## SUPPLEMENT TO "ENDOGENOUS PRODUCTION NETWORKS"

DARON ACEMOGLU
Department of Economics, MIT

PABLO D. AZAR
Department of Economics, MIT

APPENDIX B: ADDITIONAL RESULTS AND OMITTED PROOFS

*Properties of Cobb–Douglas Technologies*

LEMMA B1: THE UNIT COST *function of the Cobb–Douglas production function is*

$$K_i\big(S_i, A_i(S_i), P\big) = \frac{\prod\limits_{j \in S_i} P_j^{\alpha_{ij}}}{A_i(S_i)}.$$

PROOF OF LEMMA B1: Let $X_{ij}^*$ and $L_i^*$ be firm $i$'s optimal choices of inputs and labor when producing one unit of output. From the firm's first-order conditions, we have $P_j X_{ij}^* = \alpha_{ij} \frac{P_i}{1+\mu_i}$ and $L_i^* = (1 - \sum_{j \in S_i} \alpha_{ij}) \frac{P_i}{1+\mu_i}$. Dividing the former equation by the latter, we obtain $X_{ij}^* = \frac{\alpha_{ij} L_i^*}{(1-\sum_{j \in S_i} \alpha_{ij})P_j}$. Plugging this into the production function (and recalling that only one unit of output is produced), we obtain

$$1 = \frac{1}{\left(1 - \sum\limits_{j \in S_i} \alpha_{ij}\right)^{1-\sum_{j \in S_i} \alpha_{ij}} \prod\limits_{j \in S_i} \alpha_{ij}^{\alpha_{ij}}} A_i(S_i) \big(L_i^*\big)^{(1-\sum_{j \in S_i} \alpha_{ij})} \prod\limits_{j \in S_i} \left(\frac{\alpha_{ij} L_i^*}{\left(1 - \sum\limits_{j \in S_i} \alpha_{ij}\right) P_j}\right)^{\alpha_{ij}},$$

$$1 = \frac{1}{\left(1 - \sum\limits_{j \in S_i} \alpha_{ij}\right)^{1-\sum_{j \in S_i} \alpha_{ij}} \prod\limits_{j \in S_i} \alpha_{ij}^{\alpha_{ij}}} L_i^* A_i(S_i) \left(1 - \sum\limits_{j \in S_i} \alpha_{ij}\right)^{-\sum_{j \in S_i} \alpha_{ij}} \prod\limits_{j \in S_i} \frac{\alpha_{ij}^{\alpha_{ij}}}{P_j^{\alpha_{ij}}},$$

$$1 = \frac{L_i^* A_i(S_i)}{\left(1 - \sum\limits_{j \in S_i} \alpha_{ij}\right) \prod\limits_{j \in S_i} P_j^{\alpha_{ij}}}.$$

Therefore, $L_i^* = \frac{(1-\sum_{j \in S_i} \alpha_{ij}) \prod_{j \in S_i} P_j^{\alpha_{ij}}}{A_i(S_i)}$. Since $K_i(S_i, A_i(S_i), P) = \frac{P_i}{1+\mu_i} = \frac{L_i^*}{(1-\sum_{j \in S_i} \alpha_{ij})}$, we conclude that $K_i(S_i, A_i(S_i), P) = \frac{\prod_{j \in S_i} P_j^{\alpha_{ij}}}{A_i(S_i)}$. *Q.E.D.*

Daron Acemoglu: daron@mit.edu
Pablo D. Azar: pazar@mit.edu

COROLLARY B1: *When all industries have Cobb–Douglas production functions and the input–output network is S, equilibrium log prices are given as a solution to the following system of linear equations*:

$$p = -\big(I - \alpha(S)\big)^{-1}\big(a(S) - m\big),$$

*where* $m_i = \log(1 + \mu_i)$.

PROOF OF COROLLARY B1: From Lemma B1, $P_i = (1 + \mu_i)\frac{\prod_{j \in S_i} P_j^{\alpha_{ij}}}{A_i(S_i)}$ for each $i$. Taking logs on both sides, we obtain

$$p_i = \sum_{j \in S_i} \alpha_{ij} p_j + \log(1 + \mu_i) - a_i(S_i) \quad \text{for each } i.$$

From Assumption 1, labor is essential and thus $\sum_{j=1}^n \alpha_{ij} < 1$ for each $i$. Then, from the Perron–Frobenius theorem, the matrix $(I - \alpha(S))$ is invertible, and thus $p = -(I - \alpha(S))^{-1}(a(S) - m)$.                                                                      *Q.E.D.*

## Continuity of GDP

THEOREM B1—Continuity of GDP Without Distortions: *Suppose that* $\mu_i = 0$ *for all* $i = 1, 2, \ldots, n$. *Then,* (*real*) *GDP is continuous in the log productivity vector* $a = \{a_i(S)\}_{i \in \mathcal{N}, S_i \subset \mathcal{N}}$.

PROOF: Because the utility function is continuous, real GDP $U(C_1^*, \ldots, C_n^*)$ is a continuous function of the log price vector $p$ and nominal income of the representative household. Since distortions are zero, nominal income of the representative household is equal to labor income, which is constant and equal to 1. Thus, all we need to show is that the equilibrium log price vector $p$ varies continuously with $a$. Let $a$ be a log productivity vector and let $S$ be an input–output network (not necessarily the equilibrium one). Let $p$ be the vector of equilibrium log prices if the network is exogenously fixed to be $S$, and technology is given by $a$. Then, $p$ is the unique solution to the system of equations

$$p - k\big(S, a(S), p\big) = m,$$

where $m$ is a vector whose $i$th component is $\log(1 + \mu_i)$. The left-hand side is a continuously differentiable function of prices whose Jacobian is equal to $I - J_{k,p}$, where $J_{k,p}$ is the Jacobian of $k$ with respect to $p$. Since labor is essential, there exists $\theta < 1$ such that $\sum_{j=1}^n \frac{\partial k_i}{\partial p_j} < \theta$. Recall that a P-matrix is a matrix whose principal minors are all positive. Hawkins and Simon (1949) showed that a matrix of the form $B = I - A$ is a P-matrix if and only if $(I - A)^{-1}$ exists and all of its coefficients are nonnegative, which is true for our matrix $I - J_{k,p}$. Therefore, the matrix $I - J_{k,p}$ is a P-matrix. Then, once again from Gale and Nikaido (1965), there exists a globally defined function $p(a, S)$ that is continuously differentiable in $a$.

Now fix an arbitrary $S^0$ and let $p^0(a) = p(a, S^0)$. For any $t \geq 1$, define $S_i^t = \arg\min_{S_i^t} \log(1 + \mu_i) + k_i(S_i^t, a_i(S_i^t), p^{t-1}(a))$, $p^t(a) = p(a, S^t)$, and note that $p^t(a) \leq p^{t-1}(a)$. We have that $p^*(a) = \lim_{t \to \infty} p^t(a)$. Since each $p^t$ corresponds to a network $S^t$, and there are a finite number of possible networks, there must only be a finite number of

vectors in the sequence $\{p^t\}_{t=1}^{\infty}$.[1] Eventually, we must reach a $T$ such that $p^t = p^T$ for all $t \geq T$. This implies that $p^*(a) = p^T(a)$. Since $p^T = p(a, S^T)$ is a continuous function of $a$, we conclude that $p^*(a)$ is a continuous function of $a$. Q.E.D.

THEOREM B2—Continuity of GDP With Exogenous Production Network: *Suppose that $S$ is an exogenously fixed network. Then, (real) GDP is continuous in the log productivity vector $a = \{a_i(S)\}_{i \in \mathcal{N}, S_i \subset \mathcal{N}}$.*

PROOF: Because the utility function is continuous, real GDP, given by $U(C_1^*, \ldots, C_n^*)$, is a continuous function of the price vector $P$ and the nominal income of the representative household, $Y^N = 1 + \sum_{i=1}^{n} \lambda_i \frac{\mu_i}{1+\mu_i} P_i^* Y_i^*$. Thus, it suffices to show that $P^*$ is a continuous function of $a$, and nominal income is a continuous function of $a$.

We can use the same argument as in the proof of Theorem B1 to show that $P^*$ is continuous in $a$.

To show that nominal income is continuous in $a$, let $\widehat{Y}_i = P_i^* Y_i^*$ as in the proof of Lemma 1. We showed in Lemma 1 that $\widehat{Y}$ is a fixed point of a contraction mapping $\Phi$, and the Jacobian matrix of $\Phi$ is a P-matrix. Gale and Nikaido (1965) showed that there exists a globally defined function $\widehat{Y}(a, S)$ that is continuously differentiable in $a$ and which satisfies $\widehat{Y} = \Phi(\widehat{Y})$. Since nominal income can be written as $Y^N = 1 + \sum_{i=1}^{n} \lambda_i \frac{\mu_i}{1+\mu_i} \widehat{Y}_i^*$, we conclude that $Y^N$ is a continuous function of $a$. Q.E.D.

*Quasi-Submodularity and the Technology-Price Single-Crossing Condition*

EXAMPLE B1—Quasi-submodularity does not imply the technology-price single-crossing condition: Consider an economy with three industries. Suppose that $\mu_i = 0$ for all $i$ for simplicity. The production function in each industry is a Cobb–Douglas production function, but crucially, technology does not take a Hicks-neutral form, and the input shares of an industry depend on the set of inputs used. Namely,

$$Y_i = \frac{1}{\left(1 - \sum_{j \in S_i} \alpha_{ij}(S_i)\right)^{1 - \sum_{j \in S_i} \alpha(S_i)_{ij}} \prod_{j \in S_i} \alpha_{ij}(S_i)^{\alpha_{ij}(S_i)}} A_i(S_i) L_i^{1 - \sum_{j \in S_i} \alpha_{ij}(S_i)} \prod_{j \in S_i} X_{ij}^{\alpha_{ij}(S_i)},$$

where the conditioning of $\alpha_{ij}$'s on the set of inputs, $S_i$, emphasizes the difference from the family of Cobb–Douglas production functions with Hicks-neutral technology. Suppose also that industries 1 and 2 use only labor as input and have production functions $Y_1 = e^{-\epsilon} L_1$ and $Y_2 = e^{\epsilon} L_2$, where $\epsilon > 0$. In equilibrium, the prices for industries 1 and 2 satisfy $p_1 = -a_1 = \epsilon$ and $p_2 = -a_2 = -\epsilon$, where we have also defined $a_i$ (for $i = 1, 2$) as the log productivities of these two industries. Industry 3, on the other hand, can choose any one of $\emptyset$, $\{1\}$, $\{2\}$, or $\{1, 2\}$ as its set of inputs, with the following input shares:

$$\alpha_{31}(S) = \begin{cases} 0 & \text{if } 1 \notin S_3, \\ \frac{2}{3} & \text{if } S_3 = \{1\}, \\ \frac{1}{3} & \text{if } S_3 = \{1, 2\}, \end{cases} \quad \text{and} \quad \alpha_{32}(S) = \begin{cases} 0 & \text{if } 2 \notin S_3, \\ \frac{2}{3} & \text{if } S_3 = \{2\}, \\ \frac{1}{3} & \text{if } S_3 = \{1, 2\}. \end{cases}$$

---

[1]Note there cannot be cycles in the sequence because $p^t$ is decreasing.

The log productivity for industry 3 is given by $a_3(\emptyset) = a_3(\{1\}) = a_3(\{2\}) = 0$, and $a_3(\{1, 2\}) = \epsilon$. Quasi-submodularity then requires that, for all equilibrium prices $(p_1, p_2)$,

$$\frac{2}{3}p_2 \leq 0 \quad \implies \quad -\epsilon + \frac{1}{3}p_1 + \frac{1}{3}p_2 \leq \frac{2}{3}p_1,$$

$$\frac{2}{3}p_1 \leq 0 \quad \implies \quad -\epsilon + \frac{1}{3}p_1 + \frac{1}{3}p_2 \leq \frac{2}{3}p_2$$

(and also with strict inequalities). It is straightforward to verify that these conditions hold. In particular, because $a_1 = -\epsilon < 0$ and $a_2 = \epsilon > 0$, we have $p_1 = \epsilon > 0$ and $p_2 = -\epsilon < 0$, and thus the first condition is always satisfied as $-\epsilon + \frac{1}{3}p_2 \leq \frac{1}{3}p_1$, while the second condition is also always satisfied because we never have $p_1 \leq 0$. Hence, the unit cost function for industry 3 is quasi-submodular.

We next show that it does not satisfy the technology-price single-crossing property. First note that, given the equilibrium prices characterized so far, it is cost-minimizing for industry 3 to choose $S_3 = \{1, 2\}$, since its log unit cost with $S_3 = \emptyset$ is 0, with $S_3 = \{1\}$, it is $\frac{2}{3}\epsilon$, with $S_3 = \{2\}$, it is $-\frac{2}{3}\epsilon$, and with $S_3 = \{1, 2\}$, it takes its lowest value, $-\epsilon$. Next, consider a change in the technology of industry 2 so that $a_2$ increases to $a_2' = 3\epsilon$. This can be verified to be a positive technology shock, since we still have $p_1 > 0$ and $p_2 < 0$, and thus the quasi-submodularity condition continues to be satisfied. But following this change, the log unit cost for industry 3 from choosing $S_3 = \{2\}$ declines to $-2\epsilon$, while the log unit cost from $S_3 = \{1, 2\}$ declines only to $-\epsilon + \frac{1}{3}\epsilon - \epsilon = -\frac{5}{3}\epsilon > -2\epsilon$. Therefore, following this positive technology shock, industry 3 chooses a smaller set of input suppliers, switching from $\{1, 2\}$ to $\{2\}$.

### *Proofs of Propositions* 1–3

PROOF OF PROPOSITION 1: We first show that the technology-price single-crossing condition holds for industry $i$ when $P_{-i}' \leq P_i$, but $P_i' = P_i$. We then argue that the technology-price single-crossing condition still applies even when $P_i' < P_i$.

Because $F_i(S_i, A_i(S_i), L_i, X_i)$ is supermodular, the profit function $\Lambda_i(S_i, A_i(S_i), P, L_i, X_i) = P_i F_i(S_i, A_i(S_i), L_i, X_i) - \sum_{j=1}^{n} P_j X_{ij} - L_i$ is supermodular in $L_i, X_i, A_i(S_i), S_i$, and $-P_{-i}$. If we take $P_i$ as fixed, Topkis (1998) showed that the function

$$\widetilde{\Lambda}_i(S_i, A_i(S_i), P) = \max_{X_i, L_i} \Lambda_i(L_i, X_i, A_i(S_i), S_i, P)$$

is supermodular in $A_i(S_i)$, $S_i$ and $-P_{-i}$. Thus, $\widetilde{\Pi}_i$ will satisfy the following single-crossing condition. For all $S_i' \supset S_i$ and all $P'$ such that $P_{-i}' \leq P_{-i}$ and $P_i' = P_i$, we have

$$\widetilde{\Pi}_i(S_i', A_i(S_i'), P) \geq \widetilde{\Pi}_i(S_i, A_i(S_i), P) \quad \implies \quad \widetilde{\Pi}_i(S_i', A_i(S_i'), P') \geq \widetilde{\Pi}_i(S_i, A_i(S_i), P').$$

Let $Q_i(P)$ be the demand for good $i$ when the prices are $P$, and write $\widetilde{\Pi}_i(S_i, A_i(S_i), P) = Q_i(P)(P_i - K_i(S_i, A_i(S_i), P))$. The cost function satisfies the single-crossing condition with the following argument:

$$K_i(S_i', A_i(S_i'), P) \leq K_i(S_i, A_i(S_i), P)$$

$$\iff \quad Q_i(P)(P_i - K_i(S_i', A_i(S_i'), P)) \geq Q_i(P)(P_i - K_i(S_i, A_i(S_i), P)).$$

But the last inequality implies

$$Q_i(P')\big(P_i' - K_i(S_i', A_i(S_i'), P')\big) \geq Q_i(P')\big(P_i' - K_i(S_i, A_i(S_i), P')\big)$$

$$\Longleftrightarrow \quad K_i\big(S_i', A_i(S_i'), P'\big) \leq K_i\big(S_i, A_i(S_i), P'\big),$$

which proves that the technology-price single-crossing condition holds for industry $i$ when $P_i = P_i'$ and $P_{-i}' \leq P_{-i}$.

To see that this generalizes to cases where $P_i' < P_i$, let $P''$ be a price vector such that $P_j'' = P_j'$ for all $j \neq i$, and $P_i'' = P_i$. Assume that $K_i(S_i', A_i(S_i'), P) \leq K_i(S_i, A_i(S_i), P)$, and note that

1. Since $P'' \leq P$ and $P_i'' = P_i$, our argument above yields the inequality $K_i(S_i', A_i(S_i'), P'') \leq K_i(S_i, A_i(S_i), P'')$.

2. Since $K_i$ does not depend on the $i$th coordinate of the price vector, we have that $K_i(\cdot, \cdot, P'') = K_i(\cdot, \cdot, P')$.

From the above two observations, we conclude that $K_i(S_i', A_i(S_i'), P') \leq K_i(S_i, A_i(S_i), P')$. $\hfill$ Q.E.D.

PROOF OF PROPOSITION 2: Since the technology-price single-crossing condition is preserved by monotonic transformation, it suffices to show that it is satisfied by the log unit cost function. To show that the log unit cost function satisfies the single-crossing condition, let $S_i \subset S_i'$ and $p' \leq p$ and note that

$$k_i\big(S_i', a_i, p\big) - k_i(S_i, a_i, p) \leq 0$$

$$\Longleftrightarrow \quad \sum_{j \in S_i'} \alpha_{ij} p_j - \sum_{j \in S_i} \alpha_{ij} p_j - a_i\big(S_i'\big) + a_i(S_i) \leq 0$$

$$\Longleftrightarrow \quad \sum_{j \in S_i' - S_i} \alpha_{ij} p_j - a_i\big(S_i'\big) + a_i(S_i) \leq 0$$

$$\Longrightarrow \quad \sum_{j \in S_i' - S_i} \alpha_{ij} p_j' - a_i\big(S_i'\big) + a_i(S_i) \leq 0$$

$$\Longleftrightarrow \quad \sum_{j \in S_i'} \alpha_{ij} p_j' - \sum_{j \in S_i} \alpha_{ij} p_j' - a_i\big(S_i'\big) + a_i(S_i) \leq 0$$

$$\Longleftrightarrow \quad k_i\big(S_i', a_i, p'\big) - k_i(S_i, a_i, p') \leq 0. \hfill \text{Q.E.D.}$$

PROOF OF PROPOSITION 3: In this case, the technology function $A_i$ maps a set $S_i$ to a vector $(A_{ij})_{j \in S_i}$. Write the CES cost function for firm $i$ as

$$K_i(S_i, A_i, P) = \left(\left(1 - \sum_{j \in S_i} \alpha_{ij}\right)^{\sigma} + \sum_{j \in S_i} \alpha_{ij}^{\sigma} \left(\frac{P_j}{A_{ij}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

Since the single-crossing condition is preserved by monotone transformations, it suffices to consider a monotone transformation of $K_i$. We split the analysis into two cases:

Case 1: $\sigma < 1$

In this case, we can raise the cost function to the power $1 - \sigma$ to obtain $(K_i(S_i, A_i, P))^{1-\sigma} = (1 - \sum_{j \in S_i} \alpha_{ij})^{\sigma} + \sum_{j \in S_i} \alpha_{ij}^{\sigma} (\frac{P_j}{A_{ij}})^{1-\sigma}$. Since $1 - \sigma > 0$, minimizing $K_i$ is equivalent

to minimizing $(K_i(S_i, A_i, P))^{1-\sigma}$. We will show that $(K_i(S_i, A_i, P))^{1-\sigma}$ satisfies the single-crossing condition. Let $S_i \subset S_i'$ and $P' \leq P$. We can derive the chain of implications

$$\left(K_i\big(S_i', A_i(S_i'), P\big)\right)^{1-\sigma} - \left(K_i\big(S_i, A_i(S_i), P\big)\right)^{1-\sigma} \leq 0$$

$$\implies \left(\left(1 - \sum_{j \in S_i'} \alpha_{ij}\right)^{\sigma} - \left(1 - \sum_{j \in S_i} \alpha_{ij}\right)^{\sigma}\right) + \sum_{j \in S_i' - S_i} \alpha_{ij}^{\sigma}\left(\frac{P_j}{A_{ij}}\right)^{1-\sigma} \leq 0$$

$$\implies \left(\left(1 - \sum_{j \in S_i'} \alpha_{ij}\right)^{\sigma} - \left(1 - \sum_{j \in S_i} \alpha_{ij}\right)^{\sigma}\right) + \sum_{j \in S_i' - S_i} \alpha_{ij}^{\sigma}\left(\frac{P_j'}{A_{ij}}\right)^{1-\sigma} \leq 0$$

$$\implies \left(K_i\big(S_i', A_i(S_i'), P'\big)\right)^{1-\sigma} - \left(K_i\big(S_i, A_i(S_i), P'\big)\right)^{1-\sigma} \leq 0,$$

so the single-crossing condition is satisfied.

Case 2: $\sigma > 1$

In this case, we can raise the cost function to the power $1 - \sigma$ to obtain $(K_i(S_i, A_i, P))^{1-\sigma} = (1 - \sum_{j \in S_i} \alpha_{ij})^{\sigma} + \sum_{j \in S_i} \alpha_{ij}^{\sigma}(\frac{P_j}{A_{ij}})^{1-\sigma}$. Since $1 - \sigma < 0$, minimizing $K_i$ is equivalent to *maximizing* $(K_i(S_i, A_i, P))^{1-\sigma}$. We need to show that $(K_i(S_i, A_i, P))^{1-\sigma}$ satisfies a *reverse* single-crossing condition. For all $S_i \subset S_i'$ and $P' \leq P$, $(K_i(S_i', A_i(S_i'), P))^{1-\sigma} - (K_i(S_i, A_i(S_i), P))^{1-\sigma} \geq 0 \implies (K_i(S_i', A_i(S_i'), P'))^{1-\sigma} - (K_i(S_i, A_i(S_i'), P'))^{1-\sigma} \geq 0$.

Let $S_i \subset S_i'$ and $P' \leq P$. Since $(\frac{P_j}{A_{ij}})^{1-\sigma} \leq (\frac{P_j'}{A_{ij}})^{1-\sigma}$, we obtain the chain of implications

$$\left(K_i\big(S_i', A_i(S_i'), P\big)\right)^{1-\sigma} - \left(K_i\big(S_i, A_i(S_i), P\big)\right)^{1-\sigma} \geq 0$$

$$\implies \left(\left(1 - \sum_{j \in S_i'} \alpha_{ij}\right)^{\sigma} - \left(1 - \sum_{j \in S_i} \alpha_{ij}\right)^{\sigma}\right) + \sum_{j \in S_i' - S_i} \alpha_{ij}^{\sigma}\left(\frac{P_j}{A_{ij}}\right)^{1-\sigma} \geq 0$$

$$\implies \left(\left(1 - \sum_{j \in S_i'} \alpha_{ij}\right)^{\sigma} - \left(1 - \sum_{j \in S_i} \alpha_{ij}\right)^{\sigma}\right) + \sum_{j \in S_i' - S_i} \alpha_{ij}^{\sigma}\left(\frac{P_j'}{A_{ij}}\right)^{1-\sigma} \geq 0$$

$$\implies \left(K_i\big(S_i', A_i(S_i'), P'\big)\right)^{1-\sigma} - \left(K_i\big(S_i, A_i(S_i), P'\big)\right)^{1-\sigma} \leq 0,$$

so the single-crossing condition is satisfied.                                        *Q.E.D.*

## Borel–Cantelli Lemmas

LEMMA B2—First Borel–Cantelli Lemma: *Suppose that $\{Z_n\}_{n \in \mathbb{N}}$ is a sequence of random variables. If, for any fixed $\epsilon > 0$, we have*

$$\sum_{n=1}^{\infty} \Pr[Z_n > \epsilon] < \infty,$$

*then* $\limsup_{n \to \infty} Z_n \leq 0$ *almost surely.*

LEMMA B3—Second Borel–Cantelli Lemma: *Suppose that* $\{Z_n\}_{n\in\mathbb{N}}$ *is a sequence of independent random variables. If*

$$\sum_{n=1}^{\infty} \Pr[Z_n \geq 0] = \infty,$$

*then* $\limsup_{n\to\infty} Z_n \geq 0$ *almost surely.*

### Distributions That Satisfy Assumption 4

PROPOSITION B1: *Let* $\{a_i(S_i(t))\}_{S\subset\{1,\ldots,t\}}$ *be a random variable where each* $a_i(S_i(t))$ *is an independently drawn Gumbel random variable with cdf* $\Phi(z; \mu, \sigma) = e^{-e^{-\frac{z}{\sigma}}}$. *Then Assumption 4 is satisfied with* $D = \sigma \log 2$.

PROOF: We can write $\lim_{n\to\infty} \frac{\max_{S\subset\{1,\ldots,t\}} a_i(S_i)}{t}$ as $\limsup_{n\to\infty} \frac{Z_n}{\log_2 n + 1}$, where $n = 2^{t-1}$ and $Z_i$ is a Gumbel random variable. The probability that $Z_n$ is above $\mu + \sigma \log n$ is equal to $1 - e^{-e^{-\log n}} = 1 - e^{-\frac{1}{n}}$. Since $1 - e^{-z} = z + o(z)$, there exists a constant $\kappa > 0$ and an integer $N$ such that, for all $n \geq N$, we have $1 - e^{-\frac{1}{n}} \geq \frac{\kappa}{n}$. Since $\sum_{n=N}^{\infty} \Pr[Z_n > \mu + \sigma \log n] > \sum_{n=N}^{\infty} \frac{\kappa}{n} = \infty$ and the variables $Z_1, \ldots, Z_n$ are independent, we can use Lemma B3 to derive that $\limsup_{n\to\infty} \frac{Z_n}{\log n} \geq \sigma$ almost surely. Using the fact that $\lim_{n\to\infty} \frac{\log n}{\log_2 n + 1} = \log 2$, we conclude that $\frac{Z_n}{\log_2 n + 1} \geq \sigma \log 2$.

To prove the reverse inequality, let $\epsilon > 0$ be arbitrary. The probability that $Z_n$ is above $\mu + \sigma(1 + \epsilon) \log n$ is $1 - e^{-e^{-(1+\epsilon)\log n}} = 1 - e^{-n^{-1-\epsilon}}$. Since $1 - e^{-z} = z + o(z)$, there exists a constant $\kappa > 0$ and an integer $N$ such that, for all $n \geq N$, we have $1 - e^{-\frac{1}{n^{1+\epsilon}}} \leq \frac{\kappa}{n^{1+\epsilon}}$. Since $\epsilon > 0$ is arbitrary and $\sum_{n=N}^{\infty} \Pr[Z_n \geq \mu + \sigma(1 + \epsilon) \log n] \leq \sum_{n=N}^{\infty} \kappa n^{-1-\epsilon} < \infty$, Lemma B2 implies that $\limsup_{n\to\infty} \frac{Z_n}{\log n} \leq \sigma$ almost surely. The union of two almost-sure events occurs almost surely, so we can conclude that $\limsup_{n\to\infty} \frac{Z_n}{\log n} = \sigma$, or equivalently, $\lim_{n\to\infty} \frac{\max(Z_1,\ldots,Z_n)}{\log n} = \sigma$ almost surely. Using the fact that $\lim_{n\to\infty} \frac{\log n}{\log_2 n + 1} = \log 2$, we obtain $\lim_{n\to\infty} \frac{\max(Z_1,\ldots,Z_n)}{\log_2 n} = \lim_{n\to\infty} \frac{\max(Z_1,\ldots,Z_n)}{\log n} \log 2 = \sigma \log 2$. Q.E.D.

PROPOSITION B2: *Let* $\{a_i(S_i(t))\}_{S\subset\{1,\ldots,t\}}$ *be a random variable where each* $a_i(S_i(t))$ *is an independently drawn exponential random variable with cdf* $\Phi(z; \nu) = 1 - e^{-\nu z}$. *Then Assumption 4 is satisfied with* $D = \frac{1}{\nu} \log 2$.

PROOF: We can write $\lim_{n\to\infty} \frac{\max_{S\subset\{1,\ldots,t\}} a_i(S_i)}{t}$ as $\limsup_{n\to\infty} \frac{Z_n}{\log_2 n + 1}$, where $n = 2^{t-1}$ and $Z_i$ is an exponential random variable. The probability that $Z_n$ is above $\frac{\log n}{\nu}$ is equal to $e^{-\log n} = n^{-1}$. Since $\sum_{n=1}^{\infty} n^{-1} = \infty$ and the variables $Z_1, \ldots, Z_n$ are independent, Lemma B3 implies that $\limsup_{n\to\infty} \frac{Z_n}{\log n} \geq \frac{1}{\nu}$ almost surely. Since $\lim_{n\to\infty} \frac{\log n}{\log_2 n + 1} = \log 2$, we conclude that $\limsup_{n\to\infty} \frac{Z_n}{\log n} \geq \frac{\log 2}{\nu}$ almost surely.

To prove the reverse inequality, let $\epsilon > 0$ be arbitrary. The probability that $Z_n$ is above $\frac{\log n + \epsilon \log n}{\nu}$ is $e^{-\log n - \epsilon \log n} = n^{-1-\epsilon}$. Since $\epsilon > 0$ is arbitrary and $\sum_{n=1}^{\infty} n^{-1-\epsilon} < \infty$, Lemma B2 implies $\limsup_{n\to\infty} \frac{Z_n}{\log n} \leq \frac{1}{\nu}$ almost surely. The intersection of two almost-sure events occurs almost surely, so we can conclude that $\limsup_{n\to\infty} \frac{Z_n}{\log n} = \frac{1}{\nu}$, or equiv-

alently, $\lim_{n\to\infty} \frac{\max(Z_1,\ldots,Z_n)}{\log n} = \frac{1}{\nu}$ almost surely. Since $\lim_{n\to\infty} \frac{\log n}{\log_2 n+1} = \log 2$, we obtain $\lim_{n\to\infty} \frac{\max(Z_1,\ldots,Z_n)}{\log_2 n+1} = \lim_{n\to\infty} \frac{\max(Z_1,\ldots,Z_n)}{\log n} \log 2 = \frac{1}{\nu} \log 2$. 
                                                                                                                  *Q.E.D.*

PROPOSITION B3: *Let* $\{a_i(S_i(t))\}_{S\subset\{1,\ldots,t\}}$ *be a random variable where each* $a_i(S_i(t)) = \sum_{j\in S_i(t)} \widetilde{a}_j$ *and each* $\widetilde{a}_j$ *is an independent random variable which is equal to* $-1$ *with probability* $\frac{1}{2}$ *and equal to* $1$ *with probability* $\frac{1}{2}$. *Assumption 4 is satisfied with* $D = \frac{1}{2}$.

PROOF: We can write $a^*(t) = \max_{S\subset\{1,\ldots,t\}} a_i(S_i(t)) = |j : \widetilde{a}_j = 1| = \sum_{j=1}^{t} X_j$, where

$$X_j = \begin{cases} 1 & \text{if } \widetilde{a}_j \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is an independent Bernoulli random variable taking values 0 or 1 with probability $\frac{1}{2}$. Recall that if $X_1, \ldots, X_n$ are independent random variables in the interval $[0, 1]$, we have the following Chernoff bound:

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i]\right| \geq \epsilon\right) \leq 2e^{-2n\epsilon^2}.$$

Using this Chernoff bound, we have

$$\Pr\left(\left|\frac{1}{t}\sum_{i=1}^{t} \widetilde{a}_i - \frac{1}{2}\right| \geq \epsilon\right) \leq 2e^{-2t\epsilon^2}.$$

Since $\sum_{t=0}^{\infty} 2e^{-2t\epsilon^2}$ converges, from Lemma B2, $\limsup_t |\frac{1}{t}\sum_{i=1}^{t} X_i - \frac{1}{2}| \leq 0$ almost surely. But since absolute values cannot be negative, we also have $\liminf_t |\frac{1}{t}\sum_{i=1}^{t} X_i - \frac{1}{2}| = 0$. Therefore, the limit $\lim_{t\to\infty} \frac{\max_{S\subset\{1,\ldots,t\}} a_i(S_i(t))}{t} = \frac{1}{2}$ almost surely. 
                                                                                                                  *Q.E.D.*

PROPOSITION B4: *Let* $a_i(S_i) = \sum_{j\in S_i} b_j + \epsilon_i(S_i)$, *where each* $b_j$ *is drawn identically and independently from the same distribution, and where* $\epsilon(S)$ *is an independently drawn Gumbel random variable with cdf* $\Phi(z; \mu, \sigma) = e^{-e^{-\frac{z}{\sigma}}}$. *Assume that* $\mathbb{E}[b_j|b_j \geq 0]$ *is finite. Then*

$$\Pr(b \geq 0)\sigma \log 2 \leq \liminf_{t\to\infty} \frac{\max\limits_{S_i(t)} a_i\big(S_i(t)\big)}{t},$$

$$\limsup_{t\to\infty} \frac{\max\limits_{S_i(t)} a_i\big(S_i(t)\big)}{t} \leq \mathbb{E}[b_j|b_j \geq 0] + \sigma \log 2,$$

*almost surely.*

PROOF: Let $\mathcal{S}^+(t)$ be the collection of sets $S_i(t)$ such that $b_j \geq 0$ and $j \leq t$ for all $j \in S_i(t)$. That is, there are no negative elements $b_j$ for any set $S_i \in \mathcal{S}^+$. Let $\chi(t) = |\{j : b_j \geq 0\}|$ and note that the size of $\mathcal{S}^+(t)$ is $2^{\chi(t)}$. Applying a Chernoff bound, we obtain that $\lim_{t\to\infty} \frac{\chi(t)}{t} \geq \Pr(b_j \geq 0)$ almost surely. Using Proposition B1, we obtain that

$$\lim_{t\to\infty} \frac{\max\limits_{S_i(t)\in\mathcal{S}^+(t)} \epsilon_i\big(S_i(t)\big)}{t} \geq \Pr(b \geq 0)\sigma \log 2.$$

Since $a_i(S_i(t)) = \sum_{j \in S_i(t)} b_j + \epsilon_i(S_i(t)) \geq \epsilon_i(S_i(t))$ for every $S_i(t) \in \mathcal{S}^+$, we have that $\liminf_{t \to \infty} \frac{\max_{S_i(t) \in \mathcal{S}^+(t)} a_i(S_i(t))}{t} \geq \lim_{t \to \infty} \frac{\max_{S_i(t) \in \mathcal{S}^+(t)} \epsilon_i(S_i(t))}{t}$. We conclude that $\Pr(b \geq 0)\sigma \log 2 \leq \liminf_{t \to \infty} \frac{\max_{S_i(t)} a_i(S_i(t))}{t}$.

To prove the other side of the inequality, note that $\frac{\max_{S_i(t)} a_i(S_i(t))}{t} \leq \frac{\max_{S_i(t)} \sum_{j \in S_i(t)} b_j}{t} + \frac{\max_{S_i(t)} \epsilon_i(S_i(t))}{t}$. The first term converges to $\mathbb{E}[b_j | b_j \geq 0]$ by the law of large numbers. The second term converges to $\sigma \log 2$ by Proposition B1. Thus, $\limsup_{t \to \infty} \frac{\max_{S_i(t)} a_i(S_i(t))}{t} \leq \mathbb{E}[b_j | b_j \geq 0] + \sigma \log 2$.

<div align="right">*Q.E.D.*</div>

COROLLARY B2: *If $b_j$ defined as in Proposition B4 is drawn from a distribution satisfying $\Pr(b_j \geq 0) > 0$, $\mathbb{E}[b_j | b_j \geq 0] < \infty$, then there exist finite and positive constants $\overline{D} > \underline{D}$ such that*

$$\underline{D} \leq \lim_{t \to \infty} \inf \frac{\max\limits_{S_i(t)} a_i(S_i(t))}{t},$$

$$\limsup_{t \to \infty} \frac{\max\limits_{S_i(t)} a_i(S_i(t))}{t} \leq \overline{D},$$

*almost surely.*

### No Growth Without Choice of Input Combinations

We next state and prove a theorem that shows that, in contrast to our main growth result, Theorem 6, when new goods are introduced into the supply chain at random (or with minimal choice), there will be zero growth in the long run.

THEOREM B3—No Growth Without Selection: *Suppose that Assumptions 1′, 2′, 4, and 5 hold. At each time $t \geq 1$, a set of suppliers $S_i^O(t) \subset \{1, \ldots, t\}$ for each $i = 1, 2, \ldots, n$ is selected uniformly at random. Then each industry $i$ chooses between its existing set of suppliers, $S_i^*(t-1)$, and $S_i^O(t)$. Then $g^* = 0$ almost surely.*

PROOF OF THEOREM B3: Let $S_i^O(t)$ be the input combination available to industry $i$ at time $t$. Let $S_i^*(t)$ be the set that minimizes industry $i$'s unit cost when it chooses between $S_i^*(t-1)$ and $S_i^O(t)$. Clearly,

$$a_i(S_i^*(t)) \leq \max_{j \in \{1, \ldots, t\}} \max_{\tau \in \{1, \ldots, t\}} a_j(S_j^O(\tau)).$$

Therefore, denoting the equilibrium log productivity sequence by $a(S^*(t))$, we have

$$-\frac{\pi(t)}{t} = \frac{1}{t}\beta(t)'\mathcal{L}(t)a(S^*(t)) \leq \frac{1}{t} \max_{j \in \{1, \ldots, t\}} \max_{\tau \in \{1, \ldots, t\}} a_j(S_j^O(\tau))\beta(t)'\mathcal{L}(t)\mathbf{1}(t),$$

where $1(t)$ is a $t \times 1$ vector all of whose components are ones. Since $\beta(t)'\mathcal{L}(t)1(t) = \sum_{i,j=1}^{n} \beta_j \mathcal{L}_{ij}$ and $\sum_{j=1}^{\infty} \beta_j = 1$, this implies

$$
\begin{aligned}
\limsup_{t\to\infty}\left(-\frac{\pi(t)}{t}\right) &= \limsup_{t\to\infty} \frac{1}{t} \max_{j\in\{1,\ldots,t\}} \max_{\tau\in\{1,\ldots,t\}} a_j\big(S_j^O(\tau)\big)\beta(t)'\mathcal{L}(t)1(t) \\
&\leq \limsup_{t\to\infty} \frac{1}{1-\theta}\frac{1}{t} \max_{j\in\{1,\ldots,t\}} \max_{\tau\in\{1,\ldots,t\}} a_j\big(S_j^O(\tau)\big) \\
&= \limsup_{t\to\infty} \frac{D}{1-\theta}\frac{\log_2(t^2)}{t} = 0 \quad \text{almost surely,}
\end{aligned}
$$

where the last equality follows from Assumption 4.

Since $\liminf_{t\to\infty}(-\frac{\pi(t)}{t}) \geq 0$ (as additional technology choices cannot increase prices), the previous argument establishes that $g^* = \lim_{t\to\infty}(-\frac{\pi(t)}{t}) = 0$. *Q.E.D.*

### *Growth With Harrod-Neutral Technology and CES Production Functions*

Consider the family of (modified) constant elasticity of substitution production functions with Harrod-neutral technology:[2]

$$
F_i\big(S_i, A_i(S_i), L_i, X_i\big) = \left[\left(1 - \sum_{j\in S_i}\alpha_{ij}\right)^{\frac{1}{\sigma}} \big(A_i(S_i)L_i\big)^{\frac{\sigma-1}{\sigma}} + \sum_{j\in S_i}\alpha_{ij}^{\frac{1}{\sigma}} X_{ij}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{B1}
$$

We next state and prove a theorem that shows that, when the production functions are given by (B1), the economy grows at a constant rate. Even though in this case the asymptotic growth rate turns out to be independent of the structure of the input–output network, the level of GDP still depends on it. In this section of the Supplemental Material, we also set distortions equal to zero, that is, $\mu = 0$.

THEOREM B4: *Suppose that Assumptions 1', 2', and 4 hold, and that production functions are given by* (B1). *Assume further that distortions are zero and that each industry chooses its set of suppliers* $S_i^*(t) \subset \{1,\ldots,t\}$. *Then, for each* $i = 1, 2, \ldots, t$, *the equilibrium log price vector* $p^*(t)$ *satisfies*

$$
\lim_{t\to\infty} -\frac{p_i^*(t)}{t} = D > 0 \quad \text{almost surely,}
$$

*and thus*

$$
g^* = D \quad \text{almost surely.}
$$

PROOF OF THEOREM B4: The cost function for industry $i$ is

$$
K_i\big(S_i, A_i(S_i), P\big) = \left(\left(1 - \sum_{j\in S_i}\alpha_{ij}\right)\left(\frac{1}{A_i(S_i)}\right)^{1-\sigma} + \sum_{j\in S_i}\alpha_{ij}P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.
$$

---

[2]The qualifier "modified" refers to the fact that we are raising the distribution parameters, the $\alpha_{ij}$'s, to the power $1/\sigma$, which ensures that the unit cost functions are linear in the $\alpha_{ij}$'s.

Since distortions are equal to zero, we have $P^* = K(S^*, A(S^*), P^*)$ so that $P_i^* = ((1 - \sum_{j \in S_i} \alpha_{ij})(\frac{1}{A_i(S_i)})^{1-\sigma} + \sum_{j \in S_i} \alpha_{ij}(P_j^*)^{1-\sigma})^{\frac{1}{1-\sigma}}$. It is convenient to raise both sides in the previous equation to the $1 - \sigma$ power to obtain the following system of linear equations in $Q^* = ((P_1^*)^{1-\sigma}, \ldots, (P_n^*)^{1-\sigma})$:

$$Q_i^* = \left(1 - \sum_{j \in S_i^*} \alpha_{ij}\right)\left(\frac{1}{A_i(S_i^*)}\right)^{1-\sigma} + \sum_{j \in S_i^*} \alpha_{ij} Q_j^*.$$

The solution to this set of equations can be written as

$$Q^* = \left(I - \alpha(S^*)\right)^{-1} B,$$

where $B_i = (1 - \sum_{j \in S_i^*} \alpha_{ij})(\frac{1}{A_i(S_i^*)})^{1-\sigma}$. Write $A_i(S_i^*(t)) = e^{Dt + \epsilon_i(t)}$, where $D$ is as in Assumption 4 and $\lim_{t \to \infty} \frac{\epsilon_i(t)}{t} = 0$ almost surely. We can use this to write $Q_i^*(t)$ as

$$Q_i^*(t) = \sum_{j=1}^t \mathcal{L}_{ij}\left(S^*(t)\right)\left(1 - \sum_{k \in S_j^*(t)} \alpha_{jk}\right)\left(e^{-(1-\sigma)(Dt + \epsilon_j(t))}\right).$$

Since $1 \leq \sum_{j=1}^t \mathcal{L}_{ij}(S^*(t)) \leq \frac{1}{1-\theta}$ and $1 - \theta \leq (1 - \sum_{k \in S_j^*(t)} \alpha_{jk}) \leq 1$, we have that

$$e^{-(1-\sigma)Dt - \max_{k \leq t} |(1-\sigma)\epsilon_j(t)|}(1 - \theta) \leq Q_i^*(t) \leq e^{-(1-\sigma)Dt + \max_{k \leq t} |(1-\sigma)\epsilon_j(t)|}\frac{1}{1 - \theta}.$$

Taking logarithms, we obtain

$$-(1 - \sigma)Dt - \max_{k \leq t}\left|(1 - \sigma)\epsilon_j(t)\right| + \log(1 - \theta)$$

$$\leq (1 - \sigma)p_i^*(t) \leq -(1 - \sigma)Dt + \max_{k \leq t}\left|(1 - \sigma)\epsilon_j(t)\right| - \log(1 - \theta).$$

Dividing by $t$ and taking the limit as $t$ goes to infinity, we obtain

$$-(1 - \sigma)D \leq (1 - \sigma)\lim_{t \to \infty}\frac{p_i^*(t)}{t} \leq -(1 - \sigma)D$$

almost surely. We conclude that $\lim_{t \to \infty}\frac{p_i^*(t)}{t} = D$ almost surely, and therefore $g^* = D$ almost surely. *Q.E.D.*

## APPENDIX C: ROBUSTNESS RESULTS

In this appendix, we report four sets (see Tables C-I–C-IV) of robustness checks on the results presented in Table I in the text. First, we repeat the same regressions using alternative definitions of significant change in input structure—dummies $J_{i,10}(t)$ and $J_{i,30}(t)$ computed analogously, but with thresholds corresponding to the 10th and 30th percentiles of the distribution of the Jaccard distance in that year. Next, we report regressions that are weighted by the value added of the industry in question in 1987 to give greater weight to larger industries. Finally, we limit the sample to 1997–2002 so as to focus on the period in which the data are consistently from the NAICS classification system. The results are broadly similar to those reported in the text and imply similar counterfactual aggregate TFP growth estimates.

TABLE C-I

NEW INPUT COMBINATIONS AND TFP (10TH PERCENTILE THRESHOLD)[a]

| | (1) | (2) | (3) |
|---|---|---|---|
| Panel A: All Industries (1987–2007) | | | |
| $J_{i,10}$ | 0.023 | 0.023 | 0.054 |
| | (0.011) | (0.017) | (0.023) |
| Counterfactual TFP change | 0.72% | 0.72% | 1.66% |
| Panel B: Manufacturing (1987–2007) | | | |
| $J_{i,10}$ | 0.020 | 0.016 | 0.061 |
| | (0.013) | (0.024) | (0.031) |
| Counterfactual TFP change | 0.63% | 0.50% | 1.87% |
| Panel C: All Industries Excluding Computers (1987–2007) | | | |
| $J_{i,10}$ | 0.016 | 0.020 | 0.048 |
| | (0.011) | (0.017) | (0.023) |
| Counterfactual TFP change | 0.50% | 0.64% | 1.48% |
| Linear industry trends | No | Yes | Yes |
| Control for lagged change in TFP | No | No | Yes |

[a]The table presents OLS estimates of the regression equation $\Delta \log \mathrm{TFP}_i(t) = \beta J_{i,10}(t) + \gamma_i + \nu(t) + \epsilon_i(t)$ using a data set of five-year stacked-differences for 488 industries between 1987 and 2007. $J_{i,10}(t)$ is a dummy indicating the Jaccard distance between the sets of inputs $S_i(t)$ and $S_i(t-1)$ being above the 10th percentile of its distribution in that year. Column 1 only includes period dummies. Column 2 adds industry-specific linear trends, the $\gamma_i$'s. Column 3 adds lagged change in log TFP, $\Delta \log \mathrm{TFP}_i(t-1)$. Panel A is for the entire sample. Panel B focuses on manufacturing industries and Panel C excludes computer industries (those within the three-digit SIC industries 357 and 367). Standard errors that are robust against arbitrary heteroscedasticity and serial correlation at the level of industry are reported in parentheses.

## APPENDIX D: DETAILS OF THE QUANTITATIVE EXERCISE

We now describe the details of the quantitative exercise discussed in Section 4. We start with a disaggregated economy with Cobb–Douglas sectoral technologies and an endogenous input–output structure (with extensive margin choices about inputs) calibrated to the 2007 U.S. input–output tables from the BEA. We then compare the response of this economy to an increase in the TFP of a sector to the response of more aggregated models (using both Cobb–Douglas and CES production functions) calibrated to the same data.

In this quantitative exercise, we parameterize the sectoral production functions as follows:

$$Y_i = A_i(S_i) F_i(X_i, L_i, S_i)$$

and

$$A_i(S_i) = B_{i0} \prod_{j \in S_i} B_{ij}.$$

Then, denoting $b_{ij} = \log B_{ij}$, the log cost function for industry $i$ is

$$k_i(p, a_i(S_i)) = \sum_{j \in S_i} (p_j \alpha_{ij} - b_{ij}) - b_{i0}.$$

TABLE C-II

NEW INPUT COMBINATIONS AND TFP (30TH PERCENTILE THRESHOLD)[a]

|  | (1) | (2) | (3) |
|---|---|---|---|
| **Panel A: All Industries (1987–2007)** | | | |
| $J_{i,30}$ | 0.014 | 0.004 | 0.031 |
|  | (0.006) | (0.010) | (0.013) |
| Counterfactual TFP change | 0.23% | 0.06% | 0.50% |
| **Panel B: Manufacturing (1987–2007)** | | | |
| $J_{i,30}$ | 0.013 | 0.000 | 0.028 |
|  | (0.007) | (0.010) | (0.014) |
| Counterfactual TFP change | 0.22% | 0.01% | 0.45% |
| **Panel C: All Industries Excluding Computers (1987–2007)** | | | |
| $J_{i,30}$ | 0.006 | −0.002 | 0.023 |
|  | (0.005) | (0.009) | (0.012) |
| Counterfactual TFP change | 0.11% | −0.04% | 0.38% |
| Linear industry trends | No | Yes | Yes |
| Control for lagged change in TFP | No | No |  |

[a]The table presents OLS estimates of the regression equation $\Delta \log \mathrm{TFP}_i(t) = \beta J_{i,30}(t) + \gamma_i + \nu(t) + \epsilon_i(t)$ using a data set of five-year stacked-differences for 488 industries between 1987 and 2007. $J_{i,30}(t)$ is a dummy indicating the Jaccard distance between the sets of inputs $S_i(t)$ and $S_i(t-1)$ being above the 30th percentile of its distribution in that year. Column 1 only includes period dummies. Column 2 adds industry-specific linear trends, the $\gamma_i$'s. Column 3 adds lagged change in log TFP, $\Delta \log \mathrm{TFP}_i(t-1)$. Panel A is for the entire sample. Panel B focuses on manufacturing industries and Panel C excludes computer industries (those within the three-digit SIC industries 357 and 367). Standard errors that are robust against arbitrary heteroscedasticity and serial correlation at the level of industry are reported in parentheses.

With this parameterization, industry $i$ will adopt industry $j$ as a supplier if and only if $b_{ij} \geq p_j \alpha_{ij}$ (adopting the convention that an industry adopts an input when indifferent).[3]

We further assume that in both the disaggregated and the aggregated economies, the preferences of the representative household are Cobb–Douglas as in Assumption 1′.

### Disaggregated Economy With Endogenous Production Network

We calibrate our model economy to 2007 U.S. input–output tables from the BEA, which comprise 391 sectors. As noted in footnote 20, we exclude the government sector (consisting of nine industries in the input–output tables), privately-owned residential property, and the sector made up of custom duties (the latter two have zero labor share). Throughout, GDP refers to the sum of value added of the remaining sectors.

We choose the parameters of the model as follows: for any edge $(i, j)$ observed in the input–output matrix, $\alpha_{ij}$ is set equal to the observed $(i, j)$th entry in the input–output matrix. For any edge $(i, j)$ not observed in the data, $\alpha_{ij}$ is set equal to $\alpha_{ij} = 0.95 \cdot (1 - \sum_{j' \in S_i} \alpha_{ij'}) \frac{\sum_{i':j \in S_{i'}} \alpha_{i'j}}{\sum_{i',j':j' \in S_{i'}} \alpha_{i'j'}}$. This choice ensures that (1) all observed edges have cost

---

[3]We also note that, though we are keeping the number of industries fixed here, this specification is consistent with sustained growth when the number of industries changes as in Section 5. In particular, if $b_{i0}, b_{i1}, b_{i2}, b_{i3}, \ldots$ are drawn independently so that $\Pr(b_{ij} > \delta_i) > \epsilon_i$ for some constants $\delta_i, \epsilon_i > 0$, then Assumption 4 is satisfied (because $\liminf_{t \to \infty} a_i(S_i(t)) \geq \delta_i \epsilon_i > 0$ almost surely).

TABLE C-III

New Input Combinations and TFP (Value-Added Weighted Regressions)[a]

|  | (1) | (2) | (3) |
|---|---|---|---|
| Panel A: All Industries (1987–2007) | | | |
| $J_{i,20}$ | 0.018 | 0.018 | 0.030 |
|  | (0.021) | (0.020) | (0.023) |
| Counterfactual TFP change | 0.42% | 0.43% | 0.73% |
| Panel B: Manufacturing (1987–2007) | | | |
| $J_{i,20}$ | 0.042 | 0.036 | 0.064 |
|  | (0.024) | (0.017) | (0.022) |
| Counterfactual TFP change | 0.99% | 0.85% | 1.54% |
| Panel C: All Industries Excluding Computers (1987–2007) | | | |
| $J_{i,20}$ | 0.001 | 0.014 | 0.021 |
|  | (0.019) | (0.020) | (0.022) |
| Counterfactual TFP change | 0.02% | 0.34% | 0.51% |
| Linear industry trends | No | Yes | Yes |
| Control for lagged change in TFP | No | No | Yes |

[a]The table presents weighted OLS estimates of the regression equation $\Delta \log \mathrm{TFP}_i(t) = \beta J_{i,20}(t) + \gamma_i + \nu(t) + \epsilon_i(t)$ using a data set of five-year stacked-differences for 488 industries between 1987 and 2007 and value added of the industry in 1987 as weight. $J_{i,20}(t)$ is a dummy indicating the Jaccard distance between the sets of inputs $S_i(t)$ and $S_i(t-1)$ being above the 20th percentile of its distribution in that year. Column 1 only includes period dummies. Column 2 adds industry-specific linear trends, the $\gamma_i$'s. Column 3 adds lagged change in log TFP, $\Delta \log \mathrm{TFP}_i(t-1)$. Panel A is for the entire sample. Panel B focuses on manufacturing industries and Panel C excludes computer industries (those within the three-digit SIC industries 357 and 367). Standard errors that are robust against arbitrary heteroscedasticity and serial correlation at the level of industry are reported in parentheses.

shares equal to the cost shares in the data; (2) all edges that are absent in the 2007 input–output matrix have cost shares proportional to the observed outdegree of the supplier; and (3) the row sums of the full input–output matrix (including the edges that are absent in 2007) are less than 1 so that labor is an essential input as required by Assumption 1.

The $\beta_i$'s are set equal to each industry's consumption share.

We take distortions from the markup estimates of De Loecker, Eeckhout, and Unger (2018), which are at the two-digit level. We apply the same distortion to all sub-industries in the same two-digit industry.

We assume that the $b_{ij}$'s are drawn from truncated Normal distributions, where the truncation ensures that the productivity of the edge is consistent with its presence or absence in the input–output tables. More specifically, each $b_{i0}$ is drawn independently from a Normal prior with mean $m$ and standard deviation 1. The parameter $m$ is chosen so that equilibrium GDP matches U.S. GDP (which is computed from the BEA input–output tables as 11.563 trillion 2007 dollars, excluding the government sector, owner-occupied residential housing, and custom duties).

We implement the truncation procedure for $b_{ij}$'s as follows. Each is drawn independently from a Normal prior with mean $\frac{m}{n}$ and standard deviation $\frac{1}{n}$ (where $n = 391$ is the number of industries).

1. We draw $b_{i0}, b_{i1}, \ldots, b_{in}$ from the prior distributions described above.
2. We set $a_i(S_i) = b_{i0} + \sum_{j \in S_i} b_{ij}$.

TABLE C-IV

NEW INPUT COMBINATIONS AND TFP (1997–2007)[a]

| | (1) | (2) | (3) |
|---|---|---|---|
| Panel A: All Industries (1987–2007) | | | |
| $J_{i,20}$ | 0.007 | 0.028 | 0.070 |
| | (0.013) | (0.046) | (0.037) |
| Counterfactual TFP change | 0.64% | 2.68% | 5.05% |
| Panel B: Manufacturing (1987–2007) | | | |
| $J_{i,20}$ | 0.010 | 0.060 | 0.097 |
| | (0.014) | (0.053) | (0.042) |
| Counterfactual TFP change | 0.95% | 5.74% | 6.99% |
| Panel C: All Industries Excluding Computers (1987–2007) | | | |
| $J_{i,20}$ | 0.001 | −0.011 | 0.042 |
| | (0.009) | (0.034) | (0.033) |
| Counterfactual TFP change | 0.08% | −1.05% | 3.03% |
| Linear industry trends | No | Yes | Yes |
| Control for lagged change in TFP | No | No | Yes |

[a]The table presents OLS estimates of the regression equation $\Delta \log \text{TFP}_i(t) = \beta J_{i,20}(t) + \gamma_i + \nu(t) + \epsilon_i(t)$ using a data set of five-year stacked-differences for 488 industries between 1997 and 2007. $J_{i,20}(t)$ is a dummy indicating the Jaccard distance between the sets of inputs $S_i(t)$ and $S_i(t-1)$ being above the 20th percentile of its distribution in that year. Column 1 only includes period dummies. Column 2 adds industry-specific linear trends, the $\gamma_i$'s. Column 3 adds lagged change in log TFP, $\Delta \log \text{TFP}_i(t-1)$. Panel A is for the entire sample. Panel B focuses on manufacturing industries and Panel C excludes computer industries (those within the three-digit SIC industries 357 and 367). Standard errors that are robust against arbitrary heteroscedasticity and serial correlation at the level of industry are reported in parentheses.

3. We compute $p = (I - \alpha)^{-1}(\log(1 + \mu) - a(S))$.

4. We repeat the following steps until $b_{ij} \geq \alpha_{ij} p_j$ for all $i \in \{1, \ldots, n\}$ and all $j \in S_i$, and $b_{ij} < \alpha_{ij} p_j$ for all $i \in \{1, \ldots, n\}$ and all $j \notin S_i$:

(a) If $j \in S_i$ and $b_{ij} < \alpha_{ij} p_j$, then redraw $b_{ij}$ from a truncated Normal distribution (with the same parameters as above) with support over the interval $[\alpha_{ij} p_j, \infty)$.

(b) If $j \notin S_i$ and $b_{ij} > \alpha_{ij} p_j$, then redraw $b_{ij}$ from a truncated Normal distribution with the same parameters as above but now with support over the interval $(-\infty, \alpha_{ij} p_j]$.

(c) If $j \in S_i$ and $b_{ij} \geq \alpha_{ij} p_j$ or $j \notin S_i$ and $b_{ij} \leq \alpha_{ij} p_j$, then keep $b_{ij}$.

(d) Recompute $a_i(S_i) = b_{i0} + \sum_{j \in S_i} b_{ij}$ and $p = (I - \alpha)^{-1}(\log(1 + \mu) - a(S))$.

This procedure yields two posterior distributions, one for $b_{ij}$ conditional on $j \in S_i$ and another for $b_{ij}$ conditional on $j \notin S_i$. Figures 3 and 4 depict these conditional distributions.

Once the productivity parameters, the $a_i(S_i)$'s, have been sampled, we compute log prices from equation (10) in the text as

$$p = -(I - \alpha)^{-1}(a - \log(1 + \mu)).$$

To compute nominal GDP, we assume that all revenues generated by distortions are rebated to households (i.e., $\lambda_j = 1$ for all industries), which is consistent with our use of markup data to choose the level of distortions. Since utility and production functions are
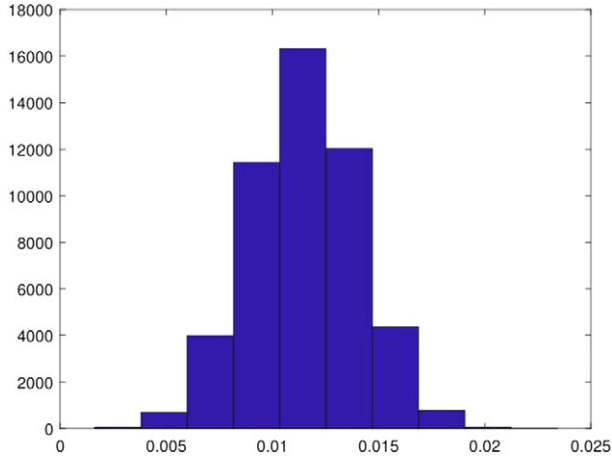
FIGURE 3.—Distribution of edge-specific productivities $b_{ij}$ conditional on edge $(i, j)$ being observed in the 2007 U.S. input–output matrix.

Cobb–Douglas, we have

$$P_i C_i = \beta_i \left( 1 + \sum_{i=1}^{n} \frac{\mu_i}{1 + \mu_i} P_i Y_i \right). \tag{D1}$$

Letting $\text{GDP}^N$ denote nominal GDP and $d_i = \frac{P_i Y_i}{\text{GDP}^N}$ denote the Domar weight for industry $i$, (D1) can be written as

$$P_i C_i = \beta_i \left( 1 + \sum_{i=1}^{n} \frac{\mu_i}{1 + \mu_i} d_i \cdot \text{GDP}^N \right).$$
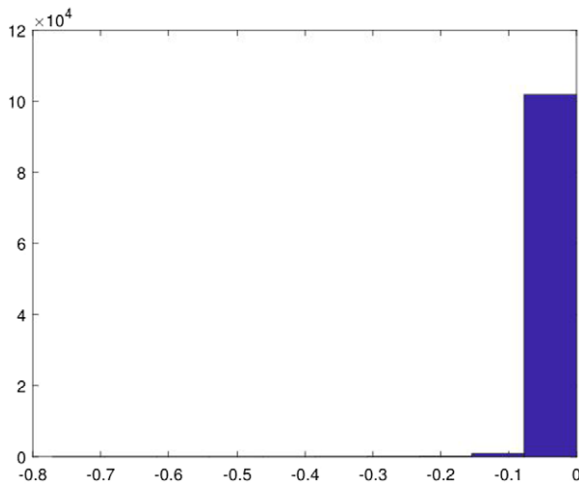


FIGURE 4.—Distribution of edge-specific productivities $b_{ij}$ conditional on edge $(i, j)$ being absent in the 2007 U.S. input–output matrix.

Summing this equation over all industries, we obtain nominal GDP in terms of Domar weights as

$$\text{GDP}^N = \frac{1}{1 - \displaystyle\sum_{i=1}^{n} \frac{\mu_i}{1 + \mu_i} d_i}. \tag{D2}$$

Moreover, we can also express the Domar weights in terms of input–output entries. Let $\widehat{\alpha}_{ji} = \frac{P_j X_{ji}}{P_i Y_i}$ denote the amount (in dollars) of good $j$ necessary to produce one dollar's worth of good $i$. Note that this is different from $\alpha_{ji}$ which is the cost share of input $i$ in the production of good $j$ (the fraction of the cost of good $j$ that goes to input $i$). In particular,

$$\widehat{\alpha}_{ji} = \frac{\alpha_{ji}}{1 + \mu_j}. \tag{D3}$$

Rearranging the market clearing condition for industry $i$,

$$P_i Y_i = P_i C_i + \sum_{j \in S_i} P_j X_{ji},$$

we can write

$$P_i Y_i = P_i C_i + \sum_{j \in S_i} \widehat{\alpha}_{ji} P_i Y_i. \tag{D4}$$

Dividing both sides of equation (D4) by nominal GDP, we get

$$d_i = \beta_i + \sum_{j \in S_i} \widehat{\alpha}_{ji} d_i,$$

$$d = (I - \widehat{\alpha}')^{-1} \beta. \tag{D5}$$

Given the Domar weights and nominal GDP, we can compute real GDP from equation (13) as

$$Y(t) = \frac{Y^N(t)}{\displaystyle\prod_{i=1}^{t} P_i(t)^{\beta_i}}.$$

Taking logarithms on both sides, and using equations (10), (D2), and (D5), this becomes

$$\log Y(t) = \beta'(I - \alpha)^{-1}(a - \log(1 + \mu)) - \log\left(1 - \sum_{i=1}^{n} \frac{\mu_i}{1 + \mu_i} d_i\right). \tag{D6}$$

We use this formula to compute real GDP and calibrate the parameter $m$ to match GDP in 2007. In this equilibrium, as in the U.S. input–output tables in 2007, 32.54% of all possible edges are present.

We then increase TFP in the computer and electronic product manufacturing sector (NAICS 334). As noted in the text, this sector makes up 1.98% of U.S. GDP in 2007. More specifically, for each one of the 20 detailed industries in the (two-digit) computer

and electronic product manufacturing sector, we increase $b_{0,i}$ by 1%. We then compute the implied increase in real GDP.

Our algorithm for computing new equilibrium prices and input–output matrix is as follows:

1. Let $\alpha(0)$ be the input–output matrix in the original economy, and let $S_i(0)$ be the original network. Let $a_i(0) = b_{i0} + \sum_{j \in S_i(0)}$ and let $p(0) = (I - \alpha(0))^{-1}(a(0) - \log(1 + \mu))$. Finally, let $\alpha$ (with no time argument) denote the full input–output matrix, including the entries for edges not observed in the 2007 data.

2. Initialize at $t = 0$ and repeat until prices converge. In particular:

(a) If $b_{ij} > p_j(t)\alpha_{ij}$, set $\alpha_{ij}(t+1) = \alpha_{ij}$. Update $S_i(t+1)$ to include $j$. Note that $S_i(t+1) \supseteq S_i(t)$ because, from Theorem 5, an increase in $b_{i0}$ (which is a positive technology shock) or a decrease in prices always expands the equilibrium production network.

(b) Set $a(t+1) = b_{i0} + \sum_{j \in S_i(t+1)} b_{ij}$.

(c) Set $p(t+1) = (I - \alpha(t+1))^{-1}(a(t+1) - \log(1 + \mu))$.

We find that the new equilibrium has 288 additional edges, so that now 32.73% of edges in the input–output matrix are present. In this new equilibrium, real GDP increases by 0.72%. Of this increase, 0.13 percentage points come from greater value added in the computer and electronic product manufacturing sector. The remaining 0.59 percentage points originate from other sectors expanding their output as they face lower prices and add additional suppliers.

### Aggregate Economies With Exogenous Production Network

We now repeat the same exercise but for three more aggregated economies with 84 industries (at the three-digit NAICS level). Crucially, these aggregated economies do not feature an extensive margin of adjustment in their input–output structure (hence "exogenous" production networks). One of those economies has Cobb–Douglas production technologies and the other two have CES technologies, with elasticity of substitution parameters $\sigma = 1/2$ and $\sigma = 2$, respectively. All three economies are calibrated to the 2007 U.S. input–output tables (at the same level of aggregation) and thus have the same baseline equilibrium as our disaggregated economy. Nevertheless, we show that they generate very different responses to the increase in the TFP of the computers and electronic product manufacturing sector—because there are no extensive margin changes in the production network.

Even though there are no such extensive margin changes in input–output linkages, in the CES economy changes in equilibrium prices will lead to changes in equilibrium input quantities and thus in the entries of the input–output matrix. Nevertheless, the increase in equilibrium GDP will be small in both the Cobb–Douglas and CES aggregated economies.

### Aggregation Procedure

We first describe how we consistently aggregate from the more disaggregated economy. Our procedure closely follows Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017), except that we adapt their formulae to include markups.[4] We use capital letters $(I, J, \ldots)$ to denote "sectors" (short for aggregated sectors) and lowercase letters $(i, j, \ldots)$ to denote "indus-

---

[4]We are grateful to Alireza Tahbaz-Salehi for help and suggestions on this point.

tries" (short for disaggregated industries), with the convention that $i$ is a disaggregated industry that is part of the aggregated sector $I$. We denote the number of aggregated sectors by $N$ and the number of industries by $n$.

We begin by aggregating the BEA input–output tables, markups, and our imputed values for $b_{i0}$, $b_{ij}$. For each sector $I$, the aggregation process should not change the following quantities: (1) households' consumption expenditure on sector $I$; (2) sector $I$'s total output; (3) sector $I$'s profits; (4) sector $I$'s expenditure on intermediate goods from sector $J$; (5) sector $I$'s expenditure on labor; and (6) real GDP. More formally, denoting value added in industry $i$ by $v_i$, these requirements imply

$$P_I C_I = \sum_{i \in I} P_i C_i, \tag{D7}$$

$$P_I Y_I = \sum_{i \in I} P_i Y_i, \tag{D8}$$

$$\Pi_I = \sum_{i \in I} \Pi_i, \tag{D9}$$

$$P_J X_{IJ} = \sum_{i \in I, j \in J} P_j X_{ij}, \tag{D10}$$

$$W L_I = \sum_{i \in I} W L_i, \tag{D11}$$

$$\sum_{I=1}^{N} v_I = \sum_{i=1}^{n} v_i. \tag{D12}$$

Because the household's utility is Cobb–Douglas, equation (D7) implies that

$$\beta_I = \sum_{i \in I} \beta_i. \tag{D13}$$

Let $d_i = \frac{P_i Y_i}{\text{GDP}}$ and let $d_I = \frac{P_I Y_I}{\text{GDP}}$ represent industry- and sector-level Domar weights. Equation (D8) then implies that

$$d_I = \sum_{i \in I} d_i. \tag{D14}$$

Denoting sectoral markups by $\mu_I$, equation (D9) yields

$$\frac{\mu_I}{1 + \mu_I} P_I Y_I = \sum_{i \in I} \frac{\mu_i}{1 + \mu_i} P_i Y_i.$$

Dividing both sides by GDP and rearranging terms, we obtain

$$\frac{\mu_I}{1 + \mu_I} = \frac{1}{d_I} \sum_{i \in I} \frac{\mu_i}{1 + \mu_i} d_i. \tag{D15}$$

To derive the aggregate input–output matrix $(\alpha_{IJ})_{I,J=1}^{N}$, begin with equation (D10) and multiply both sides by $\frac{1}{P_I Y_I} \frac{P_I Y_I}{\text{GDP}}$, which gives

$$\frac{\alpha_{IJ}}{1 + \mu_I} d_I = \sum_{i \in I, j \in J} \frac{\alpha_{ij}}{1 + \mu_i} d_i, \tag{D16}$$

where we have used the fact that $\frac{\alpha_{ij}}{1+\mu_i} = \frac{P_j X_{ij}}{P_I Y_I}$.

The labor aggregation condition (D11) implies $L_I = \sum_{i \in I} L_i$.

Finally, we need to derive sectoral-level TFPs from industry-level TFPs. In doing this, GDP and the price deflator $e^{-\beta' \mathcal{L}(a - \log(1+\mu))}$ have to be invariant to aggregation.[5] Let

$$\widetilde{d}_j = \sum_{i=1}^{n} \beta_i \mathcal{L}_{ij}, \qquad \widetilde{d}_J = \sum_{I=1}^{N} \beta_I \mathcal{L}_{IJ} \tag{D17}$$

represent the industry and sectoral *cost-based Domar weights*.[6] These can be computed from the industry and sectoral cost-based input–output matrices, $(\alpha_{ij})_{i,j=1}^{n}$, $(\alpha_{I,J})_{I,J=1}^{N}$, respectively. Then the price deflators for the disaggregated and aggregate economies are, respectively, $e^{-\sum_{i=1}^{n} \widetilde{d}_i(a_i - \log(1+\mu_i))}$ and $e^{-\sum_{I=1}^{N} \widetilde{d}_I a_I - \log(1+\mu_I)}$. Because these two expressions have to coincide, we derive our last restriction as

$$a_I = \frac{1}{\widetilde{d}_I} \sum_{i \in I} \widetilde{d}_i a_i - \frac{1}{\widetilde{d}_I} \widetilde{d}_i \log(1 + \mu_i) + \log(1 + \mu_I). \tag{D18}$$

*The Aggregated Cobb–Douglas Economy*

The above aggregation procedure conserves household and firm expenditures, firm profits, and GDP. We use it to aggregate the input–output matrix from the BEA data.[7] We also aggregate sectoral TFPs to three-digit NAICS sectoral level with the procedure described above. We then compute equilibrium prices and GDP for the aggregated Cobb–Douglas economy (which naturally coincide with GDP in the disaggregated economy).

We then treat this aggregated Cobb–Douglas economy as primitive and introduce the same 1% TFP increase in the (two-digit) computer and electronic product manufacturing sector. Following this change in TFP, there is no extensive margin change in the input–output structure of the economy (by construction), and since we have Cobb–Douglas production technologies, the entries of the input–output matrix do not change either. We then compute the resulting changes in prices and quantities and real GDP. We find that real GDP increases by 0.04% (as compared to 0.72% in the disaggregated economy with endogenous input–output linkages).

---

[5]Nominal GDP is also invariant to aggregation since $d_I \frac{\mu_I}{1+\mu_I} = \sum_{i \in I} \frac{\mu_i}{1+\mu_i} d_i$ and $\text{GDP}^N = \frac{1}{1 - \sum_{i=1}^{n} \frac{\mu_i}{1+\mu_i} d_i} = \frac{1}{1 - \sum_{I=1}^{N} \frac{\mu_I}{1+\mu_I} d_I}$.

[6]We borrow this terminology from Baqaee and Farhi (2019a, 2019b). One can show that the standard Domar weights can be computed as $\beta'(I - \widehat{\alpha})^{-1}$, where $\widehat{\alpha}_{ij} = \frac{P_j X_{ij}}{P_i Y_i}$ is the revenue-based input–output matrix. The cost-based Domar weights are computed with the analogous formula, but using the cost-based input–output matrix instead. When distortions/markups are zero, the two types of Domar weights coincide.

[7]Our markups are already at the two-digit level, so do not need to be aggregated.

*Aggregated CES Economies*

We repeat the same procedure for aggregated CES economies. In this case, we use the aggregated $\alpha_{IJ}$'s as parameters for constant elasticity of substitution sectoral production functions as in equation (11) in the text. We initialize sectoral TFPs at the levels computed for the aggregated Cobb–Douglas economy. We then raise the TFP of the computer and electronic product manufacturing sector by 1% and compute the change in real GDP in the same way.

Following the TFP shock, there is again, by construction, no extensive margin change in the input–output structure of the economy, but because the elasticity of substitution between inputs is no longer equal to 1, entries of the input–output matrix change as prices change. Nevertheless, we find that the implied increases in real GDP are again small (as in the aggregated Cobb–Douglas economy). In particular, when the elasticity of substitution between inputs is $\sigma = 1/2$, the 1% TFP increase in the computer and electronic product manufacturing sector leads to a 0.09% increase in real GDP. The same shock leads to a 0.01% increase in GDP when $\sigma = 2$.

REFERENCES

HAWKINS, D., AND H. A. SIMON (1949): "Note: Some Conditions of Macroeconomic Stability," *Econometrica*, 17 (3/4), 245–248. [2]

GALE, D., AND H. NIKAIDO (1965): "The Jacobian Matrix and Global Univalence of Mappings," *Mathematische Annalen*, 159 (2), 81–93. [2,3]

TOPKIS, D. M. (1998): *Supermodularity and Complementarity*. Princeton: Princeton University Press. [4]

DE LOECKER, J., J. EECKHOUT, AND G. UNGER (2018): "The Rise of Market Power and the Macroeconomic Implications," Manuscript. [14]

ACEMOGLU, D., A. OZDAGLAR, AND A. TAHBAZ-SALEHI (2017): "Microeconomic Origins of Macroeconomic Tail Risks," *The American Economic Review*, 107 (1), 54–108. [18]

BAQAEE, D. R., AND E. FARHI (2019a): "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," *Econometrica*, 87 (4), 1155–1203. [20]

——— (2019b): "Productivity and Misallocation in General Equilibrium," *The Quarterly Journal of Economics*, 135 (1), 105–163. [20]