Econometrica Supplementary Material

SUPPLEMENT TO "SAMPLING-BASED VERSUS DESIGN-BASED UNCERTAINTY IN REGRESSION ANALYSIS"

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S.1. PROOFS OF THE RESULTS IN SECTION 2

S.1.1. *Preliminary Calculations*

NOTICE THAT FOR ANY INTEGER $1 \leq i \leq n$ and conditional on sample size N, such that $1 \leq N \leq n$, we obtain

$$E[R_i] = \frac{N}{n}, \qquad \operatorname{var}(R_i) = \frac{N}{n}\left(1 - \frac{N}{n}\right).$$

Also, for any integers $1 \leq j < k \leq n$,

$$\operatorname{var}\left(\sum_{i=1}^{n} R_i\right) = n \operatorname{var}(R_j) + n(n-1)\operatorname{cov}(R_j, R_k) = 0.$$

This implies

$$\operatorname{cov}(R_j, R_k) = -\frac{\operatorname{var}(R_j)}{n-1} = -\frac{N}{n(n-1)}\left(1 - \frac{N}{n}\right).$$

In turn, this implies

$$E[R_i R_j] = \frac{N(N-1)}{n(n-1)}.$$

Let

$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{n} R_i Y_i \quad \text{and} \quad \mu = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

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Now,

$$E[\bar{Y}] = \frac{1}{N}\sum_{i=1}^{n} E[R_i]Y_i = \mu.$$

Let

$$S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\mu)^2.$$

Notice that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 &= \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2 \\ &= \frac{n-1}{n^2}\sum_{i=1}^{n} Y_i^2 - \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Y_iY_j.\end{aligned}$$

This implies

$$nS_Y^2 = \sum_{i=1}^{n} Y_i^2 - \frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Y_iY_j.$$

Therefore,

$$\begin{aligned}\operatorname{var}(\bar{Y}) &= \frac{1}{N^2}\sum_{i=1}^{n}\operatorname{var}(R_i)Y_i^2 + \frac{2}{N^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\operatorname{cov}(R_i,R_j)Y_iY_j \\ &= \frac{1}{N^2}\operatorname{var}(R_1)\left(\sum_{i=1}^{n} Y_i^2 - \frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Y_iY_j\right) \\ &= \frac{n}{N^2}\operatorname{var}(R_1)S_Y^2 \\ &= \frac{S_Y^2}{N}\left(1-\frac{N}{n}\right).\end{aligned}$$

Let

$$\widehat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{n} R_iY_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} R_iY_i\right)^2.$$

Then

$$\widehat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{n} R_iY_i^2 - \frac{1}{N^2}\sum_{i=1}^{n} R_iY_i^2 - \frac{2}{N^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} R_iR_jY_iY_j.$$

Therefore,

$$E\big[\widehat{\sigma}^2\big] = \frac{1}{n}\frac{N-1}{N}\sum_{i=1}^{n} Y_i^2 - \frac{2}{n(n-1)}\frac{N-1}{N}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Y_iY_j = \frac{N-1}{N}S_Y^2.$$

S.1.2. *Causal versus Descriptive Estimands*

Let

$$\widehat{\theta} = \frac{1}{N_1}\sum_{i=1}^{n} R_i X_i Y_i - \frac{1}{N_0}\sum_{i=1}^{n} R_i (1 - X_i) Y_i.$$

We will do all the analysis conditional on N_1, N_0, for $N_1 > 0$, $N_0 > 0$, $n_1 > 0$, and $n_0 > 0$. To economize notation, we will leave this conditioning implicit. Notice that

$$E[\widehat{\theta}|\mathbf{X}] = \theta^{\text{desc}},$$

where

$$\begin{aligned}\theta^{\text{desc}} &= \frac{1}{n_1}\sum_{i=1}^{n} X_i Y_i - \frac{1}{n_0}\sum_{i=1}^{n}(1 - X_i) Y_i \\ &= \frac{1}{n_1}\sum_{i=1}^{n} X_i Y_i^*(1) - \frac{1}{n_0}\sum_{i=1}^{n}(1 - X_i) Y_i^*(0).\end{aligned}$$

By the law of total variance,

$$\operatorname{var}(\widehat{\theta}) = E\big[\operatorname{var}(\widehat{\theta}|\mathbf{X})\big] + \operatorname{var}\big(\theta^{\text{desc}}\big).$$

The expectation of θ^{desc} over the randomization distribution is

$$\begin{aligned}E\big[\theta^{\text{desc}}\big] &= \frac{1}{n_1}\sum_{i=1}^{n} E[X_i] Y_i - \frac{1}{n_0}\sum_{i=1}^{n}\big(1 - E[X_i]\big) Y_i \\ &= \frac{1}{n_1}\sum_{i=1}^{n}(n_1/n) Y_i^*(1) - \frac{1}{n_0}\sum_{i=1}^{n}(n_0/n) Y_i^*(0) \\ &= \theta^{\text{causal}}.\end{aligned}$$

For the variance of θ^{desc}, we have to compute n square terms and $n(n-1)$ cross-product terms. Each square term is equal to

$$\begin{aligned}&\frac{\operatorname{var}(X_i)}{n_1^2} Y_i^*(1)^2 + \frac{\operatorname{var}(X_i)}{n_0^2} Y_i^*(0)^2 + 2\frac{\operatorname{var}(X_i)}{n_1 n_0} Y_i^*(1) Y_i^*(0) \\ &\quad= \operatorname{var}(X_i)\left(\frac{Y_i^*(1)^2}{n_1^2} + \frac{Y_i^*(0)^2}{n_0^2} + 2\frac{Y_i^*(1) Y_i^*(0)}{n_1 n_0}\right).\end{aligned}$$

Recall from previous calculations that

$$\operatorname{cov}(X_i, X_j) = -\frac{\operatorname{var}(X_i)}{n-1}.$$

Therefore, each of the cross-product terms is equal to

$$-\frac{\operatorname{var}(X_i)}{n-1}\left(\frac{Y_i^*(1) Y_j^*(1)}{n_1^2} + \frac{Y_i^*(1) Y_j^*(0)}{n_1 n_0} + \frac{Y_i^*(0) Y_j^*(1)}{n_1 n_0} + \frac{Y_i^*(0) Y_j^*(0)}{n_0^2}\right).$$

Let $\theta_i = Y_i^*(1) - Y_i^*(0)$; then

$$nS_\theta^2 = \sum_{i=1}^n \big(Y_i^*(1) - Y_i^*(0)\big)^2 - \frac{2}{n-1}\sum_{i=1}^n \sum_{j=i+1}^n \big(Y_i^*(1) - Y_i^*(0)\big)\big(Y_j^*(1) - Y_j^*(0)\big)$$
$$= nS_1^2 + nS_0^2 - 2\left(\sum_{i=1}^n Y_i^*(1)Y_i^*(0) - \frac{1}{n-1}\sum_{i=1}^n\sum_{j=i+1}^n \big(Y_i^*(1)Y_j^*(0) + Y_i^*(0)Y_j^*(1)\big)\right).$$

As a result, we obtain

$$\begin{aligned}
\operatorname{var}\big(\theta^{\text{desc}}\big) &= \operatorname{var}(X_i)\left(\frac{nS_1^2}{n_1^2} + \frac{nS_0^2}{n_0^2} + \frac{nS_1^2}{n_1 n_0} + \frac{nS_0^2}{n_1 n_0} - \frac{nS_\theta^2}{n_1 n_0}\right) \\
&= \frac{n_1 n_0}{n^2}\left(\frac{nS_1^2}{n_1^2} + \frac{nS_0^2}{n_0^2} + \frac{nS_1^2}{n_1 n_0} + \frac{nS_0^2}{n_1 n_0} - \frac{nS_\theta^2}{n_1 n_0}\right) \\
&= \frac{n_1 n_0}{n^2}\left(\frac{n^2}{n_1^2 n_0}S_1^2 + \frac{n^2}{n_1 n_0^2}S_0^2 - \frac{nS_\theta^2}{n_1 n_0}\right) \\
&= \frac{S_1^2}{n_1} + \frac{S_0^2}{n_0} - \frac{S_\theta^2}{n}.
\end{aligned}$$

Notice now that (because we condition on N_1 and N_0)

$$\begin{aligned}
\operatorname{var}(\widehat{\theta}|\mathbf{X}) &= \operatorname{var}\left(\sum_{i=1}^n R_i X_i \frac{Y_i^*(1)}{N_1} - \sum_{i=1}^n R_i(1-X_i)\frac{Y_i^*(0)}{N_0}\Big|\mathbf{X}\right) \\
&= \operatorname{var}\left(\sum_{i=1}^n R_i X_i \frac{Y_i^*(1)}{N_1}\Big|\mathbf{X}\right) + \operatorname{var}\left(\sum_{i=1}^n R_i(1-X_i)\frac{Y_i^*(0)}{N_0}\Big|\mathbf{X}\right).
\end{aligned}$$

Let us calculate the first term on the right-hand side of the last equation (the second term will be analogous):

$$\begin{aligned}
&\operatorname{var}\left(\sum_{i=1}^n R_i X_i \frac{Y_i^*(1)}{N_1}\Big|\mathbf{X}\right) \\
&\quad = \operatorname{var}(R_i|X_i = 1)\left[\sum_{i=1}^n X_i \frac{Y_i^*(1)^2}{N_1^2} - \frac{2}{n_1 - 1}\sum_{i=1}^n\sum_{j=i+1}^n X_i X_j \frac{Y_i^*(1)Y_j^*(1)}{N_1^2}\right].
\end{aligned}$$

Taking expectations, the right-hand side becomes

$$\operatorname{var}(R_i|X_i = 1)\left[\sum_{i=1}^n \frac{n_1}{n}\frac{Y_i^*(1)^2}{N_1^2} - \frac{2}{n_1-1}\sum_{i=1}^n\sum_{j=i+1}^n \frac{n_1(n_1-1)}{n(n-1)}\frac{Y_i^*(1)Y_j^*(1)}{N_1^2}\right]$$
$$= \frac{1}{nN_1}\left(\frac{n_1 - N_1}{n_1}\right)\left[\sum_{i=1}^n Y_i^*(1)^2 - \frac{2}{n-1}\sum_{i=1}^n\sum_{j=i+1}^n Y_i^*(1)Y_j^*(1)\right] = \frac{n_1 - N_1}{n_1 N_1}S_1^2.$$

This implies

$$\operatorname{var}(\widehat{\theta}) = \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} - \frac{S_\theta^2}{n}.$$

Now, notice that

$$E[\widehat{\theta}|\mathbf{R}] = \theta^{\text{causal,sample}},$$

where

$$\theta^{\text{causal,sample}} = \frac{1}{N}\sum_{i=1}^{n} R_i\big(Y_i^*(1) - Y_i^*(0)\big).$$

Therefore, by the law of total variance,

$$\operatorname{var}(\widehat{\theta}) = E\big[\operatorname{var}(\widehat{\theta}|\mathbf{R})\big] + \operatorname{var}\big(\theta^{\text{causal,sample}}\big).$$

The variance of $\theta^{\text{causal,sample}}$ is

$$\begin{aligned}
&\frac{\operatorname{var}(R_i)}{N^2}\left[\sum_{i=1}^{N}\big(Y_i^*(1) - Y_i^*(0)\big) - \frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\big(Y_i^*(1) - Y_i^*(0)\big)\big(Y_j^*(1) - Y_j^*(0)\big)\right] \\
&\quad = \frac{1}{N}\left(1 - \frac{N}{n}\right)S_\theta^2.
\end{aligned}$$

As a result,

$$\begin{aligned}
E\big[\operatorname{var}(\widehat{\theta}|\mathbf{R})\big] &= \operatorname{var}(\widehat{\theta}) - \operatorname{var}\big(\theta^{\text{causal,sample}}\big) \\
&= \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} - \frac{S_\theta^2}{N}.
\end{aligned}$$

S.1.3. *EHW Variance*

The EHW variance estimator for $\widehat{\theta}$ is

$$\widehat{V}^{\text{ehw}} = \frac{N_1 - 1}{N_1^2}\widehat{S}_1^2 + \frac{N_0 - 1}{N_0^2}\widehat{S}_0^2,$$

where

$$\widehat{S}_1^2 = \frac{1}{N_1 - 1}\sum_{i=1}^{n} R_i X_i\left(Y_i - \frac{1}{N_1}\sum_{i=1}^{n} R_i X_i Y_i\right)^2,$$

and $\widehat{S}_0^2$ is defined analogously. Using previous results, we obtain

$$\widehat{S}_1^2 = \frac{1}{N_1}\sum_{i=1}^{n} R_i X_i Y_i^2 - \frac{2}{N_1(N_1 - 1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n} R_i R_j X_i X_j Y_i Y_j.$$

Therefore,

$$E\big[\widehat{S}_1^2|\mathbf{X}\big] = \frac{1}{n_1}\sum_{i=1}^{n} X_i Y_i^2 - \frac{2}{n_1(n_1-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_i X_j Y_i Y_j$$

and

$$\begin{aligned} E\big[\widehat{S}_1^2\big] &= \frac{1}{n}\sum_{i=1}^{n} Y_i^*(1)^2 - \frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n} Y_i^*(1)Y_j^*(1) \\ &= S_1^2. \end{aligned}$$

Let

$$\widetilde{V}^{\text{ehw}} = \frac{\widehat{S}_1^2}{N_1} + \frac{\widehat{S}_0^2}{N_0}.$$

Then

$$E\big[\widetilde{V}^{\text{ehw}}\big] = \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} \geq \text{var}(\widehat{\theta}).$$

S.1.4. *Bootstrap Variance*

Consider the bootstrap variance estimator that draws N_1 treated and N_0 untreated observations separately,

$$\widehat{V}_B^{\text{boot}} = \frac{1}{B-1}\sum_{b=1}^{B}\big(\widehat{\theta}^{(b)} - \bar{\theta}_B\big)^2,$$

where

$$\widehat{\theta}^{(b)} = \frac{1}{N_1}\sum_{i=1}^{n} K_{1i}^{(b)} R_i X_i Y_i - \frac{1}{N_0}\sum_{i=1}^{n} K_{0i}^{(b)} R_i (1-X_i) Y_i$$

and

$$\bar{\theta}_B = \frac{1}{B}\sum_{b=1}^{B}\widehat{\theta}^{(b)}.$$

Conditional on $\mathbf{R}$, and $\mathbf{X}$, $K_{1i}^{(b)}$ has a multinomial distribution with paramenter $\{N_1, 1/N_1, \dots, 1/N_1\}$ for units with $R_i X_i = 1$ and $K_{0i}^{(b)}$ has a multinomial distribution with parameters $\{N_0, 1/N_0, \dots, 1/N_0\}$ for units with $R_i(1-X_i) = 1$. The variables $K_{1i}^{(b)}$ and $K_{0i}^{(b)}$ are independent of each other and independent across $b = 1, \dots, B$. As a result, for $R_i X_i = R_j X_j = 1$ with $i \neq j$, we obtain

$$\begin{aligned} E\big[K_{1i}^{(b)}|\mathbf{R}, \mathbf{X}\big] &= 1, \\ E\big[\big(K_{1i}^{(b)}\big)^2|\mathbf{R}, \mathbf{X}\big] &= (2N_1 - 1)/N_1, \end{aligned}$$

and

$$E\big[K_{1i}^{(b)} K_{1i}^{(b)}|\mathbf{R}, \mathbf{X}\big] = (N_1 - 1)/N_1.$$

Conditional on $\mathbf{Y}(1)$, $\mathbf{Y}(0)$, $\mathbf{R}$, and $\mathbf{X}$, the mean of the bootstrap variance is

$$E\big[\widehat{V}_B^{\text{boot}}|\mathbf{R},\mathbf{X}\big]=\left(\frac{B}{B-1}\right)E\big[\big(\theta^{(b)}-\bar{\theta}_B\big)^2|\mathbf{R},\mathbf{X}\big].$$

Because of independence of the bootstrap weights between treatment samples, we obtain

$$E\big[\widehat{V}_B^{\text{boot}}|\mathbf{R},\mathbf{X}\big]=\left(\frac{B}{B-1}\right)E\big[\big(\widehat{\theta}_1^{(b)}-\bar{\theta}_{1B}\big)^2|\mathbf{R},\mathbf{X}\big]+\left(\frac{B}{B-1}\right)E\big[\big(\widehat{\theta}_0^{(b)}-\bar{\theta}_{0B}\big)^2|\mathbf{R},\mathbf{X}\big],$$

where

$$\widehat{\theta}_1^{(b)}=\frac{1}{N_1}\sum_{i=1}^{n}K_{1i}^{(b)}R_iX_iY_i$$

and

$$\bar{\theta}_{1B}=\frac{1}{B}\sum_{b=1}^{B}\widehat{\theta}_1^{(b)},$$

with analogous expressions for $\widehat{\theta}_0^{(b)}$ and $\bar{\theta}_{0B}$. Now, let

$$\mathbf{K}_1=\begin{pmatrix}K_{11}^{(1)} & K_{11}^{(2)} & \cdots & K_{11}^{(B)}\\ K_{12}^{(1)} & K_{12}^{(2)} & \cdots & K_{12}^{(B)}\\ \vdots & \vdots & \cdots & \vdots\\ K_{1n}^{(1)} & K_{1n}^{(2)} & \cdots & K_{1n}^{(B)}\end{pmatrix},$$

and let $\mathbf{K}_{1\pi}$ be a random permutation of the columns of $\mathbf{K}_1$. Notice that $\bar{\theta}_{1B}$ is fixed conditional on $\mathbf{R}$, $\mathbf{X}$, and $\mathbf{K}_{1\pi}$, but $\widehat{\theta}_1^{(b)}$ is not. In addition,

$$E\big[\widehat{\theta}_1^{(b)}|\mathbf{R},\mathbf{X},\mathbf{K}_{1\pi}\big]=\bar{\theta}_{1B}.$$

Therefore,

$$E\big[\big(\widehat{\theta}_1^{(b)}-\bar{\theta}_{1B}\big)^2|\mathbf{R},\mathbf{X},\mathbf{K}_{1\pi}\big]=E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X},\mathbf{K}_{1\pi}\big]-\bar{\theta}_{1B}^2.$$

Then

$$E\big[\big(\widehat{\theta}_1^{(b)}-\bar{\theta}_{1B}\big)^2|\mathbf{R},\mathbf{X}\big]=E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X}\big]-E\big[\bar{\theta}_{1B}^2|\mathbf{R},\mathbf{X}\big].$$

Let

$$\widehat{\theta}_1=\frac{1}{N_1}\sum_{i=1}^{N}R_iX_iY_i.$$

Notice that for any b and c, such that $1\le b<c\le B$, we have

$$\begin{aligned}E\big[\bar{\theta}_{1B}^2|\mathbf{R},\mathbf{X}\big]&=\frac{1}{B}E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X}\big]+\frac{B-1}{B}E\big[\widehat{\theta}_1^{(b)}\widehat{\theta}_1^{(c)}|\mathbf{R},\mathbf{X}\big]\\&=\frac{1}{B}E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X}\big]+\frac{B-1}{B}\widehat{\theta}_1^2.\end{aligned}$$

Therefore,

$$E\big[\big(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B}\big)^2|\mathbf{R},\mathbf{X}\big] = \left(\frac{B-1}{B}\right)\big(E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X}\big] - \widehat{\theta}_1^2\big).$$

In addition,

$$E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X}\big] = \frac{1}{N_1^2}\left(\sum_{i=1}^n \frac{2N_1-1}{N_1} R_i X_i Y_i^2 + 2\sum_{i=1}^n \sum_{j=i+1}^n \frac{N_1-1}{N_1} R_i R_j X_i X_j Y_i Y_j\right).$$

Therefore,

$$\begin{aligned}
E\big[\big(\widehat{\theta}_1^{(b)}\big)^2|\mathbf{R},\mathbf{X}\big] - \widehat{\theta}_1^2 &= \frac{1}{N_1^2}\left(\sum_{i=1}^n \frac{N_1-1}{N_1} R_i X_i Y_i^2 - 2\sum_{i=1}^n \sum_{j=i+1}^n \frac{1}{N_1} R_i R_j X_i X_j Y_i Y_j\right) \\
&= \frac{N_1-1}{N_1^2}\left(\frac{1}{N_1}\sum_{i=1}^n R_i X_i Y_i^2 - \frac{2}{N_1(N_1-1)}\sum_{i=1}^n \sum_{j=i+1}^n R_i R_j X_i X_j Y_i Y_j\right) \\
&= \frac{N_1-1}{N_1^2}\widehat{S}_1
\end{aligned}$$

and

$$E\big[\big(\widehat{\theta}_1^{(b)} - \bar{\theta}_{1B}\big)^2|\mathbf{R},\mathbf{X}\big] = \left(\frac{B-1}{B}\right)\frac{N_1-1}{N_1^2}\widehat{S}_1^2,$$

with the analogous result holding for $E[(\widehat{\theta}_0^{(b)} - \bar{\theta}_{0B})^2|\mathbf{R},\mathbf{X}]$. It follows that

$$E\big[\widehat{V}_B^{\text{boot}}|\mathbf{R},\mathbf{X}\big] = \widehat{V}^{\text{ehw}}.$$

PROOF OF LEMMA 1: Let

$$W_{n,i} = \begin{pmatrix} Y_{n,i} \\ X_{n,i} \\ Z_{n,i} \end{pmatrix} \begin{pmatrix} Y_{n,i} \\ X_{n,i} \\ Z_{n,i} \end{pmatrix}',$$

and let $W_{n,i}^{(k,l)}$ be the (k,l) element of $W_{n,i}$. Similarly, let $\widetilde{W}_{n,i}^{(k,l)}$, $W_n^{(k,l)}$, $\widetilde{W}_n^{(k,l)}$, $\widetilde{\Omega}_n^{(k,l)}$, $\Omega_n^{(k,l)}$ be the (k,l) elements of $R_{n,i}W_{n,i}$, W_n, $\widetilde{W}_n$, $\widetilde{\Omega}_n$, and Ω_n, respectively. In order to have $\widetilde{W}_n^{(k,l)}$ and $\widetilde{\Omega}_n^{(k,l)}$ well-defined, let $\widetilde{W}_n^{(k,l)} = \Omega_n^{(k,l)} = 0$ when $N = 0$ (this is without loss of generality). Notice that, because $n\rho_n \to \infty$, for any fixed $0 < \varepsilon < 1$, there is n_ϵ such that for $n > n_\epsilon$, we have $n\rho_n > -\log(\varepsilon)$. Therefore, for $n > n_\varepsilon$, we obtain

$$\Pr(N=0) = \left(1 - \frac{n\rho_n}{n}\right)^n < \left(1 + \frac{\log(\varepsilon)}{n}\right)^n < e^{\log(\varepsilon)} = \varepsilon.$$

As a result, $\Pr(N=0) \to 0$ and

$$E\big[\big(\widetilde{W}_n^{(k,l)} - \Omega_n^{(k,l)}\big)^2|N=0\big]\Pr(N=0) = \big(\Omega_n^{(k,l)}\big)^2\Pr(N=0) \to 0$$

by Assumption 5 and Holder's inequality. Notice that for any integer, m, such that $1 \leq m \leq n$, we have

$$E\left[\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E\left[W_{n,i}^{(k,l)}\right]\middle|N=m\right]=0$$

and

$$\begin{aligned}
E\left[\left(\widetilde{W}_n^{(k,l)} - \Omega_n^{(k,l)}\right)^2 | N=m\right] &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E\left[W_{n,i}^{(k,l)}\right]\right)^2 \middle| N=m\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} E\left[\left(\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E\left[W_{n,i}^{(k,l)}\right]\right)^2 \middle| N=m\right] \\
&\leq \frac{1}{n^2}\sum_{i=1}^{n} E\left[\left(\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)}\right)^2 \middle| N=m\right] \\
&= \frac{1}{m}\left(\frac{1}{n}\sum_{i=1}^{n} E\left[\left(W_{n,i}^{(k,l)}\right)^2\right]\right) \\
&\leq C/m,
\end{aligned}$$

for some positive constant, C, by Assumption 5. Let

$$\widetilde{\xi}_m^{(k,l)} = \begin{cases} \left(\widetilde{W}_n^{(k,l)} - \Omega_n^{(k,l)}\right)^2 & \text{if } m > 0, \\ 0 & \text{if } m = 0, \end{cases} \quad \text{and} \quad \xi_m = \begin{cases} C/m & \text{if } m > 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Now,

$$\begin{aligned}
E\left[\left(\widetilde{W}_n^{(k,l)} - \Omega_n^{(k,l)}\right)^2 | N>0\right]\Pr(N>0) &= E\left[\widetilde{\xi}_N^{(k,l)}\right] \\
&\leq E[\xi_N].
\end{aligned}$$

Applying Chernoff's bounds, for any $\varepsilon > 0$,

$$\begin{aligned}
\Pr(\xi_N > \varepsilon) &\leq \Pr(0 < N < C/\varepsilon) \\
&< \Pr(N < C/\varepsilon) \\
&= \Pr\left(N < n\rho_n\left(1 - \frac{n\rho_n - C/\varepsilon}{n\rho_n}\right)\right) \\
&\leq e^{-\left(\frac{(n\rho_n - C/\varepsilon)^2}{2n\rho_n}\right)} \to 0,
\end{aligned}$$

which implies that ξ_N converges in probability to zero. Because ξ_N is bounded, by the portmanteau lemma we obtain $E[\xi_N] \to 0$. As a result,

$$E\left[\left(\widetilde{W}_n^{(k,l)} - \Omega_n^{(k,l)}\right)^2\right] \to 0.$$

For the second result, notice that

$$\begin{aligned} E\big[\big(\widetilde{\Omega}_n^{(k,l)} - \Omega_n^{(k,l)}\big)^2 \big| N = m\big] &= E\left[\left(\sum_{i=1}^{n}\left(\frac{R_{n,i}}{m} - \frac{1}{n}\right)E\big[W_{n,i}^{(k,l)}\big]\right)^2 \middle| N = m\right] \\ &= \sum_{i=1}^{n}\frac{1}{m^2}E\left[\left(R_{n,i} - \frac{m}{n}\right)^2 \middle| N = m\right]\big(E\big[W_{n,i}^{(k,l)}\big]\big)^2 \\ &= \sum_{i=1}^{n}\frac{1}{mn}\left(1 - \frac{m}{n}\right)\big(E\big[W_{n,i}^{(k,l)}\big]\big)^2 \\ &\leq \frac{1}{m}\left(\frac{1}{n}\sum_{i=1}^{n}\big(E\big[W_{n,i}^{(k,l)}\big]\big)^2\right). \end{aligned}$$

Now, using the same argument as above, we obtain

$$E\big[\big(\widetilde{\Omega}_n^{(k,l)} - \Omega_n^{(k,l)}\big)^2\big] \to 0.$$

The proof of the third result is analogous. *Q.E.D.*

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