 0 and variance $\sigma_Q^2 + \sigma_T^2$, the range of possible τ is $[-R_T, R_T]$, for $R_T = \sqrt{\sigma_Q^2 + \sigma_T^2} R(k)$, as in (22) and (24). Now let $\bar{\zeta}(\tau)$ be the highest possible ζ at an average test score τ, from Lemma 3, plugging in R_T and R_Z in terms of $R(k)$:

$$\bar{\zeta}(\tau) \equiv \sqrt{R(k)^2 - \frac{\tau^2}{(\sigma_Q^2 + \sigma_T^2)}}.$$

Now suppose that the agent is given full discretion to hire her favorite set of applicants in period 1 and she acts myopically. The average test score $\tau_1 \in [-R_T, R_T]$ is observable to the principal. The average z-score is not directly observable, but the principal can infer that—since the agent's payoff increases in ζ—the average z-score must have been the highest possible level consistent with τ_1, that is, $\zeta_1 = \bar{\zeta}(\tau_1)$.

If the principal knows the agent type θ, and therefore the induced quantity $\sigma_{U_P}(\theta)$, then the principal's preferences over (τ, ζ) are given by (21). The principal's optimal contract specifies that $\tau_2 = \kappa^*$, where κ^* is the τ component of the pair (τ, ζ) that optimizes (21). Hence, κ^* solves

$$\kappa^* = \arg\max_{\tau} \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} \tau + \sigma_{U_P}(\theta)\bar{\zeta}(\tau)$$

$$\Rightarrow \quad 0 = \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2} + \sigma_{U_P}(\theta)\bar{\zeta}'(\kappa^*) \tag{54}$$

$$\Rightarrow \quad \kappa^* = \frac{R(k)\sigma_Q^2\sqrt{\sigma_Q^2 + \sigma_T^2}}{\sqrt{\sigma_Q^2 + \sigma_{U_P}^2(\theta)}}.$$

Of course, $\sigma_{U_P}(\theta)$ depends on the agent's type, which the principal is trying to learn from the data.[49] But the principal knows his payoff from the first-period choices—this is exactly the average quality level ξ_1. So the principal can plug τ_1 and ξ_1 into (21) (with $V_P = \xi_1$ and $\zeta = \bar{\zeta}(\tau_1)$) to infer $\sigma_{U_P}(\theta)$:

$$\xi_1 = \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}\tau_1 + \sigma_{U_P}(\theta)\bar{\zeta}(\tau_1)$$

$$\Rightarrow \quad \sigma_{U_P}(\theta) = \frac{\xi_1(\sigma_Q^2 + \sigma_T^2) - \sigma_Q^2\tau_1}{\sqrt{(\sigma_Q^2 + \sigma_T^2)((\sigma_Q^2 + \sigma_T^2)R(k)^2 - \tau_1^2)}}. \tag{55}$$

Now plug this value of $\sigma_{U_P}(\theta)$ into (54) to get (43), that is,

$$\kappa^* = \frac{R(k)\sigma_Q^2\sqrt{\sigma_Q^2 + \sigma_T^2}}{\sqrt{\sigma_Q^2 + \dfrac{\left(\xi_1(\sigma_Q^2 + \sigma_T^2) - \sigma_Q^2\tau_1\right)^2}{(\sigma_Q^2 + \sigma_T^2)((\sigma_Q^2 + \sigma_T^2)R(k)^2 - \tau_1^2)}}}.$$

The optimal contract in the second period lets the agent accept any k applicants she wants, subject to requiring the period-2 average test score to be κ^* in (43).[50]

2. Now consider the comparative statics on κ^* with respect to τ_1 and ξ_1.

We know that τ_1 can be any value in $[0, R_T]$, with $R_T = \sqrt{\sigma_Q^2 + \sigma_T^2} \cdot R(k)$ for $R(k)$ in (22). Let us bound the range of ξ_1 consistent with an observed τ_1.

The principal's payoff in the first period, ξ_1, is equal to $\frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}\tau_1 + \sigma_{U_P}(\theta)\bar{\zeta}(\tau_1)$ from (21). And $\sigma_{U_P}(\theta)$ must be in the range $(0, \sigma_{U_A}(\theta))$ from Lemma 2 part 3.[51] So, given τ_1,

$$\frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}\tau_1 < \xi_1 < \frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}\tau_1 + \sigma_{U_A}(\theta)\bar{\zeta}(\tau_1).$$

Moreover, $\sigma_{U_A}(\theta)$ can be inferred from τ_1: the model predicts that the agent has chosen τ_1 to maximize (20) over τ, with $\zeta = \bar{\zeta}(\tau)$. Taking the first-order condition

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[49]Note that one could also solve for the optimal contract even if the observable parameters $(k, \sigma_Q^2, \sigma_T^2)$ were to change from period 1 to 2. But one would no longer plug in the period-1 value of $\sigma_{U_P}(\theta)$ into the period-2 payoff expression, since $\sigma_{U_P}(\theta)$ depends on σ_T^2 and σ_Q^2.

[50]One could also solve for the optimal contract even if the observable parameters $(k, \sigma_Q^2, \sigma_T^2)$ were to change from period 1 to 2. But one would not simply plug in the period-1 value of $\sigma_{U_P}(\theta)$ from (55) into the period-2 expression (54). The value of $\sigma_{U_P}(\theta)$ depends on σ_T^2 and σ_Q^2, which might change from period to period.

[51]The upper bound here need not be tight, depending on parameters.

and solving for $\sigma_{U_A}(\theta)$ gives

$$\sigma_{U_A}(\theta) = \frac{\sigma_Q^2}{\tau_1}\sqrt{R(k)^2 - \frac{\tau_1^2}{\sigma_Q^2 + \sigma_T^2}}.$$

Plugging this value of $\sigma_{U_A}(\theta)$ along with $\bar{\zeta}(\tau_1)$ into the above sequence of inequalities, we get (after some simplification)

$$\frac{\sigma_Q^2}{\sigma_Q^2 + \sigma_T^2}\tau_1 < \xi_1 < \frac{\sigma_Q^2 R(k)}{\tau_1}. \tag{56}$$

Now, return to the comparative statics of κ^* given by (43). The value κ^* moves in the opposite direction as the fraction $\frac{(\xi_1(\sigma_Q^2 + \sigma_T^2) - \sigma_Q^2\tau_1)^2}{(\sigma_Q^2 + \sigma_T^2)((\sigma_Q^2 + \sigma_T^2)R(k)^2 - \tau_1^2)}$ as we vary ξ_1 or τ_1. And it is immediate that the fraction is increasing in ξ_1 as long as $\xi_1(\sigma_Q^2 + \sigma_T^2) - \sigma_Q^2\tau_1 > 0$, which holds by the left inequality of (56). Next, differentiating, it is straightforward to show that the sign of the derivative of the fraction with respect to τ_1 is equal to the sign of $(\xi_1(\sigma_Q^2 + \sigma_T^2) - \sigma_Q^2\tau_1)(\tau_1\xi_1 - \sigma_Q^2 R(k)^2)$. The first parenthetical term is positive, as just described; the second parenthetical term is negative by the second inequality in (56). So κ^* decreases in ξ_1 and increases in τ_1.   *Q.E.D.*

PROOF OF LEMMA 6: From the specified joint distributions of Q and T, it follows that $\mathrm{Var}(Q) = \sigma_Q^2$, $\mathrm{Var}(T) = \sigma_Q^2 + \sigma_T^2$, and $\mathrm{Cov}(T, Q) = \sigma_Q^2$. It remains to calculate $\mathrm{Var}(U_A)$, $\mathrm{Cov}(T, U_A)$, and $\mathrm{Cov}(Q, U_A)$.

It will be helpful to note as well that $\mathrm{Cov}(S, Q) = \sigma_Q^2$, $\mathrm{Cov}(S, T) = \sigma_Q^2$, and $\mathrm{Var}(S) = \sigma_S^2 + \sigma_Q^2$. The bias term B has variance σ_B^2, and has 0 covariance with S, T, or Q.

From (7),

$$U_A = \mathbb{E}[Q|T, S] + B = \frac{\dfrac{T}{\sigma_T^2} + \dfrac{S}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} + B.$$

$\mathrm{Cov}(q, \tilde{u}_A)$ is given by

$$\mathrm{Cov}(Q, U_A) = \frac{\dfrac{\mathrm{Cov}(Q,T)}{\sigma_T^2} + \dfrac{\mathrm{Cov}(Q,S)}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} = \frac{\dfrac{\sigma_Q^2}{\sigma_T^2} + \dfrac{\sigma_Q^2}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} = \frac{\sigma_Q^4(\sigma_S^2 + \sigma_T^2)}{\sigma_Q^2\sigma_T^2 + \sigma_Q^2\sigma_S^2 + \sigma_T^2\sigma_S^2}.$$

$\mathrm{Cov}(T, U_A)$ is given by

$$\mathrm{Cov}(T, U_A) = \frac{\dfrac{\mathrm{Var}(T)}{\sigma_T^2} + \dfrac{\mathrm{Cov}(T,S)}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} = \frac{\dfrac{\sigma_Q^2 + \sigma_T^2}{\sigma_T^2} + \dfrac{\sigma_Q^2}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} = \sigma_Q^2\frac{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} = \sigma_Q^2.$$

And finally, $\text{Var}(U_A)$ is given by

$$\text{Var}(U_A) = \frac{\dfrac{\text{Var}(T)}{\sigma_T^4} + \dfrac{\text{Var}(S)}{\sigma_S^4} + 2\dfrac{\text{Cov}(T,S)}{\sigma_T^2 \sigma_S^2}}{\left(\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}\right)^2} + \sigma_B^2 = \frac{\dfrac{\sigma_Q^2 + \sigma_T^2}{\sigma_T^4} + \dfrac{\sigma_Q^2 + \sigma_S^2}{\sigma_S^4} + 2\dfrac{\sigma_Q^2}{\sigma_T^2 \sigma_S^2}}{\left(\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}\right)^2} + \sigma_B^2$$

$$= \sigma_Q^2 \frac{\left(\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}\right)\left(\dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}\right)}{\left(\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}\right)^2} + \sigma_B^2 = \sigma_Q^2 \frac{\dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}}{\dfrac{1}{\sigma_Q^2} + \dfrac{1}{\sigma_T^2} + \dfrac{1}{\sigma_S^2}} + \sigma_B^2$$

$$= \frac{\sigma_Q^4\left(\sigma_S^2 + \sigma_T^2\right)}{\sigma_Q^2 \sigma_T^2 + \sigma_Q^2 \sigma_S^2 + \sigma_T^2 \sigma_S^2} + \sigma_B^2. \qquad\qquad Q.E.D.$$

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