Econometrica Supplementary Material

SUPPLEMENT TO “DYNAMIC SPATIAL PANEL MODELS, COMMON SHOCKS, AND SEQUENTIAL EXOGENEITY”

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This supplemental appendix provides certain proofs for the results related to the examples in Section 2, a generalized version of Theorem 1, additional detail for the proofs in the main text, and an extension of the generalized Helmert transformation to multiple factors.

APPENDIX D: SUPPLEMENTAL MATERIAL

D.1. *Supplemental Material for the Example*

IN THE FOLLOWING SECTION we provide proofs for the lemmata for the example in Section 2.

PROOF OF LEMMA EX1: Confirming with $W_1^+ = [\overline{y}_1^+, Z_1^+]$, we define the partitions

$$\Gamma'_{HW} = \begin{pmatrix} \Gamma'_{HMy} \\ \Gamma'_{HZ} \end{pmatrix}, \qquad \Gamma_{WA_ru} = \begin{pmatrix} \Gamma_{yMA_ru} \\ \Gamma_{ZA_ru} \end{pmatrix}, \qquad \Gamma_{WA_rW} = \begin{pmatrix} \Gamma_{yMA_rMy} & \Gamma_{yMA_rZ} \\ \Gamma_{ZA_rMy} & \Gamma_{ZA_rZ} \end{pmatrix}.$$

Observe that $u_1^+(\delta) = y_1^+ - W_1^+\delta = u_1^+ + W_1^+(\delta_0 - \delta)$, and let $\overline{m}_l(\delta) = \operatorname{plim} n^{-1/2}\overline{m}_{n,l}(\delta)$ and $\overline{m}_{\mathrm{q}}(\delta) = \operatorname{plim} n^{-1/2}\overline{m}_{n,\mathrm{q}}(\delta)$. Then utilizing Assumption EX, the limiting objective function of the GMM estimator defined by (6) is given by

$$Q(\delta) = Q_{\mathrm{l}}(\delta) + Q_{\mathrm{q}}(\delta), \qquad Q_{\mathrm{l}}(\delta) = \overline{m}_l(\delta)'\big(V^h\big)^{-1}\overline{m}_l(\delta),$$
$$Q_{\mathrm{q}}(\delta) = \overline{m}_{\mathrm{q}}(\delta)'\big(V^a\big)^{-1}\overline{m}_{\mathrm{q}}(\delta),$$

with

$$\begin{aligned}\overline{m}_l(\delta) &= \operatorname{plim} n^{-1}H'u_1^+(\delta) = \operatorname{plim} n^{-1}H'u_1^+ + \operatorname{plim} n^{-1}H'W_1^+(\delta_0 - \delta) \\ &= \Gamma_{HW}(\delta_0 - \delta) \end{aligned} \tag{D.1}$$

and with the rth element of $\overline{m}_q(\delta)$ given by

$$\begin{aligned}\overline{m}_{q,r}(\delta) &= \operatorname{plim} n^{-1}u_1^+(\delta)'A^r u_1^+(\delta) = \operatorname{plim} n^{-1}u_1^{+\prime}A^r u_1^+ \\ &\quad + 2(\delta_0 - \delta)' \operatorname{plim} n^{-1}W_1^{+\prime}A^r u_1^+ + (\delta_0 - \delta)' \operatorname{plim} n^{-1}W_1^{+\prime}A^r W_1^+(\delta_0 - \delta) \\ &= 2(\delta_0 - \delta)'\Gamma_{WA_ru} + (\delta_0 - \delta)'\Gamma_{WA_rW}(\delta_0 - \delta). \end{aligned} \tag{D.2}$$

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Observe that $Q_l(\delta) \geq 0$, $Q_q(\delta) \geq 0$, and $Q(\delta) \geq 0$ with $Q_l(\delta_0) = Q_q(\delta_0) = Q(\delta_0) = 0$, since $\overline{m}_l(\delta_0) = 0$ and $\overline{m}_q(\delta_0) = 0$. Thus any minimizer, say δ_*, of $Q(\delta)$ not only satisfies $\partial Q(\delta_*)/\partial\delta = 0$, but also $\partial Q_l(\delta_*)/\partial\delta = 0$ and $\partial Q_q(\delta_*)/\partial\delta = 0$.

Case 1: Γ_{HW} has full column rank. In this case, $\delta = \delta_0$ is obviously the unique solution of $\overline{m}_l(\delta) = \Gamma_{HW}(\delta_0 - \delta) = 0$. Of course, since

$$\frac{\partial Q_l(\delta)}{\partial\delta} = 2\overline{m}_l(\delta)'\left(V^h\right)^{-1}\frac{\partial\overline{m}_l(\delta)}{\partial\delta} = -2\overline{m}_l(\delta)'\left(V^h\right)^{-1}\Gamma_{HW},$$

it follows further that $\delta = \delta_0$ is also the unique solution of $\partial Q_l(\delta)/\partial\delta = 0$ and of $\partial Q(\delta)/\partial\delta = 0$. Thus, in this case the unknown parameters can be identified solely from the linear moment condition. Within the context of a cross-sectional model and a general M matrix, an assumption analogous to that of assuming that Γ_{HW} has full column rank was maintained in Kelejian and Prucha (1998) and subsequent papers on instrumental variable estimators for spatial network models.

Case 2: Γ_{HZ} has full column rank, but $\Gamma_{HW} = [\Gamma_{HMy}, \Gamma_{HZ}]$ does not have full column rank. In this case, Γ_{HMy} must be contained in the linear span of Γ_{HZ}; that is, $\Gamma_{HMy} = \Gamma_{HZ}c$ for some vector c. Since Γ_{HZ} has full column rank by assumption, the vector c is unique and can be expressed as

$$c = \left(\Gamma'_{HZ}\Gamma_{HZ}\right)^{-1}\Gamma'_{HZ}\Gamma_{HMy}.$$

Observe that

$$Q_l(\delta) = (\delta_0 - \delta)'\Gamma'_{HW}\left(V^h\right)^{-1}\Gamma_{HW}(\delta_0 - \delta) = (\lambda_0 - \lambda)^2\Gamma'_{HMy}\left(V^h\right)^{-1}\Gamma_{HMy}$$
$$+ 2(\lambda_0 - \lambda)\Gamma'_{HMy}\left(V^h\right)^{-1}\Gamma_{HZ}(\beta_0 - \beta) + (\beta_0 - \beta)'\Gamma'_{HZ}\left(V^h\right)^{-1}\Gamma_{HZ}(\beta_0 - \beta).$$

Clearly

$$\partial Q_l(\delta)/\partial\lambda = -2(\lambda_0 - \lambda)\Gamma'_{HMy}\left(V^h\right)^{-1}\Gamma_{HMy} - (\beta_0 - \beta)'\Gamma'_{HZ}\left(V^h\right)^{-1}\Gamma_{HMy},$$
$$\partial Q_l(\delta)/\partial\beta = -2(\lambda_0 - \lambda)\Gamma'_{HMy}\left(V^h\right)^{-1}\Gamma_{HZ} - 2(\beta_0 - \beta)'\Gamma'_{HZ}\left(V^h\right)^{-1}\Gamma_{HZ}.$$

Utilizing that $\Gamma_{HMy} = \Gamma_{HZ}c$, we see that $\partial Q_l(\delta)/\partial\lambda = [\partial Q_l(\delta)/\partial\beta]c$; that is, the first first order condition is linearly dependent on the second set. Given that Γ_{HZ} is assumed to have full column rank, we can, for any given value of λ, solve the first order condition $\partial Q_l(\delta)/\partial\beta = 0$ for $\beta(\lambda)$, which yields

$$\beta(\lambda) - \beta_0 = -\left(\Gamma'_{HZ}\left(V^h\right)^{-1}\Gamma_{HZ}\right)^{-1}\Gamma'_{HZ}\left(V^h\right)^{-1}\Gamma_{HMy}(\lambda - \lambda_0) = -c(\lambda - \lambda_0).$$

Lee (2007, p. 493) obtains an analogous result within the context of a cross-sectional model. From this we see that β is identified if λ is identified. Now λ is identified if $\overline{m}_q(\lambda, \beta(\lambda)) = 0$ has a unique solution at λ_0, which implies that $Q_q(\beta(\lambda_0), \lambda_0)$ has a unique minimum of zero at $\lambda = \lambda_0$. Observe that in light of (D.2),

$$\overline{m}_{q,r}(\delta) = 2(\lambda - \lambda_0)\Gamma_{yMA_ru} + 2(\beta - \beta_0)'\Gamma_{ZA_ru}$$
$$+ (\lambda - \lambda_0)^2\Gamma_{yMA_rMy} + 2(\beta - \beta_0)'\Gamma_{ZA_rMy}(\lambda_0 - \lambda) + (\beta - \beta_0)'\Gamma_{ZA_rZ}(\beta - \beta_0).$$

Then utilizing $\beta(\lambda) - \beta_0 = -c(\lambda - \lambda_0)$ yields

$$\overline{m}_{q,r}\left(\lambda, \beta(\lambda)\right) = (\lambda - \lambda_0)s^*_{1r} + (\lambda - \lambda_0)^2 s^*_{2r},$$

$$s_{r1}^* = 2\Gamma_{yMA_r u} - 2c'\Gamma_{ZA_r u},$$
$$s_{r2}^* = \Gamma_{yMA_r My} - 2c'\Gamma_{ZA_r My} + c'\Gamma_{ZA_r Z}c.$$

Now let S^* be the $q \times 2$ matrix with elements s_{r1}^*, s_{r2}^* in the rth row. Then

$$\overline{m}_{\mathrm{q}}\big(\lambda, \beta(\lambda)\big) = S^*\big[\lambda - \lambda_0, (\lambda - \lambda_0)^2\big]'.$$

Substituting the above expression for c, we get

$$\begin{aligned} s_{r1}^* &= 2\big[\Gamma_{yMA_r u} - \Gamma'_{HMy}\Gamma_{HZ}\big(\Gamma'_{HZ}\Gamma_{HZ}\big)^{-1}\Gamma_{ZA_r u}\big] \\ &= 2\,\mathrm{plim}\, n^{-1}\overline{y}_1^{+\prime}Q'_H A^r u_1^+, \\ s_{r2}^* &= \Gamma_{yMA_r My} - 2\Gamma'_{HMy}\Gamma_{HZ}\big(\Gamma'_{HZ}\Gamma_{HZ}\big)^{-1}\Gamma_{ZA_r My} \\ &\quad + \Gamma'_{HMy}\Gamma_{HZ}\big(\Gamma'_{HZ}\Gamma_{HZ}\big)^{-1}\Gamma_{ZA_r Z}\big(\Gamma'_{HZ}\Gamma_{HZ}\big)^{-1}\Gamma'_{HZ}\Gamma_{HMy} \\ &= \mathrm{plim}\, n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H\overline{y}_1^+, \end{aligned}$$

with $Q_H = I - Z_1^+(Z_1^{+\prime}P_H Z_1^+)^{-1}Z_1^{+\prime}P_H$, where we utilized that $(\Gamma'_{HZ}\Phi\Gamma_{HZ})^{-1}\Gamma'_{HZ}\Phi\Gamma_{HMy} = (\Gamma'_{HZ}\Phi\Gamma_{HZ})^{-1}\Gamma'_{HZ}\Phi\Gamma_{HZ}c = c$ for any nonsingular matrix Φ.

Observing that

$$\begin{aligned} n^{-1}y_1^{+\prime}Q'_H A^r Q_H y_1^+ &= n^{-1}\big[\lambda_0\overline{y}_1^+ + Z_1^+\beta_0 + u_1^+\big]'Q'_H A^r Q_H\big[\lambda_0\overline{y}_1^+ + Z_1^+\beta_0 + u_1^+\big] \\ &= n^{-1}\big[\lambda_0\overline{y}_1^+ + u_1^+\big]'Q'_H A^r Q_H\big[\lambda_0\overline{y}_1^+ + u_1^+\big] \\ &= \lambda_0^2 n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H\overline{y}_1^+ + 2\lambda_0 n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H u_1^+ \\ &\quad + n^{-1}u_1^{+\prime}Q'_H A^r Q_H u_1^+ \\ &= \lambda_0^2 n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H\overline{y}_1^+ + 2\lambda_0 n^{-1}\overline{y}_1^{+\prime}Q'_H A^r u_1^+ + n^{-1}u_1^{+\prime}A^r u_1^+ + o_p(1) \\ &\xrightarrow{p} \lambda_0^2 s_{r2}^* + \lambda_0 s_{r1}^* \\ n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H u_1^+ &= n^{-1}\overline{y}_1^{+\prime}Q'_H A^r u_1^+ + o_p(1) \\ &\xrightarrow{p} s_{r1}^*/2, \\ n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H y_1^+ &= n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H\big[\lambda_0\overline{y}_1^+ + Z_1^+\beta_0 + u_1^+\big] \\ &= \lambda_0 n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H\overline{y}_1^+ + n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H u_1^+ \\ &\xrightarrow{p} \lambda_0 s_{r2}^* + s_{r1}^*/2, \end{aligned}$$

it follows that

$$\begin{aligned} & n^{-1}\big(y_1^+ - \lambda\overline{y}_1^+\big)'Q'_H A^r Q_H\big(y_1^+ - \lambda\overline{y}_1^+\big) \\ &= n^{-1}y_1^{+\prime}Q'_H A^r Q_H y_1^+ - 2\lambda n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H y_1^+ + \lambda^2 n^{-1}\overline{y}_1^{+\prime}Q'_H A^r Q_H\overline{y}_1^+ \\ &\xrightarrow{p} \lambda_0^2 s_{r2}^* + \lambda_0 s_{r1}^* - 2\lambda\lambda_0 s_{r2}^* - \lambda s_{r1}^* + \lambda^2 s_{r2}^* \\ &= (\lambda_0 - \lambda)s_{r1}^* + (\lambda_0 - \lambda)^2 s_{r2}^* = \overline{m}_{\mathrm{q},r}\big(\lambda, \beta(\lambda)\big). \end{aligned}$$

Let $S = [S_1', \dots, S_q']'$ with $S_r = [s_{r1}^*/2 + \lambda_0 s_{r2}^*, s_{r2}^*]$. Then

$$S = S^* \begin{pmatrix} 1/2 & 0 \\ \lambda_0 & 1 \end{pmatrix}.$$

From the latter we see that S has full column rank if and only if S^* has full column rank. *Q.E.D.*

PROOF OF LEMMA EX2: The lemma assumes for the network formation process that the elements of u are i.i.d. with $E[u_{it}] = 0$, $\mathrm{Var}(u_{it}) = 1$, conditional on z_1^1, z_2^1, τ, υ_1, υ_2, μ, ν. As discussed, in this case we may think of the matrix ζ_t to contain τ (or the subset of strictly exogenous variables not already included in z_t^1) as well as υ_t and ν. Recall that $m_{ij,1} = m_{ij,1}(\tau, \upsilon_1, \mu, \nu)$ and $m_{ij,2} = m_{ij,2}(\tau, \upsilon_1, \upsilon_2, \mu, \nu)$, and thus we can also think of ζ_t to contain M_t. With either interpretation, the conditioning information set is given by $\mathfrak{G} = \sigma(z_1, z_2, \mu)$, and M_1 and M_2 as well as h^r and A^r are measurable w.r.t. $\mathfrak{G}$. By (4) we have $E[n^{-1}h^{r\prime}u_1^+] = 0$ and $E[n^{-1}u_1^{+\prime}A^r u_1^+] = 0$. By Theorem 1, and replacing the conditioning information set by $\mathfrak{G}$, we have $\mathrm{Var}(n^{-1}h^{r\prime}u_1^+ \mid \mathfrak{G}) = n^{-2}h^{r\prime}h^r$ and $\mathrm{Var}(n^{-1}u_1^{+\prime}A^r u_1^+ \mid \mathfrak{G}) = 2n^{-2}\,\mathrm{tr}(A^r A^r)$. Observing that $E[n^{-1}h^{r\prime}u_1^+ \mid \mathfrak{G}] = 0$ and $E[n^{-1}u_1^{+\prime}A^r u_1^+ \mid \mathfrak{G}] = 0$, and using iterated expectations yields

$$\mathrm{Var}\big(n^{-1}h^{r\prime}u_1^+\big) = n^{-2}E\big[h^{r\prime}h^r\big] = n^{-2}\sum_{i=1}^n E\big[h_{ir}^2\big] \le n^{-2}\sum_{i=1}^n \|h_{ir}\|_{2+\delta}^2 \le n^{-1}K_h^2 \to 0,$$

$$\mathrm{Var}\big(n^{-1}u_1^{+\prime}A^r u_1^+\big) = n^{-2}2E\big[\mathrm{tr}\big(A^r A^r\big)\big] = n^{-2}2E\left[\sum_{i=1}^n\sum_{j=1}^n \big(a_{ij}^r\big)^2\right]$$

$$\le 2K_a n^{-2}E\left[\sum_{i=1}^n\sum_{j=1}^n \big|a_{ij}^r\big|\right] \le n^{-1}2K_a^2 \to 0.$$

Consequently $n^{-1}h^{r\prime}u_1^+ = o_p(1)$ and $n^{-1}u_1^{+\prime}A^r u_1^+ = o_p(1)$ by Chebychev's inequality, which proves part (a) of the lemma.

We next prove part (b) of the lemma. Recall that $M_t = (m_{ij,t})$ with $m_{ij,t} = d_{ij,t}/(\sum_{j=1}^N d_{ij,t})$. Let e be an $n \times 1$ vector of 1s. Then clearly $M_t e = e$ and $\sum_{j=1}^m |m_{ij,t}| = \sum_{j=1}^m m_{ij,t} = 1$. Furthermore, since $\sum_{j=1}^n 1\{s_{ij} \le c\} \le K < \infty$, we have $\sum_{i=1}^n m_{ij,t} \le K$, as shown in the proof of Lemma D.2 Let $B = (b_{ij})$ be a matrix of the form M_t^τ, $M_t^{\tau\prime}$, or $\bar{M}_t^\tau$ with $\bar{M}_t = (M_t + M_t')/2$ or more generally of the form $M_t^\sigma M_t^{\tau-\sigma\prime}$ or $\bar{M}_t^\sigma \bar{M}_t^{\tau-\sigma\prime}$ with $0 \le \sigma \le \tau$, $\tau \le s$, and $t = 1, 2$. Then it follows immediately from, for example, Kelejian and Prucha (1999, footnote 20), that $\sum_{j=1}^n |b_{ij}| \le K_b$ and $\sum_{i=1}^n |b_{ij}| \le K_b$ for some $K_b < \infty$. Let $A = B - diag(B)$. Then $\sum_{j=1}^n |a_{ij}| \le K_b$ and $\sum_{i=1}^n |a_{ij}| \le K_b$. This establishes that under the maintained assumptions, the second sufficient condition of part (a) of the lemma holds.

Suppose further that $\sum_{j=1}^n (\Pr(s_{ij} \le c))^{1/[s(2+\delta)]} \le K < \infty$. Then by Lemma D.2 and Lyapunov's inequality, $\sum_{i=1}^n \|m_{ij,t}\|_{\tau(2+\delta)} \le K$ and $\sum_{j=1}^n \|m_{ij,t}\|_{\tau(2+\delta)} \le K$ for $\tau \le s$. Let B and A be a matrix of the form defined above. Then by Lemma D.1 and the remark after that lemma, we have $\sum_{j=1}^n \|b_{ij}\|_{2+\delta} \le K^\tau$, $\sum_{i=1}^n \|b_{ij}\|_{2+\delta} \le K^\tau$ and $\sum_{j=1}^n \|a_{ij}\|_{2+\delta} \le K^\tau$, $\sum_{i=1}^n \|a_{ij}\|_{2+\delta} \le K^\tau$. This establishes the last claim of part (b) of the lemma.

To verify the claims after the lemma, suppose that $\Pr(s_{ij} \le c) = 0$ implies $1\{s_{ij} \le c\} = 0$ and $\sum_{j=1}^{n} 1\{\Pr(s_{ij} \le c) > 0\} \le K$. Then it follows from Lemma D.2 that $\sum_{j=1}^{n} 1\{s_{ij} \le c\} \le K < \infty$ and $\sum_{j=1}^{n} (\Pr(s_{ij} \le c))^{1/[s(2+\delta)]} \le K < \infty$ for all s. Consequently, for matrices B and A of the above form the $L_{2+\delta}$ summability now holds for arbitrary τ.

Let $\mathfrak{z}_t$ be an arbitrary column of z_t^1 with typical element $\mathfrak{z}_{j,t}$, let $B = M_t^r$, and let $h = (h_i) = M_t^r \mathfrak{z}_t$. Then $h_i = \sum_{j=1}^{n} b_{ij,t} \mathfrak{z}_{j,t}$. By assumption $\|\mathfrak{z}_{j,t}\|_{4+\delta_*} \le K_z$ for some $\delta_* > 0$. Let $p = 2 + \delta$ with $\delta = \delta_*/2$ and $1/p + 1/q = 1$. Then applying Hölder's and Lyapunov's inequalities yields

$$\begin{aligned}
E\big[|h_i|^p\big] &\le E\left[\left|\sum_{j=1}^{n} |b_{ij}||\mathfrak{z}_{j,t}|\right|^p\right] = E\left[\left\{\sum_{j=1}^{n} |b_{ij}|^{1/q}|b_{ij}|^{1/p}|\mathfrak{z}_{j,t}|\right\}^p\right] \\
&\le E\left[\left\{\left[\sum_{j=1}^{n} |b_{ij}|\right]^{1/q}\left[\sum_{j=1}^{n} |b_{ij}||\mathfrak{z}_{j,t}|^p\right]^{1/p}\right\}^p\right] \\
&\le K_b^{p/q} \sum_{j=1}^{n} E\big[|b_{ij}||\mathfrak{z}_{j,t}|^p\big] = K_b^{p/q}(1 + K_z)^2 \sum_{j=1}^{n} \|b_{ij}\|_2 \le \text{const} < \infty,
\end{aligned}$$

observing that $E[|b_{ij}||\mathfrak{z}_{j,t}|^p] \le \|b_{ij}\|_2 \||\mathfrak{z}_{j,t}|^p\|_2 \le \|b_{ij}\|_2 [\|\mathfrak{z}_{j,t}\|_{4+\delta_*}]^{(4+\delta_*)/(2+\delta)} \le \|b_{ij}\|_2 (1 + K_z)^2$ and observing that as shown above $\sum_{j=1}^{n} \|b_{ij}\|_2 \le \text{const} < \infty$. This shows that under the maintained assumptions the first sufficient condition of part (a) also holds, which completes the proof of part (b) of the lemma. *Q.E.D.*

LEMMA D.1: *Let A and B be symmetric $n \times n$ matrices with $\sum_{i=1}^{n} |a_{ij}| \le K < \infty$, $\sum_{i=1}^{n} \|a_{ij}\|_{p_a} \le K < \infty$, $\sum_{i=1}^{n} |b_{ij}| \le K < \infty$, and $\sum_{i=1}^{n} \|b_{ij}\|_{p_b} \le K < \infty$ for some $p_a, p_b \ge 1$, and let $C = AB$. Then $\sum_{i=1}^{n} |c_{ij}| \le K^2 < \infty$ and $\sum_{i=1}^{n} \|c_{ij}\|_p \le K^2 < \infty$ with $1/p = 1/p_a + 1/p_b$. Note that the results also hold for the row sums of C if we allow for A and B to be nonsymmetric, and place conditions both on row and column sums.*

PROOF: Since $c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$, we have

$$\sum_{i=1}^{n} |c_{ij}| \le \sum_{i=1}^{n} \sum_{l=1}^{n} |a_{il}||b_{lj}| = \sum_{l=1}^{n} |b_{lj}| \sum_{i=1}^{n} |a_{il}| \le K^2.$$

Next observe that by the generalized Hölder inequality, $\|a_{il} b_{lj}\|_p \le \|a_{il}\|_{p_a} \|b_{lj}\|_{p_b}$ and thus

$$\begin{aligned}
\sum_{i=1}^{n} \|c_{ij}\|_p &\le \sum_{i=1}^{n} \sum_{l=1}^{n} \|a_{il} b_{lj}\|_p \le \sum_{i=1}^{n} \sum_{l=1}^{n} \|a_{il}\|_{p_a} \|b_{lj}\|_{p_b} \\
&= \sum_{l=1}^{n} \|b_{lj}\|_{p_b} \sum_{i=1}^{n} \|a_{il}\|_{p_a} \le K^2.
\end{aligned}$$

Q.E.D.

REMARK: Suppose the elements of a matrix M satisfy $\sum_{i=1}^{n} \|m_{ij}\|_p \le K$ as well as $\sum_{j=1}^{n} \|m_{ij}\|_p \le K$, where w.o.l.o.g. $K \ge 1$. Then it follows for $\bar{M} = (M + M')/2$ that $\|\bar{m}_{ij}\|_p \le 2^{-1}(\|m_{ij}\|_p + \|m_{ji}\|_p)$ by the Minkowski inequality, such that $\|\bar{m}_{ij}\|_p$ is also summable in both indices. Next, for any integer $s \ge 1$, suppose that $\sum_{i=1}^{n} \|m_{ij}\|_{sp} \le K$

as well as $\sum_{j=1}^{n}\|m_{ij}\|_{sp}\leq K$. Then by applying the above lemma recursively it is readily seen that the elements of $A=M^{\tau}$ or $A=M^{\tau\prime}$ with $\tau\leq s$ satisfy $\sum_{j=1}^{n}\|a_{ij}\|_{p}\leq K^{s}$ and $\sum_{i=1}^{n}\|a_{ij}\|_{p}\leq K^{s}$. Furthermore, if $\sum_{i=1}^{n}\|m_{ij}\|_{sp}\leq K$ as well as $\sum_{j=1}^{n}\|m_{ij}\|_{sp}\leq K$, then the elements of $A=M^{\sigma}(M^{\tau-\sigma})'$ or $A=M^{\sigma\prime}M^{\tau-\sigma}$ with $0\leq\sigma\leq\tau$, $\tau\leq s$, or $A=(M'M)^{\tau/2}$ with $\tau\leq s$ even, satisfy $\sum_{j=1}^{n}\|a_{ij}\|_{p}\leq K^{s}$ and $\sum_{i=1}^{n}\|a_{ij}\|_{p}\leq K^{s}$. The results also hold if M is replaced by $\bar{M}$.

LEMMA D.2: *For $i,j=1,\ldots,n$, let $d_{ij}\in\{0,1\}$ with $d_{ii}=0$ and $\sum_{l=1}^{n}d_{il}\geq 1$, let $s_{ij}\geq 0$ be a measure of "distance" between units i and j, and let $m_{ij}=d_{ij}/(\sum_{l=1}^{n}d_{il})$. Then, obviously, $\sum_{j=1}^{n}m_{ij}=\sum_{j=1}^{n}|m_{ij}|=1$. Furthermore, consider the following situations.*

(a) *Suppose $d_{ij}=0$ for $s_{ij}>c$, where c is a finite nonnegative constant, $s_{ij}=s_{ji}$, $\sum_{j=1}^{n}1\{s_{ij}\leq c\}\leq K<\infty$, and $\sum_{j=1}^{n}(\Pr(s_{ij}\leq c))^{1/p}\leq K_{p}$ for some $p\geq 1$, where K_p is a finite constant. Then $\sum_{i=1}^{n}m_{ij}\leq K$, $\sum_{i=1}^{n}\|m_{ij}\|_{p}\leq K_{p}$, and $\sum_{j=1}^{n}\|m_{ij}\|_{p}\leq K_{p}$.*

(b) *Suppose that $\sum_{j=1}^{n}1\{\Pr(s_{ij}\leq c)>0\}\leq K$, where K is a finite constant. Then $\sum_{j=1}^{n}(\Pr(s_{ij}\leq c))^{1/p}\leq K$ for all finite $p\geq 1$.*

(c) *If $\Pr(s_{ij}\leq c)=0$ implies $1\{s_{ij}\leq c\}=0$, then $\sum_{j=1}^{n}1\{s_{ij}\leq c\}\leq\sum_{j=1}^{n}1\{\Pr(s_{ij}\leq c)>0\}$.*

PROOF: To prove part (a) of the lemma, observe that

$$\sum_{i=1}^{n}m_{ij}\leq\sum_{i=1}^{n}\left|m_{ij}1\{s_{ij}\leq c\}\right|\leq\sum_{i=1}^{n}1\{s_{ji}\leq c\}\leq K$$

by symmetry and summability of $1\{s_{ji}\leq c\}$. Further, using Minkowski's inequality,

$$\begin{aligned}\sum_{j=1}^{n}\|m_{ij}\|_{p}&\leq\sum_{j=1}^{n}\left\|m_{ij}1\{s_{ij}\leq c\}\right\|_{p}+\sum_{j=1}^{n}\left\|m_{ij}1\{s_{ij}>c\}\right\|_{p}\\&\leq\sum_{j=1}^{n}\left\|1\{s_{ij}\leq c\}\right\|_{p}=\sum_{j=1}^{n}\left(\Pr(s_{ij}\leq c)\right)^{1/p}\leq K_{p}\end{aligned}$$

as well as

$$\begin{aligned}\sum_{i=1}^{n}\|m_{ij}\|_{p}&\leq\sum_{i=1}^{n}\left\|m_{ij}1\{s_{ij}\leq c\}\right\|_{p}+\sum_{i=1}^{n}\left\|m_{ij}1\{s_{ij}>c\}\right\|_{p}\\&\leq\sum_{i=1}^{n}\left\|1\{s_{ij}\leq c\}\right\|_{p}=\sum_{i=1}^{n}\left(\Pr(s_{ji}\leq c)\right)^{1/p}\leq K_{p},\end{aligned}$$

showing summability in both indices.

To prove part (b) of the lemma observe that

$$\begin{aligned}&\sum_{j=1}^{n}\left[\Pr(s_{ij}\leq c)\right]^{1/p}\\&\quad=\sum_{j=1}^{n}\left[\Pr(s_{ij}\leq c)1\left\{\Pr(s_{ij}\leq c)>0\right\}+\Pr(s_{ij}\leq c)1\left\{\Pr(s_{ij}\leq c)=0\right\}\right]^{1/p}\end{aligned}$$

$$= \sum_{j=1}^{n} \big[\Pr(s_{ij} \le c) 1\big\{\Pr(s_{ij} \le c) > 0\big\}\big]^{1/p} \le \sum_{j=1}^{n} \big[1\big\{\Pr(s_{ij} \le c) > 0\big\}\big]^{1/p}$$

$$= \sum_{j=1}^{n} 1\big\{\Pr(s_{ij} \le c) > 0\big\} \le K.$$

For (c) note that

$$\begin{aligned} 1\{s_{ij} \le c\} &= 1\{s_{ij} \le c\} 1\big\{\Pr(s_{ij} \le c) > 0\big\} + 1\{s_{ij} \le c\}\big(1 - 1\big\{\Pr(s_{ij} \le c) > 0\big\}\big) \\ &= 1\{s_{ij} \le c\} 1\big\{\Pr(s_{ij} \le c) > 0\big\} + 1\{s_{ij} \le c\} 1\big\{\Pr(s_{ij} \le c) = 0\big\} \\ &= 1\{s_{ij} \le c\} 1\big\{\Pr(s_{ij} \le c) > 0\big\} \\ &\le 1\big\{\Pr(s_{ij} \le c) > 0\big\} \end{aligned}$$

such that

$$\sum_{j=1}^{n} 1\{s_{ij} \le c\} \le \sum_{j=1}^{n} 1\big\{\Pr(s_{ij} \le c) > 0\big\}. \qquad \textit{Q.E.D.}$$

D.2. *Variance–Covariance of Linear Quadratic Forms*

In this section we investigate the variance–covariances of linear quadratic forms based on forward differenced disturbances. We maintain the setup and notational conventions of the previous sections. For the moment we only assume that the $T^+ \times T$ transformation matrices Π and Γ are upper triangular. The following proposition gives a generalized version of Theorem 1. The extension of the proposition establishes that in general the variance–covariance matrix of linear quadratic forms is complicated and depends on higher order moments unless specific choices about Π and A_t are made. An explicit proof of the following proposition is given in Appendix D.4.

PROPOSITION D.1: *Let the disturbance process* (u_{it}) *and the information sets* $\mathcal{B}_{n,i,t}$ *be as in Assumption* 1, *and let* $\mathcal{B}_{n,t} = \sigma(\{x^o_{tj}, z^o_j, u^o_{t-1j}, \mu_j\}_{j=1}^n)$ *and* $\mathcal{Z}_n = \sigma(\{z^o_j, \mu_j\}_{j=1}^n)$. *Furthermore, assume that for all* $t = 1, \dots, T$, $i = 1, \dots, n$, $n \ge 1$, $E[u_{it}|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = 0$, $E[u_{it}^2|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \varrho_i^2 \sigma_t^2 > 0$, $E[u_{it}^3|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \mu_{3,it}$, *and* $E[u_{it}^4|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \mu_{4,it}$, *where* σ_t *is finite and measurable w.r.t.* $\mathcal{C}$, *and* ϱ_i, $\mu_{3,it}$, *and* $\mu_{4,it}$ *are finite and measurable w.r.t.* $\mathcal{Z}_n \vee \mathcal{C}$. *Define* $\Sigma_\varrho = \operatorname{diag}(\varrho_1^2, \dots, \varrho_n^2)$ *and* $\Sigma_\sigma = \operatorname{diag}(\sigma_1^2, \dots, \sigma_T^2)$. *Let* $A_t = (a_{ijt})$ *and* $B_t = (b_{ijt})$ *be* $n \times n$ *matrices, and let* $a_t = (a_{it})$ *and* $b_t = (b_{it})$ *be* $n \times 1$ *vectors, where* a_{ijt}, b_{ijt}, a_{it}, *and* b_{it} *are measurable w.r.t.* $\mathcal{B}_{n,t} \vee \mathcal{C}$. *Let* $\pi_t = [0, \dots, 0, \pi_{tt}, \dots, \pi_{tT}]$ *and* $\gamma_t = [0, \dots, 0, \gamma_{tt}, \dots, \gamma_{tT}]$ *be* $1 \times T$ *vectors where* $\pi_{t\tau}$ *and* $\gamma_{t\tau}$ *are measurable w.r.t.* $\mathcal{C}$, *and consider the forward differences* $u_t^+ = [u_{1t}^+, \dots, u_{nt}^+]'$ *and* $u_t^\times = [u_{1t}^\times, \dots, u_{nt}^\times]'$ *with*

$$u_{it}^+ = \sum_{s=t}^{T} \pi_{ts} u_{is} = \pi_t u_i', \quad \textit{and} \quad u_{it}^\times = \sum_{s=t}^{T} \gamma_{ts} u_{is} = \gamma_t u_i'.$$

Then

$$E\big[u_t^{+\prime} A_t u_t^\times + u_t^{+\prime} a_t | \mathcal{C}\big] = \pi_t \Sigma_\sigma \gamma_t \operatorname{tr}\big[E(A_t \Sigma_\varrho | \mathcal{C})\big], \tag{D.3}$$

$$
\begin{aligned}
&\operatorname{Cov}\big(u_t^{+\prime}A_t u_t^{\times}+a_t' u_t^{+}, u_t^{+\prime}B_t u_t^{\times}+b_t' u_t^{+}|\mathcal{C}\big)\\
&\quad=\big(\pi_t\Sigma_\sigma\pi_t'\big)\big(\gamma_t\Sigma_\sigma\gamma_t'\big)E\big[\operatorname{tr}\big(A_t\Sigma_\varrho B_t'\Sigma_\varrho\big)|\mathcal{C}\big]\\
&\qquad+\big(\pi_t\Sigma_\sigma\gamma_t'\big)^2E\big[\operatorname{tr}(A_t\Sigma_\varrho B_t\Sigma_\varrho)|\mathcal{C}\big]\\
&\qquad+\big(\pi_t\Sigma_\sigma\pi_t'\big)E\big[a_t'\Sigma_\varrho b_t|\mathcal{C}\big]+\mathcal{K}_1, \qquad\qquad (D.4)\\
&\operatorname{Cov}\big(u_t^{+\prime}A_t u_t^{\times}+a_t' u_t^{+}, u_s^{+\prime}B_s u_s^{\times}+b_s' u_s^{+}|\mathcal{C}\big)=\mathcal{K}_2 \quad \textit{for all } t>s, \qquad (D.5)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{K}_1={}&\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}E\big[\big(\mu_{4,iu}-3\varrho_i^4\sigma_u^4\big)a_{iit}b_{iit}|\mathcal{C}\big]\\
&+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}\sum_{i=1}^{n}E\big[\mu_{3,iu}(a_{iit}b_{it}+b_{iit}a_{it})|\mathcal{C}\big],\\
\mathcal{K}_2={}&\big(\pi_t\Sigma_\sigma\pi_s'\big)\big(\gamma_t\Sigma_\sigma\gamma_s'\big)E\big[\operatorname{tr}\big(A_t\Sigma_\varrho B_s'\Sigma_\varrho\big)|\mathcal{C}\big]+\big(\pi_t\Sigma_\sigma\gamma_s'\big)\big(\pi_s'\Sigma_\sigma\gamma_t\big)E\big[\operatorname{tr}(A_t\Sigma_\varrho B_s\Sigma_\varrho)|\mathcal{C}\big]\\
&+\big(\pi_t\Sigma_\sigma\pi_s'\big)E\big[a_t'\Sigma_\varrho b_s|\mathcal{C}\big]+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[\big(\mu_{4,iu}-3\varrho_i^4\sigma_u^4\big)a_{iit}b_{iis}|\mathcal{C}\big]\\
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}]+\sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]\\
&+\big(\pi_t\Sigma_\sigma\pi_s'\big)\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[\varrho_i^2a_{it}b_{ijs}u_{jv}|\mathcal{C}\big]\\
&+\big(\pi_t\Sigma_\sigma\gamma_s'\big)\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[\varrho_i^2a_{it}b_{jis}u_{jv}|\mathcal{C}\big]\\
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]\\
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}].
\end{aligned}
$$

(*Of course, the first covariance also yields an expression for the variance of* $u_t^{+\prime}A_tu_t^{\times}+u_t^{+\prime}a_t$ *if we take* $B_t=A_t$ *and* $b_t=a_t$.)

The standard catalogue of assumption in the spatial panel data literature resulted in formulations of the moment conditions where the coefficients of the linear quadratic forms are treated as nonstochastic. We note that in this case the expectations simplify in that the last four lines in the expression for $\mathcal{K}_2$ are zero. If furthermore $\Pi=\Gamma=I$, then $u_{it}^{+}=u_{it}^{\times}=u_{it}$ and the above proposition yields expressions consistent with those given in Kelejian and Prucha (2010).

From the above proposition we see that $E[u_t^{+\prime} A_t u_t^{\times} + u_t^{+\prime} a_t|\mathcal{C}] = 0$ if $\text{vec}_D(A_t) = 0$. Under cross-sectional homoskedasticity, that is, $\Sigma_\varrho = I$, a sufficient condition for this expectation to be zero is $\text{tr}(A_t) = 0$.

From the above proposition we see furthermore that, in general, linear quadratic forms are correlated within a time period and across time periods. We now explore the effect of using an orthogonal forward differencing transformation on the correlation structure of linear quadratic forms. More specifically, suppose that $\Pi = \Gamma$ and $\Pi\Sigma_\sigma\Pi' = I$. Then it is readily seen that the expressions for the covariances in (D.4) and (D.5) simplify as summarized in the following corollary.

COROLLARY D.1: *Under the conditions of Proposition* D.1 *and additionally imposing that* $\Pi = \Gamma$ *with* $\Pi\Sigma_\sigma\Pi' = I$, *we have*

$$\begin{aligned}
&\text{Cov}\big(u_t^{+\prime} A_t u_t^+ + u_t^{+\prime} a_t, u_t^{+\prime} B_t u_t^+ + u_t^{+\prime} b_t|\mathcal{C}\big) \\
&\quad = E\big[\text{tr}\big(A_t\Sigma_\varrho\big(B_t + B_t'\big)\Sigma_\varrho\big)|\mathcal{C}\big] + E\big[a_t'\Sigma_\varrho b_t|\mathcal{C}\big] \\
&\qquad + \sum_{u=t}^{T} \pi_{tu}^4 \sum_{i=1}^{n} E\big[\big(\mu_{4,iu} - 3\varrho_i^4\sigma_u^4\big)a_{iit}b_{iit}|\mathcal{C}\big] \\
&\qquad + \sum_{u=t}^{T} \pi_{tu}^3 \sum_{i=1}^{n} E\big[\mu_{3,iu}(a_{iit}b_{it} + b_{iit}a_{it})|\mathcal{C}\big] \qquad \text{(D.6)}
\end{aligned}$$

and for $t > s$,

$$\begin{aligned}
&\text{Cov}\big(u_t^{+\prime} A_t u_t^+ + u_t^{+\prime} a_t, u_s^{+\prime} B_s u_s^+ + u_s^{+\prime} b_s|\mathcal{C}\big) \\
&= \sum_{u=t}^{T} \pi_{tu}^2\pi_{su}^2 \sum_{i=1}^{n} E\big[\big(\mu_{4,iu} - 3\varrho_i^4\sigma_u^4\big)a_{iit}b_{iis}|\mathcal{C}\big] \\
&\quad + \sum_{u=t}^{T} \pi_{tu}^2\pi_{su} \sum_{i=1}^{n} E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}] + \sum_{u=t}^{T} \pi_{su}^2\pi_{tu} \sum_{i=1}^{n} E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T} \pi_{tu}^2\pi_{su} \sum_{v=s}^{t-1} \pi_{sv} \sum_{i=1}^{n}\sum_{j=1}^{n} E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T} \pi_{tu}^2\pi_{su} \sum_{v=s}^{t-1} \pi_{sv} \sum_{i=1}^{n}\sum_{j=1}^{n} E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}]. \qquad \text{(D.7)}
\end{aligned}$$

From this we see that in contrast to the case of pure linear forms (where $A_t = B_t = 0$), quadratic forms will generally be correlated across time, even if based on orthogonally transformed disturbances. Furthermore, when it comes to the estimation of the covariances, it seems that even if $\mu_{3,iu} = \mu_3$ and is fixed, the terms $\sum_{i=1}^n\sum_{j=1}^n E[a_{iit}b_{ijs}u_{jv}|\mathcal{C}]$ and $\sum_{i=1}^n\sum_{j=1}^n E[a_{iit}b_{jis}u_{jv}|\mathcal{C}]$ would still be difficult to handle without further assumptions on the data generating process. However, an inspection of the above expressions shows that those difficulties can be avoided if we assume that $\text{vec}_D(A_t) = \text{vec}_D(B_t) = 0$ for all t. In this case the expressions in (D.6) and (D.7) simplify further. We note that assuming only that $\text{tr}(A_t) = \text{tr}(B_t) = 0$ does not avoid those difficulties even under cross-sectional homoskedasticity.

COROLLARY D.2: *Under the conditions of Proposition D.1 and additionally imposing that* $\mathrm{vec}_D(A_t)=0$, $\mathrm{vec}_D(B_t)=0$, *and* $\Pi=\Gamma$ *with* $\Pi\Sigma_\sigma\Pi'=I$, *we have*

$$\mathrm{Cov}\big(u_t^{+\prime}A_tu_t^+ + u_t^{+\prime}a_t, u_t^{+\prime}B_tu_t^+ + u_t^{+\prime}b_t|\mathcal{C}\big) = E\big[\mathrm{tr}\big(A_t\Sigma_\varrho\big(B_t+B_t'\big)\Sigma_\varrho\big)|\mathcal{C}\big] + E\big[a_t'\Sigma_\varrho b_t|\mathcal{C}\big]$$

and

$$\mathrm{Cov}\big(u_t^{+\prime}A_tu_t^+ + u_t^{+\prime}a_t, u_s^{+\prime}B_su_s^+ + u_s^{+\prime}b_s|\mathcal{C}\big) = 0 \quad \text{for } t\neq s.$$

The linear quadratic forms are only contemporaneously correlated, but uncorrelated over time, which motivates us to restrict the class of moment conditions underlying our GMM estimator to the case where the diagonal elements of the matrices in the quadratic forms are zero, that is, $\mathrm{vec}_D(A_t)=\mathrm{vec}_D(B_t)=0$ for all t.

Next consider the case where a researcher uses both the transformation $\Gamma=F$ and the corresponding orthogonal transformation $\Pi=U^{-1}F$ with $UU'=F\Sigma_\sigma F'$. In this case $\Pi\Sigma_\sigma\Gamma'=U'$ and thus $\pi_t\Sigma_\sigma\gamma_s'=0$ for $s>t$, but generally $\pi_t\Sigma_\sigma\gamma_s'\neq 0$ for $t>s$. As a consequence, in this case the linear quadratic forms are generally autocorrelated in time due to the presence of the term $(\pi_t\Sigma_\sigma\gamma_s')\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^n\sum_{j=1}^n E[\varrho_i^2a_{it}b_{jis}u_{jv}|\mathcal{C}]$ even if $\mathrm{vec}_D(A_t)=\mathrm{vec}_D(B_t)=0$ for all t.

Finally, the next corollary covers the case where $\Pi=\Gamma$, but where the transformation is not efficient, that is, $\Pi\Sigma_\sigma\Pi'\neq I$. In that case, the moment conditions are correlated across t with the difficult to estimate term $\sum_{i=1}^n\sum_{j=1}^n E[\varrho_i^2a_{it}(b_{ijs}+b_{jis})u_{jv}|\mathcal{C}]$ present in the covariance term. This result demonstrates the importance of efficient detrending.

COROLLARY D.3: *Under the conditions of Proposition D.1, but additionally imposing that* $\mathrm{vec}_D(A_t)=0$, $\mathrm{vec}_D(B_t)=0$, *and* $\Pi=\Gamma$, *but* $\Pi\Sigma_\sigma\Pi'\neq I$, *then*

$$\begin{aligned}
&E\big[u_t^{+\prime}A_tu_t^+ + u_t^{+\prime}a_t|\mathcal{C}\big]=0,\\
&\mathrm{Cov}\big(u_t^{+\prime}A_tu_t^+ + a_t'u_t^+, u_t^{+\prime}B_tu_t^\times + b_t'u_t^+|\mathcal{C}\big)\\
&\quad = \big(\pi_t\Sigma_\sigma\pi_t'\big)^2E\big[\mathrm{tr}\big(A_t\Sigma_\varrho\big(B_t+B_t'\big)\Sigma_\varrho\big)|\mathcal{C}\big] + \big(\pi_t\Sigma_\sigma\pi_t'\big)E\big[a_t'\Sigma_\varrho b_t|\mathcal{C}\big],
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Cov}\big(u_t^{+\prime}A_tu_t^+ + a_t'u_t^+, u_s^{+\prime}B_su_s^+ + b_s'u_s^+|\mathcal{C}\big)\\
&\quad = \big(\pi_t\Sigma_\sigma\pi_s'\big)^2E\big[\mathrm{tr}\big(A_t\Sigma_\varrho\big(B_s+B_s'\big)\Sigma_\varrho\big)|\mathcal{C}\big] + \big(\pi_t\Sigma_\sigma\pi_s'\big)E\big[a_t'\Sigma_\varrho b_s|\mathcal{C}\big]\\
&\qquad + \big(\pi_t\Sigma_\sigma\pi_s'\big)\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^n\sum_{j=1}^n E\big[\varrho_i^2a_{it}(b_{ijs}+b_{jis})u_{jv}|\mathcal{C}\big].
\end{aligned}$$

D.3. *Proofs of Lemmas in Appendix C.2 and of Theorem 3*

PROOF OF LEMMA 1: In the following discussion, all bounding constants from above are w.o.l.o.g. assumed to be greater than 1. Recall from Proposition 1 that for $\pi_{ts}=\pi_{ts}(f_0,\gamma_{0,\sigma})$ we have $\pi_t\Sigma_{0,\sigma}\pi_t'=1$. Since by Assumption 1 we have $\sigma_{0,t}^2\geq c_u>0$, it follows that $\pi_t\pi_t'\leq K_\pi$ for $K_\pi=c_u^{-1}$ and, thus, $|\pi_{ts}|\leq K_\pi$. By Assumption 2(i) the $2+\delta$ absolute moments of the elements of the $p_t\times 1$ vector h_{it} are uniformly bounded by some finite constant $K_h^{2+\delta}$. Let $p_* = \max(p_t)$. Then observing that $|\lambda_{rt}| \leq 1$ since $\lambda'\lambda = 1$, it follows that

$$E\left[\left|\lambda_t' h_{it}'\right|^{2+\delta}\right] = E\left[\left|\sum_{r=1}^{p_t} \lambda_{rt} h_{irt}\right|^{2+\delta}\right] \leq p_*^{1+\delta} \sum_{r=1}^{p_t} E\left[|h_{irt}|^{2+\delta}\right] \leq p_*^{2+\delta} K_h^{2+\delta},$$

where we have also utilized inequality (1.4.3) In Bierens (1994). Hence, in light of (32),

$$E\left[|c_{it}|^{2+\delta}\right] \leq K_c$$

with $K_c = T p_*^{2+\delta} K_h^{2+\delta} K_\pi$.

By Assumption 2(ii), for all r and t we have $\sum_{i=1}^n |a_{ij,t}^r| = \sum_{j=1}^n |a_{ij,t}^r| \leq K_a$. Consequently, observing that $|\lambda_t' a_{ij,t}'| \leq \sum_{r=1}^{p_t} |\lambda_{rt}||a_{ij,t}^r| \leq \sum_{r=1}^{p_t} |a_{ij,t}^r|$, we have

$$\sum_{i=1}^n \left|\lambda_t' a_{ij,t}'\right| = \sum_{j=1}^n \left|\lambda_t' a_{ij,t}'\right| \leq \sum_{r=1}^{p_t} \sum_{j=1}^n \left|a_{ij,t}^r\right| \leq p_* K_a.$$

Thus, in light of (34) and $s \leq t$,

$$\sum_{i=1}^n |c_{ij,ts}| = \sum_{j=1}^n |c_{ij,ts}| \leq \sum_{\tau=1}^s \sum_{j=1}^n \left|\lambda_\tau' a_{ij,\tau}'\right| |\pi_{\tau s}||\pi_{\tau t}| \leq K_{cc}$$

with $K_{cc} = T p_* K_a K_\pi^2$. Furthermore for $q \geq 1$,

$$\sum_{i=1}^n |c_{ij,ts}|^q = \sum_{j=1}^n |c_{ij,ts}|^q \leq K_{cc}^{q-1} \sum_{j=1}^n \left[|c_{ij,ts}|/K_{cc}\right]^{q-1} |c_{ij,ts}| \leq K_{cc}^q.$$

By Minkowski's inequality, we have

$$\|c_{ij,ts}\|_q = \left\|\sum_{\tau=1}^s \lambda_\tau' a_{ij,\tau}' \pi_{\tau s} \pi_{\tau t}\right\|_q \leq \left\|\sum_{\tau=1}^s \sum_{r=1}^{p_\tau} \left|a_{ij,\tau}^r\right| |\pi_{\tau s}||\pi_{\tau t}|\right\|_q \leq K_\pi^2 \sum_{\tau=1}^s \sum_{r=1}^{p_\tau} \left\|a_{ij,\tau}^r\right\|_q.$$

Hence by Assumption 2(ii) and Lyapunov's inequality, we have for $1 \leq q \leq 2+\delta$,

$$\sum_{j=1}^n \|c_{ij,ts}\|_q \leq K_\pi^2 \sum_{\tau=1}^s \sum_{r=1}^{p_\tau} \sum_{j=1}^n \left\|a_{ij,\tau}^r\right\|_{2+\delta} \leq T p_* K_a K_\pi^2 = K_{cc}.$$

Also recall that by Assumption 1, observing that $\mathcal{F}_{n,(t-1)n+i} \subseteq \mathcal{B}_{n,i,t} \vee \mathcal{C}$, we have for all i and t that

$$E\left[|u_{it}|^{2+\delta} | \mathcal{F}_{n,(t-1)n+i}\right] = E\left[E\left[|u_{it}|^{2+\delta} | \mathcal{B}_{n,i,t} \vee \mathcal{C}\right] | \mathcal{F}_{n,(t-1)n+i}\right] \leq K_u,$$

and thus also $E[|u_{it}|^{2+\delta}] \leq K_u$ for some finite constant K_u. Of course, by Lyapunov's inequality we also have for $1 \leq q \leq 2+\delta$,

$$E\left[|u_{it}|^q | \mathcal{F}_{n,(t-1)n+i}\right] \leq \left\{E\left[|u_{it}|^{2+\delta} | \mathcal{F}_{n,(t-1)n+i}\right]\right\}^{q/(2+\delta)} \leq K_u^{q/(2+\delta)} \leq K_u.$$

Furthermore, observe that for $s \leq t$ and for $1 \leq q \leq 2+\delta$ and $1/q + 1/p = 1$,

$$\begin{aligned} E\left[\sum_{i=1}^{n} |u_{is}|^q |c_{ij,ts}| |\mathcal{B}_{n,s} \vee \mathcal{C}\right] &\leq \sum_{i=1}^{n} E\left[E\left[|u_{is}|^q |\mathcal{F}_{n,(s-1)n+i}\right] |c_{ij,ts}| |\mathcal{B}_{n,s} \vee \mathcal{C}\right] \\ &\leq K_u K_{cc} \\ E\left[\left(\sum_{i=1}^{n} |u_{is}||c_{ij,ts}|\right)^q |\mathcal{B}_{n,s} \vee \mathcal{C}\right] &= E\left[\left(\sum_{i=1}^{n} |u_{is}||c_{ij,ts}|^{1/q} |c_{ij,ts}|^{1/p}\right)^q |\mathcal{B}_{n,s} \vee \mathcal{C}\right] \\ &\leq E\left[\left\{\left(\sum_{i=1}^{n} |u_{is}|^q |c_{ij,ts}|\right)^{1/q} \left(\sum_{i=1}^{n} |c_{ij,ts}|\right)^{1/p}\right\}^q |\mathcal{B}_{n,s} \vee \mathcal{C}\right] \\ &\leq \left(\sum_{i=1}^{n} |c_{ij,ts}|\right)^{q/p} E\left[\sum_{i=1}^{n} |u_{is}|^q |c_{ij,ts}| |\mathcal{B}_{n,s} \vee \mathcal{C}\right] \\ &\leq K_{cc}^{q/p+1} K_u \leq K_{cc}^{2+\delta} K_u, \end{aligned}$$

where we used that $c_{ij,ts}$ is measurable w.r.t. $\mathcal{B}_{n,s} \vee \mathcal{C}$ and $\mathcal{F}_{n,(s-1)n+i}$. Recall that w.o.l.o.g. the constants K_c, K_{cc}, K_a, and K_u are assumed to be greater than 1. Also let K_σ and K_ϱ be finite constants such that $\sigma_{0,t}^2 \leq K_\sigma$ and $\varrho_{0,i}^2 \leq K_\varrho$. In the following discussion we take

$$K = \max\left\{K_\sigma, K_\varrho, K_c, K_{cc}, K_{cc}^{2+\delta}, p_* K_a, K_u\right\} \geq 1. \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA 2: Probability Limit of $n^{-1}\sum_{i=1}^n \varsigma_{it}^{(1)}$. Let

$$\varsigma_t^{(1)} = \operatorname*{plim}_{n\to\infty} n^{-1} \sum_{i=1}^{n} \sigma_{0,t}^2 E\left[c_{it}^2 |\mathcal{C}\right].$$

Then recalling the definition of c_{it} in (32), it follows from Assumption 3 that $n^{-1} \times \sum_{i=1}^n \varsigma_{it}^{(1)} = n^{-1}\sum_{i=1}^n c_{it}^2 \xrightarrow{p} \varsigma_t^{(1)}$ as $n \to \infty$.

Probability Limit of $n^{-1}\sum_{i=1}^n \varsigma_{it}^{(2)}$. Let $\varsigma_t^{(2)} = \operatorname{plim}_{n\to\infty} 2\sigma_{0,t}^2 n^{-1} \sum_{i=1}^n \sum_{j=1}^n E[c_{ij,tt}^2|\mathcal{C}]$. Recalling that $c_{ij,tt} = c_{ji,tt}$ and $c_{ii,tt} = 0$, it is readily seen that

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} \varsigma_{it}^{(2)} &= 4n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \sum_{l=1}^{i-1} c_{ij,tt} c_{il,tt} u_{jt} u_{lt} \\ &= 4n^{-1} \sum_{i=1}^{n-1} u_{it}^2 \sum_{j=i+1}^{n} c_{ji,tt}^2 + 8n^{-1} \sum_{i=2}^{n-1} \sum_{l=1}^{i-1} u_{it} u_{lt} \sum_{j=i+1}^{n} c_{ji,tt} c_{jl,tt}. \end{aligned}$$

By observing furthermore that $2\sum_{i=1}^{n-1}\sum_{j=i+1}^n c_{ji,tt}^2 = \sum_{i=1}^n \sum_{j=1}^n c_{ij,tt}^2$, we have

$$n^{-1}\sum_{i=1}^{n} \varsigma_{it}^{(2)} = \sum_{i=1}^{n-1} Y_{n,i} + 2\sigma_{0,t}^2 n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij,tt}^2,$$

where

$$Y_{n,i} = n^{-1}\left[4\left(u_{it}^2 - \sigma_{0,t}^2\right)\sum_{j=i+1}^{n} c_{ji,tt}^2 + 8u_{it}\sum_{l=1}^{i-1} u_{lt}\sum_{j=i+1}^{n} c_{ji,tt}c_{jl,tt}\right].$$

Given Assumption 3 and recalling the definition of $c_{ij,tt}$ in (34) it now follows that

$$2\sigma_{0,t}^2 n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij,tt}^2 \xrightarrow{p} \varsigma_t^{(2)} \quad \text{as } n\to\infty$$

and, consequently, $n^{-1}\sum_{i=1}^{n}\varsigma_{it}^{(2)} \xrightarrow{p} \varsigma_t^{(2)}$, provided we show that for given t, $\sum_{i=1}^{n-1} Y_{n,i} \xrightarrow{p} 0$. Let $\mathcal{G}_{n,i-1} = \mathcal{F}_{n,(t-1)n+i}$ and $\mathcal{G}_{n,0} = \mathcal{F}_{n,(t-2)n+n}$. Then clearly $E[Y_{n,i}|\mathcal{G}_{n,i-1}] = 0$ and $Y_{n,i}$ is $\mathcal{G}_{n,i}$-measurable, and thus

$$\{Y_{n,i}, \mathcal{G}_{n,i}, i = 1, \dots, n\}$$

is a martingale difference array. Next observe that with $d_{ni} = n^{-1}$, $q = 1+\delta/2$, and $1/p + 1/q = 1$, it follows upon utilizing the triangle and Hölder's inequalities and the bounds in Lemma 1(ii), (iii), (v), and (vii) that

$$\begin{aligned}
E\left[|Y_{n,i}/d_{ni}|^q\right] &\le 2^q E\left[\left|4\left|u_{it}^2 - \sigma_{0,t}^2\right|\sum_{j=i+1}^{n}|c_{ji,tt}|^2\right|^q\right] + 2^q E\left[|8|u_{it}|\sum_{l=1}^{i-1}|u_{lt}|\sum_{j=i+1}^{n}|c_{ji,tt}||c_{jl,tt}|^q\right] \\
&\le 2^q 4^q E\left[\left|u_{it}^2 - \sigma_{0,t}^2\right|^q\left(\sum_{j=1}^{n}|c_{ji,tt}|^2\right)^q\right] \\
&\quad + 2^q 8^q E\left[|u_{it}|^q\left(\sum_{l=1}^{i-1}|u_{lt}|\sum_{j=i+1}^{n}|c_{ji,tt}||c_{jl,tt}|\right)^q\right] \\
&\le 2^q 4^q K^q E\left\{E\left[\left|u_{it}^2 - \sigma_{0,t}^2\right|^q|\mathcal{F}_{n,(t-1)n+i}\right]\right\} \\
&\quad + 2^q 8^q E\left[E\left[|u_{it}|^q|\mathcal{F}_{n,(t-1)n+i}\right]\left(\sum_{l=1}^{i-1}|u_{lt}|\sum_{j=i+1}^{n}|c_{ji,tt}||c_{jl,tt}|\right)^q\right] \\
&\le 2^q 4^q 2^q K^{q+1} + 2^q 8^q K E\left[\left(\sum_{j=1}^{n}|c_{ji,tt}|\sum_{l=1}^{n}|u_{lt}||c_{jl,tt}|\right)^q\right] \\
&\le 2^q 4^q 2^q K^{q+1} + 2^q 8^q K^{q+2} < \infty.
\end{aligned}$$

Consequently, $|Y_{n,i}/d_{ni}|$ is uniformly integrable. Observing furthermore that $\sum_{i=1}^{n} d_{in} = 1$ and $\sum_{i=1}^{n} d_{in}^2 \to 0$, it follows from the law of large numbers for martingale difference arrays given as Theorem 19.7 in Davidson (1994) that indeed $\sum_{i=1}^{n} Y_{n,i} \xrightarrow{p} 0$.

Probability Limit of $n^{-1}\sum_{i=1}^{n}\varsigma_{it}^{(3)}$. Let

$$\varsigma_t^{(3)} = \operatorname{plim}_{n\to\infty}\sum_{s=1}^{t-1} 4\sigma_{0,s}^2 n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left[c_{ji,ts}^2|\mathcal{C}\right].$$

Recalling that $c_{ij,tt} = c_{ji,tt}$ and $c_{ii,tt} = 0$, and that $\sum_{j=1}^{n}\sum_{l=1}^{n} c_{ij,ts}c_{il,t\tau}u_{js}u_{l\tau}$ is symmetric in s and τ, it is readily seen that

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} \mathfrak{s}_{it}^{(3)} &= n^{-1}\sum_{i=1}^{n} 4\left[\sum_{s=1}^{t-1}\sum_{j=1}^{n} c_{ij,ts}u_{js}\right]^2 \\ &= 4\sum_{s=1}^{t-1} n^{-1}\sum_{i=1}^{n} u_{is}^2 \sum_{j=1}^{n} c_{ji,ts}^2 + 8\sum_{s=1}^{t-1} n^{-1}\sum_{i=1}^{n} u_{is}\sum_{l=1}^{i-1} u_{ls}\sum_{j=1}^{n} c_{ji,ts}c_{jl,ts} \\ &\quad + 8\sum_{s=1}^{t-1} n^{-1}\sum_{i=1}^{n} u_{is}\sum_{\tau=1}^{s-1}\sum_{l=1}^{n} u_{l\tau}\sum_{j=1}^{n} c_{ji,ts}c_{jl,t\tau} \end{aligned}$$

and, thus,

$$n^{-1}\sum_{i=1}^{n} \mathfrak{s}_{it}^{(3)} = \sum_{s=1}^{t-1}\sum_{i=1}^{n} Y_{i,n}^{(s)} + \sum_{s=1}^{t-1} 4\sigma_{0,s}^2 n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji,ts}^2$$

with

$$Y_{i,n}^{(s)} = n^{-1}\left\{4\left(u_{is}^2 - \sigma_{0,s}^2\right)\sum_{j=1}^{n} c_{ji,ts}^2 + 8u_{is}\sum_{l=1}^{i-1} u_{ls}\sum_{j=1}^{n} c_{ji,ts}c_{jl,ts} + 8u_{is}\sum_{\tau=1}^{s-1}\sum_{l=1}^{n} u_{l\tau}\sum_{j=1}^{n} c_{ji,ts}c_{jl,t\tau}\right\}.$$

Given Assumption 3 and recalling the definition of $c_{ij,ts}$ in (34), it now follows that for $n \to \infty$, we have $\sum_{s=1}^{t-1} 4\sigma_{0,s}^2 n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji,ts}^2 \xrightarrow{p} \mathfrak{s}_t^{(3)}$ and, consequently, $n^{-1}\sum_{i=1}^{n} \mathfrak{s}_{it}^{(3)} \xrightarrow{p} \mathfrak{s}_t^{(3)}$, provided we show that $\sum_{i=1}^{n} Y_{n,i}^{(s)} \xrightarrow{p} 0$. Fix s with $1 \le s \le t-1$, and let $\mathcal{G}_{n,i-1} = \mathcal{F}_{n,(s-1)n+i}$ and $\mathcal{G}_{n,0} = \mathcal{F}_{n,(s-2)n+n}$. Then clearly, observing that $c_{ji,t\tau}$ with $\tau \le s$ is measurable w.r.t. $\mathcal{F}_{n,(s-1)n+i}$, we have $E[Y_{n,i}^{(s)}|\mathcal{G}_{n,i-1}] = 0$ and $Y_{n,i}^{(s)}$ is $\mathcal{G}_{n,i}$-measurable, and thus $\{Y_{n,i}^{(s)}, \mathcal{G}_{n,i}, i = 1, \dots, n\}$ is a martingale difference array. Next observe that with $d_{ni} = n^{-1}$ and with $q = 1 + \delta/2$, it follows upon utilizing the triangle inequality and inequality (1.4.3) in Bierens (1994) that

$$\begin{aligned} E\left[\left|Y_{j,n}^{(s)}/d_{nj}\right|^q\right] &= E\left[\left|4\left(u_{is}^2 - \sigma_{0,s}^2\right)\sum_{j=1}^{n} c_{ji,ts}^2 + 8u_{is}\sum_{l=1}^{i-1} u_{ls}\sum_{j=1}^{n} c_{ji,ts}c_{jl,ts}\right.\right. \\ &\qquad \left.\left. + 8u_{is}\sum_{\tau=1}^{s-1}\sum_{l=1}^{n} u_{l\tau}\sum_{j=1}^{n} c_{ji,ts}c_{jl,t\tau}\right|^q\right] \\ &\le 3^q 4^q E\left[\left|u_{is}^2 - \sigma_{0,s}^2\right|^q \left|\sum_{j=1}^{n} c_{ji,ts}^2\right|^q\right] \\ &\quad + 3^q 8^q E\left[|u_{is}|^q \left|\sum_{l=1}^{i-1} |u_{ls}| \sum_{j=1}^{n} |c_{ji,ts}||c_{jl,ts}|\right|^q\right] \\ &\quad + 3^q 8^q E\left[|u_{is}|^q \left|\sum_{\tau=1}^{s-1}\sum_{l=1}^{n} |u_{l\tau}| \sum_{j=1}^{n} |c_{ji,ts}||c_{jl,t\tau}|\right|^q\right]. \end{aligned}$$

Next observe that in light of the bounds in Lemma 1(iii) and (v),

$$E\left[\left|u_{is}^2-\sigma_{0,s}^2\right|^q\left|\sum_{j=1}^n c_{ji,ts}^2\right|^q\right]\le 2^q K^q E\left[E\left[|u_{is}|^q+\sigma_{0,s}^{2q}|\mathcal{F}_{n,(s-1)n+i}\right]\right]\le 2^{q+1}K^{q+1}.$$

Furthermore, in light of the bounds in Lemma 1(ii)–(vii), and observing that $c_{ij,ts}$ is measurable w.r.t. $\mathcal{B}_{n,s}\vee\mathcal{C}$ and $\mathcal{F}_{n,(s-1)n+i}$,

$$\begin{aligned}
&E\left[|u_{is}|^q\left|\sum_{l=1}^{i-1}|u_{ls}|\sum_{j=1}^n|c_{ji,ts}||c_{jl,ts}|\right|^q\right]\\
&\le E\left[E\left[|u_{is}|^q|\mathcal{F}_{n,(s-1)n+i}\right]\left(\sum_{j=1}^n|c_{ji,ts}|\sum_{l=1}^{i-1}|u_{ls}||c_{jl,ts}|\right)^q\right]\\
&\le KE\left[\left(\sum_{j=1}^n|c_{ji,ts}|^{1/p}|c_{ji,ts}|^{1/q}\sum_{l=1}^n|u_{ls}||c_{jl,ts}|\right)^q\right]\\
&\le KE\left[\left(\sum_{j=1}^n|c_{ji,ts}|\right)^{q/p}\sum_{j=1}^n|c_{ji,ts}|\left(\sum_{l=1}^n|u_{ls}||c_{jl,ts}|\right)^q\right]\\
&\le K^{1+q/p}E\left[\sum_{j=1}^n|c_{ji,ts}|E\left[\left(\sum_{l=1}^n|u_{ls}||c_{jl,ts}|\right)^q\,\middle|\,\mathcal{B}_{n,s}\vee\mathcal{C}\right]\right]\\
&\le K^{2+q/p}E\left[\sum_{j=1}^n|c_{ji,ts}|\right]\le K^{3+q/p}
\end{aligned}$$

and

$$\begin{aligned}
&E\left[|u_{is}|^q\left|\sum_{\tau=1}^{s-1}\sum_{l=1}^n|u_{l\tau}|\sum_{j=1}^n|c_{ji,ts}||c_{jl,t\tau}|\right|^q\right]\\
&\le E\left[E\left[|u_{is}|^q|\mathcal{F}_{n,(s-1)n+i}\right]T^q\sum_{\tau=1}^{s-1}\left(\sum_{j=1}^n|c_{ji,ts}|\sum_{l=1}^n|u_{l\tau}||c_{jl,t\tau}|\right)^q\right]\\
&\le KT^q\sum_{\tau=1}^{s-1}E\left[\left(\sum_{j=1}^n|c_{ji,ts}|\sum_{l=1}^n|u_{l\tau}||c_{jl,t\tau}|\right)^q\right]\le K^{2q+2}T^{q+1}
\end{aligned}$$

since

$$\begin{aligned}
E\left[\left(\sum_{j=1}^n|c_{ji,ts}|\sum_{l=1}^n|u_{l\tau}||c_{jl,t\tau}|\right)^q\right]&=\left\|\sum_{j=1}^n|c_{ji,ts}|\sum_{l=1}^n|u_{l\tau}||c_{jl,t\tau}|\right\|_q^q\\
&\le\left[\sum_{j=1}^n\sum_{l=1}^n\left\||c_{ji,ts}||u_{l\tau}||c_{jl,t\tau}|\right\|_q\right]^q
\end{aligned}$$

$$\leq \left[\sum_{j=1}^{n} \sum_{l=1}^{n} \|c_{ji,ts}\|_{2q} \sum_{l=1}^{n} \big\| |u_{l\tau}||c_{jl,t\tau}| \big\|_{2q} \right]^q$$

$$\leq \left[K^{1/(2q)} \sum_{j=1}^{n} \|c_{ji,ts}\|_{2+\delta} \sum_{l=1}^{n} \|c_{jl,t\tau}\|_{2+\delta} \right]^q \leq K^{2q+1/2},$$

where we used that in light of Assumption 1,

$$\big\| |u_{l\tau}||c_{jl,t\tau}| \big\|_{2q} = \big(E\big[|u_{l\tau}|^{2q}|c_{jl,t\tau}|^{2q}\big]\big)^{1/(2q)} = \big(E\big[|c_{jl,t\tau}|^{2q}E\big[|u_{l\tau}|^{2q}|\mathcal{B}_{n,\tau} \vee \mathcal{C}\big]\big]\big)^{1/(2q)}$$
$$\leq K^{1/(2q)}\|c_{jl,t\tau}\|_{2q}.$$

Consequently, $|Y_{n,j}/d_{nj}|$ is uniformly integrable. Observing furthermore that $\sum_{j=1}^n d_{jn} = 1$ and $\sum_{j=1}^n d_{jn}^2 \to 0$, it follows from the law of large numbers for martingale difference arrays given as Theorem 19.7 in Davidson (1994) that indeed $\sum_{i=1}^n Y_{n,i}^{(s)} \xrightarrow{p} 0$.

Probability Limit of $n^{-1}\sum_{i=1}^n s_{it}^{(4)} = 4n^{-1}\sum_{i=1}^n c_{it}\sum_{j=1}^{i-1} c_{ij,tt}u_{jt}$. It is readily seen that $n^{-1}\sum_{i=1}^n s_{it}^{(4)} = \sum_{i=1}^n Y_{i,n}$ with

$$Y_{i,n} = n^{-1}4u_{it}\sum_{j=i+1}^{n} c_{jt}c_{ji,tt}.$$

Let $\mathcal{G}_{n,i-1} = \mathcal{F}_{n,(t-1)n+i}$ and $\mathcal{G}_{n,0} = \mathcal{F}_{n,(t-2)n+n}$. Then clearly $E[Y_{n,i}|\mathcal{G}_{n,i-1}] = 0$ and $Y_{n,i}$ is $\mathcal{G}_{n,i}$-measurable, and thus $\{Y_{n,i}, \mathcal{G}_{n,i}, i = 1, \ldots, n\}$ is a martingale difference array. Next observe that with $d_{ni} = n^{-1}$, $q = 2+\delta$, and $1/p + 1/q = 1$, it follows upon utilizing the triangle and Hölder's inequalities and the bounds in Lemma 1(i)–(iii) and (v) that

$$E\big[|Y_{n,i}/d_{ni}|^{1+\delta/2}\big] \leq 4^{1+\delta/2}E\left[\left(\sum_{j=i+1}^{n}|c_{ji,tt}||c_{jt}|\right)^{1+\delta/2} E\big[|u_{it}|^{1+\delta/2}|\mathcal{F}_{n,(t-1)n+i}\big]\right]$$

$$\leq 4^{1+\delta/2}KE\left[\left(\sum_{j=i+1}^{n}|c_{ji,tt}|^{\frac{q-2}{q}}|c_{ji,tt}|^{\frac{2}{q}}|c_{jt}|\right)^{q/2}\right]$$

$$\leq 4^{1+\delta/2}KE\left[\left[\sum_{j=i+1}^{n}|c_{ji,tt}|\right]^{\frac{q}{2}\frac{q-2}{q}}\left[\sum_{j=i+1}^{n}|c_{ji,tt}||c_{jt}|^{q/2}\right]\right]$$

$$\leq 4^{1+\delta/2}K^{1+\delta/2}E\left[\left(\sum_{j=i+1}^{n}|c_{ij,tt}||c_{jt}|^{q/2}\right)\right]$$

$$\leq 4^{1+\delta/2}K^{1+\delta/2}\left(\sum_{j=i+1}^{n}E\big[|c_{ij,tt}|^2\big]E\big[|c_{jt}|^q\big]\right)^{1/2}$$

$$\leq 4^{1+\delta/2}K^{\frac{3}{2}+\delta/2}\left(E\left[\sum_{j=i+1}^{n}|c_{ij,tt}|^2\right]\right)^{1/2} \leq K^{2+\delta/2} < \infty.$$

Consequently, $|Y_{n,i}/d_{ni}|$ is uniformly integrable. Observing furthermore that $\sum_{i=1}^{n} d_{in} = 1$ and $\sum_{i=1}^{n} d_{in}^2 \to 0$, it follows from the law of large numbers for martingale difference arrays given as Theorem 19.7 in Davidson (1994) that $n^{-1}\sum_{i=1}^{n} \varsigma_{it}^{(4)} = \sum_{i=1}^{n} Y_{n,i} \xrightarrow{p} 0$.

Probability Limit of $n^{-1}\sum_{i=1}^{n} \varsigma_{it}^{(5)}$. Note that, in general, $E[c_{it}c_{ij,ts}u_{js}] \neq 0$ for $s < t$, which suggests that, in general, $\text{plim}_{n\to\infty} n^{-1}\sum_{i=1}^{n} \varsigma_{it}^{(5)} \neq 0$. However, recall that ultimately the probability limit of $n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n} \varsigma_{it}^{(5)}$ is of interest. We next show that when our moment conditions are based on forward orthogonalized innovations, the terms involving nonzero expectations sum to zero, and that in that case $\text{plim}_{n\to\infty} n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n} \varsigma_{it}^{(5)} = 0$.

Recalling the expressions for c_{it} and $c_{ij,ts}$ in (32) and (34), respectively, we obtain

$$
\begin{aligned}
\Delta_n^{(5)} &= n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n} \varsigma_{it}^{(5)} \\
&= n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n} 4c_{it} \sum_{s=1}^{t-1}\sum_{j=1}^{n} c_{ij,ts}u_{js} \\
&= 4n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n}\sum_{\tau=1}^{t} \lambda_\tau' h_{i\tau}' \pi_{\tau t} \sum_{v=1}^{t-1}\sum_{j=1}^{n} c_{ij,tv}u_{jv} \\
&= 4n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n}\sum_{\tau=1}^{t} \lambda_\tau' h_{i\tau}' \pi_{\tau t} \sum_{v=1}^{t-1}\sum_{j=1}^{n}\sum_{s=1}^{v} \lambda_s' a_{ijs}' \pi_{sv}\pi_{st}u_{jv} \\
&= 4n^{-1}\sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{\tau=1}^{t}\sum_{v=1}^{t-1}\sum_{s=1}^{v} \pi_{\tau t}\pi_{sv}\pi_{st} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_\tau' h_{i\tau}' \lambda_s' a_{ijs}' u_{jv} \\
&= 4n^{-1}\sum_{u=1}^{T}\sum_{\tau=1}^{u}\sum_{v=1}^{u-1}\sum_{s=1}^{v} \sigma_{0,u}^2 \pi_{\tau u}\pi_{su}\pi_{sv} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_\tau' h_{i\tau}' \lambda_s' a_{ijs}' u_{jv} \\
&= 4n^{-1}\sum_{u=1}^{T}\sum_{t=1}^{u}\sum_{v=1}^{u-1}\sum_{s=1}^{v} \sigma_{0,u}^2 \pi_{tu}\pi_{su}\pi_{sv} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ijs}' u_{jv} \\
&= \Delta_n^{(5,1)} + \Delta_n^{(5,2)}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_n^{(5,1)} &= 4n^{-1}\sum_{u=1}^{T}\sum_{t=1}^{u}\sum_{v=1}^{t-1}\sum_{s=1}^{v} \sigma_{0,u}^2 \pi_{tu}\pi_{su}\pi_{sv} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ijs}' u_{jv}, \\
\Delta_n^{(5,2)} &= 4n^{-1}\sum_{u=1}^{T}\sum_{t=1}^{u}\sum_{v=t}^{u-1}\sum_{s=1}^{v} \sigma_{0,u}^2 \pi_{tu}\pi_{su}\pi_{sv} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ijs}' u_{jv}.
\end{aligned}
$$

Observing that $\pi_{tu} = 0$ for $t > u$, we have

$$\begin{aligned}\Delta_n^{(5,1)} &= 4n^{-1} \sum_{u=1}^{T} \sum_{t=1}^{T-1} \sum_{v=1}^{t-1} \sum_{s=1}^{v} \sigma_{0,u}^2 \pi_{tu} \pi_{su} \pi_{sv} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ij,s}' u_{jv} \\ &= 4n^{-1} \sum_{t=1}^{T-1} \sum_{v=1}^{t-1} \sum_{s=1}^{v} (\pi_t \Sigma_{0,\sigma} \pi_s') \pi_{sv} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ij,s}' u_{jv} = 0\end{aligned}$$

since $\pi_t \Sigma_{0,\sigma} \pi_s' = 0$ for $t \neq s$ under the maintained orthogonal forward differencing transformation. To determine the probability limit of $\Delta_n^{(5,2)}$, we can focus on the summands $n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ij,s}' u_{jv}$ for given u, t, ν, and s. Observe that in the above sum, $v \geq t$ and $v \geq s$. Define $n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_t' h_{it}' \lambda_s' a_{ij,s}' u_{jv} = \sum_{i=1}^{n} Y_{i,n}$ with

$$Y_{i,n} = n^{-1} u_{iv} \sum_{j=1}^{n} \lambda_t' h_{jt}' \lambda_s' a_{ji,s}'.$$

Let $\mathcal{G}_{n,i-1} = \mathcal{F}_{n,(\nu-1)n+i}$ and $\mathcal{G}_{n,0} = \mathcal{F}_{n,(\nu-2)n+n}$. Then clearly $E[Y_{n,i} | \mathcal{G}_{n,i-1}] = 0$ and $Y_{n,i}$ is $\mathcal{G}_{n,i}$-measurable and, thus, $\{Y_{n,i}, \mathcal{G}_{n,i}, i = 1, \dots, n\}$ is a martingale difference array. Next observe that with $d_{ni} = n^{-1}$, $q = 1 + \delta/2$, and $1/p + 1/q = 1$, it follows, upon utilizing the triangle and Hölder's inequalities as well as Assumption 2(ii), that

$$\begin{aligned}E\left[|Y_{i,n}/d_{nj}|^q\right] &\leq E\left[|u_{iv}|^q \left(\sum_{j=1}^{n} |\lambda_t' h_{jt}'| |\lambda_s' a_{ji,s}'|\right)^q\right] \\ &\leq E\left[E\left[|u_{iv}|^q | \mathcal{F}_{n,(\nu-1)n+i}\right] \left(\sum_{j=1}^{n} |\lambda_t' h_{jt}'| |\lambda_s' a_{ji,s}'|\right)^q\right] \\ &\leq K \left\| \sum_{j=1}^{n} |\lambda_t' h_{jt}'| |\lambda_s' a_{ji,s}'| \right\|_q^q \leq K \left(\sum_{j=1}^{n} \left\| |\lambda_t' h_{jt}'| |\lambda_s' a_{ji,s}'| \right\|_q \right)^q \\ &\leq K \left(\sum_{j=1}^{n} \|\lambda_t' h_{jt}'\|_{2q} \|\lambda_s' a_{ji,s}'\|_{2q} \right)^q \leq p_*^{1+\delta/2} K^{2+\delta/2} \left(\sum_{j=1}^{n} \|\lambda_s' a_{ji,s}'\|_{2+\delta} \right)^{1+\delta/2} \\ &\leq p_*^{1+\delta/2} K^{2+\delta/2} \left(\sum_{r=1}^{p_t} \sum_{j=1}^{n} \|a_{ji,s}^r\|_{2+\delta} \right)^{1+\delta/2} \leq p_*^{2+\delta} K^{3+\delta} < \infty.\end{aligned}$$

Consequently, $|Y_{n,i}/d_{ni}|$ is uniformly integrable. Observing furthermore that $\sum_{i=1}^{n} d_{in} = 1$ and $\sum_{i=1}^{n} d_{in}^2 \to 0$, it follows from the law of large numbers for martingale difference arrays given as Theorem 19.7 in Davidson (1994) that $\sum_{i=1}^{n} Y_{n,i} \xrightarrow{p} 0$ and, thus, $\Delta_n^{(5,2)} \xrightarrow{p} 0$. Since we have shown that $\Delta_n^{(5,1)} = 0$, it follows that $\Delta_n^{(5)} = n^{-1} \sum_{t=1}^{T} \sigma_{0,t}^2 \sum_{i=1}^{n} s_{it}^{(5)} \xrightarrow{p} 0$.

Probability Limit of $n^{-1}\sum_{i=1}^{n} s_{it}^{(6)}$. Clearly $n^{-1}\sum_{i=1}^{n} s_{it}^{(6)} \xrightarrow{p} 0$ provided we show that for given t,

$$\Delta_{nt}^{(6)} = n^{-1}\sum_{i=1}^{n} 8 \sum_{j=1}^{i-1} c_{ij,tt}u_{jt}\sum_{s=1}^{t-1}\sum_{l=1}^{n} c_{il,ts}u_{ls} = \sum_{j=1}^{n-1} Y_{j,n} \xrightarrow{p} 0,$$

with $Y_{j,n} = 8n^{-1}u_{jt}\sum_{i=j+1}^{n} c_{ij,tt}\sum_{s=1}^{t-1}\sum_{l=1}^{n} c_{il,ts}u_{ls}$. Let $\mathcal{G}_{n,j-1} = \mathcal{F}_{n,(t-1)n+j}$ and $\mathcal{G}_{n,0} = \mathcal{F}_{n,(t-2)n+n}$. Then observing that $\sum_{i=j+1}^{n} c_{ij,tt}\sum_{s=1}^{t-1}\sum_{l=1}^{n} c_{il,ts}u_{ls}$ is measurable w.r.t. $\mathcal{G}_{n,j-1}$, it follows from Assumption 1 that $E[Y_{n,j}|\mathcal{G}_{n,j-1}] = 0$. Since $Y_{j,n}$ is $\mathcal{G}_{n,j}$-measurable, it follows further that $\{Y_{n,j}, \mathcal{G}_{n,j}, i = 1, \ldots, n\}$ is a martingale difference array.

Next observe that with $d_{jn} = n^{-1}$ and with $q = 1 + \delta/2$, it follows upon utilizing the Minkowski and Hölder inequalities and the bound in Lemma 1(v) that

$$\begin{aligned}
E\big[|Y_{j,n}/d_{jn}|^q\big] &= 8^q E\left[E\big[|u_{j,t}|^q|\mathcal{F}_{n,(t-1)n+j}\big]\left|\sum_{i=j+1}^{n} c_{ij,tt}\sum_{s=1}^{t-1}\sum_{l=1}^{n} c_{il,ts}u_{ls}\right|^q\right] \\
&\le 8^q K E\left[\left|\sum_{i=j+1}^{n} c_{ij,tt}\vartheta_{it-1}\right|^q\right] = 8^q K\left(\left\|\sum_{i=j+1}^{n} c_{ij,tt}\vartheta_{it-1}\right\|_q\right)^q \\
&\le 8^q K\left(\sum_{i=j+1}^{n} \|c_{ij,tt}\vartheta_{it-1}\|_q\right)^q \le 8^q K\left(\sum_{i=j+1}^{n} \|c_{ij,tt}\|_{2q}\|\vartheta_{it-1}\|_{2q}\right)^q,
\end{aligned}$$

with $\vartheta_{it-1} = \sum_{s=1}^{t-1}\sum_{l=1}^{n} c_{il,ts}u_{ls}$. Next observe that with $\overline{q} = 2q = 2 + \delta$ and $1/\overline{q} + 1/\overline{p} = 1$, we have upon utilizing the triangle and Hölder inequalities and the bounds in Lemma 1(ii) and (v),

$$\begin{aligned}
E\big[|\vartheta_{it-1}|^{\overline{q}}\big] &= E\left[\left|\sum_{s=1}^{t-1}\sum_{l=1}^{n} c_{il,ts}u_{ls}\right|^{\overline{q}}\right] \\
&\le E\left[\left(\sum_{s=1}^{t-1}\sum_{l=1}^{n} |c_{il,ts}|^{1/\overline{p}}|c_{il,ts}|^{1/\overline{q}}|u_{ls}|\right)^{\overline{q}}\right] \\
&\le T^{\overline{q}-1}\sum_{s=1}^{t-1} E\left[\left(\sum_{l=1}^{n} |c_{il,ts}|^{1/\overline{p}}|c_{il,ts}|^{1/\overline{q}}|u_{ls}|\right)^{\overline{q}}\right] \\
&\le T^{\overline{q}-1}\sum_{s=1}^{t-1} E\left[\left(\sum_{l=1}^{n} |c_{il,ts}|\right)^{\overline{q}//\overline{p}}\left(\sum_{l=1}^{n} |c_{il,ts}||u_{ls}|^{\overline{q}}\right)\right] \\
&\le T^{\overline{q}-1}K^{\overline{q}/\overline{p}}\sum_{s=1}^{t-1} E\left[\sum_{l=1}^{n} |c_{il,ts}|E\big[|u_{ls}|^{\overline{q}}|\mathcal{F}_{n,(s-1)n+l}\big]\right] \\
&\le T^{\overline{q}-1}K^{\overline{q}/\overline{p}+1}\sum_{s=1}^{t-1} E\left[\sum_{l=1}^{n} |c_{il,ts}|\right] \le T^{\overline{q}}K^{\overline{q}+1}.
\end{aligned}$$

Hence,

$$E\big[|Y_{j,n}/d_{nj}|^q\big] \leq 8^q K\left(\sum_{i=j+1}^{n} \|c_{ij,tt}\|_{2q}\|\vartheta_{it-1}\|_{2q}\right)^q$$
$$\leq 8^q K\left(\big(T^{\overline{q}}K^{\overline{q}+1}\big)^{1/\overline{q}} \sum_{i=j+1}^{n} \|c_{ij,tt}\|_{2+\delta}\right)^q \leq 8^q T^q K^{2q+3/2} < \infty$$

in light of the bound in Lemma 1(iv). Consequently $|Y_{j,n}/d_{jn}|$ is uniformly integrable. Observing furthermore that $\sum_{j=1}^{n} d_{jn} = 1$ and $\sum_{j=1}^{n} d_{jn}^2 \to 0$, it follows from the law of large numbers for martingale difference arrays given as Theorem 19.7 in Davidson (1994) that indeed $\Delta_{nt}^{(6)} = \sum_{j=1}^{n} Y_{j,n} \xrightarrow{p} 0$. *Q.E.D.*

PROOF OF LEMMA 3: The proof utilizes a structure similar to that of the proof of Lemma 3.2 in Pötscher and Prucha (1997) and employs Lemma A1 in that book. We first prove the a.s. version of the proposition. Let $\Omega_0 \subseteq \Omega$ be a set of probability 1 such that for $\omega \in \Omega_0$, the following statement holds: $\mathcal{R}(\omega, \theta)$ is uniformly continuous on $\underline{\Theta}_\theta$, and as $n \to \infty$,

$$\widehat{\theta}_n(\omega) \to \theta_*(\omega), \tag{D.8}$$

$$\sup_{\theta\in\underline{\Theta}_\theta}\big|\mathcal{R}_n(\omega, \theta) - \mathcal{R}(\omega, \theta)\big| \to 0. \tag{D.9}$$

To apply Lemma A1, define $\Phi = \underline{\Theta}_\theta$, $\Lambda = \{1\}$, $g_n(\cdot) = \mathcal{R}_n(\omega, \cdot)$, $\overline{g}_n(\cdot) = \mathcal{R}(\omega, \cdot)$, $\varphi_n(\lambda) = \widehat{\theta}_n(\omega)$, and $\overline{\varphi}_n(\lambda) = \theta_*(\omega)$. Then (D.8) translates into condition (1) of Lemma A1. The assumption that $\mathcal{R}(\omega, \theta)$ is uniformly continuous on $\underline{\Theta}_\theta$ translates into condition (2)(i) of Lemma A1, and (D.9) translates into condition (2)(ii) of Lemma A1. It then follows from part (i) of Lemma A1 that as $n \to \infty$,

$$\mathcal{R}_n\big(\omega, \widehat{\theta}_n(\omega)\big) - \mathcal{R}\big(\omega, \theta_*(\omega)\big) \to 0,$$

which proves the a.s. version of the proposition. The convergence i.p. version of the proposition follows again from a standard subsequence argument. *Q.E.D.*

PROOF OF LEMMA 4: The proof utilizes a structure similar to that of the proof of Lemma 3.3 in Pötscher and Prucha (1997), and employs Lemma A1 in that book. We first prove the a.s. version of the proposition. Let $\Omega_0 \subseteq \Omega$ be a set of probability 1 on which $\mathfrak{m}(\omega, \theta) \in K$, $\Xi(\omega)$ is finite, $\Xi_n(\omega) - \Xi(\omega) \to 0$ as $n \to \infty$, and (36) holds. Fix $\omega \in \Omega_0$. To apply Lemma A1, define $\Phi = \mathbb{R}^m$, $\Lambda = \underline{\Theta}_\theta$, $\lambda = \theta$, $g_n(\varphi) = \varphi'\Xi_n(\omega)\varphi$, $\overline{g}_n(\varphi) = \overline{g}(\varphi) = \varphi'\Xi(\omega)\varphi$, $\varphi_n(\lambda) = \mathfrak{m}_n(\omega, \theta)$ and $\overline{\varphi}_n(\lambda) = \overline{\varphi}(\lambda) = \mathfrak{m}(\omega, \theta)$. In light of (36) we have

$$\sup_{\lambda\in\Lambda}\big\|n^{-1/2}\varphi_n(\lambda) - \overline{\varphi}(\lambda)\big\| = \sup_{\theta\in\underline{\Theta}_\theta}\big\|\mathfrak{m}_n(\omega, \theta) - \mathfrak{m}(\omega, \theta)\big\| \to 0 \quad \text{as } n \to \infty,$$

which verifies condition (1) of Lemma A1. Condition (2′)(i) of Lemma A1 regarding $\overline{g}(\varphi) = \varphi'\Xi(\omega)\varphi$ is clearly satisfied with $\Phi_* = K$. Next observe that clearly there exists sets

Φ_{**} and $\underline{\Phi}_{**}$ which are, respectively, open and compact, and satisfy $\Phi_* \subseteq \Phi_{**} \subseteq \underline{\Phi}_{**} \subseteq \Phi$. Then

$$\begin{aligned}\sup_{\varphi\in\Phi_{**}} \left|g_n(\varphi) - \overline{g}(\varphi)\right| &= \sup_{\varphi\in\Phi_{**}} \left|\varphi'\left[\Xi_n(\omega) - \Xi(\omega)\right]\varphi\right| \\ &\leq \sup_{\varphi\in\underline{\Phi}_{**}} \|\varphi\|^2 \left\|\Xi_n(\omega) - \Xi(\omega)\right\| \to 0\end{aligned}$$

as $n \to \infty$, since $\sup_{\varphi\in\underline{\Phi}_{**}} \|\varphi\|^2$ is finite given that $\underline{\Phi}_{**}$ is compact, and since $\|\Xi_n(\omega) - \Xi(\omega)\| \to 0$ by assumption. Thus also condition (2′)(ii) of Lemma A1 is satisfied. Thus by that lemma,

$$\sup_{\lambda\in\Lambda}\left|g_n\left(\varphi_n(\lambda)\right) - \overline{g}\left(\varphi(\lambda)\right)\right| = \sup_{\theta\in\underline{\Theta}_\theta}\left|\mathfrak{m}_n(\omega,\theta)'\Xi_n\mathfrak{m}_n(\omega,\theta) - \mathfrak{m}(\omega,\theta)'\Xi\mathfrak{m}(\omega,\theta)\right| \to 0$$

as $n \to \infty$, which proves the a.s. version of the proposition. The convergence i.p. version of the proposition follows again from a standard subsequence argument. *Q.E.D.*

PROOF OF LEMMA 5: We first prove part (i) of the lemma. In light of (19), (20), (23), and (24) it is readily seen that the typical element $n^{-1/2}\overline{m}_{rt,n}(\theta,\gamma)$ of $n^{-1/2}\overline{m}_n(\theta,\gamma)$ is given by

$$\begin{aligned}&n^{-1/2}\overline{m}_{rt,n}(\theta,\gamma) \\ &\quad= \sum_{s=t}^{T} \pi_{ts}(f,\gamma_\sigma) n^{-1} h'_{r,t} \Sigma_\varrho(\gamma_\varrho)^{-1/2} \underline{R}_s(\rho)(y_s - W_s\delta) \\ &\qquad+ \sum_{\tau=t}^{T}\sum_{s=t}^{T} \pi_{t\tau}(f,\gamma_\sigma)\pi_{ts}(f,\gamma_\sigma) n^{-1}(y_\tau - W_\tau\delta)'\underline{R}_\tau(\rho)'\Sigma_\varrho(\gamma_\varrho)^{-1/2}A_{r,t} \\ &\qquad\times \Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{R}_s(\rho)(y_s - W_s\delta),\end{aligned} \tag{D.10}$$

with $h_{r,t} = [h_{1rt}, \dots, h_{nrt}]'$ and $A_{r,t} = [a^r_{ij,t}]_{i,j=1,\dots,n}$. Observe furthermore that

$$\begin{aligned}&\underline{R}_\tau(\rho)'\Sigma_\varrho(\gamma_\varrho)^{-1/2}A_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{R}_s(\rho) \\ &\quad= \sum_{q=0}^{Q}\sum_{p=0}^{Q} \rho_q\rho_p \underline{M}_{q,\tau}\Sigma_\varrho(\gamma_\varrho)^{-1/2}A_{r.t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{M}_{p,s},\end{aligned} \tag{D.11}$$

with $\rho_0 = -1$ and $\underline{M}_{0,\tau} = \underline{M}_{0,s} = I$. By Assumption 5(i) the elements of $\Sigma_\varrho(\bar{\gamma}_{\varrho*})^{-1/2}$ are uniformly bounded and measurable w.r.t. $\mathcal{Z}_n \vee \mathcal{C}$. Thus in light of (D.10) and (D.11), it is readily seen that $n^{-1/2}\overline{m}_{rt,n}(\theta,\bar{\gamma}_*)$ is a sum composed of terms of the form considered in Assumption 5(ii) in part multiplied by elements of δ and $\pi_{t\tau}(f,\bar{\gamma}_{\sigma*})$. By Assumption 5(ii), the probability limits of those terms exist, are measurable w.r.t. $\mathcal{C}$, and bounded in absolute value. Given the continuity of $\pi_{t\tau}(\cdot,\cdot)$ it is then readily seen that

$$\mathfrak{m}_{rt}(\theta) = \operatorname*{plim}_{n\to\infty} n^{-1/2}\overline{m}_{rt,n}(\theta,\bar{\gamma}_*)$$

and, thus,

$$\mathfrak{m}(\theta) = \operatorname*{plim}_{n\to\infty} n^{-1/2}\overline{m}_n(\theta, \bar{\gamma}_*)$$

exists, and that $\mathfrak{m}(\theta)$ is measurable w.r.t. $\mathcal{C}$ for each $\theta \in \underline{\Theta}_\theta$.

By Assumption 1 we have $\Theta_{\gamma_\sigma} \subseteq [c_u, K_u/c_u]$ with $c_u > 0$. Recall further that $\underline{\Theta}_\lambda$, $\underline{\Theta}_\beta$, $\underline{\Theta}_\rho$, and $\underline{\Theta}_f$ are compact. In light of the boundedness conditions maintained by Assumption 5, and since $|\pi_{ts}(f, \gamma_\sigma)| \le C_\pi$ for $f \in \underline{\Theta}_f$ and $\gamma_\sigma \in [c_u, K_u/c_u]$ for some constant $C_\pi < \infty$ by Weierstrass' extreme value theorem, it then follows that $\mathfrak{m}(\theta) \in K$, where K is a compact subset of $\mathbb{R}^p$.

We next prove part (ii) of the lemma. Consider the typical element $n^{-1/2}\overline{m}_{rt,n}(\theta, \gamma)$ of $n^{-1/2}\overline{m}_n(\theta, \gamma)$ as defined in (D.10) and (D.11). Let $G_n(\theta, \gamma) = \partial n^{-1/2}\overline{m}_n(\theta, \gamma)/\partial\theta$ and let $g_{rt,n}(\theta, \gamma) = n^{-1/2}\partial\overline{m}_{rt,n}(\theta, \gamma)/\partial\theta$ denote the typical row of $G_n(\theta, \gamma)$. Then in light of (D.10) and (D.11) we see that $g_{rt,n}(\theta, \gamma)$ is composed of the terms

$$\begin{aligned}
g^{\delta}_{rt,n}(\theta, \gamma) &= n^{-1/2}\partial\overline{m}_{rt,n}(\theta, \gamma)/\partial\delta \\
&= -n^{-1}\sum_{s=t}^{T} h'_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{R}_s(\rho)W_s\pi_{ts}(f, \gamma_\sigma) \\
&\quad - 2n^{-1}\sum_{s=t}^{T}\sum_{\tau=t}^{T}(y_\tau - W_\tau\delta)'\underline{R}_\tau(\rho)'\Sigma_\varrho(\gamma_\varrho)^{-1/2}A_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{R}_s(\rho)W_s \\
&\quad \times \pi_{t\tau}(f, \gamma_\sigma)\pi_{ts}(f, \gamma_\sigma), \\
g^{\rho q}_{rt,n}(\theta, \gamma) &= n^{-1/2}\partial\overline{m}_{rt,n}(\theta, \gamma)/\partial\rho_q \\
&= -n^{-1}\sum_{s=t}^{T} h'_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{M}_{q,s}(y_s - W_s\delta)\pi_{ts}(f, \gamma_\sigma) \\
&\quad - 2n^{-1}\sum_{\tau=t}^{T}\sum_{s=t}^{T}(y_\tau - W_\tau\delta)'\underline{R}_\tau(\rho)'\Sigma_\varrho(\gamma_\varrho)^{-1/2}A_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{M}_{q,s}(y_s - W_s\delta) \\
&\quad \times \pi_{t\tau}(f, \gamma_\sigma)\pi_{ts}(f, \gamma_\sigma) \\
g^{f}_{rt,n}(\theta, \gamma) &= n^{-1/2}\partial\overline{m}_{rt,n}(\theta, \gamma)/\partial f \\
&= n^{-1}\sum_{s=t}^{T} h'_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{R}_s(\rho)(y_s - W_s\delta)\big[\partial\pi_{ts}(f, \gamma_\sigma)/\partial f\big] \\
&\quad + 2n^{-1}\sum_{\tau=t}^{T}\sum_{s=t}^{T}(y_\tau - W_\tau\delta)'\underline{R}_\tau(\rho)'\Sigma_\varrho(\gamma_\varrho)^{-1/2}A_{r,t}\Sigma_\varrho(\gamma_\varrho)^{-1/2}\underline{R}_s(\rho)(y_s - W_s\delta) \\
&\quad \times \pi_{t\tau}(f, \gamma_\sigma)\big[\partial\pi_{ts}(f, \gamma_\sigma)/\partial f\big].
\end{aligned}$$

From the above, we see that the derivatives $g^{\delta}_{rt,n}(\theta, \gamma_*)$, $g^{\rho q}_{rt,n}(\theta, \gamma_*)$, and $g^{f}_{rt,n}(\theta, \gamma_*)$ are again sums composed of terms of the form defined in Assumption 5, in part multiplied by elements of δ, ρ, $\Pi(f, \gamma_{\sigma*})$, and $\partial\Pi(f, \gamma_{\sigma*})/\partial f$, as well as products thereof. By Assumption 5, the probability limits of those terms exist, and they are measurable w.r.t. $\mathcal{C}$ and are bounded in absolute value. From this we see that $G(\theta) = \operatorname{plim}_{n\to\infty} G_n(\theta, \gamma_*)$ exists and is finite, and that $G(\theta)$ is $\mathcal{C}$-measurable for all $\theta \in \underline{\Theta}_\theta$. Furthermore, it is readily seen that $G(\theta)$ is continuous for each $\theta \in \underline{\Theta}_\theta$. Since $\underline{\Theta}_\theta$ is compact, this implies in turn that $G(\theta)$ is also uniformly continuous on $\underline{\Theta}_\theta$. *Q.E.D.*

PROOF OF THEOREM 3: As above, let $G_n(\theta, \gamma) = \partial n^{-1/2}\overline{m}_n(\theta, \gamma)/\partial\theta$ and $G(\theta) = \operatorname{plim}_{n\to\infty} \partial n^{-1/2}\overline{m}_n(\theta, \gamma_*)/\partial\theta$. It then follows from Lemma 5 that $G(\theta)$ exists and is finite for each $\theta \in \underline{\Theta}_\theta$, that $G(\theta)$ is $\mathcal{C}$-measurable for each $\theta \in \underline{\Theta}$, and that $G(\theta)$ is uniformly continuous on $\underline{\Theta}_\theta$. By Assumption 6,

$$\sup_{\theta\in\underline{\Theta}_\theta} \left\| G_n(\theta, \tilde{\gamma}_n) - G(\theta) \right\| = o_p(1)$$

since $\tilde{\gamma}_n \xrightarrow{p} \gamma_*$. Consequently, in light of Lemma 3,

$$\operatorname*{plim}_{n\to\infty} G_n(\tilde{\theta}_n, \tilde{\gamma}_n) = G(\theta_*) = G,$$

since $\tilde{\theta}_n \xrightarrow{p} \theta_*$ by Theorem 2. (Of course the above result continues to hold for any consistent estimator for θ_*.)

Given the definition of $\tilde{\theta}_n = \tilde{\theta}_n(\tilde{\gamma}_n)$ by (25), and given that $\tilde{\theta}_n$ was shown to be consistent by Theorem 2, the estimator satisfies the first order condition

$$\overline{m}_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n \frac{\partial \overline{m}_n(\tilde{\theta}_n, \tilde{\gamma}_n)}{\partial\theta} = 0.$$

By the mean value theorem, we have

$$\overline{m}_n(\tilde{\theta}_n, \tilde{\gamma}_n) = \overline{m}_n(\theta_{n,0}, \gamma_{n,0}) + \frac{\partial \overline{m}_n(\bar{\theta}_n, \bar{\gamma}_n)}{\partial\theta}(\tilde{\theta}_n - \theta_{n,0}) + \frac{\partial \overline{m}_n(\bar{\theta}_n, \bar{\gamma}_n)}{\partial\gamma}(\tilde{\gamma}_n - \gamma_{n,0}),$$

where $\bar{\theta}_n$ and $\bar{\gamma}_n$ are some between value (abusing notation in an obvious way) with $\bar{\theta}_n \xrightarrow{p} \theta_*$ and $\bar{\gamma}_n \xrightarrow{p} \gamma_*$. Substitution of the above expression into the first order condition yields

$$\begin{aligned}
&\frac{\partial n^{-1/2}\overline{m}_n(\tilde{\theta}_n, \tilde{\gamma}_n)}{\partial\theta'}\tilde{\Xi}_n \frac{\partial n^{-1/2}\overline{m}_n(\bar{\theta}_n, \bar{\gamma}_n)}{\partial\theta} n^{1/2}(\tilde{\theta}_n - \theta_{n,0}) \\
&\quad = -\frac{\partial n^{-1/2}\overline{m}_n(\tilde{\theta}_n, \tilde{\gamma}_n)}{\partial\theta'}\tilde{\Xi}_n \overline{m}_n(\theta_{n,0}, \gamma_{n,0}) \\
&\qquad - \frac{\partial n^{-1/2}\overline{m}_n(\tilde{\theta}_n, \tilde{\gamma}_n)}{\partial\theta'}\tilde{\Xi}_n \frac{\partial n^{-1/2}\overline{m}_n(\bar{\theta}_n, \bar{\gamma}_n)}{\partial\gamma} n^{1/2}(\tilde{\gamma}_n - \gamma_{n,0}).
\end{aligned}$$

From this we have

$$\begin{aligned}
&n^{1/2}(\tilde{\theta}_n - \theta_{n,0}) \\
&\quad = -\left[G_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n G_n(\bar{\theta}_n, \bar{\gamma}_n)\right]^+ G_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n \overline{m}_n(\theta_{n,0}, \gamma_{n,0}) \\
&\qquad - \left[G_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n G_n(\bar{\theta}_n, \bar{\gamma}_n)\right]^+ G_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n \frac{\partial n^{-1/2}\overline{m}_n(\bar{\theta}_n, \bar{\gamma}_n)}{\partial\gamma} \\
&\qquad \times n^{1/2}(\tilde{\gamma}_n - \gamma_{n,0}).
\end{aligned} \tag{D.12}$$

As shown above $G_n(\tilde{\theta}_n, \tilde{\gamma}_n) \xrightarrow{p} G$ and similarly $G_n(\bar{\theta}_n, \bar{\gamma}_n) \xrightarrow{p} G$. Hence under the maintained assumptions $[G_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n G_n(\bar{\theta}_n, \bar{\gamma}_n)]^+ \xrightarrow{p} (G'\Xi G)^{-1}$, observing that $G'\Xi G$ is positive definite a.s. and $G_n(\tilde{\theta}_n, \tilde{\gamma}_n)'\tilde{\Xi}_n \xrightarrow{p} G'\Xi$. Since $\partial n^{-1/2}\overline{m}_n(\bar{\theta}_n, \bar{\gamma}_n)/\partial\gamma \xrightarrow{p} 0$ by Assumption 6 and $n^{1/2}(\tilde{\gamma}_n - \gamma_{n,0}) = O_p(1)$, it follows that the term in the second line of (D.12) is $o_p(1)$. The limiting distribution of $\overline{m}_n(\theta_{n,0}, \gamma_{n,0})$ follows from Theorem 3 with $\gamma_\varrho = \gamma_{0,\varrho}$ and, thus, $\varrho_i^2 = \varrho_{0,i}^2 = \varrho_i^2(\gamma_{0,\varrho})$. Part (i) of that theorem establishes that $\overline{m}_n = \overline{m}_n(\theta_{n,0}, \gamma_{n,0})$ converges $\mathcal{C}$-stably in distribution (and hence in distribution) to a random vector to $V^{1/2}\xi$, where $V = \operatorname{diag}_{t=1}^{T-1}(V_t^h + 2V_t^a)$ and $n^{-1}\sum_{i=1}^n E[h_{it}'h_{it}|\mathcal{C}] \xrightarrow{p} V_t^h$, $n^{-1}\sum_{i=1}^n \sum_{j=1}^n E[a_{ij,t}'a_{ij,t}|\mathcal{C}] \xrightarrow{p} V_t^a$, and where $\xi \sim N(0, I_p)$, and ξ and $\mathcal{C}$ are independent. Thus utilizing (D.12),

$$n^{1/2}(\tilde{\theta}_n - \theta_{n,0}) = -\big(G'\Xi G\big)^{-1} G'\Xi\overline{m}_n(\theta_{n,0}, \gamma_{n,0}) + o_p(1),$$

$$Bn^{1/2}(\tilde{\theta}_n - \theta_{n,0}) = -B\big(G'\Xi G\big)^{-1} G'\Xi\overline{m}_n(\theta_{n,0}, \gamma_{n,0}) + o_p(1).$$

Observing further that by assumption Ξ, G, and B are $\mathcal{C}$-measurable and that $\Psi = (G'\Xi G)^{-1}G'\Xi V\Xi G(G'\Xi G)^{-1}$ and $B\Psi B'$ are positive definite a.s., it follows from part (ii) of Theorem 3 that

$$n^{1/2}(\tilde{\theta}_n - \theta_{n,0}) \xrightarrow{d} \Psi^{1/2}\xi_*,$$

$$Bn^{1/2}(\tilde{\theta}_n - \theta_{n,0}) \xrightarrow{d} (B\Psi B)^{1/2}\xi_{**},$$

where ξ_* and ξ_{**} are independent of $\mathcal{C}$, and $\xi_* \sim N(0, I_{p_\theta})$ and $\xi_{**} \sim N(0, I_q)$. *Q.E.D.*

D.4. *Proof of the Proposition on Variance–Covariance of Linear Quadratic Forms*

PROOF OF PROPOSITION D.1: Note that $\mathcal{Z}_n \subseteq \mathcal{B}_{n,t} \subseteq \mathcal{B}_{n,i,t}$ for all n, i, and t. It proves helpful to define

$$Q_t^A = u_t^{+\prime} A_t u_t^\times + a_t' u_t^+,$$

$$Q_t^B = u_t^{+\prime} B_t u_t^\times + b_t' u_t^+.$$

Recalling that $u_t^+ = [u_{1t}^+, \dots, u_{nt}^+]'$ and $u_t^\times = [u_{1t}^\times, \dots, u_{nt}^\times]'$ with

$$u_{it}^+ = \sum_{u=t}^T \pi_{tu} u_{iu} \quad \text{and} \quad u_{it}^\times = \sum_{u=t}^T \gamma_{tu} u_{iu}$$

we have

$$\begin{aligned} Q_t^A &= \sum_{i=1}^n \sum_{j=1}^n a_{ijt} u_{it}^+ u_{jt}^\times + \sum_{i=1}^n a_{it} u_{it}^+ \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ijt} \sum_{u=t}^T \sum_{v=t}^T \pi_{tu} u_{iu} \gamma_{tv} u_{jv} + \sum_{i=1}^n a_{it} \sum_{u=t}^T \pi_{tu} u_{iu}. \end{aligned} \tag{D.13}$$

From this we see that we can express Q_t^A and Q_t^B as

$$\begin{aligned} Q_t^A &= \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{a}_{ijt} \sum_{u=t}^{T}\sum_{v=t}^{T} \underline{\pi}_{tu}\underline{u}_{iu}\underline{\gamma}_{tv}\underline{u}_{jv} + \sum_{i=1}^{n} \underline{a}_{it} \sum_{u=t}^{T} \underline{\pi}_{tu}\underline{u}_{iu}, \\ Q_t^B &= \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{b}_{ijt} \sum_{u=t}^{T}\sum_{v=t}^{T} \underline{\pi}_{tu}\underline{u}_{iu}\underline{\gamma}_{tv}\underline{u}_{jv} + \sum_{i=1}^{n} \underline{b}_{it} \sum_{u=t}^{T} \underline{\pi}_{tu}\underline{u}_{iu}, \end{aligned} \tag{D.14}$$

where

$$\begin{aligned} &\underline{\pi}_{tu} = \sigma_u \pi_{tu}, && \underline{\gamma}_{tv} = \sigma_v \gamma_{tv}, \\ &\underline{a}_{ijt} = \varrho_i \varrho_j a_{ijt}, && \underline{a}_{it} = \varrho_i a_{it}, \\ &\underline{b}_{ijt} = \varrho_i \varrho_j b_{ijt}, && \underline{b}_{it} = \varrho_i b_{it}, \\ &\underline{u}_{iu} = u_{iu}/(\varrho_i \sigma_u), \end{aligned} \tag{D.15}$$

and thus

$$\begin{aligned} &E[\underline{u}_{it}|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = 0, && E[\underline{u}_{it}^2|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = 1, \\ &E[\underline{u}_{it}^3|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \mu_{3,it}/(\varrho_i^3 \sigma_t^3), && E[\underline{u}_{it}^4|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \mu_{4,it}/(\varrho_i^4 \sigma_t^4). \end{aligned} \tag{D.16}$$

In proving the theorem it will be convenient to work with the expressions for Q_t^A and Q_t^B as given in (D.14), and to switch notation by writing π_{tu}, γ_{tv}, a_{ijt}, a_{it}, b_{ijt}, b_{it}, u_{iu}, $\mu_{3,it}$, and $\mu_{4,it}$ for $\underline{\pi}_{tu}$, $\underline{\gamma}_{tv}$, $\underline{a}_{ijt}$, $\underline{a}_{it}$, $\underline{b}_{ijt}$, $\underline{b}_{it}$, $\underline{u}_{iu}$, $\mu_{3,it}/(\varrho_i^3\sigma_t^3)$, and $\mu_{4,it}/(\varrho_i^4\sigma_t^4)$. We note that the maintained assumptions on π_{tu}, γ_{tu}, a_{ijt}, a_{it}, b_{ijt}, and b_{it} are sufficiently general to allow for the absorption of the scaling factors σ_u into the weights π_{tu} and γ_{tu}, and for the absorption of the scaling factors ϱ_i and ϱ_j into a_{ijt}, a_{it}, b_{ijt}, and b_{it}. In the following discussion, we thus first proceed with

$$\begin{aligned} &E[u_{it}|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = 0, && E[u_{it}^2|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = 1, \\ &E[u_{it}^3|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \mu_{3,it}, && E[u_{it}^4|\mathcal{B}_{n,i,t} \vee \mathcal{C}] = \mu_{4,it}. \end{aligned}$$

Next observe that

$$\begin{aligned} &\sum_{u=t}^{T}\sum_{v=t}^{T} \pi_{tu}u_{iu}\gamma_{tv}u_{jv} \\ &\quad = \sum_{u=t}^{T} \pi_{tu}\gamma_{tu}u_{iu}u_{ju} + \sum_{u=t}^{T}\sum_{v=t}^{u-1} [\pi_{tu}\gamma_{tv}u_{iu}u_{jv} + \pi_{tv}\gamma_{tu}u_{iv}u_{ju}], \end{aligned} \tag{D.17}$$

where we adopt the convention that a sum is zero if the upper limit is less than the lower limit. With (D.17) the quadratic term in (D.13) can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}u_{iu}\gamma_{tv}u_{jv} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}u_{iu}u_{ju} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}\sum_{u=t}^{T}\sum_{v=t}^{u-1}[\pi_{tu}\gamma_{tv}u_{iu}u_{jv} + \pi_{tv}\gamma_{tu}u_{iv}u_{ju}],$$

where

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}u_{iu}u_{ju}\\
&= \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}u_{iu}u_{ju}\\
&= (1/2)\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ijt}+a_{jit})u_{iu}u_{ju}\\
&= \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n} a_{iit}u_{iu}^{2} + \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{i-1}(a_{ijt}+a_{jit})u_{iu}u_{ju}\\
&= \sum_{u=t}^{T}\sum_{i=1}^{n}\pi_{tu}\gamma_{tu}a_{iit}u_{iu}^{2} + \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\pi_{tu}\gamma_{tu}(a_{ijt}+a_{jit})u_{iu}u_{ju}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}\sum_{u=t}^{T}\sum_{v=t}^{u-1}[\pi_{tu}\gamma_{tv}u_{iu}u_{jv} + \pi_{tv}\gamma_{tu}u_{iv}u_{ju}]\\
&= \sum_{u=t}^{T}\sum_{v=t}^{u-1}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}[\pi_{tu}\gamma_{tv}u_{iu}u_{jv} + \pi_{tv}\gamma_{tu}u_{iv}u_{ju}]\\
&= \sum_{u=t}^{T}\sum_{v=t}^{u-1}\pi_{tu}\gamma_{tv}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}u_{iu}u_{jv} + \sum_{u=t}^{T}\sum_{v=t}^{u-1}\pi_{tv}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}u_{iv}u_{ju}\\
&= \sum_{u=t}^{T}\sum_{v=t}^{u-1}\pi_{tu}\gamma_{tv}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ijt}u_{iu}u_{jv} + \sum_{u=t}^{T}\sum_{v=t}^{u-1}\pi_{tv}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{jit}u_{iu}u_{jv}\\
&= \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=t}^{u-1}\sum_{j=1}^{n}[\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit}]u_{iu}u_{jv}.
\end{aligned}$$

Hence

$$Q_t^A = \sum_{u=t}^{T}\sum_{i=1}^{n} a_{it}\pi_{tu}u_{iu} + \sum_{u=t}^{T}\sum_{i=1}^{n} \pi_{tu}\gamma_{tu}a_{iit}u_{iu}^2$$
$$+ \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{j=1}^{i-1} \pi_{tu}\gamma_{tu}(a_{ijt}+a_{jit})u_{iu}u_{ju}$$
$$+ \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=t}^{u-1}\sum_{j=1}^{n}[\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit}]u_{iu}u_{jv}.$$

For $u \geq t$, let

$$Y_{iu}^{(t,A)} = \pi_{tu}\gamma_{tu}a_{iit}\left(u_{iu}^2-1\right) + a_{it}\pi_{tu}u_{iu} + \sum_{j=1}^{i-1}(a_{ijt}+a_{jit})\pi_{tu}\gamma_{tu}u_{iu}u_{ju}$$
$$+ \sum_{v=t}^{u-1}\sum_{j=1}^{n}[\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit}]u_{iu}u_{jv}.$$

Then

$$Q_t^A - \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n}a_{iit} = \sum_{u=t}^{T}\sum_{i=1}^{n}Y_{iu}^{(t,A)}.$$

It furthermore proves convenient to rewrite $Y_{iu}^{(t,A)}$ as

$$Y_{iu}^{(t,A)} = \left(u_{iu}^2-1\right)z_{iu}^{(t,A)} + u_{iu}Z_{iu}^{(t,A)}$$

with

$$z_{iu}^{(t,A)} = \pi_{tu}\gamma_{tu}a_{iit},$$
$$Z_{iu}^{(t,A)} = a_{it}\pi_{tu} + \sum_{j=1}^{i-1}(a_{ijt}+a_{jit})\pi_{tu}\gamma_{tu}u_{ju}$$
$$+ \sum_{v=t}^{u-1}\sum_{j=1}^{n}[\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit}]u_{jv}.$$

Similarly we have

$$Q_t^B - \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n}b_{iit} = \sum_{u=t}^{T}\sum_{i=1}^{n}Y_{iu}^{(t,B)},$$

where

$$Y_{iu}^{(t,B)} = \left(u_{iu}^2-1\right)z_{iu}^{(t,B)} + u_{iu}Z_{iu}^{(t,B)}$$

with

$$z_{iu}^{(t,B)} = \pi_{tu}\gamma_{tu}b_{iit},$$

$$Z_{iu}^{(t,B)} = b_{it}\pi_{tu} + \sum_{j=1}^{i-1}(b_{ijt}+b_{jit})\pi_{tu}\gamma_{tu}u_{ju} + \sum_{v=t}^{u-1}\sum_{j=1}^{n}[\pi_{tu}\gamma_{tv}b_{ijt}+\pi_{tv}\gamma_{tu}b_{jit}]u_{jv}$$

for $u \geq t$.

Observe that $z_{iu}^{(t,A)}$, $Z_{iu}^{(t,A)}$, $z_{iu}^{(t,B)}$, and $Z_{iu}^{(t,B)}$ as well as $z_{lu}^{(t,A)}$, $Z_{lu}^{(t,A)}$, $z_{lu}^{(t,B)}$, and $Z_{lu}^{(t,B)}$ for $l<i$ and $z_{l\tau}^{(t,A)}$, $Z_{l\tau}^{(t,A)}$, and $z_{l\tau}^{(t,B)}$, $Z_{l\tau}^{(t,B)}$ for all l and $t \leq \tau < u$ are measurable w.r.t. $\mathcal{B}_{n,i,u}$. Hence

$$\begin{aligned}
&E\big[Y_{iu}^{(t,A)}|\mathcal{C}\big]\\
&\quad = E\big[E\big[\big(u_{iu}^2-1\big)z_{iu}^{(t,A)} + u_{iu}Z_{iu}^{(t,A)}|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\quad = E\big[z_{iu}^{(t,A)}E\big[\big(u_{iu}^2-1\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big] + Z_{iu}^{(t,A)}E[u_{iu}|\mathcal{B}_{n,i,u}\vee\mathcal{C}]|\mathcal{C}\big] = 0,\\
&E\big[Y_{iu}^{(t,A)}Y_{lu}^{(t,B)}|\mathcal{C}\big]\\
&\quad = E\big[E\big[\big(\big(u_{iu}^2-1\big)z_{iu}^{(t,A)} + u_{iu}Z_{iu}^{(t,A)}\big)\big(\big(u_{lu}^2-1\big)z_{lu}^{(t,B)} + u_{lu}Z_{lu}^{(t,B)}\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\quad = E\big[z_{iu}^{(t,A)}\big(u_{lu}^2-1\big)z_{lu}^{(t,B)}E\big[\big(u_{iu}^2-1\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\qquad + E\big[z_{iu}^{(t,A)}u_{lu}Z_{lu}^{(t,B)}E\big[\big(u_{iu}^2-1\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\qquad + E\big[Z_{iu}^{(t,A)}u_{lu}Z_{lu}^{(t,B)}E[u_{iu}|\mathcal{B}_{n,i,u}\vee\mathcal{C}]|\mathcal{C}\big]]\\
&\qquad + E\big[Z_{iu}^{(t,A)}u_{lu}Z_{lu}^{(t,B)}E[u_{iu}|\mathcal{B}_{n,i,u}\vee\mathcal{C}]|\mathcal{C}\big]\\
&\quad = 0 \quad \text{for } l<i,
\end{aligned}$$

and

$$\begin{aligned}
&E\big[Y_{iu}^{(t,A)}Y_{l\tau}^{(t,B)}|\mathcal{C}\big]\\
&\quad = E\big[E\big[\big(\big(u_{iu}^2-1\big)z_{iu}^{(t,A)} + u_{iu}Z_{iu}^{(t,A)}\big)\big(\big(u_{l\tau}^2-1\big)z_{l\tau}^{(t,B)} + u_{l\tau}Z_{l\tau}^{(t,B)}\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\quad = E\big[z_{iu}^{(t,A)}\big(\big(u_{l\tau}^2-1\big)z_{l\tau}^{(t,B)} + u_{l\tau}Z_{l\tau}^{(t,B)}\big)E\big[\big(u_{iu}^2-1\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\qquad + E\big[Z_{iu}^{(t,A)}\big(\big(u_{l\tau}^2-1\big)z_{l\tau}^{(t,B)} + u_{l\tau}Z_{l\tau}^{(t,B)}\big)E[u_{iu}|\mathcal{B}_{n,i,u}\vee\mathcal{C}]|\mathcal{C}\big]\\
&\quad = 0 \quad \text{for all } l \text{ and } t\leq\tau<u.
\end{aligned}$$

Consequently,

$$E\big[Q_t^A|\mathcal{C}\big] - \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\sum_{i=1}^{n}E[a_{iit}|\mathcal{C}] = \sum_{u=t}^{T}\sum_{i=1}^{n}E\big[Y_{iu}^{(t,A)}|\mathcal{C}\big] = 0. \tag{D.18}$$

After rescaling, this will, as shown at the end of the proof, establish the claim in (D.3).

We next proceed to prove the claim in (D.4). In light of the above discussion, we have

$$\mathrm{Cov}\big(Q_t^A, Q_t^B|\mathcal{C}\big) = \sum_{u=t}^{T}\sum_{i=1}^{n} E\big[Y_{iu}^{(t,A)}Y_{iu}^{(t,B)}|\mathcal{C}\big].$$

Now observe that

$$\begin{aligned}
&E\big[Y_{iu}^{(t,A)}Y_{iu}^{(t,B)}|\mathcal{C}\big]\\
&\quad= E\big[\big(\big(u_{iu}^2-1\big)z_{iu}^{(t,A)} + u_{iu}Z_{iu}^{(t,A)}\big)\big(\big(u_{iu}^2-1\big)z_{iu}^{(t,B)} + u_{iu}Z_{iu}^{(t,B)}\big)|\mathcal{C}\big]\\
&\quad= E\big[\big(u_{iu}^2-1\big)^2 z_{iu}^{(t,A)}z_{iu}^{(t,B)}|\mathcal{C}\big]\\
&\qquad+ E\big[\big(u_{iu}^3-u_{iu}\big)z_{iu}^{(t,A)}Z_{iu}^{(t,B)}|\mathcal{C}\big] + E\big[\big(u_{iu}^3-u_{iu}\big)z_{iu}^{(t,B)}Z_{iu}^{(t,A)}|\mathcal{C}\big]\\
&\qquad+ E\big[u_{iu}^2 Z_{iu}^{(t,A)}Z_{iu}^{(t,B)}|\mathcal{C}\big]\\
&\quad= E\big[z_{iu}^{(t,A)}z_{iu}^{(t,B)}E\big[\big(u_{iu}^2-1\big)^2|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\qquad+ E\big[\big(z_{iu}^{(t,A)}Z_{iu}^{(t,B)} + z_{iu}^{(t,B)}Z_{iu}^{(t,A)}\big)E\big[\big(u_{iu}^3-u_{iu}\big)|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\qquad+ E\big[Z_{iu}^{(t,A)}Z_{iu}^{(t,B)}E\big[u_{iu}^2|\mathcal{B}_{n,i,u}\vee\mathcal{C}\big]|\mathcal{C}\big]\\
&\quad= E\big[(\mu_{4,iu}-1)z_{iu}^{(t,A)}z_{iu}^{(t,B)}|\mathcal{C}\big]\\
&\qquad+ E\big[\mu_{3,iu}z_{iu}^{(t,A)}Z_{iu}^{(t,B)} + \mu_{3,iu}z_{iu}^{(t,B)}Z_{iu}^{(t,A)}|\mathcal{C}\big]\\
&\qquad+ E\big[Z_{iu}^{(t,A)}Z_{iu}^{(t,B)}|\mathcal{C}\big].
\end{aligned}$$

Since the respective terms in $Z_{iu}^{(t,A)}$ and $Z_{iu}^{(t,B)}$ corresponding to u_{ju} and $u_{\bar{j}\bar{u}}$ are uncorrelated if either $j\neq\bar{j}$ or $u\neq\bar{u}$, and given that $E[u_{iu}^2|\mathcal{C}]=1$, we have

$$\begin{aligned}
E\big[(\mu_{4,iu}-1)z_{iu}^{(t,A)}z_{iu}^{(t,B)}|\mathcal{C}\big] &= E\big[(\mu_{4,iu}-1)a_{iit}b_{iit}|\mathcal{C}\big]\pi_{tu}^2\gamma_{tu}^2,\\
E\big[\mu_{3,iu}z_{iu}^{(t,A)}Z_{iu}^{(t,B)}|\mathcal{C}\big] &= E[\mu_{3,iu}a_{iit}b_{it}|\mathcal{C}]\pi_{tu}^2\gamma_{tu},\\
E\big[\mu_{3,iu}z_{iu}^{(t,B)}Z_{iu}^{(t,A)}|\mathcal{C}\big] &= E[\mu_{3,iu}b_{iit}a_{it}|\mathcal{C}]\pi_{tu}^2\gamma_{tu},\\
E\big[Z_{iu}^{(t,A)}Z_{iu}^{(t,B)}|\mathcal{C}\big] &= E[a_{it}b_{it}|\mathcal{C}]\pi_{tu}^2 + \sum_{j=1}^{i-1}E\big[(a_{ijt}+a_{jit})(b_{ijt}+b_{jit})|\mathcal{C}\big]\pi_{tu}^2\gamma_{tu}^2\\
&\quad+ \sum_{v=t}^{u-1}\sum_{j=1}^{n}E\big[(\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit})(\pi_{tu}\gamma_{tv}b_{ijt}+\pi_{tv}\gamma_{tu}b_{jit})|\mathcal{C}\big]
\end{aligned}$$

and thus

$$\begin{aligned}
&\mathrm{Cov}\big(Q_t^A, Q_t^B|\mathcal{C}\big)\\
&\quad= \sum_{u=t}^{T}\sum_{i=1}^{n}E\big[(\mu_{4,iu}-1)a_{iit}b_{iit}|\mathcal{C}\big]\pi_{tu}^2\gamma_{tu}^2
\end{aligned}$$

$$+\sum_{u=t}^{T}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{it}|\mathcal{C}]\pi_{tu}^2\gamma_{tu}+\sum_{u=t}^{T}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iit}a_{it}|\mathcal{C}]\pi_{tu}^2\gamma_{tu}$$

$$+\sum_{u=t}^{T}\sum_{i=1}^{n}E[a_{it}b_{it}|\mathcal{C}]\pi_{tu}^2$$

$$+\sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{j=1}^{i-1}E\big[(a_{ijt}+a_{jit})(b_{ijt}+b_{jit})|\mathcal{C}\big]\pi_{tu}^2\gamma_{tu}^2$$

$$+\sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=t}^{u-1}\sum_{j=1}^{n}E[(\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit})(\pi_{tu}\gamma_{tv}b_{ijt}+\pi_{tv}\gamma_{tu}b_{jit})|\mathcal{C}$$

which can be rewritten as

$$\begin{aligned}
&\operatorname{Cov}\big(Q_t^A,Q_t^B|\mathcal{C}\big)\\
&=\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}E\big[(\mu_{4,iu}-1)a_{iit}b_{iit}|\mathcal{C}\big]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}\sum_{i=1}^{n}E\big[\mu_{3,iu}(a_{iit}b_{it}+b_{iit}a_{it})|\mathcal{C}\big]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}^2\sum_{i=1}^{n}E[a_{it}b_{it}|\mathcal{C}]\\
&\quad+(1/2)\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[(a_{ijt}+a_{jit})(b_{ijt}+b_{jit})|\mathcal{C}\big]\\
&\quad-2\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}E[a_{iit}b_{iit}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=t}^{u-1}\sum_{j=1}^{n}E\big[\pi_{tu}^2\gamma_{tv}^2a_{ijt}b_{ijt}+\pi_{tu}\gamma_{tv}\pi_{tv}\gamma_{tu}(a_{ijt}b_{jit}+a_{jit}b_{ijt})+\pi_{tv}^2\gamma_{tu}^2a_{jit}b_{jit}|\mathcal{C}\big].
\end{aligned}$$

Now collecting terms gives

$$\begin{aligned}
&\operatorname{Cov}\big(Q_t^A,Q_t^B|\mathcal{C}\big)\\
&=\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}(\mu_{4,iu}-3)E[a_{iit}b_{iit}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}\sum_{i=1}^{n}\mu_{3,iu}E[a_{iit}b_{it}+b_{iit}a_{it}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}^2\sum_{i=1}^{n}E[a_{it}b_{it}|\mathcal{C}]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijt}+a_{ijt}b_{jit}|\mathcal{C}]\right)\\
&+\sum_{u=t}^{T}\sum_{v=t}^{u-1}\left(\pi_{tu}^2\gamma_{tv}^2+\pi_{tv}^2\gamma_{tu}^2\right)\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijt}|\mathcal{C}]\\
&+\sum_{u=t}^{T}\sum_{v=t}^{u-1}\pi_{tu}\gamma_{tv}\pi_{tv}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{jit}+a_{jit}b_{ijt}|\mathcal{C}],
\end{aligned}
$$

which can be further simplified as

$$
\begin{aligned}
&\operatorname{Cov}\left(Q_t^A,Q_t^B|\mathcal{C}\right)\\
&=\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}E\left[(\mu_{4,iu}-3)a_{iit}b_{iit}|\mathcal{C}\right]\\
&+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}\sum_{i=1}^{n}E\left[\mu_{3,iu}(a_{iit}b_{it}+b_{iit}a_{it})|\mathcal{C}\right]\\
&+\sum_{u=t}^{T}\pi_{tu}^2\sum_{i=1}^{n}E[a_{it}b_{it}|\mathcal{C}]\\
&+(1/2)\sum_{u=t}^{T}\sum_{v=t}^{T}\left(\pi_{tu}^2\gamma_{tv}^2+\pi_{tv}^2\gamma_{tu}^2\right)\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijt}|\mathcal{C}]\\
&+\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\gamma_{tv}\pi_{tv}\gamma_{tu}\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{jit}|\mathcal{C}].
\end{aligned}
$$

Observing that

$$
\sum_{u=t}^{T}\sum_{v=t}^{T}\left(\pi_{tu}^2\gamma_{tv}^2+\pi_{tv}^2\gamma_{tu}^2\right)=2\left(\pi_t\pi_t'\right)\left(\gamma_t\gamma_t'\right),\qquad \sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\gamma_{tv}\pi_{tv}\gamma_{tu}=\left(\pi_t\gamma_t'\right)^2,
$$

the above expression can be written more compactly as

$$
\begin{aligned}
&\operatorname{Cov}\left(Q_t^A,Q_t^B|\mathcal{C}\right)\\
&=\left(\pi_t\pi_t'\right)\left(\gamma_t\gamma_t'\right)E\left[\operatorname{tr}\left(A_tB_t'\right)|\mathcal{C}\right]+\left(\pi_t\gamma_t'\right)^2E\left[\operatorname{tr}(A_tB_t)|\mathcal{C}\right]+\left(\pi_t\pi_t'\right)E\left[a_t'b_t|\mathcal{C}\right]\\
&+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}E\left[(\mu_{4,iu}-3)a_{iit}b_{iit}|\mathcal{C}\right]\\
&+\sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}\sum_{i=1}^{n}E\left[\mu_{3,iu}(a_{iit}b_{it}+b_{iit}a_{it})|\mathcal{C}\right].
\end{aligned}
\tag{D.19}
$$

After rescaling, this will, as shown at the end of the proof, establish the claim in (D.4).

We now proceed to prove the claim in (D.5). In light of the above discussion, clearly

$$
\begin{aligned}
Q_s^B - \sum_{\bar{u}=s}^{T} \pi_{s\bar{u}} \gamma_{s\bar{u}} \sum_{i=1}^{n} b_{iis} &= \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ijs} u_{si}^{+} u_{sj}^{\times} + \sum_{i=1}^{n} b_{is} u_{si}^{+} \\
&= \sum_{\bar{u}=s}^{T} \sum_{i=1}^{n} Y_{i\bar{u}}^{(s,B)}
\end{aligned}
$$

with ($\overline{u} \geq s$)

$$
Y_{i\bar{u}}^{(s,B)} = \left(u_{i\bar{u}}^2 - 1\right) z_{i\bar{u}}^{(s,B)} + u_{i\bar{u}} Z_{i\bar{u}}^{(s,B)},
$$

where

$$
\begin{aligned}
z_{i\bar{u}}^{(s,B)} &= \pi_{s\bar{u}} \gamma_{s\bar{u}} b_{iis}, \\
Z_{i\bar{u}}^{(s,B)} &= b_{is} \pi_{s\bar{u}} + \sum_{j=1}^{i-1} (b_{ijs} + b_{jis}) \pi_{s\bar{u}} \gamma_{s\bar{u}} u_{j\bar{u}} \\
&\quad + \sum_{\bar{v}=s}^{\bar{u}-1} \sum_{j=1}^{n} [\pi_{s\bar{u}} \gamma_{s\bar{v}} b_{ijs} + \pi_{s\bar{v}} \gamma_{s\bar{u}} b_{jis}] u_{j\bar{v}}.
\end{aligned}
$$

In the following discussion, let $s < t$, $s \leq \bar{u} \leq T$ and $t \leq u \leq T$. Observe that for $\bar{u} < u$, and all i and l we have

$$
\begin{aligned}
&E\left[Y_{iu}^{(t,A)} Y_{l\bar{u}}^{(s,B)} | \mathcal{C}\right] \\
&\quad = E\left[E\left[Y_{iu}^{(t,A)} Y_{l\bar{u}}^{(s,B)} | \mathcal{B}_{n,i,u} \vee \mathcal{C}\right] | \mathcal{C}\right] = E\left[Y_{l\bar{u}}^{(s,B)} E\left[Y_{iu}^{(t,A)} | \mathcal{B}_{n,i,u} \vee \mathcal{C}\right] | \mathcal{C}\right] \\
&\quad = E\left[Y_{l\bar{u}}^{(s,B)} z_{iu}^{(t,A)} E\left[\left(u_{iu}^2 - 1\right) | \mathcal{B}_{n,i,u} \vee \mathcal{C}\right] | \mathcal{C}\right] + E\left[Y_{l\bar{u}}^{(s,B)} Z_{iu}^{(t,A)} E[u_{iu} | \mathcal{B}_{n,i,u} \vee \mathcal{C}] | \mathcal{C}\right] \\
&\quad = 0,
\end{aligned}
$$

and for $u < \bar{u}$ and all i and l we have

$$
\begin{aligned}
&E\left[Y_{iu}^{(t,A)} Y_{l\bar{u}}^{(s,B)} | \mathcal{C}\right] \\
&\quad = E\left[E\left[Y_{iu}^{(t,A)} Y_{l\bar{u}}^{(s,B)} | \mathcal{B}_{n,l,\bar{u}} \vee \mathcal{C}\right] | \mathcal{C}\right] = E\left[Y_{iu}^{(t,A)} E\left[Y_{l\bar{u}}^{(s,B)} | \mathcal{B}_{n,l,\bar{u}} \vee \mathcal{C}\right] | \mathcal{C}\right] \\
&\quad = E\left[Y_{iu}^{(t,A)} z_{i\bar{u}}^{(s,B)} E\left[\left(u_{i\bar{u}}^2 - 1\right) | \mathcal{B}_{n,l,\bar{u}} \vee \mathcal{C}\right] | \mathcal{C}\right] + E\left[Y_{iu}^{(t,A)} Z_{i\bar{u}}^{(s,B)} E[u_{i\bar{u}} | \mathcal{B}_{n,l,\bar{u}} \vee \mathcal{C}] | \mathcal{C}\right] \\
&\quad = 0.
\end{aligned}
$$

Furthermore, observe that for $u = \bar{u}$ and $l < i$,

$$
\begin{aligned}
&E\left[Y_{iu}^{(t,A)} Y_{lu}^{(s,B)} | \mathcal{C}\right] \\
&\quad = E\left[E\left[Y_{iu}^{(t,A)} Y_{lu}^{(s,B)} | \mathcal{B}_{n,i,u} \vee \mathcal{C}\right] | \mathcal{C}\right] = E\left[Y_{lu}^{(s,B)} E\left[Y_{iu}^{(t,A)} | \mathcal{B}_{n,i,u} \vee \mathcal{C}\right] | \mathcal{C}\right] \\
&\quad = E\left[Y_{lu}^{(s,B)} z_{iu}^{(t,A)} E\left[\left(u_{iu}^2 - 1\right) | \mathcal{B}_{n,i,u} \vee \mathcal{C}\right] | \mathcal{C}\right] + E\left[Y_{lu}^{(s,B)} Z_{iu}^{(t,A)} E[u_{iu} | \mathcal{B}_{n,i,u} \vee \mathcal{C}] | \mathcal{C}\right] \\
&\quad = 0,
\end{aligned}
$$

and for $u = \bar{u}$ and $l > i$,

$$
\begin{aligned}
&E\big[Y_{iu}^{(t,A)} Y_{lu}^{(s,B)} | \mathcal{C}\big] \\
&\quad = E\big[E\big[Y_{iu}^{(t,A)} Y_{lu}^{(s,B)} | \mathcal{B}_{n,l,u} \vee \mathcal{C}\big] | \mathcal{C}\big] = E\big[Y_{iu}^{(t,A)} E\big[Y_{lu}^{(s,B)} | \mathcal{B}_{n,l,u} \vee \mathcal{C}\big] | \mathcal{C}\big] \\
&\quad = E\big[Y_{iu}^{(t,A)} z_{lu}^{(s,B)} E\big[\big(u_{lu}^2 - 1\big) | \mathcal{B}_{n,l,u} \vee \mathcal{C}\big] | \mathcal{C}\big] + E\big[Y_{iu}^{(t,A)} Z_{lu}^{(s,B)} E[u_{lu} | \mathcal{B}_{n,l,u} \vee \mathcal{C}] | \mathcal{C}\big] \\
&\quad = 0.
\end{aligned}
$$

Thus, w.o.l.o.g. for $t > s$,

$$
E\big[Q_t^A Q_s^B | \mathcal{C}\big] = \sum_{u=t}^{T} \sum_{i=1}^{n} \sum_{\bar{u}=s}^{T} \sum_{l=1}^{n} E\big[Y_{iu}^{(t,A)} Y_{l\bar{u}}^{(s,B)} | \mathcal{C}\big] = \sum_{u=t}^{T} \sum_{i=1}^{n} E\big[Y_{iu}^{(t,A)} Y_{iu}^{(s,B)} | \mathcal{C}\big].
$$

We next calculate $E[Y_{iu}^{(t,A)} Y_{iu}^{(s,B)} | \mathcal{C}]$. Observe that since $u \geq t > s$, we have

$$
\begin{aligned}
&E\big[Y_{iu}^{(t,A)} Y_{iu}^{(s,B)} | \mathcal{C}\big] \\
&\quad = E\big[\big(\big(u_{iu}^2 - 1\big) z_{iu}^{(t,A)} + u_{iu} Z_{iu}^{(t,A)}\big)\big(\big(u_{iu}^2 - 1\big) z_{iu}^{(s,B)} + u_{iu} Z_{iu}^{(s,B)}\big) | \mathcal{C}\big] \\
&\quad = E\big[\big(u_{iu}^2 - 1\big)^2 z_{iu}^{(t,A)} z_{iu}^{(s,B)} | \mathcal{C}\big] \\
&\qquad + E\big[\big(u_{iu}^3 - u_{iu}\big) z_{iu}^{(t,A)} Z_{iu}^{(s,B)} | \mathcal{C}\big] + E\big[\big(u_{iu}^3 - u_{iu}\big) z_{iu}^{(s,B)} Z_{iu}^{(t,A)} | \mathcal{C}\big] \\
&\qquad + E\big[u_{iu}^2 Z_{iu}^{(t,A)} Z_{iu}^{(s,B)} | \mathcal{C}\big] \\
&\quad = E\big[z_{iu}^{(t,A)} z_{iu}^{(s,B)} E\big[\big(u_{iu}^2 - 1\big)^2 | \mathcal{B}_{n,i,u} \vee \mathcal{C}\big] | \mathcal{C}\big] \\
&\qquad + E\big[\big(z_{iu}^{(t,A)} Z_{iu}^{(s,B)} + z_{iu}^{(s,B)} Z_{iu}^{(t,A)}\big) E\big[\big(u_{iu}^3 - u_{iu}\big) | \mathcal{B}_{n,i,u} \vee \mathcal{C}\big] | \mathcal{C}\big] \\
&\qquad + E\big[Z_{iu}^{(t,A)} Z_{iu}^{(s,B)} E\big[u_{iu}^2 | \mathcal{B}_{n,i,u} \vee \mathcal{C}\big] | \mathcal{C}\big] \\
&\quad = E\big[(\mu_{4,iu} - 1) z_{iu}^{(t,A)} z_{iu}^{(s,B)} | \mathcal{C}\big] \\
&\qquad + E\big[\mu_{3,iu}\big(z_{iu}^{(t,A)} Z_{iu}^{(s,B)} + z_{iu}^{(s,B)} Z_{iu}^{(t,A)}\big) | \mathcal{C}\big] \\
&\qquad + E\big[Z_{iu}^{(t,A)} Z_{iu}^{(s,B)} | \mathcal{C}\big].
\end{aligned}
$$

Observe that

$$
E\big[(\mu_{4,iu} - 1) z_{iu}^{(t,A)} z_{iu}^{(s,B)} | \mathcal{C}\big] = E\big[(\mu_{4,iu} - 1) a_{iit} b_{iis} | \mathcal{C}\big] \pi_{tu} \gamma_{tu} \pi_{su} \gamma_{su}.
$$

To calculate $E[\mu_{3,iu} z_{iu}^{(t,A)} Z_{iu}^{(s,B)} | \mathcal{C}]$, observe that

$$
\begin{aligned}
z_{iu}^{(t,A)} Z_{iu}^{(s,B)} = {} & \pi_{tu} \gamma_{tu} a_{iit} b_{is} \pi_{su} \\
& + \sum_{j=1}^{i-1} \pi_{tu} \gamma_{tu} a_{iit} (b_{ijs} + b_{jis}) \pi_{su} \gamma_{su} u_{ju} \\
& + \sum_{v=s}^{u-1} \sum_{j=1}^{n} \pi_{tu} \gamma_{tu} a_{iit} [\pi_{su} \gamma_{sv} b_{ijs} + \pi_{sv} \gamma_{su} b_{jis}] u_{jv}.
\end{aligned}
$$

We next consider the conditional expectation of all the products composing $\mu_{3,iu} z_{iu}^{(t,A)} Z_{iu}^{(s,B)}$ with $u \geq t > s$:

$$E[\mu_{3,iu}\pi_{tu}\gamma_{tu}a_{iit}b_{is}\pi_{su}|\mathcal{C}] = \pi_{tu}\gamma_{tu}\pi_{su}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}].$$

For all j,

$$\begin{aligned} &E\big[\mu_{3,iu}\pi_{tu}\gamma_{tu}a_{iit}(b_{ijs}+b_{jis})\pi_{su}\gamma_{su}u_{ju}|\mathcal{C}\big] \\ &\quad = E\big[\mu_{3,iu}\pi_{tu}\gamma_{tu}a_{iit}(b_{ijs}+b_{jis})\pi_{su}\gamma_{su}E[u_{ju}|\mathcal{B}_{n,j,u}\vee\mathcal{C}]|\mathcal{C}\big] = 0. \end{aligned}$$

For $v \geq t > s$ and all j,

$$\begin{aligned} &E\big[\mu_{3,iu}\pi_{tu}\gamma_{tu}a_{iit}(\pi_{su}\gamma_{sv}b_{ijs}+\pi_{sv}\gamma_{su}b_{jis})u_{jv}\big] \\ &\quad = E\big[\mu_{3,iu}\pi_{tu}\gamma_{tu}a_{iit}(\pi_{su}\gamma_{sv}b_{ijs}+\pi_{sv}\gamma_{su}b_{jis})E[u_{jv}|\mathcal{B}_{n,j,v}\vee\mathcal{C}]\big] = 0 \end{aligned}$$

and for $t > v \geq s$ and all j,

$$\begin{aligned} &E\big[\mu_{3,iu}\pi_{tu}\gamma_{tu}a_{iit}(\pi_{su}\gamma_{sv}b_{ijs}+\pi_{sv}\gamma_{su}b_{jis})u_{jv}|\mathcal{C}\big] \\ &\quad = E\big[\mu_{3,iu}\pi_{tu}\gamma_{tu}(\pi_{su}\gamma_{sv}b_{ijs}+\pi_{sv}\gamma_{su}b_{jis})E[a_{iit}u_{jv}|\mathcal{B}_{n,j,v}\vee\mathcal{C}]|\mathcal{C}\big] \\ &\quad = \pi_{tu}\gamma_{tu}\pi_{su}\gamma_{sv}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}] + \pi_{tu}\gamma_{tu}\pi_{sv}\gamma_{su}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}] \neq 0. \end{aligned}$$

Thus,

$$\begin{aligned} E\big[\mu_{3,iu}z_{iu}^{(t,A)}Z_{iu}^{(s,B)}|\mathcal{C}\big] = {} & \pi_{tu}\gamma_{tu}\pi_{su}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}] \\ & + \sum_{v=s}^{t-1}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{sv}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}] \\ & + \sum_{v=s}^{t-1}\pi_{tu}\gamma_{tu}\pi_{sv}\gamma_{su}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}]. \end{aligned}$$

To calculate $E[\mu_{3,iu}z_{iu}^{(s,B)}Z_{iu}^{(t,A)}|\mathcal{C}]$, observe that

$$\begin{aligned} z_{iu}^{(s,B)}Z_{iu}^{(t,A)} = {} & \pi_{su}\gamma_{su}b_{iis}a_{it}\pi_{tu} \\ & + \sum_{j=1}^{i-1}\pi_{su}\gamma_{su}b_{iis}(a_{ijt}+a_{jit})\pi_{tu}\gamma_{tu}u_{ju} \\ & + \sum_{v=t}^{u-1}\sum_{j=1}^{n}\pi_{su}\gamma_{su}b_{iis}[\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit}]u_{jv}. \end{aligned}$$

Next we consider the conditional expectation of all the products composing $\mu_{3,iu} z_{iu}^{(s,B)} Z_{iu}^{(t,A)}$ with $u \geq t > s$:

$$E[\mu_{3,iu}\pi_{su}\gamma_{su}b_{iis}a_{it}\pi_{tu}|\mathcal{C}] = \pi_{su}\gamma_{su}\pi_{tu}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}].$$

For all j,

$$E\big[\mu_{3,iu}\pi_{su}\gamma_{su}b_{iis}(a_{ijt}+a_{jit})\pi_{tu}\gamma_{tu}u_{ju}|\mathcal{C}\big]$$
$$=E\big[\mu_{3,iu}\pi_{su}\gamma_{su}b_{iis}(a_{ijt}+a_{jit})\pi_{tu}\gamma_{tu}E[u_{ju}|\mathcal{B}_{n,j,v}\vee\mathcal{C}]|\mathcal{C}\big]=0.$$

For $v\geq t>s$ and all j,

$$E\big[\mu_{3,iu}\pi_{su}\gamma_{su}b_{iis}(\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit})u_{jv}|\mathcal{C}\big]$$
$$=E\big[\mu_{3,iu}\pi_{su}\gamma_{su}b_{iis}(\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit})E[u_{jv}|\mathcal{B}_{n,j,v}\vee\mathcal{C}]|\mathcal{C}\big]=0.$$

The case $t>v\geq s$ does not arise for this term. Thus

$$E\big[\mu_{3,iu}z_{iu}^{(s,B)}Z_{iu}^{(t,A)}|\mathcal{C}\big]=\pi_{su}\gamma_{su}\pi_{tu}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}].$$

We next calculate $E[Z_{iu}^{(t,A)}Z_{iu}^{(s,B)}|\mathcal{C}]$ for $u\geq t>s$. Recall that

$$Z_{iu}^{(t,A)}=a_{it}\pi_{tu}+\sum_{j=1}^{i-1}(a_{ijt}+a_{jit})\pi_{tu}\gamma_{tu}u_{ju}$$
$$+\sum_{v=t}^{u-1}\sum_{j=1}^{n}[\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit}]u_{jv},$$
$$Z_{iu}^{(s,B)}=b_{is}\pi_{su}+\sum_{l=1}^{i-1}(b_{ils}+b_{lis})\pi_{su}\gamma_{su}u_{lu}$$
$$+\sum_{\bar{v}=s}^{u-1}\sum_{l=1}^{n}[\pi_{su}\gamma_{s\bar{v}}b_{ils}+\pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\bar{v}}.$$

In the following discourse we consider the conditional expectation of all the products composing $Z_{iu}^{(t,A)}Z_{iu}^{(s,B)}$:

$$E[a_{it}\pi_{tu}b_{is}\pi_{su}|\mathcal{C}]=\pi_{tu}\pi_{su}E[a_{it}b_{is}|\mathcal{C}].$$

For all l,

$$E\big[a_{it}\pi_{tu}(b_{ils}+b_{lis})\pi_{su}\gamma_{su}u_{lu}|\mathcal{C}\big]$$
$$=E\big[E\big[a_{it}\pi_{tu}(b_{ils}+b_{lis})\pi_{su}\gamma_{su}u_{lu}|\mathcal{B}_{n,l,u}\vee\mathcal{C}\big]|\mathcal{C}\big]$$
$$=E\big[a_{it}\pi_{tu}(b_{ils}+b_{lis})\pi_{su}\gamma_{su}E[u_{lu}|\mathcal{B}_{n,l,u}\vee\mathcal{C}]|\mathcal{C}\big]=0.$$

For $\bar{v}\geq t\geq s$ and all l,

$$E\big[a_{it}\pi_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils}+\pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\bar{v}}|\mathcal{C}\big]$$
$$=E\big[E\big[a_{it}\pi_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils}+\pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\bar{v}}|\mathcal{B}_{n,l,\bar{v}}\vee\mathcal{C}\big]|\mathcal{C}\big]$$
$$=E\big[a_{it}\pi_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils}+\pi_{s\bar{v}}\gamma_{su}b_{lis}]E[u_{l\bar{v}}|\mathcal{B}_{n,l,\bar{v}}\vee\mathcal{C}]|\mathcal{C}\big]=0,$$

and for $t > \overline{v} \geq s$ and all l,

$$\begin{aligned}
&E\big[a_{it}\pi_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\overline{v}}|\mathcal{C}\big] \\
&\quad = E\big[E\big[a_{it}\pi_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\overline{v}}|\mathcal{B}_{n,l,\overline{v}} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = E\big[\pi_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis}]E[a_{it}u_{l\overline{v}}|\mathcal{B}_{n,l,\overline{v}} \vee \mathcal{C}]|\mathcal{C}\big] \\
&\quad = \pi_{tu}\pi_{su}\gamma_{s\bar{v}}E[a_{it}b_{ils}u_{l\overline{v}}|\mathcal{C}] + \pi_{tu}\pi_{s\bar{v}}\gamma_{su}E[a_{it}b_{lis}u_{l\overline{v}}|\mathcal{C}] \neq 0
\end{aligned}$$

in general.

For all j,

$$\begin{aligned}
&E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}b_{is}\pi_{su}|\mathcal{C}\big] \\
&\quad = E\big[E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}b_{is}\pi_{su}|\mathcal{B}_{n,j,u} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}b_{is}\pi_{su}E[u_{ju}|\mathcal{B}_{n,j,u} \vee \mathcal{C}]|\mathcal{C}\big] = 0.
\end{aligned}$$

For $j = l$,

$$\begin{aligned}
&E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}(b_{ijs} + b_{jis})\pi_{su}\gamma_{su}u_{ju}|\mathcal{C}\big] \\
&\quad = E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}(b_{ijs} + b_{jis})\pi_{su}\gamma_{su}E\big[u_{ju}^2|\mathcal{B}_{n,j,u} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = \pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}E\big[(a_{ijt} + a_{jit})(b_{ijs} + b_{jis})|\mathcal{C}\big],
\end{aligned}$$

and for $j \neq l$, say $j > l$,

$$\begin{aligned}
&E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}(b_{ils} + b_{lis})\pi_{su}\gamma_{su}u_{lu}|\mathcal{C}\big] \\
&\quad = E\big[E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}(b_{ils} + b_{lis})\pi_{su}\gamma_{su}u_{lu}|\mathcal{B}_{n,j,u} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}(b_{ils} + b_{lis})\pi_{su}\gamma_{su}u_{lu}E[u_{ju}|\mathcal{B}_{n,j,u} \vee \mathcal{C}]|\mathcal{C}\big] = 0.
\end{aligned}$$

For $u > \overline{v}$ and all j and l,

$$\begin{aligned}
&E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}[\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\bar{v}}|\mathcal{C}\big] \\
&\quad = E\big[E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}u_{ju}[\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\bar{v}}|\mathcal{B}_{n,j,u} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = E\big[(a_{ijt} + a_{jit})\pi_{tu}\gamma_{tu}[\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis}]u_{l\bar{v}}E[u_{ju}|\mathcal{B}_{n,j,u} \vee \mathcal{C}]|\mathcal{C}\big] = 0.
\end{aligned}$$

For all j and $v \geq t \geq s$,

$$\begin{aligned}
&E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}b_{is}\pi_{su}|\mathcal{C}\big] \\
&\quad = E\big[E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}b_{is}\pi_{su}|\mathcal{B}_{n,j,v} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})b_{is}\pi_{su}E[u_{jv}|\mathcal{B}_{n,j,v} \vee \mathcal{C}]|\mathcal{C}\big] = 0.
\end{aligned}$$

For $u > v$ and all j and l,

$$\begin{aligned}
&E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(b_{ils} + b_{lis})\pi_{su}\gamma_{su}u_{lu}|\mathcal{C}\big] \\
&\quad = E\big[E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(b_{ils} + b_{lis})\pi_{su}\gamma_{su}u_{lu}|\mathcal{B}_{n,l,u} \vee \mathcal{C}\big]|\mathcal{C}\big] \\
&\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(b_{ils} + b_{lis})\pi_{su}\gamma_{su}E[u_{lu}|\mathcal{B}_{n,l,u} \vee \mathcal{C}]|\mathcal{C}\big] = 0.
\end{aligned}$$

For $v \neq \bar{v}$, say $v > \bar{v}$, and all i and j, observing that $v \geq t$,

$$\begin{aligned} &E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis})u_{l\bar{v}}|\mathcal{C}\big] \\ &\quad = E\big[E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis})u_{l\bar{v}}|\mathcal{B}_{n,j,v} \vee \mathcal{C}\big]|\mathcal{C}\big] \\ &\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis})u_{l\bar{v}}E[u_{jv}|\mathcal{B}_{n,j,v} \vee \mathcal{C}]|\mathcal{C}\big] = 0. \end{aligned}$$

For $v \neq \bar{v}$, say $v < \bar{v}$, and all i and j, observing that $v \geq t$,

$$\begin{aligned} &E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis})u_{l\bar{v}}|\mathcal{C}\big] \\ &\quad = E\big[E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis})u_{l\bar{v}}|\mathcal{B}_{n,j,\bar{v}} \vee \mathcal{C}\big]|\mathcal{C}\big] \\ &\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{s\bar{v}}b_{ils} + \pi_{s\bar{v}}\gamma_{su}b_{lis})u_{jv}E[u_{l\bar{v}}|\mathcal{B}_{n,j,\bar{v}} \vee \mathcal{C}]|\mathcal{C}\big] = 0. \end{aligned}$$

For $v = \bar{v}$ and $j \neq l$, say $j > l$, observing that $v \geq t$,

$$\begin{aligned} &E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{sv}b_{ils} + \pi_{sv}\gamma_{su}b_{lis})u_{lv}|\mathcal{C}\big] \\ &\quad = E\big[E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{sv}b_{ils} + \pi_{sv}\gamma_{su}b_{lis})u_{lv}|\mathcal{B}_{n,j,v} \vee \mathcal{C}\big]|\mathcal{C}\big] \\ &\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{sv}b_{ils} + \pi_{sv}\gamma_{su}b_{lis})u_{lv}E[u_{jv}|\mathcal{B}_{n,j,v} \vee \mathcal{C}]|\mathcal{C}\big] = 0. \end{aligned}$$

For $v = \bar{v}$ and $j = l$, observing that $v \geq t$,

$$\begin{aligned} &E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{sv}b_{ijs} + \pi_{sv}\gamma_{su}b_{jis})u_{jv}|\mathcal{C}\big] \\ &\quad = E\big[E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})u_{jv}(\pi_{su}\gamma_{sv}b_{ijs} + \pi_{sv}\gamma_{su}b_{jis})u_{jv}|\mathcal{B}_{n,j,v} \vee \mathcal{C}\big]|\mathcal{C}\big] \\ &\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{sv}b_{ijs} + \pi_{sv}\gamma_{su}b_{jis})E\big[u_{jv}^2|\mathcal{B}_{n,j,v} \vee \mathcal{C}\big]|\mathcal{C}\big] \\ &\quad = E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{sv}b_{ijs} + \pi_{sv}\gamma_{su}b_{jis})|\mathcal{C}\big]. \end{aligned}$$

Thus,

$$\begin{aligned} &E\big(Z_{iu}^{(t,A)}Z_{iu}^{(s,B)}|\mathcal{C}\big) \\ &\quad = \pi_{tu}\pi_{su}E[a_{it}b_{is}|\mathcal{C}] \\ &\qquad + \sum_{j=1}^{i-1}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}E\big[(a_{ijt} + a_{jit})(b_{ijs} + b_{jis})|\mathcal{C}\big] \\ &\qquad + \sum_{v=t}^{u-1}\sum_{j=1}^{n}E\big[(\pi_{tu}\gamma_{tv}a_{ijt} + \pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{sv}b_{ijs} + \pi_{sv}\gamma_{su}b_{jis})|\mathcal{C}\big] \\ &\qquad + \pi_{tu}\pi_{su}\sum_{\bar{v}=s}^{t-1}\gamma_{s\bar{v}}\sum_{l=1}^{n}E[a_{it}b_{ils}u_{l\bar{v}}|\mathcal{C}] \\ &\qquad + \pi_{tu}\gamma_{su}\sum_{\bar{v}=s}^{t-1}\pi_{s\bar{v}}\sum_{l=1}^{n}E[a_{it}b_{lis}u_{l\bar{v}}|\mathcal{C}]. \end{aligned}$$

Recall that

$$\begin{aligned}
E\big[Q_t^A Q_s^B|\mathcal{C}\big] &= \sum_{u=t}^{T}\sum_{i=1}^{n} E\big[Y_{iu}^{(t,A)} Y_{iu}^{(s,B)}|\mathcal{C}\big] \\
&= \sum_{u=t}^{T}\sum_{i=1}^{n} E\big[(\mu_{4,iu}-1) z_{iu}^{(t,A)} z_{iu}^{(s,B)}|\mathcal{C}\big] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n} E\big[\mu_{3,iu}\big(z_{iu}^{(t,A)} Z_{iu}^{(s,B)} + z_{iu}^{(s,B)} Z_{iu}^{(t,A)}\big)|\mathcal{C}\big] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n} E\big[Z_{iu}^{(t,A)} Z_{iu}^{(s,B)}|\mathcal{C}\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
&E\big[Q_t^A Q_s^B|\mathcal{C}\big] \\
&= \sum_{u=t}^{T}\sum_{i=1}^{n} E\big[(\mu_{4,iu}-1) a_{iit} b_{iis}|\mathcal{C}\big]\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su} \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n} \pi_{tu}\gamma_{tu}\pi_{su} E[\mu_{3,iu} a_{iit} b_{is}|\mathcal{C}] + \sum_{u=t}^{T}\sum_{i=1}^{n} \pi_{su}\gamma_{su}\pi_{tu} E[\mu_{3,iu} b_{iis} a_{it}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=s}^{t-1} \pi_{tu}\gamma_{tu}\pi_{su}\gamma_{sv} \sum_{j=1}^{n} E[\mu_{3,iu} a_{iit} b_{ijs} u_{jv}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=s}^{t-1} \pi_{tu}\gamma_{tu}\pi_{sv}\gamma_{su} \sum_{j=1}^{n} E[\mu_{3,iu} a_{iit} b_{jis} u_{jv}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n} \pi_{tu}\pi_{su} E[a_{it} b_{is}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{j=1}^{i-1} \pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su} E\big[(a_{ijt}+a_{jit})(b_{ijs}+b_{jis})|\mathcal{C}\big] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n}\sum_{v=t}^{u-1}\sum_{j=1}^{n} E\big[(\pi_{tu}\gamma_{tv} a_{ijt} + \pi_{tv}\gamma_{tu} a_{jit})(\pi_{su}\gamma_{sv} b_{ijs} + \pi_{sv}\gamma_{su} b_{jis})|\mathcal{C}\big] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n} \pi_{tu}\pi_{su} \sum_{\bar{v}=s}^{t-1} \gamma_{s\bar{v}} \sum_{l=1}^{n} E[a_{it} b_{ils} u_{l\bar{v}}|\mathcal{C}] \\
&\quad + \sum_{u=t}^{T}\sum_{i=1}^{n} \pi_{tu}\gamma_{su} \sum_{\bar{v}=s}^{t-1} \pi_{s\bar{v}} \sum_{l=1}^{n} E[a_{it} b_{lis} u_{l\bar{v}}|\mathcal{C}]
\end{aligned}$$

and

$$
\begin{aligned}
&E\big[Q_t^A Q_s^B|\mathcal{C}\big]\\
&\quad=\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[(\mu_{4,iu}-1)a_{iit}b_{iis}|\mathcal{C}\big]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}]+\sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{i=1}^{n}E[a_{it}b_{is}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}\sum_{j=1}^{i-1}E\big[(a_{ijt}+a_{jit})(b_{ijs}+b_{jis})|\mathcal{C}\big]\\
&\qquad+\sum_{u=t}^{T}\sum_{v=t}^{u-1}\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[(\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{sv}b_{ijs}+\pi_{sv}\gamma_{su}b_{jis})|\mathcal{C}\big]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{\bar{v}=s}^{t-1}\gamma_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{ils}u_{l\bar{v}}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{su}\sum_{\bar{v}=s}^{t-1}\pi_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{lis}u_{l\bar{v}}|\mathcal{C}].
\end{aligned}
$$

From this we have

$$
\begin{aligned}
&E\big(Q_t^A Q_s^B|\mathcal{C}\big)\\
&\quad=\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[(\mu_{4,iu}-1)a_{iit}b_{iis}|\mathcal{C}\big]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}]+\sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{i=1}^{n}E[a_{it}b_{is}|\mathcal{C}]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}\sum_{j=1}^{i-1}E\big[(a_{ijt}+a_{jit})(b_{ijs}+b_{jis})|\mathcal{C}\big]\\
&+\sum_{u=t}^{T}\sum_{v=t}^{u-1}\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[(\pi_{tu}\gamma_{tv}a_{ijt}+\pi_{tv}\gamma_{tu}a_{jit})(\pi_{su}\gamma_{sv}b_{ijs}+\pi_{sv}\gamma_{su}b_{jis})|\mathcal{C}\big]\\
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]\\
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}]\\
&+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{\bar{v}=s}^{t-1}\gamma_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{ils}u_{l\bar{v}}|\mathcal{C}]\\
&+\sum_{u=t}^{T}\pi_{tu}\gamma_{su}\sum_{\bar{v}=s}^{t-1}\pi_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{lis}u_{l\bar{v}}|\mathcal{C}],
\end{aligned}
$$

$$
\begin{aligned}
&E\big[Q_t^A Q_s^B|\mathcal{C}\big]\\
&\quad=\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[(\mu_{4,iu}-1)a_{iit}b_{iis}|\mathcal{C}\big]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}]+\sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{i=1}^{n}E[a_{it}b_{is}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijs}+a_{jit}b_{jis}|\mathcal{C}]\\
&\quad-2\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E[a_{iit}b_{iis}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\sum_{v=t}^{u-1}(\pi_{tu}\gamma_{tv}\pi_{su}\gamma_{sv}+\pi_{tv}\gamma_{tu}\pi_{sv}\gamma_{su})\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijs}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\sum_{v=t}^{u-1}(\pi_{tu}\gamma_{tv}\pi_{sv}\gamma_{su}+\pi_{tv}\gamma_{tu}\pi_{su}\gamma_{sv})\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{jis}|\mathcal{C}]\\
&\quad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]
\end{aligned}
$$

$$+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{\bar{v}=s}^{t-1}\gamma_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{ils}u_{l\bar{v}}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\gamma_{su}\sum_{\bar{v}=s}^{t-1}\pi_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{lis}u_{l\bar{v}}|\mathcal{C}],$$

and, furthermore,

$$E\big[Q_t^A Q_s^B|\mathcal{C}\big]$$
$$=\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[(\mu_{4,iu}-3)a_{iit}b_{iis}|\mathcal{C}\big]$$
$$+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}]+\sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{i=1}^{n}E[a_{it}b_{is}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\gamma_{tv}\pi_{su}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijs}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\gamma_{tv}\pi_{sv}\gamma_{su}\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{jis}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{\bar{v}=s}^{t-1}\gamma_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{ils}u_{l\bar{v}}|\mathcal{C}]$$
$$+\sum_{u=t}^{T}\pi_{tu}\gamma_{su}\sum_{\bar{v}=s}^{t-1}\pi_{s\bar{v}}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{lis}u_{l\bar{v}}|\mathcal{C}].$$

Observing that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{ijs}|\mathcal{C}]=E\big[\mathrm{tr}\big(A_tB_s'\big)|\mathcal{C}\big],$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}E[a_{ijt}b_{jis}|\mathcal{C}]=E\big[\operatorname{tr}(A_tB_s)|\mathcal{C}\big],$$

$$\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\gamma_{tv}\pi_{su}\gamma_{sv}=\sum_{u=t}^{T}\pi_{tu}\pi_{su}\sum_{v=t}^{T}\gamma_{tv}\gamma_{sv}=\big(\pi_t\pi_s'\big)\big(\gamma_t\gamma_s'\big),$$

$$\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\gamma_{tv}\pi_{sv}\gamma_{su}=\sum_{u=t}^{T}\pi_{tu}\gamma_{su}\sum_{v=t}^{T}\gamma_{tv}\pi_{sv}=\big(\pi_t\gamma_s'\big)\big(\pi_s\gamma_t'\big),$$

we have

$$\begin{aligned}
&E\big[Q_t^AQ_s^B|\mathcal{C}\big]\\
&\quad=\big(\pi_t\pi_s'\big)\big(\gamma_t\gamma_s'\big)E\big[\operatorname{tr}\big(A_tB_s'\big)|\mathcal{C}\big]+\big(\pi_t\gamma_s'\big)\big(\pi_s'\gamma_t\big)E\big[\operatorname{tr}(A_tB_s)|\mathcal{C}\big]\\
&\qquad+\big(\pi_t\pi_s'\big)E\big[a_t'b_s|\mathcal{C}\big]\\
&\qquad+\big(\pi_t\pi_s'\big)\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{ils}u_{lv}|\mathcal{C}]\\
&\qquad+\big(\pi_t\gamma_s'\big)\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{l=1}^{n}E[a_{it}b_{lis}u_{lv}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[(\mu_{4,iu}-3)a_{iit}b_{iis}|\mathcal{C}\big]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}]+\sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]\\
&\qquad+\sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}].
\end{aligned}\tag{D.20}$$

After rescaling this will, as shown at the end of the proof, establish the claim in (D.5).

The claims in (D.3)–(D.5) are readily obtained from (D.18)–(D.20) after reversing the rescaling operations of (D.14)–(D.16). More specifically, to reverse the rescaling operation of (D.14)–(D.16), we make replacements in (D.18)–(D.20),

$$\begin{aligned}
\pi_{tu}&\longmapsto\sigma_u\pi_{tu}, & \gamma_{tv}&\longmapsto\sigma_v\gamma_{tv},\\
a_{ijt}&\longmapsto\varrho_i\varrho_ja_{ijt}, & a_{it}&\longmapsto\varrho_ia_{it},\\
b_{ijt}&\longmapsto\varrho_i\varrho_jb_{ijt}, & b_{it}&\longmapsto\varrho_ib_{it},\\
\mu_{3,it}&\longmapsto\mu_{3,it}/\big(\varrho_i^3\sigma_t^3\big), & \mu_{4,it}&\longmapsto\mu_{4,it}/\big(\varrho_i^4\sigma_t^4\big),\\
u_{it}&\longmapsto u_{it}/(\varrho_i\sigma_t),
\end{aligned}$$

or in corresponding matrix notation,

$$\pi_t \longmapsto \pi_t \Sigma_\sigma^{1/2}, \qquad \gamma_t \longmapsto \gamma_t \Sigma_\sigma^{1/2},$$
$$A_t \longmapsto \Sigma_\varrho^{1/2} A_t \Sigma_\varrho^{1/2}, \qquad a_t \longmapsto \Sigma_\varrho^{1/2} a_t,$$
$$B_t \longmapsto \Sigma_\varrho^{1/2} B_t \Sigma_\varrho^{1/2}, \qquad b_t \longmapsto \Sigma_\varrho^{1/2} b_t.$$

Applying those replacements to (D.18) yields

$$E\big[Q_t^A|\mathcal{C}\big] = \pi_t \Sigma_\sigma \gamma_t \operatorname{tr}\big[E(A_t \Sigma_\varrho|\mathcal{C})\big],$$

which proves the claim in (D.3). Applying the replacements to (D.19) yields

$$\begin{aligned}
&\operatorname{Cov}\big(Q_t^A, Q_t^B|\mathcal{C}\big)\\
&\quad= (\pi_t\Sigma_\sigma\pi_t')(\gamma_t\Sigma_\sigma\gamma_t')E\big[\operatorname{tr}\big(A_t\Sigma_\varrho B_t'\Sigma_\varrho\big)|\mathcal{C}\big] + (\pi_t\Sigma_\sigma\gamma_t')^2 E\big[\operatorname{tr}(A_t\Sigma_\varrho B_t\Sigma_\varrho)|\mathcal{C}\big]\\
&\qquad + (\pi_t\Sigma_\sigma\pi_t')E\big[a_t'\Sigma_\varrho b_t|\mathcal{C}\big] + \sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}^2\sum_{i=1}^{n}E\big[\big(\mu_{4,iu} - 3\varrho_i^4\sigma_u^4\big)a_{iit}b_{iit}|\mathcal{C}\big]\\
&\qquad + \sum_{u=t}^{T}\pi_{tu}^2\gamma_{tu}\sum_{i=1}^{n}E\big[\mu_{3,iu}(a_{iit}b_{it} + b_{iit}a_{it})|\mathcal{C}\big],
\end{aligned}$$

which proves the claim in (D.4). Applying the replacements to (D.20) yields

$$\begin{aligned}
&E\big[Q_t^A Q_s^B|\mathcal{C}\big]\\
&\quad= (\pi_t\Sigma_\sigma\pi_s')(\gamma_t\Sigma_\sigma\gamma_s')E\big[\operatorname{tr}\big(A_t\Sigma_\varrho B_s'\Sigma_\varrho\big)|\mathcal{C}\big] + (\pi_t\Sigma_\sigma\gamma_s')(\pi_s'\Sigma_\sigma\gamma_t)E\big[\operatorname{tr}(A_t\Sigma_\varrho B_s\Sigma_\varrho)|\mathcal{C}\big]\\
&\qquad + (\pi_t\Sigma_\sigma\pi_s')E\big[a_t'\Sigma_\varrho b_s|\mathcal{C}\big] + (\pi_t\Sigma_\sigma\pi_s')\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[\varrho_i^2 a_{it}b_{ijs}u_{jv}|\mathcal{C}\big]\\
&\qquad + (\pi_t\Sigma_\sigma\gamma_s')\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E\big[\varrho_i^2 a_{it}b_{jis}u_{jv}|\mathcal{C}\big]\\
&\qquad + \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\gamma_{su}\sum_{i=1}^{n}E\big[\big(\mu_{4,iu} - 3\varrho_i^4\sigma_u^4\big)a_{iit}b_{iis}|\mathcal{C}\big]\\
&\qquad + \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{i=1}^{n}E[\mu_{3,iu}a_{iit}b_{is}|\mathcal{C}] + \sum_{u=t}^{T}\pi_{su}\gamma_{su}\pi_{tu}\sum_{i=1}^{n}E[\mu_{3,iu}b_{iis}a_{it}|\mathcal{C}]\\
&\qquad + \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\pi_{su}\sum_{v=s}^{t-1}\gamma_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{ijs}u_{jv}|\mathcal{C}]\\
&\qquad + \sum_{u=t}^{T}\pi_{tu}\gamma_{tu}\gamma_{su}\sum_{v=s}^{t-1}\pi_{sv}\sum_{i=1}^{n}\sum_{j=1}^{n}E[\mu_{3,iu}a_{iit}b_{jis}u_{jv}|\mathcal{C}].
\end{aligned}$$

which proves the claim in (D.5). *Q.E.D.*

D.5. *Forward Differencing Transformations*

In this section we provide some further background and generalizations of the forward differencing transformation considered in Proposition 1. This discussion will utilize the following general observations. For $T^+ \leq T$, let F be a $T^+ \times T$ upper triangular matrix and let U be an upper triangular $T^+ \times T^+$ matrix such that $UU' = F\Sigma_\sigma F'$, where $\Sigma_\sigma = \operatorname{diag}(\sigma_1^2, \dots, \sigma_T^2)$ is some diagonal matrix. Then, as is known from standard linear algebra, U^{-1} is also upper triangular and, thus, so is the $T^+ \times T$ matrix $\Pi = (\pi_{ts}) = U^{-1}F$. Furthermore observe that

$$\Pi\Sigma_\sigma\Pi' = \left[\pi_t \Sigma \pi_s'\right] = U^{-1}F\Sigma F' U'^{-1} = U^{-1}UU'U'^{-1} = I.$$

Let $\pi_t = [0, \dots, 0, \pi_{tt}, \dots, \pi_{tT}]$ denote the tth row of Π. Then

$$\pi_t \Sigma_\sigma \pi_s' = \sum_{\tau=\max(t,s)}^{T} \pi_{t\tau}\pi_{s\tau}\sigma_\tau^2 = \begin{cases} 1 & \text{for } t = s \\ 0 & \text{for } t \neq s. \end{cases}$$

Now let f be some $T \times 1$ vector, and suppose that $F = F(f)$ and $Ff = 0$; then also $\Pi f = U^{-1}Ff = 0$. Furthermore note that $U = F\Sigma_\sigma\Pi'$ in that $\Pi\Sigma_\sigma F' = U^{-1}F\Sigma_\sigma F' = U^{-1}UU' = U'$. To denote the dependence of Π on f and $\gamma_\sigma = [\sigma_1^2, \dots, \sigma_T^2]'$, we sometimes write $\Pi = \Pi(f, \gamma_\sigma)$.

D.5.1. *Single Factor*

Consider the disturbance process $\eta_{it} = \mu_i f_t + u_{it}$ in (14), and let

$$\underset{T-1\times T}{F} = \begin{pmatrix} 1 & -f_1/f_2 & 0 & \dots & 0 & 0 \\ 0 & 1 & -f_2/f_3 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -f_{T-2}/f_{T-1} & 0 \\ 0 & 0 & \dots & 0 & 1 & -f_{T-1}/f_T \end{pmatrix}$$

and $f = [f_1, \dots, f_T]$. Then clearly $Ff = 0$. Now, corresponding to the F above, let $\Pi = U^{-1}F = (\pi_{ts})$ be an upper triangular $T-1 \times T$ matrix with $UU' = F\Sigma_\sigma F'$ as above. Then by construction $\Pi f = 0$ and $\Pi\Sigma_\sigma\Pi' = I_{T-1}$. Furthermore, as claimed by Proposition 1, the elements of Π can be expressed as

$$\begin{aligned} \pi_{tt} &= \pi_{tt}(f, \gamma_\sigma) = (\sqrt{\phi_{t+1}/\phi_t})/\sigma_t, \\ \pi_{ts} &= \pi_{ts}(f, \gamma_\sigma) = -f_t f_s(\sqrt{\phi_{t+1}/\phi_t})/\left(\phi_{t+1}\sigma_t\sigma_s^2\right) \quad \text{for } s > t, \\ \pi_{ts} &= 0 \quad \text{for } s < t, \end{aligned}$$

where $\phi_t = \sum_{\tau=t}^{T}(f_\tau/\sigma_\tau)^2$ and, thus, $\phi_t = (f_t/\sigma_t)^2 + \phi_{t+1}$. We now verify that indeed $\pi_t\Sigma_\sigma\pi_t' = 1$ and $\pi_t\Sigma_\sigma\pi_s' = 0$ for $t \neq s$. To see this observe that (with $s > t$)

$$\begin{aligned} \pi_t\Sigma_\sigma\pi_t' &= \sum_{\tau=t}^{T} \pi_{t\tau}^2\sigma_\tau^2 = \frac{\phi_{t+1}}{\phi_t}\frac{1}{\sigma_t^2}\sigma_t^2 + \frac{f_t^2}{\sigma_t^2}\frac{1}{\phi_t\phi_{t+1}}\sum_{\tau=t+1}^{T}\frac{f_\tau^2}{\sigma_\tau^4}\sigma_\tau^2 \\ &= \frac{\phi_{t+1}}{\phi_t} + \frac{f_t^2}{\sigma_t^2}\frac{1}{\phi_t} = \frac{\phi_{t+1} + (f_t/\sigma_t)^2}{\phi_t} = 1, \end{aligned}$$

$$
\begin{aligned}
\pi_t\pi_s' &= \sum_{\tau=s}^{T} \pi_{t\tau}\pi_{s\tau}\sigma_\tau^2 \\
&= -f_t f_s \sqrt{\frac{\phi_{t+1}}{\phi_t}} \frac{1}{\phi_{t+1}\sigma_t\sigma_s^2} \sqrt{\frac{\phi_{s+1}}{\phi_s}} \frac{1}{\sigma_s}\sigma_s^2 \\
&\quad + f_t \sqrt{\frac{\phi_{t+1}}{\phi_t}} \frac{1}{\phi_{t+1}\sigma_t} f_s \sqrt{\frac{\phi_{s+1}}{\phi_s}} \frac{1}{\phi_{s+1}\sigma_s} \sum_{\tau=s+1}^{T} f_\tau^2 \frac{1}{\sigma_\tau^4}\sigma_\tau^2 \\
&= \frac{f_t}{\sigma_t}\frac{f_s}{\sigma_s} \sqrt{\frac{\phi_{t+1}}{\phi_t}} \frac{1}{\phi_{t+1}} \sqrt{\frac{\phi_{s+1}}{\phi_s}} \left\{ -1 + \frac{1}{\phi_{s+1}} \sum_{\tau=s+1}^{T} (f_\tau/\sigma_\tau)^2 \right\} = 0.
\end{aligned}
$$

Now let $\eta_{it}^+ = \sum_{s=t}^{T} \pi_{ts}\eta_{is}$ and $u_{it}^+ = \sum_{s=t}^{T} \pi_{ts}u_{is}$ denote the corresponding forward differences. Then

$$
\eta_{it}^+ = \left(\sum_{s=t}^{T} \pi_{ts} f_s \right) \mu_i + \sum_{s=t}^{T} \pi_{ts} u_{it} = u_{it}^+,
$$

since $\Pi f = 0$. Furthermore, the property that $\Pi\Sigma_\sigma\Pi' = I_{T-1}$ is, in light of Proposition D.1, crucial to achieve that linear quadratic forms based on forward differenced disturbances are uncorrelated across time.

Now consider $\overline{\Sigma}_\sigma = \operatorname{diag}(\overline{\sigma}_1^2, \ldots, \overline{\sigma}_T^2) \neq \Sigma_\sigma$ and $\overline{\Pi} = (\overline{\pi}_{ts}) = \overline{U}^{-1}F$, where $\overline{U}\overline{U}' = F\overline{\Sigma}_\sigma F'$ and thus $\overline{\pi}_{ts} = \pi_{ts}(f, \overline{\gamma}_\sigma)$. Then we still have $\overline{\Pi} f = 0$ and thus this transformation removes the time-varying unit. However, in general, $\overline{\Pi}\Sigma_\sigma\overline{\Pi}' \neq I$, and in light of Proposition D.1, we see that this transformation will generally not deliver that linear quadratic forms based on forward differenced disturbances are uncorrelated across time.

D.5.2. *Multiple Factors*

We next discuss how the generalized Helmert transformation can be extended to the case of multiple factors, where the disturbance process $\eta_{it} = \mu_i f_t + u_{it}$ in (14) is generalized to

$$
\eta_{it} = \mu_{i1} f_{t1} + \mu_{i2} f_{t2} + \cdots + \mu_{iP} f_{tP} + u_{it}, \tag{D.21}
$$

where f_{tp} denotes the pth factor and μ_{ip} denotes the corresponding factor loading. Part of the discussion will focus on how we can impose a needed normalization of the factors, while ensuring that the weights π_{ts} of the forward differencing transformation remain well defined for all realizations of the factors.

Let $f_t = [f_{t1}, \ldots, f_{tP}]$ and $f = [f_1', \ldots, f_T']'$ be the $T \times P$ matrix containing the P factors for period $t = 1, \ldots, T$. Similar to Ahn, Lee, and Schmidt (2013, p. 3) we assume that

$$
\underset{T\times P}{f} = \begin{pmatrix} \underset{T-P\times P}{f_U} \\ \underset{P\times P}{f_C} \end{pmatrix} \quad \text{with } \underset{P\times P}{f_C} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{D.22}
$$

as a normalization, where f_U denotes the matrix of unconstrained elements and f_C denotes the matrix of constrained elements of f.[1]

Consider the forward differences $\eta_{it}^{+} = \sum_{s=t}^{T} \pi_{ts}\eta_{is}$ and $u_{it}^{+} = \sum_{s=t}^{T} \pi_{ts}u_{is}$ corresponding to an upper triangular $T-P \times T$ transformation matrix $\Pi = (\pi_{ts})$ with $\pi_{ts} = 0$ for $s < t$. Then any transformation matrix Π with $\Pi f = 0$ removes the common factors in that

$$\eta_{it}^{+} = \left[\sum_{s=t}^{T} \pi_{ts} f_{s1}\right]\mu_{i1} + \left[\sum_{s=t}^{T} \pi_{ts} f_{s2}\right]\mu_{i2} + \cdots + \left[\sum_{s=t}^{T} \pi_{ts} f_{sP}\right]\mu_{i,P} + \sum_{s=t}^{T} \pi_{ts} u_{it} = u_{it}^{+}.$$

In the following we present a simple approach to finding expressions for Π such that $\Pi f = 0$ and $\Pi \Sigma_\sigma \Pi' = I_{T-P}$. More specifically, we will give expressions on how to compute Π recursively as

$$\Pi = \Pi_P \ldots \Pi_2 \Pi_1,$$

where the matrices Π_p are of dimension $(T-p) \times (T-p+1)$, and $\Pi_1 \Sigma_\sigma \Pi_1' = I_{T-1}$ and $\Pi_p \Pi_p' = I_{T-p}$. This in turn implies that $\Pi \Sigma_\sigma \Pi' = I_{T-P}$. We shall also use the notation $f_{.p} = [f_{1p}, \ldots, f_{Tp}]'$ for the pth column of f and to simplify the exposition, we focus on the case where $\Sigma_\sigma = I$.[2]

Computation of Π_1. As a first step we apply the generalized Helmert transformation to remove the $\mu_{i1} f_{t1}$. Consider the $(T-1) \times T$ matrix $\Pi_1 = (\pi_{ts,1})$ with $\pi_{ts,1} = 0$ for $s < t$, and

$$\pi_{tt,1} = \pi_{tt}(f_{.1}, \mathbf{1}) = \left(\sqrt{\phi_{t+1,1}/\phi_{t1}}\right),$$
$$\pi_{ts,1} = \pi_{ts}(f_{.1}, \mathbf{1}) = -f_{t1} f_{s1}\left(\sqrt{\phi_{t+1,1}/\phi_{t1}}\right)/\phi_{t+1,1} \quad \text{for } s > t,$$

with $\phi_{t1} = \sum_{\tau=t}^{T} f_{\tau 1}^2$. Recall that $f_{T,1} = 1$ and $f_{t1} = 0$ for $t = T-P+1, \ldots, T-1$ given the normalization in (D.22). The normalization $f_{T,1} = 1$ ensures that all elements of Π_1 are well defined for all realizations of the factors. By Proposition 1 it follows that $\Pi_1 \Pi_1' = I_{T-1}$ and $\Pi_1 f_{.1} = 0$. Furthermore, utilizing Lemma D.3 below we have

$$\underset{T-1\times T}{\Pi_1} = \begin{pmatrix} \pi_{11,1} & \pi_{12,1} & \cdots & \pi_{1,T-P,1} & \pi_{1,T-P+1,1} & \cdots & \pi_{1,T-1,1} & \pi_{1T,1} \\ 0 & \pi_{22,1} & \cdots & \pi_{2,T-P,1} & \pi_{2,T-P+1,1} & \cdots & \pi_{2,T-1,1} & \pi_{2T,1} \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & & \cdots & \pi_{T-P,T-P,1} & \pi_{T-P,T-P+1,1} & & \pi_{T-P,T-1,1} & \pi_{T-P,T,1} \\ 0 & & & 0 & 1 & & 0 & 0 \\ \vdots & & & \vdots & & \ddots & 0 & 0 \\ 0 & & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{D.23}$$

Let $f^{+1} = [f_{.1}^{+1}, f_{.2}^{+1}, \ldots, f_{.P}^{+1}] = \Pi_1 f$ be the matrix of transformed factors based on the transformation Π_1. Then in light of the derived structure for the elements of Π_1 in (D.23)

[1]Of course, we could also adopt the normalization $f_C = I_P$. The reason for the above specification is that it simplifies the notation in the subsequent discussion.

[2]The discussion can be readily extended to the case $\Sigma_\sigma \neq I$.

and the adopted normalization for f_C in (D.22), we have

$$\underset{T-1\times P}{f^{+1}} = \begin{pmatrix} \underset{T-P\times 1}{0} & \underset{T-P\times P-1}{f_U^{+1}} \\ \underset{P-1\times 1}{0} & \underset{P-1\times P-1}{f_C^{+1}} \end{pmatrix} \quad \text{with} \quad \underset{P-1\times P-1}{f_C^{+1}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{D.24}$$

From this we see that the adopted normalization is preserved by the transformed factors. Applying the transformation Π_1 to (D.21) yields

$$\eta_{it}^{+1} = \mu_{i2} f_{t2}^{+1} + \dots + \mu_{iP} f_{tP}^{+1} + u_{it}^{+1} \tag{D.25}$$

with $\eta_{it}^{+1} = \sum_{s=t}^{T} \pi_{ts,1}\eta_{s1}$, $f_{tp}^{+1} = \sum_{s=t}^{T} \pi_{ts,1} f_{sp}$, and $u_{it}^{+1} = \sum_{s=t}^{T} \pi_{ts,1} u_{is}$.

Computation of Π_2. Having eliminated $\mu_{i1} f_{t1}$, as a next step we apply the generalized Helmert transformation to (D.25) to remove the $\mu_{i2} f_{t2}^{+1}$. Consider the $(T-2)\times(T-1)$ matrix $\Pi_2 = (\pi_{ts,2})$ with $\pi_{ts,2} = 0$ for $s < t$, and

$$\pi_{tt,2} = \pi_{tt}(f_{.2}^{+1}, \mathbf{1}) = (\sqrt{\phi_{t+1,2}/\phi_{t2}}),$$
$$\pi_{ts,2} = \pi_{ts}(f_{.2}^{+1}, \mathbf{1}) = -f_{t2}^{+1} f_{s2}^{+1}(\sqrt{\phi_{t+1,2}/\phi_{t2}})/\phi_{t+1,2} \quad \text{for } s > t,$$

with $\phi_{t2} = \sum_{\tau=t}^{T}(f_{\tau 2}^{+1})^2$. Observe that $f_{T-1,2}^{+1} = 1$ and $f_{t2}^{+1} = 0$ for $t = T-P+1, \dots, T-2$ given the normalization in (D.24). The normalization $f_{T-1,2}^{+1} = 1$ ensures that all elements of Π_2 are well defined for all realizations of the factors. By Proposition 1 it follows that $\Pi_2 \Pi_2' = I_{T-2}$ and $\Pi_2 f_{.2}^{+1} = 0$. Furthermore, utilizing Lemma D.3 below we have

$$\underset{T-2\times T-1}{\Pi_2} = \begin{pmatrix} \pi_{11,2} & \pi_{12,2} & \dots & \pi_{1,T-P,2} & \pi_{1,T-P+1,2} & \dots & \pi_{1,T-2,2} & \pi_{1,T-1,2} \\ 0 & \pi_{22,2} & \dots & \pi_{2,T-P,2} & \pi_{2,T-P+1,2} & \dots & \pi_{2,T-2,2} & \pi_{2,T-1,2} \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & & \dots & \pi_{T-P,T-P,2} & \pi_{T-P,T-P+1,2} & & \pi_{T-P,T-2,2} & \pi_{T-P,T-1,2} \\ 0 & & & 0 & 1 & & 0 & 0 \\ \vdots & & & \vdots & & \ddots & 0 & 0 \\ 0 & & \dots & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{D.26}$$

Let $f^{+2} = [f_{.1}^{+2}, f_{.2}^{+2}, \dots, f_{.P}^{+2}] = \Pi_2 f^{+1}$ be the matrix of transformed factors based on the transformation Π_2. Then in light of the derived structure for the elements of Π_2 in (D.26) and the adopted normalization for f_C^{+1} in (D.24), we have

$$\underset{T-2\times P}{f^{+2}} = \begin{pmatrix} \underset{T-P\times 2}{0} & \underset{T-P\times P-2}{f_U^{+1}} \\ \underset{P-2\times 2}{0} & \underset{P-2\times P-2}{f_C^{+1}} \end{pmatrix} \quad \text{with} \quad \underset{P-2\times P-2}{f_C^{+2}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

From this we see that adopted normalization is again preserved by the transformed factors. Applying the transformation Π_2 to (D.25) yields

$$\eta_{it}^{+2} = \mu_{i3} f_{t3}^{+2} + \dots + \mu_{iP} f_{tP}^{+2} + u_{it}^{+2}$$

with $\eta_{it}^{+2} = \sum_{s=t}^{T} \pi_{ts,2}\eta_{s1}$, $f_{tp}^{+2} = \sum_{s=t}^{T} \pi_{ts,2} f_{sp}$, and $u_{it}^{+2} = \sum_{s=t}^{T} \pi_{ts,2} u_{is}$.

Computation of Π_p, $p > 2$. The matrices Π_p for $p = 3, \ldots, P$ can be computed by applying the above approach recursively. By construction, it follows that the total transformation $\Pi = \Pi_P \ldots \Pi_1$ eliminates all time-varying unit-specific components $\mu_{ip} f_{tp}$ in that $\Pi f = 0$, and the transformation is orthogonal in the sense that $\Pi \Sigma_\sigma \Pi' = I_{T-P}$.

The above discussion utilized the following lemma.

LEMMA D.3: *Let $B = (b_{ij})$ be some upper triangular $(M + m - 1) \times (M + m)$ matrix with $b_{ij} = 0$ for $j < i$, let $D = diag(d_i^2)$ be a diagonal $(M + m) \times (M + m)$ matrix, and let $b = [b_1, \ldots, b_M, 0, \ldots, 0, \tau]'$ be a $(M + m) \times 1$ vector with $\tau \neq 0$. Suppose that $BDB' = I_{M+m-1}$ and $Bb = 0$. Then B must be of the form*

$$B = \begin{pmatrix} b_{11} & & b_{1M} & b_{1,M+1} & \ldots & b_{1,M+m-1} & b_{1,M+m} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & b_{MM} & b_{M,M+1} & & b_{M,M+m-1} & b_{M,M+m} \\ 0 & & 0 & 1/d_{M+1} & & 0 & 0 \\ \vdots & & \vdots & & \ddots & 0 & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 1/d_{M+m-1} & 0 \end{pmatrix}.$$

That is, for $i = M + 1, \ldots, M + m - 1$, we have $b_{ii} = 1/d_i$ and $b_{ij} = 0$ for all $j \neq i$.

PROOF: To see this, observe that $Bb = 0$ implies that $b_{i,M+m} = 0$ for $i = M + 1, \ldots, M + m$. Since with $b_{ij} = 0$ for $j < i$, by assumption it follows that the $(M + m - 1)$th row of B is given by $[0, \ldots, 0, b_{M+m-1,M+m-1}, 0]$, and, thus, the $(M + m - 1)$th column of B' is given by $[0, \ldots, 0, b_{M+m-1,M+m-1}, 0]'$. Consequently, for $i = M + 1, \ldots, M + m - 1$,

$$c_{i,M+m-1} = d_{M+m-1}^2 b_{i,M+m-1} b_{M+m-1,M+m-1}. \tag{D.27}$$

For $i = M + m - 1$, we have from (D.27) that $c_{M+m-1,M+m-1} = d_{M+m-1}^2 b_{M+m-1,M+m-1}^2$. Recalling that, by assumption, $C = BDB' = I_{M+m-1}$ and, thus, $c_{M+m-1,M+m-1} = 1$, we have $b_{M+m-1,M+m-1} = 1/d_{M+m-1}$. Since $c_{i,M+m-1} = 0$ for $i \neq M + m - 1$, it follows furthermore from (D.27) that $b_{i,M+m-1} = 0$ for $i = M + 1, \ldots, M + m - 2$. This in turn implies that for $i = M + 1, \ldots, M + m - 1$,

$$c_{i,M+m-2} = d_{M+m-2}^2 b_{i,M+m-2} b_{M+m-2,M+m-2}.$$

Since $c_{M+m-2,M+m-2} = 1$, we have $b_{M+m-2,M+m-2} = 1/d_{M+m-2}$, and since $c_{i,M+m-2} = 0$ for $i \neq M + m - 2$, we have $b_{i,M+m-2} = 0$ for $i = M + 1, \ldots, M + m - 3$ and $i = M + m - 1$. The remainder of the claim is proven by recursive argumentation. *Q.E.D.*

APPENDIX E: ADDITIONAL RESULTS

Below we provide some additional results and detailed derivations for the benefit of the referee. This material is not intended for publication.

E.1. *Detailed Derivation of* S_2

In the following discussion we verify the expression for $S_2 = \sum_{i=1}^{n} \sum_{t=1}^{T-1} \lambda_t' \sum_{j=1}^{n} a_{ij,t}' u_{it}^{+} u_{jt}^{+}$ given in (33). Let $\lambda_T = 0$, and recalling that $\pi_{ts} = 0$ for $s < t$ and that $a_{ii,t} = 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{t=1}^{T-1}\lambda_t'\sum_{j=1}^{n}a_{ij,t}'\sum_{u=t}^{T}\sum_{v=t}^{T}\pi_{tu}\pi_{tv}u_{iu}u_{jv} \\
&\quad=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\lambda_t'a_{ij,t}'\sum_{u=1}^{T}\sum_{v=1}^{T}\pi_{tu}\pi_{tv}u_{iu}u_{jv} \\
&\quad=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\lambda_t'a_{ij,t}'\sum_{u=1}^{T}\pi_{tu}\pi_{tu}u_{iu}u_{ju}+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\lambda_t'a_{ij,t}'\sum_{u=2}^{T}\sum_{v=1}^{u-1}\pi_{tu}\pi_{tv}(u_{iu}u_{jv}+u_{iv}u_{ju}) \\
&\quad=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{u=1}^{T}u_{iu}u_{ju}\sum_{t=1}^{T}\lambda_t'a_{ij,t}'\pi_{tu}^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{u=2}^{T}\sum_{v=1}^{u-1}(u_{iu}u_{jv}+u_{iv}u_{ju})\sum_{t=1}^{T}\lambda_t'a_{ij,t}'\pi_{tu}\pi_{tv} \\
&\quad=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{u=1}^{T}u_{iu}u_{ju}\sum_{t=1}^{u}\lambda_t'a_{ij,t}'\pi_{tu}^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{u=2}^{T}\sum_{v=1}^{u-1}(u_{iu}u_{jv}+u_{iv}u_{ju})\sum_{t=1}^{v}\lambda_t'a_{ij,t}'\pi_{tu}\pi_{tv} \\
&\quad=\sum_{u=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}u_{iu}u_{ju}\sum_{t=1}^{u}\lambda_t'a_{ij,t}'\pi_{tu}^2+\sum_{u=2}^{T}\sum_{v=1}^{u-1}\sum_{i=1}^{n}\sum_{j=1}^{n}(u_{iu}u_{jv}+u_{iv}u_{ju})\sum_{t=1}^{v}\lambda_t'a_{ij,t}'\pi_{tu}\pi_{tv} \\
&\quad=\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}u_{it}u_{jt}\sum_{\tau=1}^{t}\lambda_\tau'a_{ij,\tau}'\pi_{\tau t}^2+\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{n}\sum_{j=1}^{n}(u_{it}u_{js}+u_{is}u_{jt})\sum_{\tau=1}^{s}\lambda_\tau'a_{ij,\tau}'\pi_{\tau t}\pi_{\tau s} \\
&\quad=\sum_{t=1}^{T}\sum_{i=1}^{n}2\sum_{j=1}^{i-1}u_{it}u_{jt}\sum_{\tau=1}^{t}\lambda_\tau'a_{ij,\tau}'\pi_{\tau t}^2+\sum_{t=2}^{T}\sum_{i=1}^{n}2\sum_{s=1}^{t-1}\sum_{j=1}^{n}u_{it}u_{js}\sum_{\tau=1}^{s}\lambda_\tau'a_{ij,\tau}'\pi_{\tau t}\pi_{\tau s} \\
&\quad=\sum_{t=1}^{T}\sum_{i=1}^{n}2\left\{\sum_{j=1}^{i-1}u_{it}u_{jt}c_{ij,tt}\sum_{\tau=1}^{t}\lambda_\tau'a_{ij,\tau}'\pi_{\tau t}^2+\sum_{s=1}^{t-1}\sum_{j=1}^{n}u_{it}u_{js}c_{ij,ts}\right\},
\end{aligned}
$$

with $c_{ij,ts} = \sum_{\tau=1}^{s} \lambda_\tau' a_{ij,\tau}' \pi_{\tau s}\pi_{\tau t}$.

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