Econometrica Supplementary Material

SUPPLEMENT TO "BAYESIAN LEARNING, SMOOTH APPROXIMATE OPTIMAL BEHAVIOR, AND CONVERGENCE TO ε-NASH EQUILIBRIUM"

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APPENDIX A PROVIDES A KEY MATHEMATICAL RESULT—the definitions of $\tilde{T}_s$ and $\mathcal{P}^j_{s-1}(\tilde{T}_s, 3, i)$—and the construction of $\{\tilde{\mathcal{P}}^i_s\}_{i,s}$. Appendix B gives the proofs for Lemmas 3 and 5 and the definition of $\hat{r}(\eta, u, \delta)$. Appendix C provides the proofs for Lemmas 6–8 and Proposition 1.B, and shows two claims for the proof of Lemma 9. Appendix D demonstrates that Theorem 1 is obtained from Proposition 1 and $\bar{r}(\varepsilon, \bar{\delta}, u, \delta)$ is derived from $\hat{r}(\eta, u, \delta)$.

APPENDIX A

A.1. *Mathematical Preparation*

For the argument in the appendix, I define additional notations. An *infinite history* is generically denoted by h_∞. If a finite history h is an initial segment of a (finite or infinite) history h', it is denoted by $h \leq h'$. In particular, if $h \leq h'$ and $h \neq h'$, it is denoted by $h < h'$. The *joint* of two finite partitions $\mathcal{P}$ and $\mathcal{Q}$ is denoted by $\mathcal{P} \wedge \mathcal{Q}$, that is, $\mathcal{P} \wedge \mathcal{Q} := \{\alpha \cap \beta \mid \alpha \in \mathcal{P}, \beta \in \mathcal{Q}\}$.[1] Moreover, I will often define a subset of H and call it a *class* although the subset may not be an element of any given conditioning rule (CR).

Next, I prepare a key mathematical result: *a conditional extension of large deviations*. Given a class α, let $\mathbf{S}^\alpha_m$ denote the event that state S occurs between the mth α-active period and the $(m+1)$th α-active period.[2] Let $\mathcal{T}^\alpha_m(h_\infty)$ denote the calendar time of the mth α-active period in h_∞; $\mathcal{T}^\alpha_m(h_\infty) < \infty$ means that α is active at least m times in h_∞. Let $\mathbf{d}^\alpha_m[S](h_\infty)$ denote the number of times that S has occurred between two subsequent α-active periods up to the mth α-active period in h_∞.

PROPOSITION A: *Take any history $h_T \in H$ and any class α such that for all $h < h_T$, $h \notin \alpha$. Suppose that a strategy profile σ and events $\{\mathbf{S}^\alpha_m\}_m$ satisfy the following condition*: *for all $h \in \alpha$ and all $m = 1, 2, \dots$ such that $h \geq h_T$, $\mu_\sigma(h) > 0$, and α has been active exactly $(m-1)$ times in h,*

$$c_1 \leq \mu_\sigma\big(\mathbf{S}^\alpha_m | h\big) \leq c_2,$$

[1]Note that $\mathcal{P} \wedge \mathcal{Q}$ is also a finite partition, and is finer than $\mathcal{P}$ and $\mathcal{Q}$, that is, $\mathcal{P}, \mathcal{Q} \leq \mathcal{P} \wedge \mathcal{Q}$.

[2]In one example, a particular pure action a_i is realized in the mth α-active period. In another example, AES$(\hat{\sigma})$ occurs between the mth α-active period and the $(m+1)$th α-active period.

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where c_1 and c_2 are nonnegative constants. Then, for all $\varepsilon > 0$ and all $m = 1, 2, \dots,$

$$\mu_\sigma\left(\mathcal{T}_m^\alpha < \infty, \frac{\mathbf{d}_m^\alpha[S]}{m} \le c_1 - \varepsilon \text{ or } \frac{\mathbf{d}_m^\alpha[S]}{m} \ge c_2 + \varepsilon \middle| h_T\right)$$
$$\le 2\exp(-2m\varepsilon^2).$$

PROOF: This is a straightforward generalization of Proposition A in Noguchi (2015), and the proof is just the same as that of Proposition A in Noguchi (2015). *Q.E.D.*

A.2. *Periodic Conditioning Rules*

Given $\mathcal{P}_s^j$ and $\tilde{T}$, I define a conditioning rule $\mathcal{P}_s^j(\tilde{T}, 3, i)$ to construct temporary beliefs. First, I define a $\tilde{T}$-periodic conditioning rule $\mathcal{P}_s^j(\tilde{T})$ by partitioning each class α in $\mathcal{P}_s^j$ into two subclasses α_{A} and α_{B}: $h_T \in \alpha_{\mathrm{A}}$ if $h_T \in \alpha$ and $T = n\tilde{T}$ for some integer n, and $h_T \in \alpha_{\mathrm{B}}$ otherwise, that is, $h_T \in \alpha$ and $T \neq n\tilde{T}$ for any integer n. Let $\mathcal{P}_s^j(\tilde{T}) := \{\alpha_{\mathrm{A}}, \alpha_{\mathrm{B}} \mid \alpha \in \mathcal{P}_s^j\}$. Next, I construct a $(\tilde{T}, 3)$-periodic conditioning rule $\mathcal{P}_s^j(\tilde{T}, 3)$ by partitioning each α_{A} and α_{B} in $\mathcal{P}_s^j(\tilde{T})$ into three subclasses: $h_T \in \alpha_{\mathrm{AF}}$ if $h_T \in \alpha_{\mathrm{A}}$ and $\#\{T' \mid h_{T'} < h_T, h_{T'} \in \alpha_{\mathrm{A}}\} = 3n$ for some integer n, $h_T \in \alpha_{\mathrm{AS}}$ if $h_T \in \alpha_{\mathrm{A}}$ and $\#\{T' \mid h_{T'} < h_T, h_{T'} \in \alpha_{\mathrm{A}}\} = 3n + 1$ for some integer n, and $h_T \in \alpha_{\mathrm{AT}}$ if $h_T \in \alpha_{\mathrm{A}}$ and $\#\{T' \mid h_{T'} < h_T, h_{T'} \in \alpha_{\mathrm{A}}\} = 3n + 2$ for some integer n. Let $\mathrm{AF}_s^j := \bigcup_\alpha \alpha_{\mathrm{AF}}$, $\mathrm{AS}_s^j := \bigcup_\alpha \alpha_{\mathrm{AS}}$, and $\mathrm{AT}_s^j := \bigcup_\alpha \alpha_{\mathrm{AT}}$. Define $\mathrm{A}_s^j := \mathrm{AF}_s^j \cup \mathrm{AS}_s^j \cup \mathrm{AT}_s^j$; $h_T \in \mathrm{A}_s^j$ if and only if $T = n\tilde{T}$ for some integer n. I also define α_{BF} as follows: $h_T \in \alpha_{\mathrm{BF}}$ if $h_T \in \alpha_{\mathrm{B}}$ and the most recent A_s^j-active period is AF_s^j-active in h_T, that is, $h_{M(T)} \in \mathrm{AF}_s^j$, where $M(T) := \max\{T' \mid T' < T, T' = n\tilde{T}$ for some integer $n\}$; α_{BS} and α_{BT} are defined similarly. Then let $\mathcal{P}_s^j(\tilde{T}, 3) := \{\alpha_{\mathrm{AF}}, \alpha_{\mathrm{AS}}, \alpha_{\mathrm{AT}}, \alpha_{\mathrm{BF}}, \alpha_{\mathrm{BS}}, \alpha_{\mathrm{BT}} \mid \alpha \in \mathcal{P}_s^j\}$, and let $\mathrm{BF}_s^j := \bigcup_\alpha \alpha_{\mathrm{BF}}$, $\mathrm{BS}_s^j := \bigcup_\alpha \alpha_{\mathrm{BS}}$, and $\mathrm{BT}_s^j := \bigcup_\alpha \alpha_{\mathrm{BT}}$. Define $\mathrm{B}_s^j := \mathrm{BF}_s^j \cup \mathrm{BS}_s^j \cup \mathrm{BT}_s^j$; $h_T \in \mathrm{B}_s^j$ if and only if $T \neq n\tilde{T}$ for any integer n. Finally, I define an action-based conditioning rule $\mathcal{P}_s^j(\tilde{T}, 3, i)$ by partitioning α_{BF} and α_{BS} in $\mathcal{P}_s^j(\tilde{T}, 3)$ according to player i's $(\neq j)$ action: $h_T \in \alpha_{\mathrm{BF}}(a_i)$ (resp. $\alpha_{\mathrm{BS}}(a_i)$) if $h_T \in \alpha_{\mathrm{BF}}$ (resp. α_{BS}) and player i played a_i in the most recent A_s^j-active period in h_T. Then let $\mathcal{P}_s^j(\tilde{T}, 3, i) := \{\alpha_{\mathrm{AF}}, \alpha_{\mathrm{AS}}, \alpha_{\mathrm{AT}}, \alpha_{\mathrm{BF}}(a_i), \alpha_{\mathrm{BS}}(a_i), \alpha_{\mathrm{BT}} \mid \alpha \in \mathcal{P}_s^j, a_i \in A_i\}$. Clearly, $\mathcal{P}_s^j \le \mathcal{P}_s^j(\tilde{T}) \le \mathcal{P}_s^j(\tilde{T}, 3) \le \mathcal{P}_s^j(\tilde{T}, 3, i)$ for all j, all $i \neq j$, all s, and all $\tilde{T}$.

Next, I arbitrarily take positive integers $\{\tilde{T}_s\}_s$ such that (T.1) $\tilde{T}_{s-1} \le \tilde{T}_s$ for all s and (T.2) $s(1 - \frac{1}{s})^{\tilde{T}_s} \to 0$ as $s \to \infty$.

Finally, given any $\{\mathcal{P}_s^i\}_{i,s}$, I construct CRs $\{\tilde{\mathcal{P}}_s^i\}_{i,s}$ such that $\{\tilde{\mathcal{P}}_s^i\}_{i,s}$ satisfy (P.1)–(P.3), and for all i, $\mathcal{P}_s^i \le \tilde{\mathcal{P}}_s^i$ for all s. Let $\tilde{\mathcal{P}}_0^i := \mathcal{P}_0^i$ for all i. For any $s \ge 1$, define $\{\tilde{\mathcal{P}}_s^i\}_i$ inductively as follows: for all i, let $\tilde{\mathcal{P}}_s^i := \mathcal{P}_s^i \wedge (\bigwedge_{\substack{(m,n) \\ m \neq n}} \tilde{\mathcal{P}}_{s-1}^m(\tilde{T}_s, 3, n)) \wedge$

$(\bigwedge_{j\neq i}\mathcal{F}_s\tilde{\mathcal{P}}^j_{s-1})$. Clearly, from the definition of $\tilde{\mathcal{P}}^i_s$ and the joint property, $\mathcal{P}^i_s \leq \tilde{\mathcal{P}}^i_s$ for all i and all s. Furthermore, by the definitions of $\tilde{\mathcal{P}}^i_s$ and $\tilde{\mathcal{P}}^m_{s-1}(\tilde{T}_s, 3, n)$ and the joint property, $\tilde{\mathcal{P}}^i_{s-1} \leq \tilde{\mathcal{P}}^i_{s-1}(\tilde{T}_s, 3, j) \leq \tilde{\mathcal{P}}^i_s$ for all i, all $j \neq i$, and all s. Hence, $\{\tilde{\mathcal{P}}^i_s\}_{i,s}$ satisfy (P.1). Next, take any i, any $j \neq i$, any s, and any T. Then let $k := \max[s, T]$. From (P.1), the property of $\mathcal{F}_T\mathcal{P}$, the definition of $\tilde{\mathcal{P}}^i_s$, and the joint property, it follows that $\mathcal{F}_T\tilde{\mathcal{P}}^j_s \leq \mathcal{F}_k\tilde{\mathcal{P}}^j_k \leq \mathcal{F}_{k+1}\tilde{\mathcal{P}}^j_k \leq \tilde{\mathcal{P}}^i_{k+1}$. Thus, $\{\tilde{\mathcal{P}}^i_s\}_{i,s}$ satisfy (P.2). Finally, by the definitions of $\tilde{\mathcal{P}}^i_s$ and $\tilde{\mathcal{P}}^m_{s-1}(\tilde{T}_s, 3, n)$ and the joint property, for all i, all $j \neq i$, and all s, $\tilde{\mathcal{P}}^j_{s-1} \leq \tilde{\mathcal{P}}^j_{s-1}(\tilde{T}_s, 3, i) \leq \tilde{\mathcal{P}}^i_s, \tilde{\mathcal{P}}^j_s$. Therefore, $\{\tilde{\mathcal{P}}^i_s\}_{i,s}$ satisfy (P.3).

APPENDIX B

B.1. *Temporary Belief Leading to Opponent's Belief Rejection*

For the proof of Lemma 3, I first construct player i's temporary beliefs and strategies according to her stage game payoff and discount factor. I assume that $\tilde{T}$ is large. Let U^*_i denote player i's maximum stage game payoff, that is, $U^*_i := \max_a u_i(a)$, and let $\underline{U}_i$ denote player i's minmax stage game payoff, that is, $\underline{U}_i := \min_{\pi_{-i}} \max_{a_i} u_i(a_i, \pi_{-i})$. Clearly, $U^*_i \geq \underline{U}_i$.

The Case of a Unique Weakly Dominant Action. Let a^*_i denote a unique weakly dominant action. Note that for some a^*_{-i}, $(a^*_i, a^*_{-i}) \in \arg\max_a u_i(a)$ and $U^*_i = u_i(a^*_i, a^*_{-i})$. Furthermore, there exists a *pure* minmax action profile: $\underline{a}_{-i} \in \arg\min_{\pi_{-i}} \max_{a_i} u_i(a_i, \pi_{-i})$. Let $\bar{a}_{-i} \in \arg\min_{a_{-i}}[u_i(a^*_i, a_{-i}) - \max_{a_i\neq a^*_i} u_i(a_i, a_{-i})]$. Define $u^*_i := \max_{a_i} u_i(a_i, \bar{a}_{-i})$ $(= u_i(a^*_i, \bar{a}_{-i}))$ and $\bar{u}_i := \max_{a_i\neq a^*_i} u_i(a_i, \bar{a}_{-i})$; let $\bar{a}_i \in \arg\max_{a_i\neq a^*_i} u_i(a_i, \bar{a}_{-i})$. Clearly, $u^*_i \geq \bar{u}_i$.

Case 1. $\delta_i > 0$ and $U^*_i > \underline{U}_i$.[3] Given any a_{-i}, let π^a_j denote the mixed action of player j $(\neq i)$ such that $\pi^a_j[b_j] = 1$ if $b_j = a_j$ and $\pi^a_j[b_j] = 0$ otherwise, that is, $b_j \neq a_j$, and let $\pi^a_{-i} := (\pi^a_j)_{j\neq i}$. Then define $\pi_{-i}(t) := t\pi^{a^*}_{-i} + (1-t)\pi^{\bar{a}}_{-i}$ $(:= (t\pi^{a^*}_j + (1-t)\pi^{\bar{a}}_j)_{j\neq i})$. I consider three subcases.

Subcase 1. $(1-\delta_i)\bar{u}_i + \delta_i U^*_i > (1-\delta_i)u^*_i + \delta_i\underline{U}_i$. Given $\mathcal{P}^j_{s-1}(\tilde{T}, 3, i)$, define player i's temporary belief $\rho^i_{(t,t',\pi'')}$ as follows: let π''_{-i} denote a mixed action profile of player i's opponents. Then, for any $0 \leq t, t' \leq 1$ and any π''_{-i}, let

$$\rho^i_{(t,t',\pi'')}(h) := \begin{cases} \pi_{-i}(t) & \text{in any AF}^j_{s-1}\text{-active period,} \\ \pi_{-i}(t') & \text{in any AS}^j_{s-1}\text{-active period,} \\ \pi''_{-i} & \text{in any AT}^j_{s-1}\text{-active period.} \end{cases}$$

[3]The prisoner's dilemma stage game payoff is one example.

Furthermore, in any BF^j_{s-1}-active period (resp. any BS^j_{s-1}-active period), all players other than i play $t\pi^{a^*}_{-i} + (1-t)\pi^{\underline{a}}_{-i}$ (resp. $t'\pi^{a^*}_{-i} + (1-t')\pi^{\underline{a}}_{-i}$) if player i took a^*_i in the most recent A^j_{s-1}-active period, they play $t\pi^{\underline{a}}_{-i} + (1-t)\pi^{a^*}_{-i}$ (resp. $t'\pi^{\underline{a}}_{-i} + (1-t')\pi^{a^*}_{-i}$) if player i took $\bar{a}_i$ in the most recent A^j_{s-1}-active period, and they play $\pi^{\underline{a}}_{-i}$ (resp. $\pi^{\underline{a}}_{-i}$) if player i took any other action than a^*_i or $\bar{a}_i$ in the most recent A^j_{s-1}-active period. In any BT^j_{s-1}-active period, they play π''_{-i}. Thus, $\rho^i_{(t,t',\pi'')}$ is generated by $\mathcal{P}^j_{s-1}(\tilde{T}, 3, i)$.

From the first-order condition and Lemma 1, there exists an upper bound $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that[4] for any $0 < r_i \leq \hat{r}_i(\eta, u_i, \delta_i)$ and any large $\tilde{T}$, player i's best response $\sigma_i^{t,t',\pi''}$ to $\rho^i_{(t,t',\pi'')}$ has the following property: for all t' and all π''_{-i}, $\sigma_i^{0,t',\pi''}(h)[a^*_i] \approx 0$ and $\sigma_i^{1,t',\pi''}(h)[a^*_i] \approx 1$ in any AF^j_{s-1}-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,0,\pi''}(h)[a^*_i] \approx 0$ and $\sigma_i^{t,1,\pi''}(h)[a^*_i] \approx 1$ in any AS^j_{s-1}-active period. Therefore, for any $0 < c, c' < 1$, there exist $0 \leq t_c, t'_{c'} \leq 1$ such that for all π''_{-i}, $\sigma_i^{t_c,t'_{c'},\pi''}(h)[a^*_i] \approx c$ in any AF^j_{s-1}-active period and $\sigma_i^{t_c,t'_{c'},\pi''}(h)[a^*_i] \approx c'$ in any AS^j_{s-1}-active period.

Subcase 2.[5] $(1-\delta_i)\bar{u}_i + \delta_i U^*_i = (1-\delta_i)u^*_i + \delta_i \underline{U}_i$. Given $\mathcal{P}^j_{s-1}(\tilde{T}, 3, i)$, consider the same belief $\rho^i_{(t,t',\pi'')}$ as in Subcase 1. From the first-order condition and Lemma 1, there exists $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that, for any $0 < r_i \leq \hat{r}_i(\eta, u_i, \delta_i)$ and any large $\tilde{T}$, $\sigma_i^{t,t',\pi''}$ has the following property:[6] for all t' and all π''_{-i}, $\sigma_i^{0,t',\pi''}(h)[a^*_i] \approx 1/2$ and $\sigma_i^{1,t',\pi''}(h)[a^*_i] \approx 1$ in any AF^j_{s-1}-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,0,\pi''}(h)[a^*_i] \approx 1/2$ and $\sigma_i^{t,1,\pi''}(h)[a^*_i] \approx 1$ in any AS^j_{s-1}-active period. Hence, for any $1/2 < c, c' < 1$, there exist $0 \leq t_c, t'_{c'} \leq 1$ such that for all π''_{-i}, $\sigma_i^{t_c,t'_{c'},\pi''}(h)[a^*_i] \approx c$ in any AF^j_{s-1}-active period and $\sigma_i^{t_c,t'_{c'},\pi''}(h)[a^*_i] \approx c'$ in any AS^j_{s-1}-active period.

Subcase 3. $(1-\delta_i)\bar{u}_i + \delta_i U^*_i < (1-\delta_i)u^*_i + \delta_i \underline{U}_i$. There exists $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that for all $0 < r_i \leq \hat{r}_i(\eta, u_i, \delta_i)$, player i always plays almost the same

[4]Strictly speaking, the upper bound $\hat{r}_i(\eta, u_i, \delta_i)$ does not depend on η in this subcase (and Subcases 2, 4, 6, and 7 and the case of no weakly dominant action). However, it depends on η in the case in which player i always plays almost the same mixed action regardless of her prior belief. See Subcases 3, 5, and 8.

[5]As noted in footnote 9 in the paper, there are several *nongeneric* cases where the property of assigning almost equal probability is used. Subcase 2 is one of them. See Subcases 4, 7, and 8 and the case of no weakly dominant action for the other cases.

[6]By the property of assigning almost equal probability, for all t' and all π''_{-i}, $\sigma_i^{0,t',\pi''}(h)[a^*_i] \approx \frac{1}{2}$ and $\sigma_i^{0,t',\pi''}(h)[\bar{a}_i] \approx \frac{1}{2}$ in any AF^j_{s-1}-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,0,\pi''}(h)[a^*_i] \approx \frac{1}{2}$ and $\sigma_i^{t,0,\pi''}(h)[\bar{a}_i] \approx \frac{1}{2}$ in any AS^j_{s-1}-active period.

mixed action, which puts almost all the weight on a_i^*, regardless of her prior belief. This allows us to ignore player i throughout my argument.[7]

Case 2. Either $\delta_i = 0$ *or* $\delta_i > 0$ *and* $U_i^* = \underline{U}_i$. I consider two subcases.

Subcase 4. $u_i^* = \bar{u}_i$. Let $\bar{A}_i := \arg\max_{a_i \neq a_i^*} u_i(a_i, \bar{a}_{-i})$, and take an arbitrary action $\bar{a}_i$ in $\bar{A}_i$. Since $\bar{a}_i \neq a_i^*$, there exists $\tilde{a}_{-i}$ such that $u_i(\bar{a}_i, \tilde{a}_{-i}) < \max_{a_i} u_i(a_i, \tilde{a}_{-i}) = u_i(a_i^*, \tilde{a}_{-i})$. Then let $\bar{\pi}_{-i}(t) := t\pi_{-i}^{\bar{a}} + (1-t)\pi_{-i}^{\tilde{a}}$. Given $\mathcal{P}_{s-1}^j(\tilde{T}, 3, i)$, define $\rho_{(t,t',\pi'')}^i$ as follows: for any $0 \leq t, t' \leq 1$ and any π''_{-i}, let $\rho_{(t,t',\pi'')}^i(h) := \bar{\pi}_{-i}(t)$ in any AF_{s-1}^j-active period, let $\rho_{(t,t',\pi'')}^i(h) := \bar{\pi}_{-i}(t')$ in any AS_{s-1}^j-active period, and let $\rho_{(t,t',\pi'')}^i := \pi''_{-i}$ in any other period. Clearly, $\rho_{(t,t',\pi'')}^i$ is generated by $\mathcal{P}_{s-1}^j(\tilde{T}, 3, i)$. Then, from the first-order condition and Lemma 1, there exists $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that for any $0 < r_i \leq \hat{r}_i(\eta, u_i, \delta_i)$ and any large $\tilde{T}$, $\sigma_i^{t,t',\pi''}$ has the following property:[8] for all t' and all π''_{-i}, $\sigma_i^{0,t',\pi''}(h)[\bar{a}_i] \approx 0$ and $\sigma_i^{1,t',\pi''}(h)[\bar{a}_i] \approx 1/(\#\bar{A}_i + 1)$ in any AF_{s-1}^j-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,0,\pi''}(h)[\bar{a}_i] \approx 0$ and $\sigma_i^{t,1,\pi''}(h)[\bar{a}_i] \approx 1/(\#\bar{A}_i + 1)$ in any AS_{s-1}^j-active period. Hence, for any $0 < c, c' < 1/(\#\bar{A}_i + 1)$, there exist $0 \leq t_c, t'_{c'} \leq 1$ such that for all π''_{-i}, $\sigma_i^{t_c,t'_{c'},\pi''}(h)[\bar{a}_i] \approx c$ in any AF_{s-1}^j-active period and $\sigma_i^{t_c,t'_{c'},\pi''}(h)[\bar{a}_i] \approx c'$ in any AS_{s-1}^j-active period.

Subcase 5. $u_i^* > \bar{u}_i$. This subcase is similar to Subcase 3.

The Case of Multiple Weakly Dominant Actions. Let A_i^* denote the set of weakly dominant actions, and take any two (weakly dominant) actions a_i^* and b_i^* in A_i^*.

Case 3. $\delta_i > 0$ *and* $U_i^* > \underline{U}_i$.

Subcase 6. Note that for some a_{-i}^*, $U_i^* = u_i(a_i^*, a_{-i}^*) = u_i(b_i^*, a_{-i}^*)$. Given $\mathcal{P}_{s-1}^j(\tilde{T}, 3, i)$, define $\rho_{(t,t',\pi'')}^i$ as follows: for any $0 \leq t, t' \leq 1$ and any π''_{-i}, let $\rho_{(t,t',\pi'')}^i(h) := a_{-i}^*$ in any AF_{s-1}^j-active period and in any AS_{s-1}^j-active period, and let $\rho_{(t,t',\pi'')}^i(h) := \pi''_{-i}$ in any AT_{s-1}^j-active period. Further, in any BF_{s-1}^j-active period (resp. any BS_{s-1}^j-active period), all players other than i play $t\pi_{-i}^{a^*} + (1-t)\pi_{-i}^{\underline{a}}$ (resp. $t'\pi_{-i}^{a^*} + (1-t')\pi_{-i}^{\underline{a}}$) if player i took a_i^* in the most recent A_{s-1}^j-active period, they play $t\pi_{-i}^{\underline{a}} + (1-t)\pi_{-i}^{a^*}$ (resp. $t'\pi_{-i}^{\underline{a}} + (1-t')\pi_{-i}^{a^*}$) if player i took b_i^* in the most recent A_{s-1}^j-active period, and they play $\pi_{-i}^{\underline{a}}$ (resp. $\pi_{-i}^{\underline{a}}$)

[7]Given player i's opponents' tolerance levels $(\bar{\xi}^j)_{j \neq i}$, I can take a small $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that the difference between player i's (mixed) actions is far less than any $\bar{\xi}^j$ $(\leq \eta/3)$ regardless of her prior belief. Hence, the opponents are statistically convinced that player i always plays (almost) the same action.

[8]Note that $\bar{A}_i \cup \{a_i^*\} = \arg\max_{a_i} u_i(a_i, \bar{a}_{-i})$. Hence, by the property of assigning almost equal probability, for all t' and all π''_{-i}, $\sigma_i^{1,t',\pi''}(h)[a_i] \approx 1/(\#\bar{A}_i + 1)$ for all $a_i \in \bar{A}_i$ in any AF_{s-1}^j-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,1,\pi''}(h)[a_i] \approx 1/(\#\bar{A}_i + 1)$ for all $a_i \in \bar{A}_i$ in any AS_{s-1}^j-active period.

if player i took any other action than a_i^* or b_i^* in the most recent A_{s-1}^j-active period. In any BT_{s-1}^j-active period, they play π''_{-i}. Thus, $\rho^i_{(t,t',\pi'')}$ is generated by $\mathcal{P}_{s-1}^j(\tilde{T}, 3, i)$. The rest of the argument is the same as in Subcase 1.

Case 4. Either $\delta_i = 0$ *or* $\delta_i > 0$ *and* $U_i^* = \underline{U}_i$.[9]

Subcase 7. $u_i^* = \bar{u}_i$. Note that $\bar{A}_i \cup A_i^* = \arg\max_{a_i} u_i(a_i, \bar{a}_{-i})$, where $\bar{A}_i := \arg\max_{a_i \notin A_i^*} u_i(a_i, \bar{a}_{-i})$. Take any $\bar{a}_i \in \bar{A}_i$. Then an argument similar to Subcase 4 holds.[10] Accordingly, there exists $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that for any $0 < r_i \le \hat{r}_i(\eta, u_i, \delta_i)$ and any large $\tilde{T}$, $\sigma_i^{t,t',\pi''}$ has the following property: for all t' and all π''_{-i}, $\sigma_i^{0,t',\pi''}(h)[\bar{a}_i] \approx 0$ and $\sigma_i^{1,t',\pi''}(h)[\bar{a}_i] \approx 1/(\#\bar{A}_i + \#A_i^*)$ in any AF_{s-1}^j-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,0,\pi''}(h)[\bar{a}_i] \approx 0$ and $\sigma_i^{t,1,\pi''}(h)[\bar{a}_i] \approx 1/(\#\bar{A}_i + \#A_i^*)$ in any AS_{s-1}^j-active period. Hence, for any $0 < c, c' < 1/(\#\bar{A}_i + \#A_i^*)$, there exist $0 \le t_c, t'_{c'} \le 1$ such that for all π''_{-i}, $\sigma_i^{t_c,t'_{c'},\pi''}(h)[\bar{a}_i] \approx c$ in any AF_{s-1}^j-active period and $\sigma_i^{t_c,t'_{c'},\pi''}(h)[\bar{a}_i] \approx c'$ in any AS_{s-1}^j-active period.

Subcase 8. $u_i^* > \bar{u}_i$. This subcase is similar to Subcases 3 and 5: player i always plays almost the same mixed action that assigns almost equal probability to each weakly dominant action.

The Case of No Weakly Dominant Action.[11] Let $(a_i^*, a_{-i}^*) \in \arg\max_a u_i(a)$; thus, $U_i^* = u_i(a_i^*, a_{-i}^*)$. Further, let $\hat{A}_i := \arg\max_{a_i} u_i(a_i, a_{-i}^*)$.[12] Clearly, $a_i^* \in \hat{A}_i$. Since a_i^* is not a weakly dominant action, there exists $\tilde{a}_{-i}$ such that $u_i(a_i^*, \tilde{a}_{-i}) < \max_{a_i} u_i(a_i, \tilde{a}_{-i})$. Define $\tilde{\pi}_{-i}(t) := t\pi_{-i}^{a^*_i} + (1-t)\pi_{-i}^{\tilde{a}}$. Then replace $\bar{\pi}_{-i}(\cdot)$ by $\tilde{\pi}_{-i}(\cdot)$ in the definition of $\rho^i_{(t,t',\pi'')}$ in Subcase 4. Accordingly, an argument similar to Subcase 4 holds. That is, there exists $\hat{r}_i(\eta, u_i, \delta_i) > 0$ such that for any $0 < r_i \le \hat{r}_i(\eta, u_i, \delta_i)$ and any large $\tilde{T}$, $\sigma_i^{t,t',\pi''}$ has the following property: for all t' and all π''_{-i}, $\sigma_i^{0,t',\pi''}(h)[a_i^*] \approx 0$ and $\sigma_i^{1,t',\pi''}(h)[a_i^*] \approx 1/\#\hat{A}_i$ in any AF_{s-1}^j-active period. Similarly, for all t and all π''_{-i}, $\sigma_i^{t,0,\pi''}(h)[a_i^*] \approx 0$ and $\sigma_i^{t,1,\pi''}(h)[a_i^*] \approx 1/\#\hat{A}_i$ in any AS_{s-1}^j-active period. Hence, for any $0 < c, c' < 1/\#\hat{A}_i$, there exist $0 \le t_c, t'_{c'} \le 1$ such that for all π''_{-i}, $\sigma_i^{t_c,t'_{c'},\pi''}(h)[a_i^*] \approx c$ in any AF_{s-1}^j-active period and $\sigma_i^{t_c,t'_{c'},\pi''}(h)[a_i^*] \approx c'$ in any AS_{s-1}^j-active period.

From the above argument, define $\hat{r}(\eta, u, \delta)$ (in Proposition 1) as $\hat{r}(\eta, u, \delta) := \min_i \hat{r}_i(\eta, u_i, \delta_i)$.

[9]Let $\bar{a}_{-i} \in \arg\min_{a_{-i}}[u_i(a_i^*, a_{-i}) - \max_{a_i \notin A_i^*} u_i(a_i, a_{-i})]$, $u_i^* := \max_{a_i} u_i(a_i, \bar{a}_{-i})$ $(= u_i(a_i^*, \bar{a}_{-i}))$, and $\bar{u}_i := \max_{a_i \notin A_i^*} u_i(a_i, \bar{a}_{-i})$, as in the case of a unique weakly dominant action.

[10]Recall that the weakly dominant action is unique in Subcase 4; that is, $A_i^* = \{a_i^*\}$.

[11]The matching pennies stage game payoff is one example.

[12]As noted in footnote 5, the case that $\#\hat{A}_i \ge 2$ is *nongeneric*.

B.2. *Proof of Lemma 3*

I can form player i's belief $f^i[j]$ on the basis of Appendix B.1. I only consider Subcase 1 in Appendix B.1. All other cases are similar, and I have omitted them. I use the following facts to prove Lemma 3.

Given $\mathcal{P}^j_{s-1}(\tilde{T}_s, 3, i)$, consider player i's belief $\rho^i_{(t,t',\pi'')}$ defined in Subcase 1 in Appendix B.1. Thus, $\rho^i_{(t,t',\pi'')}$ is generated by $\mathcal{P}^j_{s-1}(\tilde{T}_s, 3, i)$. Then, from Appendix B.1, Lemma 1, and (T.2), player i's best response $\sigma_i^{t,t',\pi''}$ to $\rho^i_{(t,t',\pi'')}$ has the following property: (b.1) there exists s_1 such that for any i, any $j \neq i$, any $s \geq s_1$, any $1/8(\#A_i + 1) < c_0 < 1/4(\#A_i + 1)$, and any $3/4(\#A_i + 1) < c'_0 < 7/8(\#A_i + 1)$, there exist $0 \leq t_0, t'_0 \leq 1$ such that for all π''_{-i}, $c_0 - \frac{1}{6}\bar{\xi}^j \leq \sigma_i^{t_0,t'_0,\pi''}(h)[a^*_i] \leq c_0 + \frac{1}{6}\bar{\xi}^j$ in any AF^j_{s-1}-active period and $c'_0 - \frac{1}{6}\bar{\xi}^j \leq \sigma_i^{t_0,t'_0,\pi''}(h)[a^*_i] \leq c'_0 + \frac{1}{6}\bar{\xi}^j$ in any AS^j_{s-1}-active period, where a^*_i is the unique weakly dominant action in Subcase 1 in Appendix B.1. Furthermore, for each i, let $\pi''_{0,-i}$ be any mixed action profile such that for any $j \neq i$, $\pi''_{0,j}[a^*_j] := 1$ if $\hat{\pi}_j[a^*_j] \leq 1/2$ and $\pi''_{0,j}[a^*_j] := 0$ otherwise, where a^*_{-i} is taken from Subcase 1 in Appendix B.1. Therefore, (b.2) for any i, any $j \neq i$, any s, and any $0 \leq t, t' \leq 1$, $\|\hat{\sigma}_j(h) - \rho^i_{(t,t',\pi''_0),j}(h)\| \geq |\hat{\sigma}_j(h)[a^*_j] - \rho^i_{(t,t',\pi''_0),j}(h)[a^*_j]| = |\hat{\pi}_j[a^*_j] - \pi''_{0,j}[a^*_j]| \geq 1/2$ in any AT^j_{s-1}-active period.

I also use the following facts: from (P.1) and (P.3), it follows that (b.3) for all i, all $j \neq i$, and all s, $\mathcal{P}^j_{s-1}(\tilde{T}_s, 3, i) \leq \mathcal{P}^i_{s'}, \mathcal{P}^j_{s''}$ for all $s', s'' \geq s$. Thus, $\rho^i_{(t,t',\pi'')}$ is also generated by $\mathcal{P}^i_{s'}$ and $\mathcal{P}^j_{s''}$. Since $\lim_{s\to\infty} \underline{n}^i_s = \infty$ for all i, (b.4) there exists s_2 such that for all $s \geq s_2$, all i, and all $j \neq i$, $1/\underline{n}^i_s \leq \bar{\xi}^j/12B_i$, where $B_i := \max[1, C_i/(1-\delta_i)^2]$ and C_i is taken from Lemma 1. Since $\lim_{m\to\infty} K^i(m) = \infty$ for all i, Lemma 2 induces that (b.5) there exists s_3 such that for any $s \geq s_3$, any i, any f^i, and any $h \in H_{f^i}(s)$, $\|\sigma^f_i(h) - \sigma^*_i(h)\| \leq \bar{\xi}^j/16$ for all $j \neq i$.[13] Finally, it is obvious that (b.6) there exists s_4 such that for all $s \geq s_4$, $\#\bar{A}\exp(-s) \leq 1$, where $\#\bar{A} := \max_i \#A_i$.

PROOF OF LEMMA 3: Let $\bar{s}_1 := \max_{1\leq k\leq 4} s_k$. Suppose that $s^i \geq s^j$ and $s^i, s^j + q^j \geq \bar{s}_1$. Then let $\hat{s} := \min[s^i, s^j + q^j]$. Thus, $\bar{s}_1, s^j \leq \hat{s} \leq s^i, s^j + q^j$. Given $\mathcal{P}^j_{\hat{s}-1}(\tilde{T}_{\hat{s}}, 3, i)$, consider $\rho^i_{(t_0,t'_0,\pi''_0)}$, where t_0 and t'_0 are taken from (b.1) and $\pi''_{0,-i}$ is taken from (b.2).[14] Then, from the $1/\underline{n}^i_s$-density of $G(\mathcal{P}^i_s, \underline{n}^i_s)$, (b.3), and (b.4), it follows that player i can form $f^i[j]$ in any formation phase during epoch s^i such that (b.7) for all h, all $j' \neq i$, and all $j'' \neq i$, $\|\rho^i_{(t_0,t'_0,\pi''_0),j'}(h) - f^i_{j'}[j](h)\| \leq 1/\underline{n}^i_{s^i} \leq \bar{\xi}^{j''}/12B_i$, where $B_i := \max[1, C_i/(1-\delta_i)^2]$: there exists $h_{-i,N^i_{s^i}}$ such that

[13]Define $H_{f^i}(s)$ as follows: $h_T \in H_{f^i}(s)$ if and only if $h_T \in H_{f^i}$ and time $T+1$ is in epoch s of player i.

[14]I arbitrarily fix c_0 and c'_0 such that $1/8(\#A_i+1) < c_0 < 1/4(\#A_i+1)$ and $3/4(\#A_i+1) < c'_0 < 7/8(\#A_i+1)$.

$f^i[j] = \mathcal{B}^i_{s^i}(h_{-i,N^i_{s^i}})$. Furthermore, this, along with Lemma 1, implies that for all h, $\|\sigma_i^{t_0,t_0',\pi_0''}(h) - \sigma_i^{f[j]}(h)\| \le \bar{\xi}^{j''}/12$ for all $j'' \neq i$. As for σ_i^*, from (b.5), for all $h \in H_{f^i[j]}(s^i)$, $\|\sigma_i^{f[j]}(h) - \sigma_i^*(h)\| \le \bar{\xi}^{j'}/16$ for all $j' \neq i$. From these, it follows that (b.8) for all $h \in H_{f^i[j]}(s^i)$, $\|\sigma_i^{t_0,t_0',\pi_0''}(h) - \sigma_i^*(h)\| \le \bar{\xi}^{j'}/6$ for all $j' \neq i$.

Now, I prove Lemma 3(i). Consider any player j's test phase in epoch s^j in which player j employs f^j that is generated by $\mathcal{P}^j_{s^j}$, and let $(\mathcal{P}^j_{s^j+q^j}, \underline{m}^j_{s^j+q^j} + d)$ denote the CR and the smallest sample size used in the test phase. Suppose that player i employs $f^i[j]$ at the beginning of the test phase. Let $\bar{h}_{\bar{T}}$ be the finite history realized just before the above test phase.

For each $\beta \in \mathcal{P}^j_{\hat{s}-1}$, define a class β^f_{AF} such that $h_T \in \beta^f_{\mathrm{AF}}$ if and only if (a) $h_T \in \beta_{\mathrm{AF}}$, (b) $\bar{h}_{\bar{T}} \le h_T$, and (c) time $T+1$ is in the test phase; β^f_{AS} is defined similarly.[15] Let $L_f^{\beta_{\mathrm{AF}}}[a_i^*] := \sup_{h \in \beta^f_{\mathrm{AF}}} \sigma_i^*(h)[a_i^*]$ and $l_f^{\beta_{\mathrm{AF}}}[a_i^*] := \inf_{h \in \beta^f_{\mathrm{AF}}} \sigma_i^*(h)[a_i^*]$; $L_f^{\beta_{\mathrm{AS}}}[a_i^*]$ and $l_f^{\beta_{\mathrm{AS}}}[a_i^*]$ are defined similarly. Furthermore, for each $\alpha \in \mathcal{P}^j_{s^j+q^j}$, define a class α_f such that $h_T \in \alpha_f$ if and only if (d) $h_T \in \alpha$, (e) $\bar{h}_{\bar{T}} \le h_T$, and (f) time $T+1$ is in the test phase. Let $\mathbf{d}_m^{\alpha_f}[a_i^*]$ denote the number of times that a_i^* has been realized in the first m α_f-active periods.

CLAIM B.1: *With (conditional) probability at least* $1/2$, *there exist* $\alpha', \alpha'' \in \mathcal{P}^j_{s^j+q^j}$ *and* $\bar{\beta} \in \mathcal{P}^j_{\hat{s}-1}$ *such that* (i) $\alpha' \subseteq \bar{\beta}_{\mathrm{AF}}$, (ii) $\alpha'' \subseteq \bar{\beta}_{\mathrm{AS}}$, (iii) *both* α' *and* α'' *obtain enough samples during the test phase, that is,* $\tilde{m}^{\alpha'}, \tilde{m}^{\alpha''} \ge \underline{m}^j_{s^j+q^j} + d$, *and* (iv) $l_f^{\bar{\beta}_{\mathrm{AF}}}[a_i^*] - \frac{1}{4}\bar{\xi}^j \le \tilde{D}^j_i(\alpha')[a_i^*] \le L_f^{\bar{\beta}_{\mathrm{AF}}}[a_i^*] + \frac{1}{4}\bar{\xi}^j$ *and* $l_f^{\bar{\beta}_{\mathrm{AS}}}[a_i^*] - \frac{1}{4}\bar{\xi}^j \le \tilde{D}^j_i(\alpha'')[a_i^*] \le L_f^{\bar{\beta}_{\mathrm{AS}}}[a_i^*] + \frac{1}{4}\bar{\xi}^j$, *where* $\tilde{D}^j(\alpha')$ $(:= (\tilde{D}^j_k(\alpha'))_{k\neq j})$ (*resp.* $\tilde{D}^j(\alpha'')$) *denotes the empirical distributions of the samples collected in the* α'*-active periods* (*resp. the* α''*-active periods*) *during the test phase.*

PROOF: Let $\mathbf{M}(\alpha', \alpha'', \bar{\beta}) := \{h_\infty \mid \text{(i)–(iv) hold in } h_\infty\}$. Furthermore, let $\mathbf{M} := \bigcup_{\alpha' \in \mathcal{P}^j_{s^j+q^j}} \bigcup_{\alpha'' \in \mathcal{P}^j_{s^j+q^j}} \bigcup_{\bar{\beta} \in \mathcal{P}^j_{\hat{s}-1}} \mathbf{M}(\alpha', \alpha'', \bar{\beta})$. Then it suffices to prove that $\mu_{\sigma^*}(\mathbf{M}|\bar{h}_{\bar{T}}) \ge 1/2$. Note that $\mathcal{P}^j_{\hat{s}-1} \le \mathcal{P}^j_{\hat{s}-1}(\tilde{T}_{\hat{s}}, 3, i) \le \mathcal{P}^j_{s^j+q^j}$ by (P.1) and (P.3). Furthermore, for each $\alpha \in \mathcal{P}^j_{s^j+q^j}$, if $\alpha \subseteq \beta_{\mathrm{AF}}$ for some (unique) $\beta \in \mathcal{P}^j_{\hat{s}-1}$, let $\mathbf{N}^\alpha_m := \{h_\infty \mid \mathcal{T}^{\alpha_f}_m < \infty, \mathbf{d}^{\alpha_f}_m[a_i^*]/m \le l_f^{\beta_{\mathrm{AF}}}[a_i^*] - \bar{\xi}^j/4 \text{ or } \mathbf{d}^{\alpha_f}_m[a_i^*]/m \ge L_f^{\beta_{\mathrm{AF}}}[a_i^*] + \bar{\xi}^j/4\}$. If $\alpha \subseteq \beta_{\mathrm{AS}}$ for some (unique) $\beta \in \mathcal{P}^j_{\hat{s}-1}$, let $\mathbf{N}^\alpha_m := \{h_\infty \mid \mathcal{T}^{\alpha_f}_m < \infty, \mathbf{d}^{\alpha_f}_m[a_i^*]/m \le l_f^{\beta_{\mathrm{AS}}}[a_i^*] - \bar{\xi}^j/4 \text{ or } \mathbf{d}^{\alpha_f}_m[a_i^*]/m \ge L_f^{\beta_{\mathrm{AS}}}[a_i^*] + \bar{\xi}^j/4\}$. Otherwise, let $\mathbf{N}^\alpha_m := \emptyset$. Then, from Proposition A, it follows that $\mu_{\sigma^*}(\mathbf{N}^\alpha_m|\bar{h}_{\bar{T}}) \le 2\exp(-\frac{1}{8}m(\bar{\xi}^j)^2)$ for

[15] Note that when $h_T \in \beta^f_{\mathrm{AF}}$, player i employs $f^i[j]$ at time $T+1$. Similarly, when $h_T \in \beta^f_{\mathrm{AS}}$, player i employs $f^i[j]$ at time $T+1$.

all m and all $\alpha \in \mathcal{P}^j_{s^j+q^j}$. Therefore,

$$\begin{aligned}
&\mu_{\sigma^*}\Bigg(\bigcup_{\alpha\in\mathcal{P}^j_{s^j+q^j}}\ \bigcup_{m\geq \underline{m}^j_{s^j+q^j}+d} \mathbf{N}^\alpha_m \Big| \bar{h}_{\bar{T}}\Bigg)\\
&\quad\leq \sum_{\alpha\in\mathcal{P}^j_{s^j+q^j}}\ \sum_{m\geq \underline{m}^j_{s^j+q^j}+d} \mu_{\sigma^*}\big(\mathbf{N}^\alpha_m|\bar{h}_{\bar{T}}\big)\\
&\quad\leq \sum_{\alpha\in\mathcal{P}^j_{s^j+q^j}}\ \sum_{m\geq \underline{m}^j_{s^j+q^j}+d} 2\exp\Big(-\frac{1}{8}m\big(\bar{\xi}^j\big)^2\Big)\\
&\quad\leq \frac{1}{2}\Bigg[4\big(\#\mathcal{P}^j_{s^j+q^j}\big)\sum_{m\geq \underline{m}^j_{s^j+q^j}} \exp\Big(-\frac{1}{8}m\big(\bar{\xi}^j\big)^2\Big)\Bigg]\\
&\quad\leq \frac{1}{2}\exp\big(-s^j-q^j\big)\leq \frac{1}{2}.
\end{aligned}$$

The fourth inequality holds by the LS condition. All other inequalities are obvious. Let $\mathbf{N} := \bigcap_{\alpha\in\mathcal{P}^j_{s^j+q^j}}\bigcap_{m\geq \underline{m}^j_{s^j+q^j}+d}(\mathbf{N}^\alpha_m)^c$, where $(\mathbf{N}^\alpha_m)^c$ is the complement of $\mathbf{N}^\alpha_m$. Then $\mu_{\sigma^*}(\mathbf{N}|\bar{h}_{\bar{T}}) \geq 1/2$.

I show that $\mathbf{N} \subseteq \mathbf{M}$. Suppose that $h_\infty \in \mathbf{N}$. Note that the length of the test phase is at least $3\tilde{T}_{s^j+q^j}(\underline{m}^j_{s^j+q^j} + d)(\#\mathcal{P}^j_{s^j+q^j})^2$ periods. Furthermore, $\tilde{T}_{\hat{s}} \leq \tilde{T}_{s^j+q^j}$ by (T.1). Recall that $\mathcal{P}^j_{\hat{s}-1} \leq \mathcal{P}^j_{\hat{s}-1}(\tilde{T}_{\hat{s}}, 3, i) \leq \mathcal{P}^j_{s^j+q^j}$. Therefore, the test phase is long enough so that there always exist $\alpha', \alpha'' \in \mathcal{P}^j_{s^j+q^j}$ and $\bar{\beta} \in \mathcal{P}^j_{\hat{s}-1}$ such that (i) $\alpha' \subseteq \bar{\beta}_{\mathrm{AF}}$, (ii) $\alpha'' \subseteq \bar{\beta}_{\mathrm{AS}}$, and (iii) α' and α'' obtain *enough* samples *during* the test phase, that is, $\tilde{m}^{\alpha'}, \tilde{m}^{\alpha''} \geq \underline{m}^j_{s^j+q^j} + d$. Since $h_\infty \in \mathbf{N}$, these imply that $l^{\bar{\beta}_{\mathrm{AF}}}_f[a^*_i] - \bar{\xi}^j/4 \leq \mathbf{d}^{\alpha'_f}_{m'}[a^*_i]/m' \leq L^{\bar{\beta}_{\mathrm{AF}}}_f[a^*_i] + \bar{\xi}^j/4$ and $l^{\bar{\beta}_{\mathrm{AS}}}_f[a^*_i] - \bar{\xi}^j/4 \leq \mathbf{d}^{\alpha''_f}_{m''}[a^*_i]/m'' \leq L^{\bar{\beta}_{\mathrm{AS}}}_f[a^*_i] + \bar{\xi}^j/4$, where $m' := \tilde{m}^{\alpha'}$ and $m'' := \tilde{m}^{\alpha''}$. By the definitions of $\tilde{D}^j_i(\alpha')[a^*_i]$ and $\mathbf{d}^{\alpha'_f}_{m'}[a^*_i]/m'$, $\tilde{D}^j_i(\alpha')[a^*_i] = \mathbf{d}^{\alpha'_f}_{m'}[a^*_i]/m'$; similarly, $\tilde{D}^j_i(\alpha'')[a^*_i] = \mathbf{d}^{\alpha''_f}_{m''}[a^*_i]/m''$. Thus, I obtain (iv). Hence, $h_\infty \in \mathbf{M}(\alpha', \alpha'', \bar{\beta})$. Therefore, $\mathbf{N} \subseteq \mathbf{M}$. Let $\mathbf{C}(\bar{h}_{\bar{T}})$ denote the cylinder set based on $\bar{h}_{\bar{T}}$: $\mathbf{C}(\bar{h}_{\bar{T}}) := \{h_\infty \mid \bar{h}_{\bar{T}} < h_\infty\}$. Then $\mu_{\sigma^*}(\mathbf{M}|\bar{h}_{\bar{T}}) = \mu_{\sigma^*}(\mathbf{M} \cap \mathbf{C}(\bar{h}_{\bar{T}}))/\mu_{\sigma^*}(\mathbf{C}(\bar{h}_{\bar{T}})) \geq \mu_{\sigma^*}(\mathbf{N} \cap \mathbf{C}(\bar{h}_{\bar{T}}))/\mu_{\sigma^*}(\mathbf{C}(\bar{h}_{\bar{T}})) = \mu_{\sigma^*}(\mathbf{N}|\bar{h}_{\bar{T}}) \geq 1/2$. This completes the proof. *Q.E.D.*

From (b.1) and (b.8), it follows that $L^{\bar{\beta}_{\mathrm{AF}}}_f[a^*_i] \leq c_0 + \frac{1}{3}\bar{\xi}^j$ and $l^{\bar{\beta}_{\mathrm{AS}}}_f[a^*_i] \geq c'_0 - \frac{1}{3}\bar{\xi}^j$. Since $c'_0 - c_0 > \frac{1}{2(\#A_i+1)}$, this implies that $l^{\bar{\beta}_{\mathrm{AS}}}_f[a^*_i] - L^{\bar{\beta}_{\mathrm{AF}}}_f[a^*_i] \geq c'_0 - c_0 - \frac{1}{3}\bar{\xi}^j -$

$\frac{1}{3}\bar{\xi}^j > \frac{1}{2(\#A_i+1)} - \frac{2}{3}\bar{\xi}^j$. Recall that $\bar{\xi}^j \leq \min[\eta/3, 1/8(\#\bar{A}+1)]$. From these and Claim B.1(iv), it follows that $|\tilde{D}_i^j(\alpha')[a_i^*] - \tilde{D}_i^j(\alpha'')[a_i^*]| > \frac{1}{2(\#A_i+1)} - \frac{2}{3}\bar{\xi}^j - \frac{1}{4}\bar{\xi}^j - \frac{1}{4}\bar{\xi}^j = \frac{1}{2(\#A_i+1)} - \frac{7}{6}\bar{\xi}^j \geq \frac{1}{2(\#A_i+1)} - \frac{7}{6}\frac{1}{8(\#A_i+1)} = \frac{17}{6}\frac{1}{8(\#A_i+1)} > \bar{\xi}^j$. From Claim B.1(i)–(ii), it follows that there exists $\beta' \in \mathcal{P}^j_{s^j-1}$ such that $\alpha', \alpha'' \subseteq \beta'$ because $\mathcal{P}^j_{s^j-1} \leq \mathcal{P}^j_{\hat{s}-1}$ by (P.1). These, along with Claim B.1(iii), imply that the complexity test rejects f^j at the end of the test phase with (conditional) probability at least 1/2. This completes the proof of Lemma 3(i).

Next, I prove Lemma 3(ii). As an additional assumption, let $s^j \geq \bar{s}_1$. Then consider any player i's test phase in epoch s^i in which player i employs $f^i[j]$ that is generated by $\mathcal{P}^i_{s^i}$, and let $(\mathcal{P}^i_{s^i+q^i}, \underline{m}^i_{s^i+q^i} + d')$ denote the CR and the smallest sample size used in the test phase. Suppose that player j employs an AEB $\hat{g}^j$ (near $\hat{\sigma}$) at the beginning of the test phase. Let $\hat{h}_{\hat{T}}$ be the (finite) history realized just before the above test phase.

By (4.1) in Section 4.3, for all h, $\|\sigma_j^{\hat{g}}(h) - \hat{\sigma}_j(h)\| \leq \bar{\xi}^{i'}/18$ for all $i' \neq j$. However, then, since $s^j \geq \bar{s}_1$, (b.5) induces that for all $h \in H_{\hat{g}^j}(s^j)$, $\|\sigma_j^*(h) - \sigma_j^{\hat{g}}(h)\| \leq \bar{\xi}^{i'}/16$ for all $i' \neq j$. These imply that for all $h \in H_{\hat{g}^j}(s^j)$, $\|\sigma_j^*(h) - \hat{\sigma}_j(h)\| \leq \bar{\xi}^{i'}/8$ for all $i' \neq j$. Accordingly, for each $\beta \in \mathcal{P}^j_{\hat{s}-1}$, define a class $\beta^{\hat{g}}_{\mathrm{AT}}$ such that $h_T \in \beta^{\hat{g}}_{\mathrm{AT}}$ if and only if (a) $h_T \in \beta_{\mathrm{AT}}$, (b) $\hat{h}_{\hat{T}} \leq h_T$, and (c) time $T+1$ is in the test phase.[16] Let $L^{\beta_{\mathrm{AT}}}_{\hat{g}}[a_j] := \sup_{h\in\beta^{\hat{g}}_{\mathrm{AT}}} \sigma_j^*(h)[a_j]$ and $l^{\beta_{\mathrm{AT}}}_{\hat{g}}[a_j] := \inf_{h\in\beta^{\hat{g}}_{\mathrm{AT}}} \sigma_j^*(h)[a_j]$ for all a_j. Furthermore, for each $\alpha \in \mathcal{P}^i_{s^i+q^i}$, define a class $\alpha_{\hat{g}}$ such that $h_T \in \alpha_{\hat{g}}$ if and only if (d) $h_T \in \alpha$, (e) $\hat{h}_{\hat{T}} \leq h_T$, and (f) time $T+1$ is in the test phase. Let $\mathbf{d}^{\alpha_{\hat{g}}}_m[a_j]$ denote the number of times that a_j has been realized in the first m $\alpha_{\hat{g}}$-active periods.

CLAIM B.2: *With (conditional) probability at least* 1/2, *there exist* $\alpha''' \in \mathcal{P}^i_{s^i+q^i}$ *and* $\hat{\beta} \in \mathcal{P}^j_{\hat{s}-1}$ *such that* (i) $\alpha''' \subseteq \hat{\beta}_{\mathrm{AT}}$, (ii) α''' *obtains enough samples during the test phase, that is,* $\tilde{m}^{\alpha'''} \geq \underline{m}^i_{s^i+q^i} + d'$, *and* (iii) $l^{\hat{\beta}_{\mathrm{AT}}}_{\hat{g}}[a_j] - \frac{1}{4}\bar{\xi}^i \leq \tilde{D}^i_j(\alpha''')[a_j] \leq L^{\hat{\beta}_{\mathrm{AT}}}_{\hat{g}}[a_j] + \frac{1}{4}\bar{\xi}^i$ *for all* a_j, *where* $\tilde{D}^i(\alpha''')$ $(:= (\tilde{D}^i_k(\alpha'''))_{k\neq i})$ *denotes the empirical distributions of the samples collected in the* α'''*-active periods during the test phase.*

PROOF: Let $\mathbf{Q}(\alpha''', \hat{\beta}) := \{h_\infty \mid \text{(i)–(iii) hold in } h_\infty\}$. Furthermore, let $\mathbf{Q} := \bigcup_{\alpha'''\in\mathcal{P}^i_{s^i+q^i}} \bigcup_{\hat{\beta}\in\mathcal{P}^j_{\hat{s}-1}} \mathbf{Q}(\alpha''', \hat{\beta})$. Then it suffices to prove that $\mu_{\sigma^*}(\mathbf{Q}|\hat{h}_{\hat{T}}) \geq 1/2$. Note that $\mathcal{P}^j_{\hat{s}-1} \leq \mathcal{P}^j_{\hat{s}-1}(\tilde{T}_{\hat{s}}, 3, i) \leq \mathcal{P}^i_{s^i+q^i}$ by (P.1) and (P.3). Then, for each $\alpha \in \mathcal{P}^i_{s^i+q^i}$, if $\alpha \subseteq \beta_{\mathrm{AT}}$ for some (unique) $\beta \in \mathcal{P}^j_{\hat{s}-1}$, let $\mathbf{R}^\alpha_m[a_j] := \{h_\infty \mid \mathcal{T}^{\alpha_{\hat{g}}}_m < \infty, \mathbf{d}^{\alpha_{\hat{g}}}_m[a_j]/m \leq l^{\beta_{\mathrm{AT}}}_{\hat{g}}[a_j] - \bar{\xi}^i/4 \text{ or } \mathbf{d}^{\alpha_{\hat{g}}}_m[a_j]/m \geq L^{\beta_{\mathrm{AT}}}_{\hat{g}}[a_j] + \bar{\xi}^i/4\}$ for all a_j.

[16] Note that when $h_T \in \beta^{\hat{g}}_{\mathrm{AT}}$, player j employs $\hat{g}^j$ at time $T+1$.

Otherwise, let $\mathbf{R}_m^{\alpha}[a_j] := \emptyset$ for all a_j. The remainder of the proof is quite similar to that of Claim B.1. Indeed, by Proposition A, the LS condition, and (b.6), $\mu_{\sigma^*}(\bigcup_{\alpha\in\mathcal{P}^i_{s^i+q^i}}\bigcup_{m\geq \underline{m}^i_{s^i+q^i}+d'}\bigcup_{a_j\in A_j}\mathbf{R}_m^{\alpha}[a_j]|\hat{h}_{\hat{T}}) \leq 1/2$. Then let $\mathbf{R} := \bigcap_{\alpha\in\mathcal{P}^i_{s^i+q^i}}\bigcap_{m\geq \underline{m}^i_{s^i+q^i}+d'}\bigcap_{a_j\in A_j}(\mathbf{R}_m^{\alpha}[a_j])^c$, where $(\mathbf{R}_m^{\alpha}[a_j])^c$ is the complement of $\mathbf{R}_m^{\alpha}[a_j]$. Thus, $\mu_{\sigma^*}(\mathbf{R}|\hat{h}_{\hat{T}}) \geq 1/2$. Furthermore, it is easy to show that $\mathbf{R}\subseteq\mathbf{Q}$. Hence, $\mu_{\sigma^*}(\mathbf{Q}|\hat{h}_{\hat{T}}) \geq \mu_{\sigma^*}(\mathbf{R}|\hat{h}_{\hat{T}}) \geq 1/2$. This completes the proof. *Q.E.D.*

Note that $\hat{\sigma}_j$ takes the *same* (mixed) action $\hat{\sigma}_j(\hat{\beta})$ $(=\hat{\pi}_j)$ in *all* $\hat{\beta}$-active periods. Since $\|\sigma_j^*(h)-\hat{\sigma}_j(h)\| \leq \bar{\xi}^i/8$ for all $h\in H_{\hat{g}^j}(s^j)$, this means that $L_{\hat{g}}^{\hat{\beta}\mathrm{AT}}[a_j] \leq \hat{\sigma}_j(\hat{\beta})[a_j] + \frac{1}{8}\bar{\xi}^i$ and $\hat{\sigma}_j(\hat{\beta})[a_j] - \frac{1}{8}\bar{\xi}^i \leq l_{\hat{g}}^{\hat{\beta}\mathrm{AT}}[a_j]$ for all a_j. Furthermore, $\hat{\sigma}_j(\alpha''') = \hat{\sigma}_j(\hat{\beta})$ $(=\hat{\pi}_j)$ because of Claim B.2(i). These, along with Claim B.2(iii), imply that $\|\hat{\sigma}_j(\alpha''') - \tilde{D}^i_j(\alpha''')\| \leq \frac{1}{8}\bar{\xi}^i + \frac{1}{4}\bar{\xi}^i = \frac{3}{8}\bar{\xi}^i$. Next, recall from (b.7) that for all h, $\|f_j^i[j](h) - \rho^i_{(t_0,t_0',\pi_0''),j}(h)\| \leq \bar{\xi}^j/12B_i$. Therefore, $\|f_j^i[j](\alpha''') - \rho^i_{(t_0,t_0',\pi_0''),j}(\alpha''')\| \leq \bar{\xi}^j/12B_i$.[17] In addition, from (b.2), $\|\hat{\sigma}_j(h) - \rho^i_{(t_0,t_0',\pi_0''),j}(h)\| \geq 1/2$ in any $\mathrm{AT}^j_{\hat{s}-1}$-active period. Thus, by Claim B.2(i), $\|\hat{\sigma}_j(\alpha''') - \rho^i_{(t_0,t_0',\pi_0''),j}(\alpha''')\| \geq 1/2 > 1/2(\#\bar{A}+1)$. Finally, recall that $\bar{\xi}^i, \bar{\xi}^j \leq \min[\eta/3, 1/8(\#\bar{A}+1)]$. These imply that

$$
\begin{aligned}
&\|\tilde{D}^i_j(\alpha''') - f_j^i[j](\alpha''')\| \\
&\quad \geq \|\hat{\sigma}_j(\alpha''') - \rho^i_{(t_0,t_0',\pi_0''),j}(\alpha''')\| - \|f_j^i[j](\alpha''') - \rho^i_{(t_0,t_0',\pi_0''),j}(\alpha''')\| \\
&\qquad - \|\hat{\sigma}_j(\alpha''') - \tilde{D}^i_j(\alpha''')\| \\
&\quad > \frac{1}{2(\#\bar{A}+1)} - \frac{\bar{\xi}^j}{12B_i} - \frac{3}{8}\bar{\xi}^i \\
&\quad \geq \frac{1}{2(\#\bar{A}+1)} - \frac{11}{24}\frac{1}{8(\#\bar{A}+1)} \\
&\quad > \frac{3}{8(\#\bar{A}+1)} > \bar{\xi}^i.
\end{aligned}
$$

This, along with Claim B.2(ii), implies that (even if the complexity test is passed) $f^i[j]$ is rejected by some current class (i.e., class α''') at the end of the test phase with (conditional) probability at least 1/2. This completes the proof of Lemma 3(ii). *Q.E.D.*

[17] Since $f^i[j]$ and $\rho^i_{(t_0,t_0',\pi_0'')}$ are generated by $\mathcal{P}^i_{s^i+q^i}$, $f^i[j](\alpha''')$ and $\rho^i_{(t_0,t_0',\pi_0'')}(\alpha''')$ are well defined.

B.3. *Proof of Lemma 5*

The (Disjoint) ER(s) *Interval in the General Case.* First, I define the (i, j) subinterval for $i \neq j$. Given any formation phase of player i, consider the *shortest* time interval such that (a) it begins in the first period of the given formation phase, (b) it includes at least one active interval of player j $(\neq i)$, and (c) it ends in the last period of a test phase of player i. The time interval is called an (i, j) *subinterval*. Then I define the ER(s) interval. Suppose that the first belief rejection by the maximum epoch player in maximum epoch s has occurred; assume that (maximum epoch) player i has made the belief rejection. Then consider the *shortest* time interval such that (a) it begins in the next period following player i's belief rejection, (b) it consists of a sequence of disjoint (i, j) subintervals (i.e., it includes exactly one (i, j) subinterval for each $j \neq i$), a formation phase, and a transition phase of player i, and (c) all players' epoch stages are no more than s through the interval regardless of the realized history in the interval. The time interval is called the first ER(s) interval. Inductively, suppose that belief rejection by the maximum epoch player has occurred for the first time after the mth ER(s) interval.[18] Then the shortest time interval satisfying (a)–(c) is called the $(m+1)$th ER(s) interval.

PROOF OF LEMMA 5: Take $\bar{s}$ such that for all $s \geq \bar{s}$, $\underline{p}_s = (\frac{1}{s})^{s\bar{N}_s} \leq (\frac{1}{2})^{2(I-1)} \times (\prod_i \underline{l}_i)^{(\bar{c}I+1)(I-1)\bar{N}_s}$. Let $\bar{s}_3 := \max[\bar{s}_1, \bar{s}_2, \bar{s}]$. I may assume that the ER(s) interval is initiated by (maximum epoch) player 1 and that the ER(s) interval consists of $(1, 2), (1, 3), \ldots, (1, I)$ subintervals, a formation phase, and a transition phase of player 1. Accordingly, let $\bar{s}_3 \leq s^j \leq s^1 = s$ for all $j \geq 2$.

Step 1. First, from Lemma 3, player 1 forms $f^1[2]$ in the initial formation phase during the $(1, 2)$ subinterval with probability at least $(\prod_{i\neq 1} \underline{l}_i)^{N_s^1}$. Then player 1 retains $f^1[2]$ until the last test phase in the $(1, 2)$ subinterval with probability $(\prod_{i\neq 1} \underline{l}_i)^{(\bar{c}-1)N_s^1}$; even if player 1 rejects $f^1[2]$ in any interim test phase, she forms it again in the next formation phase with probability $(\prod_{i\neq 1} \underline{l}_i)^{N_s^1}$ and, by the RB condition, there are at most $(\bar{c} - 1)$ interim test phases of player 1 in the $(1, 2)$ subinterval. Hence, by Lemma 3(i), player 2 rejects her current belief f^2 in her first test phase in the $(1, 2)$ subinterval with probability $\frac{1}{2}$. Furthermore, from Lemma 4, player 2 forms an AEB $\hat{g}^2$ in the next formation phase with probability $(\prod_{i\neq 2} \underline{l}_i)^{N^2_{s^2}}$. Then player 2 retains $\hat{g}^2$ until the end of the $(1, 2)$ subinterval with probability $(\prod_{i\neq 2} \underline{l}_i)^{(\bar{c}-1)N^2_{s^2}}$ for the same reason as in the case of $f^1[2]$. By Lemma 3(ii), this, in turn, leads player 1 to reject $f^1[2]$ in the last test phase with probability $\frac{1}{2}$. That is, player 1 rejects $f^1[2]$ at the end of the $(1, 2)$ subinterval. By the AS condition, the probability of the sequential events is at least the product of the probabilities of

[18]Note that a maximum epoch player other than player i, for example, player k $(\neq i)$, may make the belief rejection. In that case, player i is replaced by player k in (a) and (b).

those events, that is, $(\prod_{i\neq 1}\underline{l}_i)^{N_s^1}(\prod_{i\neq 1}\underline{l}_i)^{(\bar{c}-1)N_s^1}(\frac{1}{2})(\prod_{i\neq 2}\underline{l}_i)^{N_{s^2}^2}(\prod_{i\neq 2}\underline{l}_i)^{(\bar{c}-1)N_{s^2}^2}(\frac{1}{2})$ $(\geq (\frac{1}{2})^2(\prod_i \underline{l}_i)^{\bar{c}(N_s^1+N_{s^2}^2)})$. In addition, player 2 retains $\hat{g}^2$ *until* the end of the ER(s) interval with probability $(\prod_{i\neq 2}\underline{l}_i)^{2\bar{c}(I-2)N_{s^2}^2}$ $(\geq (\prod_i \underline{l}_i)^{2\bar{c}(I-2)N_{s^2}^2})$ because, by the RB condition, there are at most $2\bar{c}$ test phases of player 2 in the $(1, j)$ subinterval for any $j \geq 3$, and $(I-2)$ subintervals remain after the $(1,2)$ subinterval.

Step 2. I consider the remaining subintervals inductively: for any $j \geq 3$, from Lemma 3, player 1 forms $f^1[j]$ in the first formation phase during the $(1, j)$ subinterval with probability $(\prod_{i\neq 1}\underline{l}_i)^{N_s^1}$. The remainder is the same as in the case of the $(1,2)$ subinterval by replacing player 2 with player j. Again, from the AS condition, the probability of the sequential events is at least $(\prod_{i\neq 1}\underline{l}_i)^{N_s^1}(\prod_{i\neq 1}\underline{l}_i)^{(\bar{c}-1)N_s^1}(\frac{1}{2})(\prod_{i\neq j}\underline{l}_i)^{N_{s^j}^j}(\prod_{i\neq j}\underline{l}_i)^{(\bar{c}-1)N_{s^j}^j}(\frac{1}{2})$ $(\geq (\frac{1}{2})^2(\prod_i \underline{l}_i)^{\bar{c}(N_s^1+N_{s^j}^j)})$; player j retains an AEB $\hat{g}^j$ and player 1 rejects $f^1[j]$ at the end of the $(1, j)$ subinterval. By the inductive hypothesis, players 2 to $j-1$ retain their AEBs $(\hat{g}^i)_{2\leq i\leq j-1}$ until the end of the ER(s) interval. Furthermore, player j also retains $\hat{g}^j$ until the end of the ER(s) interval with probability $(\prod_{i\neq j}\underline{l}_i)^{2\bar{c}(I-j)N_{s^j}^j}$ $(\geq (\prod_i \underline{l}_i)^{2\bar{c}(I-j)N_{s^j}^j})$ since, by the RB condition, there are at most $2\bar{c}$ test phases of player j in the $(1, j')$ subinterval for all $j' > j$, and $(I-j)$ subintervals remain after the $(1, j)$ subinterval.

Step 3. By the inductive argument, all players other than 1 employ their AEBs $(\hat{g}^j)_{j\neq 1}$ and player 1 rejects $f^1[I]$ at the end of the $(1, I)$ subinterval with probability at least $(\frac{1}{2})^{2(I-1)}(\prod_i \underline{l}_i)^{\theta}$, where $\theta := \bar{c}(I-1)N_s^1 + \bar{c}\sum_{j=2}^{I} N_{s^j}^j + 2\bar{c}\sum_{j=2}^{I}(I-j)N_{s^j}^j$. Finally, player 1 also forms an AEB $\hat{g}^1$ in the final formation phase with probability $(\prod_{i\neq 1}\underline{l}_i)^{N_s^1}$ and changes $f^1[I]$ to $\hat{g}^1$ in the final transition phase: all players employ their AEBs $\hat{g}$ at the end of the ER(s) interval. That is, AES($\hat{\sigma}$) is reached at the end of the ER(s) interval with probability at least $(\frac{1}{2})^{2(I-1)}(\prod_i \underline{l}_i)^{\theta}(\prod_{i\neq 1}\underline{l}_i)^{N_s^1}$ $(\geq (\frac{1}{2})^{2(I-1)}(\prod_i \underline{l}_i)^{\theta+N_s^1})$. Note that $s^i \leq s$ for all i. Thus, $N_{s^i}^i \leq N_s^i \leq \bar{N}_s$ for all i. From these, it follows that for all $s \geq \bar{s}_3$ $(\geq \bar{s})$,

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{2(I-1)}\left(\prod_i \underline{l}_i\right)^{\theta+N_s^1}\\
&\quad\geq \left(\frac{1}{2}\right)^{2(I-1)}\left(\prod_i \underline{l}_i\right)^{(\bar{c}+1)(I-1)\bar{N}_s+\bar{c}(I-1)\bar{N}_s+2\bar{c}(1/2)(I-1)(I-2)\bar{N}_s}\\
&\quad= \left(\frac{1}{2}\right)^{2(I-1)}\left(\prod_i \underline{l}_i\right)^{(\bar{c}I+1)(I-1)\bar{N}_s} \geq \left(\frac{1}{s}\right)^{s\bar{N}_s} = \underline{p}_s. \qquad \text{Q.E.D.}
\end{aligned}$$

APPENDIX C

C.1. *Proof of Lemma 6*

The (i, j) *Belief-Rejectable Interval for* $i \neq j$. Suppose that player i has rejected her belief for the first time in epoch s^i. Then consider the time interval such that (a) it begins in the next period following the rejection, (b) it ends in the last period of the first test phase of player j ($\neq i$) after the rejection, and (c) the epoch stage of player i is no more than s^i through the interval regardless of the realized history in the interval. This time interval is called *the first* (i, j) *belief-rejectable interval in epoch* s^i *of player* i. It is abbreviated as the first $\mathrm{BR}^i_j(s^i)$ interval. Inductively, suppose that in epoch s^i, player i has rejected her belief for the first time after the mth $\mathrm{BR}^i_j(s^i)$ interval. Then the time interval satisfying (a)–(c) is called the $(m+1)$th $\mathrm{BR}^i_j(s^i)$ interval.

I prepare one claim to prove Lemma 6.

CLAIM C.1: *For any* i, j *with* $i \neq j$, *any* $s^i \geq \bar{s}_1$, *and any* $\mathrm{BR}^i_j(s^i)$ *interval in which* $s^i \geq s^j$ *and* $s^j + q^j \geq \bar{s}_1$, *player* j *rejects her belief at the end of the* $\mathrm{BR}^i_j(s^i)$ *interval with* (*conditional*) *probability at least* $\frac{1}{2}(\prod_k \underline{l}_k)^{\bar{c}N^i_{s^i}}$.[19]

PROOF: Consider any $\mathrm{BR}^i_j(s^i)$ interval and let $\mathcal{P}^j_{s^j+q^j}$ be the CR used in the only test phase of player j during the $\mathrm{BR}^i_j(s^i)$ interval. Suppose that $s^i \geq s^j$, and $s^i, s^j + q^j \geq \bar{s}_1$, where $\bar{s}_1$ is taken from Lemma 3. Then Lemma 3 ensures that player i forms $f^i[j]$ in the first formation phase during the $\mathrm{BR}^i_j(s^i)$ interval with probability $(\prod_{k\neq i} \underline{l}_k)^{N^i_{s^i}}$. Then player i retains $f^i[j]$ until the end of the $\mathrm{BR}^i_j(s^i)$ interval with probability $(\prod_{k\neq i} \underline{l}_k)^{(\bar{c}-1)N^i_{s^i}}$ since, by the RB condition, there are at most $(\bar{c}-1)$ test phases of player i in any $\mathrm{BR}^i_j(s^i)$ interval. Thus, since player i retains $f^i[j]$ through the only test phase of player j, by Lemma 3(i), player j rejects her current belief f^j at the end of the test phase with probability $\frac{1}{2}$. From the AS condition, the (conditional) probability of the sequential events is at least $(\prod_k \underline{l}_k)^{N^i_{s^i}}(\prod_k \underline{l}_k)^{(\bar{c}-1)N^i_{s^i}}(\frac{1}{2}) = \frac{1}{2}(\prod_k \underline{l}_k)^{\bar{c}N^i_{s^i}}$. *Q.E.D.*

For any $s^i \geq \bar{s}_1$, define a class $\gamma(s^i, i, j)$ such that $h_T \in \gamma(s^i, i, j)$ if and only if (a) a $\mathrm{BR}^i_j(s^i)$ interval starts at time $T+1$, (b) $s^i \geq s^j$ in the $\mathrm{BR}^i_j(s^i)$ interval, and (c) player j uses a CR $\mathcal{P}^j_{s^j+q^j}$ whose index is no less than $\bar{s}_1$, that is, $s^j + q^j \geq \bar{s}_1$, in her only test phase during the $\mathrm{BR}^i_j(s^i)$ interval. Furthermore, for any $s^i \geq \bar{s}_1$, let $\mathbf{d}^{\gamma(s^i,i,j)}_m(h_\infty)$ denote the number of times up to the mth

[19]This is the probability conditional on the finite history realized just before the $\mathrm{BR}^i_j(s^i)$ interval.

$\gamma(s^i, i, j)$-active period that player j has rejected her belief in a $\mathrm{BR}^i_j(s^i)$ interval in which $s^i \geq s^j$, and $s^j + q^j \geq \bar{s}_1$. Then define $\mathbf{A}_m(s^i, i, j) := \{h_\infty \mid \mathcal{T}_m^{\gamma(s^i,i,j)} < \infty, \mathbf{d}_m^{\gamma(s^i,i,j)}/m < \underline{p}_{s^i} - \frac{1}{2}\underline{p}_{s^i}\}$. Note that $\frac{1}{2}(\prod_k \underline{l}_k)^{\bar{c}N^i_{s^i}} \geq \underline{p}_{s^i}$ for any large s^i. From this, Claim C.1, and Proposition A, it follows that, for any large s^i $(\geq \bar{s}_1)$, $\mu_{\sigma^*}(\mathbf{A}_m(s^i, i, j)) \leq 2\exp(-2m(\frac{1}{2}\underline{p}_{s^i})^2) = 2\exp(-\frac{1}{2}m(\underline{p}_{s^i})^2)$ for all i, all $j \neq i$, and all m. Furthermore, since $w^i_{s^i} = \frac{1}{s^i}(p^i_{s^i})^{s^i(I-1)}$, for all i, $w^i_{s^i}R^i_{s^i} \leq R^i_{s^i}/\bar{c}$ for any large s^i. From the RB condition, for all i, $\bar{N}_{s^i} \leq \bar{n}N^i_{s^i}$ for all s^i. Therefore, for any large s^i, $(p^i_{s^i})^{s^i} \leq (p^i_{s^i})^{\bar{n}} = (\frac{1}{s^i})^{s^i\bar{n}N^i_{s^i}} \leq (\frac{1}{s^i})^{s^i\bar{N}_{s^i}} = \underline{p}_{s^i}$ for all i. From these inequalities and the MR condition, it follows that for any i, any $j \neq i$, and any large s' $(\geq \bar{s}_1)$,

$$\begin{aligned}
&\mu_{\sigma^*}\Bigg(\bigcup_{s^i \geq s'} \bigcup_{m \geq R^i_{s^i}/\bar{c}} \mathbf{A}_m(s^i, i, j)\Bigg) \\
&\quad \leq \sum_{s^i \geq s'} \sum_{m \geq w^i_{s^i}R^i_{s^i}} 2\exp\left(-\frac{1}{2}m(p^i_{s^i})^{2s^i}\right) \\
&\quad \leq 2\sum_{s^i \geq s'} \exp(-s^i) = 2(1 - \exp(-1))^{-1}\exp(-s').
\end{aligned}$$

The first inequality holds because, for any large s^i, $(p^i_{s^i})^{s^i} \leq \underline{p}_{s^i}$, and $w^i_{s^i}R^i_{s^i} \leq R^i_{s^i}/\bar{c}$. The second inequality is obtained by the MR condition. Hence, for all i and all $j \neq i$, $\mu_{\sigma^*}(\bigcap_{s' \geq \bar{s}_1} \bigcup_{s^i \geq s'} \bigcup_{m \geq R^i_{s^i}/\bar{c}} \mathbf{A}_m(s^i, i, j)) = 0$. Therefore,

$$\mu_{\sigma^*}\Bigg(\bigcup_i \bigcup_{j \neq i} \bigcap_{s' \geq \bar{s}_1} \bigcup_{s^i \geq s'} \bigcup_{m \geq R^i_{s^i}/\bar{c}} \mathbf{A}_m(s^i, i, j)\Bigg) = 0.$$

From this, I obtain Lemma 6.

PROOF OF LEMMA 6: Let $\mathbf{A} := \bigcap_i \bigcap_{j \neq i} \bigcup_{s' \geq \bar{s}_1} \bigcap_{s^i \geq s'} \bigcap_{m \geq R^i_{s^i}/\bar{c}} (\mathbf{A}_m(s^i, i, j))^c$, where $(\mathbf{A}_m(s^i, i, j))^c$ is the complement of $\mathbf{A}_m(s^i, i, j)$. Then $\mu_{\sigma^*}(\mathbf{A}) = 1$. Suppose that there are infinite belief rejections but that some player, for example, player j_1, only rejects her belief a finite number of times along $h_\infty \in \mathbf{A}$. On the one hand, since player j_1 does not reject her belief from some period on, she stays in some epoch, for example, epoch $s^{j_1}_0$, forever from some period. Moreover, she continues switching to finer CRs in her test phases throughout epoch $s^{j_1}_0$, as defined in Section 4.1.2: $s^{j_1}_0 + q^{j_1} \to \infty$ as $T \to \infty$. On the other hand, since there are infinite belief rejections, there exists i_1 $(\neq j_1)$ such that player i_1 rejects her belief infinitely many times; her epoch stage goes to infinity, that is, $s^{i_1}_T \to \infty$ as $T \to \infty$. These imply that for any large s^{i_1} $(\geq \bar{s}_1)$,

$s^{i_1} \geq s_0^{j_1}$ and $s_0^{j_1} + q^{j_1} \geq \bar{s}_1$ in any $\text{BR}_{j_1}^{i_1}(s^{i_1})$ interval during epoch s^{i_1}. In addition, there are at least $(R_{s^{i_1}}^{i_1}/\bar{c})$ $\text{BR}_{j_1}^{i_1}(s^{i_1})$ intervals in each epoch s^{i_1} because, by the RB condition, there are at most $(\bar{c}-1)$ test phases of player i_1 in any $\text{BR}_{j_1}^{i_1}(s^{i_1})$ interval. Since $h_\infty \in \mathbf{A}$, these imply that, for any large s^{i_1} $(\geq s' \geq \bar{s}_1)$, (i) for all m, $\mathbf{d}_m^{\gamma(s^{i_1},i_1,j_1)}(h_\infty)$ equals the number of times that player j_1 has rejected her belief in the first m $\text{BR}_{j_1}^{i_1}(s^{i_1})$ intervals and (ii) $\mathbf{d}_{R_{s^{i_1}}^{i_1}/\bar{c}}^{\gamma(s^{i_1},i_1,j_1)}(h_\infty) \geq \frac{1}{2}\underline{p}_{s^{i_1}}(R_{s^{i_1}}^{i_1}/\bar{c})$. This means that player j_1 rejects her belief at least $\frac{1}{2}\underline{p}_{s^{i_1}}(R_{s^{i_1}}^{i_1}/\bar{c})$ times in epoch s^{i_1} for any large s^{i_1}. However, Remark 1 implies that $\underline{p}_{s^{i_1}} R_{s^{i_1}}^{i_1} \to \infty$ as $s^{i_1} \to \infty$. Therefore, player j_1 rejects her belief infinitely many times along h_∞. This is a contradiction. *Q.E.D.*

C.2. *Proof of Lemma 7*

For all $s \geq 1$, define a class $\omega(s)$ such that $h_T \in \omega(s)$ if and only if (a) time $T+1$ is the first period of an $\text{ER}_{\text{mod}\, I}(s)$ interval and (b) all players' epoch stages are no less than $\bar{s}_3$ at the beginning of the $\text{ER}_{\text{mod}\, I}(s)$ interval, where $\bar{s}_3$ is taken from Lemma 5. Let $\mathbf{d}_m^{\omega(s)}(h_\infty)$ denote the number of times that $\text{AES}(\hat{\sigma})$ has been reached in the first m $\text{ER}_{\text{mod}\, I}(s)$ intervals in which all players' epoch stages are no less than $\bar{s}_3$. Then define $\mathbf{B}_m^s := \{h_\infty \mid \mathcal{T}_m^{\omega(s)} < \infty, \mathbf{d}_m^{\omega(s)}/m < \underline{p}_s - \frac{1}{2}\underline{p}_s\}$. By Lemma 5 and Proposition A, $\mu_{\sigma^*}(\mathbf{B}_m^s) \leq 2\exp(-2m(\frac{1}{2}\underline{p}_s)^2) = 2\exp(-\frac{1}{2}m(\underline{p}_s)^2)$ for all s and all m. Recall that there are at least $(\underline{R}_s/2\bar{c}I \times (I-1))$ $\text{ER}_{\text{mod}\, I}(s)$ intervals in each maximum epoch s. Note that for any large s, there exists i_s such that $R_s^{i_s}/s \leq R_s^{i_s}/2\bar{c}I(I-1) = \underline{R}_s/2\bar{c}I(I-1)$. Therefore, $w_s^{i_s} R_s^{i_s} := \frac{1}{s}(p_s^{i_s})^{s(I-1)} R_s^{i_s} \leq \underline{R}_s/2\bar{c}I(I-1)$ for any large s. Furthermore, from the RB condition, it follows that for any large s, $(p_s^i)^s \leq (p_s^i)^{\bar{n}} = (\frac{1}{s})^{s\bar{n}N_s^i} \leq (\frac{1}{s})^{s\bar{N}_s} = \underline{p}_s$ for all i. From these inequalities and the MR condition, the same computation as in Appendix C.1 induces that

$$\mu_{\sigma^*}\left(\bigcap_{s'\geq 1}\bigcup_{s\geq s'}\bigcup_{m\geq \underline{R}_s/2\bar{c}I(I-1)} \mathbf{B}_m^s\right) = 0.$$

Let $\mathbf{B} := \bigcup_{s'\geq 1}\bigcap_{s\geq s'}\bigcap_{m\geq \underline{R}_s/2\bar{c}I(I-1)}(\mathbf{B}_m^s)^c$, where $(\mathbf{B}_m^s)^c$ is the complement of $\mathbf{B}_m^s$. Then $\mu_{\sigma^*}(\mathbf{B}) = 1$. From this, I obtain Lemma 7.

PROOF OF LEMMA 7: Consider $\mathbf{B} \cap \mathbf{Z}$, where $\mathbf{Z} := \{h_\infty \mid$ if there are infinite belief rejections, then every player rejects her belief infinitely many times in $h_\infty\}$; $\mu_{\sigma^*}(\mathbf{Z}) = 1$ by Lemma 6. Then $\mu_{\sigma^*}(\mathbf{B} \cap \mathbf{Z}) = 1$. Take any $h_\infty \in \mathbf{B} \cap \mathbf{Z}$ and suppose that there are infinite belief rejections in h_∞. Since $h_\infty \in \mathbf{Z}$, every player rejects her belief infinitely many times in h_∞. This means that there exists $\bar{s}_3'$ $(\geq \bar{s}_3)$ such that for all $s \geq \bar{s}_3'$, all players' epoch stages are no less than $\bar{s}_3$ through maximum epoch s in h_∞. Therefore, for all $s \geq \bar{s}_3'$ and all m, $\mathbf{d}_m^{\omega(s)}(h_\infty)$ equals the number of times that $\mathrm{AES}(\hat{\sigma})$ has been reached in the first m $\mathrm{ER}_{\mathrm{mod}\, I}(s)$ intervals. In addition, since $h_\infty \in \mathbf{B}$, there exists $\bar{s}_4$ $(\geq \bar{s}_3')$ such that for all $s \geq \bar{s}_4$, $\mathbf{d}_m^{\omega(s)}(h_\infty) \geq \frac{1}{2}\underline{p}_s m$ for all $m \geq \underline{R}_s/2\bar{c}I(I-1)$. Since there are at least $\underline{R}_s/2\bar{c}I(I-1)\mathrm{ER}_{\mathrm{mod}\, I}(s)$ intervals in maximum epoch s for all s, this means that for all $s \geq \bar{s}_4$, $\mathrm{AES}(\hat{\sigma})$ is reached at least $\frac{1}{2}\underline{p}_s(\underline{R}_s/2\bar{c}I(I-1))$ times in the first $\underline{R}_s/2\bar{c}I(I-1)\mathrm{ER}_{\mathrm{mod}\, I}(s)$ intervals in maximum epoch s in h_∞. *Q.E.D.*

C.3. *Proof of Lemma 8*

For convenience, I introduce the α-sampling. Given $(\mathcal{P}_{s+q}^i, \underline{m}_{s+q}^i + d)$ and $\alpha \in \mathcal{P}_{s+q}^i$, by the $(d+1)$th α-sampling in epoch s of player i, I mean that player i collects samples (i.e., her opponents' realized actions) in α-active periods for the statistical test that uses $(\mathcal{P}_{s+q}^i, \underline{m}_{s+q}^i + d)$. Further, the $(d+1)$th α-sampling (in epoch s of player i) is *effective* at time T if player i continues collecting samples in α-active periods at time T for the corresponding statistical test. Then, for all i, all $s \geq 1$, all $q \geq 0$, all $\alpha \in \mathcal{P}_{s+q}^i$, and all $d \geq 0$, define a class $\alpha(s,q,d)$ such that $h_T \in \alpha(s,q,d)$ if and only if (a) $h_T \in \alpha$, that is, time $T+1$ is α-active, (b) the $(d+1)$th α-sampling is effective at time $T+1$, and (c) for any $h_{T'} \leq h_T$ such that time $T'+1$ is α-active and the $(d+1)$th α-sampling is effective at time $T'+1$, $\|\hat{g}_j^i(h_{T'}) - \sigma_j^*(h_{T'})\| \leq \bar{\xi}^i/8$ for all $j \neq i$, where player i's AEB $\hat{g}^i$ has been formed just after the most recent belief rejection by player i in h_T. Let $\mathbf{d}_{j,m}^{\alpha(s,q,d)}[a_j]$ denote the number of times that a_j has been realized in the first m $\alpha(s,q,d)$-active periods, and let $\mathbf{d}_{j,m}^{\alpha(s,q,d)} := (\mathbf{d}_{j,m}^{\alpha(s,q,d)}[a_j])_{a_j}$. Recall that for all i, $\mathcal{P}_{\mathrm{Id}} \leq \mathcal{P}_s^i$ for all s. Accordingly, for all i, all $s \geq 1$, all $q \geq 0$, all $\alpha \in \mathcal{P}_{s+q}^i$, all $j \neq i$, and all a_j, let $\overline{\hat{\sigma}}_j^i(\alpha)[a_j] := \hat{\sigma}_j(\alpha)[a_j] + \frac{1}{4}\bar{\xi}^i$ and $\underline{\hat{\sigma}}_j^i(\alpha)[a_j] := \hat{\sigma}_j(\alpha)[a_j] - \frac{1}{4}\bar{\xi}^i$. Then define

$$\mathbf{C}_m^{\alpha(s,q,d)}[a_j] := \left\{ h_\infty \,\middle|\, \mathcal{T}_m^{\alpha(s,q,d)} < \infty, \frac{\mathbf{d}_{j,m}^{\alpha(s,q,d)}[a_j]}{m} < \underline{\hat{\sigma}}_j^i(\alpha)[a_j] - \frac{\bar{\xi}^i}{4} \text{ or } \frac{\mathbf{d}_{j,m}^{\alpha(s,q,d)}[a_j]}{m} > \overline{\hat{\sigma}}_j^i(\alpha)[a_j] + \frac{\bar{\xi}^i}{4} \right\}$$

and $\mathbf{C}_m^{\alpha(s,q,d)} := \bigcup_{j\neq i}\bigcup_{a_j} \mathbf{C}_m^{\alpha(s,q,d)}[a_j]$. From the definition of $\alpha(s,q,d)$ and (4.1) in Section 4.3, it follows that for all i, all $s \geq 1$, all $q \geq 0$, all $\alpha \in \mathcal{P}_{s+q}^i$, all $d \geq 0$, all $h \in \alpha(s,q,d)$, and all $j \neq i$, $\underline{\hat{\sigma}}_j^i(\alpha)[a_j] \leq \sigma_j^*(h)[a_j] \leq \overline{\hat{\sigma}}_j^i(\alpha)[a_j]$ for all a_j. From this and Proposition A, it follows that for all i, all $s \geq 1$, all $q \geq 0$, all $\alpha \in \mathcal{P}_{s+q}^i$, and all $d \geq 0$, $\mu_{\sigma^*}(\mathbf{C}_m^{\alpha(s,q,d)}) \leq (\sum_{j\neq i} \# A_j) 2\exp(-\frac{1}{8}m(\bar{\xi}^i)^2)$ for all m.

Hence, for all i and all $s' \geq 1$,

$$\mu_{\sigma^*}\left(\bigcup_{s\geq s'}\bigcup_{q\geq 0}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{d\geq 0}\bigcup_{m\geq \underline{m}^i_{s+q}+d}\mathbf{C}^{\alpha(s,q,d)}_m\right)$$

$$\leq \sum_{s\geq s'}\sum_{q\geq 0}\sum_{\alpha\in\mathcal{P}^i_{s+q}}\sum_{d\geq 0}\sum_{m\geq \underline{m}^i_{s+q}+d}\left(\sum_{j\neq i}\#A_j\right)2\exp\left(-\frac{1}{8}m(\bar{\xi}^i)^2\right)$$

$$=2\left(\sum_{j\neq i}\#A_j\right)\sum_{s\geq s'}\sum_{q\geq 0}\sum_{\alpha\in\mathcal{P}^i_{s+q}}\sum_{d\geq 0}\left(1-\exp\left(-\frac{1}{8}(\bar{\xi}^i)^2\right)\right)^{-1}$$

$$\times\exp\left(-\frac{1}{8}(\underline{m}^i_{s+q}+d)(\bar{\xi}^i)^2\right)$$

$$=2\left(\sum_{j\neq i}\#A_j\right)\left(1-\exp\left(-\frac{1}{8}(\bar{\xi}^i)^2\right)\right)^{-1}$$

$$\times\sum_{s\geq s'}\sum_{q\geq 0}(\#\mathcal{P}^i_{s+q})\sum_{m\geq \underline{m}^i_{s+q}}\exp\left(-\frac{1}{8}m(\bar{\xi}^i)^2\right)$$

$$\leq 2\left(\sum_{j\neq i}\#A_j\right)\left(1-\exp\left(-\frac{1}{8}(\bar{\xi}^i)^2\right)\right)^{-1}\sum_{s\geq s'}\sum_{q\geq 0}\exp(-s-q)$$

$$=2\left(\sum_{j\neq i}\#A_j\right)\left(1-\exp\left(-\frac{1}{8}(\bar{\xi}^i)^2\right)\right)^{-1}(1-\exp(-1))^{-2}\exp(-s').$$

The fourth inequality results from the LS condition. All other (in-) equalities are obvious. Therefore, for all i,

$$\mu_{\sigma^*}\left(\bigcap_{s'\geq 1}\bigcup_{s\geq s'}\bigcup_{q\geq 0}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{d\geq 0}\bigcup_{m\geq \underline{m}^i_{s+q}+d}\mathbf{C}^{\alpha(s,q,d)}_m\right)=0.$$

Then letting $\mathbf{C} := \bigcup_i\bigcap_{s'\geq 1}\bigcup_{s\geq s'}\bigcup_{q\geq 0}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{d\geq 0}\bigcup_{m\geq \underline{m}^i_{s+q}+d}\mathbf{C}^{\alpha(s,q,d)}_m$, $\mu_{\sigma^*}(\mathbf{C})=0$. From this, I obtain Lemma 8.

PROOF OF LEMMA 8: Define $\mathbf{W} := \{h_\infty \mid$ there are infinite belief rejections in AES$(\hat{\sigma})$ under accurate testing along $h_\infty\}$. Then it suffices to prove that $\mathbf{W}\subseteq\mathbf{C}$ because $\mu_{\sigma^*}(\mathbf{C})=0$. Suppose that $h_\infty\in\mathbf{W}$. This means that some player, for example, player i, infinitely rejects her AEB in AES$(\hat{\sigma})$ under accurate testing along h_∞. Moreover, from Lemma 2 and (4.1) in Section 4.3, it follows that there exists $\bar{T}$ such that for all $T\geq\bar{T}$, if time $T+1$ is in AES$(\hat{\sigma})$, then $\|\hat{g}^i_j(h_T)-$

$\sigma_j^*(h_T)\| \le \bar{\xi}^i/8$ for all $j \neq i$ and all i. Clearly, in infinitely many epochs $\{s_n\}_n$ of player i after time $\bar{T}$, player i rejects her belief in AES($\hat{\sigma}$) under accurate testing. Then, for each s_n, *either* there exist q_n, α'_n, α''_n, β_n, d_n, and $\hat{g}_n$ such that (i) $\alpha'_n, \alpha''_n \in \mathcal{P}^i_{s_n+q_n}$ and $\beta_n \in \mathcal{P}^i_{s_n-1}$, (ii) $\alpha'_n, \alpha''_n \subseteq \beta_n$, (iii) α'_n and α''_n obtain enough samples during the test phase of using $(\mathcal{P}^i_{s_n+q_n}, \underline{m}^i_{s_n+q_n} + d_n)$, that is, $\tilde{m}^{\alpha'_n}, \tilde{m}^{\alpha''_n} \ge \underline{m}^i_{s_n+q_n} + d_n$, (iv) AEBs $\hat{g}_n$ are employed during the test phase, (v) $\|\hat{g}^i_{n,j}(h_T) - \sigma_j^*(h_T)\| \le \bar{\xi}^i/8$ for all $j \neq i$ during the test phase, and (vi) $\hat{g}^i_n$ is rejected by the complexity test, $\|\tilde{D}^i_j(\alpha'_n) - \tilde{D}^i_j(\alpha''_n)\| > \bar{\xi}^i$ for some $j \neq i$, *or* there exist q_n, α_n, d_n, and $\hat{g}_n$ such that (vii) $\alpha_n \in \mathcal{P}^i_{s_n+q_n}$, (viii) α_n obtains enough samples, that is, $m^{\alpha_n} \ge \underline{m}^i_{s_n+q_n} + d_n$, (ix) AEBs $\hat{g}_n$ are employed as long as the (d_n+1)th α_n-sampling is effective, (x) for all $h_T < h_\infty$ such that time $T+1$ is α_n-active and the (d_n+1)th α_n-sampling is effective at time $T+1$, $\|\hat{g}^i_{n,j}(h_T) - \sigma_j^*(h_T)\| \le \bar{\xi}^i/8$ for all $j \neq i$, (xi) for all h, $\|\hat{g}^i_{n,j}(h) - \hat{\sigma}_j(h)\| \le \bar{\xi}^i/18$ for all $j \neq i$, and (xii) $\hat{g}^i_n$ is rejected by α_n, $\|D^i_j(\alpha_n) - \hat{g}^i_{n,j}(\alpha_n)\| > \bar{\xi}^i$ for some $j \neq i$.

First, I consider the former case: $\|\tilde{D}^i_j(\alpha'_n) - \tilde{D}^i_j(\alpha''_n)\| > \bar{\xi}^i$ for some $j \neq i$ by (vi). From (i), (iv), (v), and the definition of $\alpha(s,q,d)$, it follows that $\tilde{D}^i_j(\alpha'_n) = \mathbf{d}^{\alpha'_n(s_n,q_n,d_n)}_{j,m'}/m'$ and $\tilde{D}^i_j(\alpha''_n) = \mathbf{d}^{\alpha''_n(s_n,q_n,d_n)}_{j,m''}/m''$, where $m' := \tilde{m}^{\alpha'_n}$ and $m'' := \tilde{m}^{\alpha''_n}$. Furthermore, $\tilde{m}^{\alpha'_n}, \tilde{m}^{\alpha''_n} \ge \underline{m}^i_{s_n+q_n} + d_n$ by (iii). However, then if $\|\tilde{D}^i_j(\alpha'_n) - \hat{\sigma}_j(\alpha'_n)\| \le \frac{1}{2}\bar{\xi}^i$ and $\|\tilde{D}^i_j(\alpha''_n) - \hat{\sigma}_j(\alpha''_n)\| \le \frac{1}{2}\bar{\xi}^i$, then $\|\tilde{D}^i_j(\alpha'_n) - \tilde{D}^i_j(\alpha''_n)\| \le \bar{\xi}^i$ because $\hat{\sigma}(\alpha'_n) = \hat{\sigma}(\alpha''_n)$ $(= \hat{\sigma}(\beta_n) = \hat{\pi})$ by (i) and (ii). This contradicts (vi). Hence, for some a_j, either $|\tilde{D}^i_j(\alpha'_n)[a_j] - \hat{\sigma}_j(\alpha'_n)[a_j]| > \frac{1}{2}\bar{\xi}^i$ or $|\tilde{D}^i_j(\alpha''_n)[a_j] - \hat{\sigma}_j(\alpha''_n)[a_j]| > \frac{1}{2}\bar{\xi}^i$. This implies that *either* $\tilde{D}^i_j(\alpha'_n)[a_j] < \underline{\hat{\sigma}}^i_j(\alpha'_n)[a_j] - \frac{1}{4}\bar{\xi}^i$ or $\tilde{D}^i_j(\alpha'_n)[a_j] > \overline{\hat{\sigma}}^i_j(\alpha'_n)[a_j] + \frac{1}{4}\bar{\xi}^i$, *or* $\tilde{D}^i_j(\alpha''_n)[a_j] < \underline{\hat{\sigma}}^i_j(\alpha''_n)[a_j] - \frac{1}{4}\bar{\xi}^i$ or $\tilde{D}^i_j(\alpha''_n)[a_j] > \overline{\hat{\sigma}}^i_j(\alpha''_n)[a_j] + \frac{1}{4}\bar{\xi}^i$.

I proceed to the latter case: $\|D^i_j(\alpha_n) - \hat{g}^i_{n,j}(\alpha_n)\| > \bar{\xi}^i$ for some $j \neq i$ by (xii). However, then, from (xi), it follows that $\hat{\sigma}_j(\alpha_n)[a_j] - \frac{1}{16}\bar{\xi}^i \le \hat{g}^i_{n,j}(\alpha_n)[a_j] \le \hat{\sigma}_j(\alpha_n)[a_j] + \frac{1}{16}\bar{\xi}^i$ for all a_j.[20] These imply that for some a_j, either $D^i_j(\alpha_n)[a_j] < \hat{g}^i_{n,j}(\alpha_n)[a_j] - \bar{\xi}^i \le (\hat{\sigma}_j(\alpha_n)[a_j] + \frac{1}{16}\bar{\xi}^i) - \bar{\xi}^i = \underline{\hat{\sigma}}^i_j(\alpha_n)[a_j] - \frac{11}{16}\bar{\xi}^i < \underline{\hat{\sigma}}^i_j(\alpha_n)[a_j] - \frac{1}{4}\bar{\xi}^i$ or $D^i_j(\alpha_n)[a_j] > \hat{g}^i_{n,j}(\alpha_n)[a_j] + \bar{\xi}^i \ge (\hat{\sigma}_j(\alpha_n)[a_j] - \frac{1}{16}\bar{\xi}^i) + \bar{\xi}^i = \overline{\hat{\sigma}}^i_j(\alpha_n)[a_j] + \frac{11}{16}\bar{\xi}^i > \overline{\hat{\sigma}}^i_j(\alpha_n)[a_j] + \frac{1}{4}\bar{\xi}^i$. Moreover, from (vii), (viii), (ix), (x), and the definition of $\alpha(s,q,d)$, it follows that $D^i_j(\alpha_n) = \mathbf{d}^{\alpha_n(s_n,q_n,d_n)}_{j,m'''}/m'''$, where $m''' := m^{\alpha_n} \ge \underline{m}^i_{s_n+q_n} + d_n$.

Both cases show that $h_\infty \in \mathbf{C}$. Therefore, $\mathbf{W} \subseteq \mathbf{C}$. *Q.E.D.*

[20]Note that $\hat{g}^i_n$ and $\hat{\sigma}$ are generated by $\mathcal{P}^i_{s_n}$, $\mathcal{P}^i_{s_n} \le \mathcal{P}^i_{s_n+q_n}$, and $\alpha_n \in \mathcal{P}^i_{s_n+q_n}$. Hence, $\hat{g}^i_n(\alpha_n)$ and $\hat{\sigma}(\alpha_n)$ are well defined; see Section 4.1.1 for details.

C.4. *Two Claims for the Proof of Lemma 9*

As mentioned in Section 4.3.2, I can obtain the following claim.

CLAIM C.2: *With* μ_{σ^*}*-probability* 1, *if there are infinite belief rejections*, *there exists* $\bar{s}_5$ *such that for each* $s \geq \bar{s}_5$, $\mathrm{AES}(\hat{\sigma})$, *which has been reached at the end of an* $\mathrm{ER}_{\mathrm{mod}\, I}(s)$ *interval*, *survives the first* $(I-1)$ *belief rejections* (*after the* $\mathrm{ER}_{\mathrm{mod}\, I}(s)$ *interval*) *at least* $(\frac{1}{2}\underline{p}_s)^{I-1} \cdot \frac{1}{2}\underline{p}_s(\underline{R}_s/2\bar{c}I(I-1))$ *times in maximum epoch* s.

Since the proof of Claim C.2 is the same as that of Lemma 7, I omit it. Next, player i's AEB $\hat{g}^i$ is *under accurate testing* in an $\mathrm{AES}(\hat{\sigma})$ at time T if all effective statistical tests of player i at time T have begun after the $\mathrm{AES}(\hat{\sigma})$ has been reached. Therefore, an $\mathrm{AES}(\hat{\sigma})$ is under accurate testing at time T if and only if all players' AEBs are under accurate testing in the $\mathrm{AES}(\hat{\sigma})$ at time T. Moreover, the proof of Lemma 8 (in Appendix C.3) implies that (i) *with* μ_{σ^*}*-probability 1, from some period on, if a player's AEB is under accurate testing in* $\mathrm{AES}(\hat{\sigma})$*, her AEB is never rejected.* In addition, from the definition of statistical testing, it is obvious that (ii) *for all* i, *whenever player* i *rejects her belief, all of her effective statistical tests that began before the rejection are terminated.* From (i) and (ii), I obtain the following claim.

CLAIM C.3: *With* μ_{σ^*}*-probability* 1, *from some period on*, $\mathrm{AES}(\hat{\sigma})$ *is under accurate testing just after it survives subsequent* $(I-1)$ *belief rejections*.

PROOF: Suppose that $\mathrm{AES}(\hat{\sigma})$ survives the first $(I-1)$ belief rejections after the $\mathrm{AES}(\hat{\sigma})$ has been reached. Then consider the final step to reach the $\mathrm{AES}(\hat{\sigma})$ (e.g., see the proof of Lemma 5 in Appendix B.3): the corresponding player, for example, player i_1, rejects her (wrong) belief and forms an AEB $\hat{g}^{i_1}$ in the test and formation phases, respectively, while all the other players have already been employing their AEBs $(\hat{g}^j)_{j\neq i_1}$. Thus, $\mathrm{AES}(\hat{\sigma})$ is reached at the end of the next transition phase. Hence, from (ii), it follows that player i_1's AEB $\hat{g}^{i_1}$ is under accurate testing in the $\mathrm{AES}(\hat{\sigma})$ just after the $\mathrm{AES}(\hat{\sigma})$ is reached. This, along with (i), implies that some other player, for example, player i_2 ($\neq i_1$), must make the first belief rejection after the $\mathrm{AES}(\hat{\sigma})$ is reached. However, since the $\mathrm{AES}(\hat{\sigma})$ survives the $(I-1)$ belief rejections, player i_2 forms $\hat{g}^{i_2}$ again just after the first belief rejection. Then, from (ii), player i_2's AEB $\hat{g}^{i_2}$ is also under accurate testing in the $\mathrm{AES}(\hat{\sigma})$ after the first belief rejection, and this, along with (i), implies that the second belief rejection must be made by a player other than i_1 or i_2. I repeat this argument so that all players' AEBs $\hat{g}$ are under accurate testing in the $\mathrm{AES}(\hat{\sigma})$ after the $\mathrm{AES}(\hat{\sigma})$ survives the $(I-1)$ belief rejections; that is, the $\mathrm{AES}(\hat{\sigma})$ is under accurate testing. Clearly, this implies Claim C.3. *Q.E.D.*

C.5. *Proof of Proposition 1.B*

For any i and any positive integer L (≥ 2), let $\Delta_L^i := \{\pi_i \in \Delta(A_i) \mid \forall a_i \in A_i\ \exists l \in \mathbb{N}(\pi_i[a_i] = l/L)\}$ and $S_L^i(\pi_i) := \{\pi_i' \in \Delta(A_i) \mid \|\pi_i' - \pi_i\| \leq 2/L\}$. Note that (for all i), (c.1) $\bigcup_{\pi_i \in \Delta_L^i} S_L^i(\pi_i) = \Delta(A_i)$ and (c.2) for any subset Δ of $\Delta(A_i)$ with its diameter no larger than $1/2L$, that is, $\mathrm{diam}(\Delta) \leq 1/2L$,[21] there exists $\pi_i \in \Delta_L^i$ such that $\Delta \subseteq S_L^i(\pi_i)$.

Step 1. Let $L_\eta := \min\{L \mid 2/L \leq \eta/6\}$; for convenience, let $\eta \leq 1$. Then for all i, all $j \neq i$, all $s \geq 1$, all $q \geq 0$, all $\alpha \in \mathcal{P}_{s+q}^i$, and all $\pi_j \in \Delta_{L_\eta}^j$, define a class $\alpha(\pi_j)$ as follows: $h_T \in \alpha(\pi_j)$ if and only if (a) $h_T \in \alpha$, that is, time $T+1$ is α-active, (b) the *first* α-sampling in epoch s of player i is effective at time $T+1$,[22] and (c) for all $h_{T'} \leq h_T$ such that time $T'+1$ is α-active and the first α-sampling is effective at time $T'+1$, $\pi_j[a_j] - \frac{1}{6}\eta \leq \sigma_j^*(h_{T'})[a_j] \leq \pi_j[a_j] + \frac{1}{6}\eta$ for all a_j. (See Appendix C.3 for the α-sampling.) Moreover, let $\mathbf{d}_{j,m}^{\alpha(\pi_j)}[a_j]$ denote the number of times that a_j has been realized in the first m $\alpha(\pi_j)$-active periods, and let $\mathbf{d}_{j,m}^{\alpha(\pi_j)} := (\mathbf{d}_{j,m}^{\alpha(\pi_j)}[a_j])_{a_j}$. Let $\bar{\pi}_j[a_j] := \pi_j[a_j] + \frac{1}{6}\eta$ and $\underline{\pi}_j[a_j] := \pi_j[a_j] - \frac{1}{6}\eta$ for all a_j. Then for all i, all $j \neq i$, all s, all q, all $\alpha \in \mathcal{P}_{s+q}^i$, all $\pi_j \in \Delta_{L_\eta}^j$, and all m, define

$$\begin{aligned}
&\mathbf{D}_j^i(s, q, \alpha, \pi_j, m) \\
&\quad := \Bigg\{ h_\infty \ \Bigg|\ \mathcal{T}_m^{\alpha(\pi_j)} < \infty, \\
&\qquad \exists a_j \left(\frac{\mathbf{d}_{j,m}^{\alpha(\pi_j)}[a_j]}{m} < \underline{\pi}_j[a_j] - \frac{\eta}{6} \text{ or } \frac{\mathbf{d}_{j,m}^{\alpha(\pi_j)}[a_j]}{m} > \bar{\pi}_j[a_j] + \frac{\eta}{6} \right) \Bigg\}.
\end{aligned}$$

Recall that $\bar{\xi}^i \leq \min[\eta/3, 1/8(\#\bar{A}+1)]$ for all i. From this, the LS condition, and Proposition A, it follows that for all i, all $j \neq i$, all s, and all $\bar{q}$,

$$\begin{aligned}
&\mu_{\sigma^*}\left(\bigcup_{q \geq \bar{q}} \bigcup_{\alpha \in \mathcal{P}_{s+q}^i} \bigcup_{\pi_j \in \Delta_{L_\eta}^j} \bigcup_{m \geq \underline{m}_{s+q}^i} \mathbf{D}_j^i(s, q, \alpha, \pi_j, m) \right) \\
&\quad \leq \sum_{q \geq \bar{q}} \#\mathcal{P}_{s+q}^i \#\Delta_{L_\eta}^j \sum_{m \geq \underline{m}_{s+q}^i} (\#A_j) 2 \exp\left(-2m \left(\frac{\eta}{6} \right)^2 \right) \\
&\quad \leq 2\#\Delta_{L_\eta}^j \#A_j \sum_{q \geq \bar{q}} \#\mathcal{P}_{s+q}^i \sum_{m \geq \underline{m}_{s+q}^i} \exp\left(-\frac{1}{8} m (\bar{\xi}^i)^2 \right)
\end{aligned}$$

[21]The diameter of Δ is defined as $\mathrm{diam}(\Delta) := \sup\{\|\pi - \pi'\| \mid \pi, \pi' \in \Delta\}$.
[22]That is, $d = 0$.

$$\leq 2\#\Delta^j_{L_\eta}\#A_j \sum_{q\geq\bar{q}} \exp(-s-q)$$

$$= 2\#\Delta^j_{L_\eta}\#A_j\big(1-\exp(-1)\big)^{-1}\exp(-s-\bar{q}).$$

Therefore, $\mu_{\sigma^*}(\bigcap_{\bar{q}\geq 0}\bigcup_{q\geq\bar{q}}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{\pi_j\in\Delta^j_{L_\eta}}\bigcup_{m\geq\underline{m}^i_{s+q}}\mathbf{D}^i_j(s,q,\alpha,\pi_j,m)) = 0$ for all i, all $j\neq i$, and all s. Thus,

$$\mu_{\sigma^*}\Bigg(\bigcup_i\bigcup_{j\neq i}\bigcup_{s\geq 1}\bigcap_{\bar{q}\geq 0}\bigcup_{q\geq\bar{q}}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{\pi_j\in\Delta^j_{L_\eta}}\bigcup_{m\geq\underline{m}^i_{s+q}}\mathbf{D}^i_j(s,q,\alpha,\pi_j,m)\Bigg)=0.$$

Step 2. Let $\mathbf{U} := \{h_\infty \mid$ there are at most a finite number of belief rejections in $h_\infty\}$. I say that ρ_* are η-different from σ^* infinitely many times in h_∞ if for infinitely many h ($< h_\infty$), there exist i and j ($\neq i$) such that $\|\rho^i_{*,j}(h) - \sigma^*_j(h)\| > \eta$. Let $\mathbf{V} := \{h_\infty \mid \rho_*$ are η-different from σ^* infinitely many times in $h_\infty\}$. Then I obtain that

$$\mathbf{U}\cap\mathbf{V}\subseteq\bigcup_i\bigcup_{j\neq i}\bigcup_{s\geq 1}\bigcup_{\bar{q}\geq 0}\bigcap_{q\geq\bar{q}}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{\pi_j\in\Delta^j_{L_\eta}}\bigcup_{m\geq\underline{m}^i_{s+q}}\mathbf{D}^i_j(s,q,\alpha,\pi_j,m)$$

$$\subseteq\bigcup_i\bigcup_{j\neq i}\bigcup_{s\geq 1}\bigcap_{\bar{q}\geq 0}\bigcup_{q\geq\bar{q}}\bigcup_{\alpha\in\mathcal{P}^i_{s+q}}\bigcup_{\pi_j\in\Delta^j_{L_\eta}}\bigcup_{m\geq\underline{m}^i_{s+q}}\mathbf{D}^i_j(s,q,\alpha,\pi_j,m).$$

The second inclusion is obvious. I show the first inclusion from Step 3 to Step 6.

Step 3. Suppose that $h_\infty\in\mathbf{U}\cap\mathbf{V}$. On the one hand, since $h_\infty\in\mathbf{U}$, there exists $\bar{T}_0$ such that (i) no belief rejection occurs from time $\bar{T}_0$, (ii) there exist beliefs $(\bar{f}^i)_i$ such that each player i retains $\bar{f}^i$ forever from time $\bar{T}_0$, and (iii) there exist $(s^i_0)_i$ such that each player i stays in epoch s^i_0 forever from time $\bar{T}_0$. Hence, for all i, $\bar{f}^i$ is generated by $\mathcal{P}^i_{s^i_0}$, and $\rho^i_*(h_T)=\bar{f}^i(h_T)$ for all $T\geq\bar{T}_0$. On the other hand, since $h_\infty\in\mathbf{V}$, there exist i_0 and j_0 ($\neq i_0$) such that $\|\bar{f}^{i_0}_{j_0}(h_{T_k})-\sigma^*_{j_0}(h_{T_k})\| = \|\rho^{i_0}_{*,j_0}(h_{T_k})-\sigma^*_{j_0}(h_{T_k})\|>\eta$ for infinitely many $h_{T_k}<h_\infty$.

Step 4. From Step 3, (P.1), (P.2), footnote 29 (in Section 4.1.2), and Lemma 1, it follows that for all i, all $j\neq i$, and all s, there exists $\hat{q}(i,j,s)$ such that for all $q\geq\hat{q}(i,j,s)$, (c.3) $\mathcal{P}^i_{s+q}$ is a CR of $\bar{f}^i$ and (c.4) $\mathcal{P}^i_{s+q}$ is an $\eta/96$-ACR of $\sigma^{\bar{f}}_j$: for all $\alpha\in\mathcal{P}^i_{s+q}$ and all $h,h'\in\alpha$, $\bar{f}^i(h)=\bar{f}^i(h')$ and $\|\sigma^{\bar{f}}_j(h)-\sigma^{\bar{f}}_j(h')\|\leq\eta/96$; hence, $\bar{f}^i(\alpha)$ is well defined for all $\alpha\in\mathcal{P}^i_{s+q}$. Moreover, since $\lim_{m\to\infty}K^i(m)=\infty$ for all i, it follows from Lemma 2 that, for all i, $\|\sigma^{\bar{f}}_i(h_T)-\sigma^*_i(h_T)\|\to 0$ as $T\to\infty$. Thus, there exists $\bar{T}_1$ ($\geq\bar{T}_0$) such that (c.5) for all i, all $j\neq i$, all s, all $q\geq\hat{q}(i,j,s)$, all $\alpha\in\mathcal{P}^i_{s+q}$, and all $h_T,h'_{T'}\in H$, if $h_T,h'_{T'}\in\alpha$, $h_T,h'_{T'}<h_\infty$, and

$T, T' \geq \bar{T}_1$, then $\|\sigma_j^*(h_T) - \sigma_j^*(h'_{T'})\| \leq \eta/48$: for each $\alpha \in \mathcal{P}^i_{s+q}$, $\mathrm{diam}(\{\sigma_j^*(h_T) \mid h_T \in \alpha, h_T < h_\infty, T \geq \bar{T}_1\}) \leq \eta/48 \leq 1/2L_\eta$. Therefore, by (c.2) and (c.5), for all i, all $j \neq i$, all s, all $q \geq \hat{q}(i, j, s)$, and all $\alpha \in \mathcal{P}^i_{s+q}$, there exists $\pi_j \in \Delta^j_{L_\eta}$ such that $\{\sigma_j^*(h_T) \mid h_T \in \alpha, h_T < h_\infty, T \geq \bar{T}_1\} \subseteq S^j_{L_\eta}(\pi_j)$.

Step 5. In addition, since no belief rejection occurs from time $\bar{T}_0$ and $\{\mathcal{P}^{i_0}_{s_0^{i_0}+q}\}_q$ is a set of *finite* partitions, it is clear that there exists $\bar{q} \geq \hat{q}(i_0, j_0, s_0^{i_0})$ such that for all $q \geq \bar{q}$, there exists $\alpha_q \in \mathcal{P}^{i_0}_{s_0^{i_0}+q}$ such that (c.6) the first α_q-sampling begins after time $\bar{T}_1$[23] and (c.7) $h_{T_{k_n}} \in \alpha_q$ for all n, where $\{h_{T_{k_n}}\}_n$ is an infinite subsequence of $\{h_{T_k}\}_k$. Furthermore, from Step 3, (c.3), (c.5), and (c.7), it follows that (c.8) $\|\bar{f}^{i_0}_{j_0}(h_T) - \sigma^*_{j_0}(h_T)\| > \eta - \frac{1}{48}\eta = \frac{47}{48}\eta$ for all $h_T < h_\infty$ such that $T \geq \bar{T}_1$ and $h_T \in \alpha_q$.

Step 6. From Steps 4 and 5, it is derived that for all $q \geq \bar{q}$, there exists $\pi^q_{j_0} \in \Delta^{j_0}_{L_\eta}$ such that (c.9) $\{\sigma^*_{j_0}(h_T) \mid h_T \in \alpha_q, h_T < h_\infty, T \geq \bar{T}_1\} \subseteq S^{j_0}_{L_\eta}(\pi^q_{j_0})$. From (c.6), (c.8), and the definition of $\alpha_q(\pi^q_{j_0})$, it then follows that (c.10) $\|\bar{f}^{i_0}_{j_0}(h) - \sigma^*_{j_0}(h)\| > \frac{47}{48}\eta$ for all $h < h_\infty$ such that $h \in \alpha_q(\pi^q_{j_0})$. Clearly, (c.6) and (c.9) imply that (c.11) in any α_q-active period in which the first α_q-sampling is effective, $\pi^q_{j_0}[a_{j_0}] - \frac{1}{6}\eta \leq \sigma^*_{j_0}(h)[a_{j_0}] \leq \pi^q_{j_0}[a_{j_0}] + \frac{1}{6}\eta$ for all a_{j_0}. Furthermore, from (c.6), (c.7), (c.11), and no belief rejection from time $\bar{T}_0$, it follows that $\#\{h \mid h \in \alpha_q(\pi^q_{j_0}), h < h_\infty\} = m^{\alpha_q} \geq \underline{m}^{i_0}_{s_0^{i_0}+q}$, where m^{α_q} is the number of the samples obtained for α_q in h_∞. This implies that α_q obtains enough samples but does not reject $\bar{f}^{i_0}$ in h_∞ and that $D^{i_0}_{j_0}(\alpha_q) = \mathbf{d}^{\alpha_q(\pi^q_{j_0})}_{j_0, m^{\alpha_q}}/m^{\alpha_q}$. Therefore, $h_\infty \in \mathbf{D}^{i_0}_{j_0}(s_0^{i_0}, q, \alpha_q, \pi^q_{j_0}, m^{\alpha_q})$. Indeed, if not, then $\|\mathbf{d}^{\alpha_q(\pi^q_{j_0})}_{j_0, m^{\alpha_q}}/m^{\alpha_q} - \pi^q_{j_0}\| \leq \eta/3$. However, then α_q does not reject $\bar{f}^{i_0}$, which means that $\|\bar{f}^{i_0}_{j_0}(h) - \mathbf{d}^{\alpha_q(\pi^q_{j_0})}_{j_0, m^{\alpha_q}}/m^{\alpha_q}\| = \|\bar{f}^{i_0}_{j_0}(\alpha_q) - D^{i_0}_{j_0}(\alpha_q)\| \leq \bar{\xi}^{i_0} \leq \eta/3$ for all $h \in \alpha_q$. Furthermore, by the definition of $\alpha_q(\pi^q_{j_0})$, for all $h \in \alpha_q(\pi^q_{j_0})$, $\pi^q_{j_0}[a_{j_0}] - \frac{1}{6}\eta \leq \sigma^*_{j_0}(h)[a_{j_0}] \leq \pi^q_{j_0}[a_{j_0}] + \frac{1}{6}\eta$ for all a_{j_0}. Therefore, $\|\bar{f}^{i_0}_{j_0}(h) - \sigma^*_{j_0}(h)\| \leq \|\bar{f}^{i_0}_{j_0}(h) - \mathbf{d}^{\alpha_q(\pi^q_{j_0})}_{j_0, m^{\alpha_q}}/m^{\alpha_q}\| + \|\mathbf{d}^{\alpha_q(\pi^q_{j_0})}_{j_0, m^{\alpha_q}}/m^{\alpha_q} - \pi^q_{j_0}\| + \|\pi^q_{j_0} - \sigma^*_{j_0}(h)\| \leq \frac{1}{3}\eta + \frac{1}{3}\eta + \frac{1}{6}\eta = \frac{5}{6}\eta = \frac{40}{48}\eta$ for all $h < h_\infty$ such that $h \in \alpha_q(\pi^q_{j_0})$. This is a contradiction to (c.10).

Step 7. Finally, from all the above steps, it follows that $\mu_{\sigma^*}(\mathbf{U} \cap \mathbf{V}) = 0$. Since $\mu_{\sigma^*}(\mathbf{U}) = 1$ by Proposition 1.A, $\mu_{\sigma^*}(\mathbf{U} \cap \mathbf{V}^c) = 1$, where $\mathbf{V}^c$ is the complement

[23]Since no belief rejection occurs from time $\bar{T}_0$, for any large q, $\mathcal{P}^{i_0}_{s_0^{i_0}+q}$ is used only once in the test phases: $d = 0$. This means that for each $\alpha \in \mathcal{P}^{i_0}_{s_0^{i_0}+q}$, the first α-sampling starts from the beginning of the only test phase of using $\mathcal{P}^{i_0}_{s_0^{i_0}+q}$.

of $\mathbf{V}$. Therefore, $\mu_{\sigma^*}(\mathbf{V}^c) = 1$. Moreover, for each $h_\infty \in \mathbf{V}^c$, there exists $\bar{T}$ such that for all i and all $j \neq i$, $\|\rho^i_{*,j}(h_T) - \sigma^*_j(h_T)\| \leq \eta$ for all $T \geq \bar{T}$.

APPENDIX D

I prepare one lemma to derive Theorem 1 from Proposition 1.

LEMMA D: *For any i, any σ_i, and any $\sigma_{-i}, \sigma'_{-i}$,*

$$\left|\bar{V}_i^v(\sigma_{-i}) - \bar{V}_i^v(\sigma'_{-i})\right|, \left|V_i^v(\sigma_i, \sigma_{-i}) - V_i^v(\sigma_i, \sigma'_{-i})\right|$$
$$\leq D_i \sum_{T=1}^{\infty} \delta_i^{T-1} \max_{h \in H_{T-1}} \max_{j \neq i} \left\|\sigma_j(h) - \sigma'_j(h)\right\|,$$

where $D_i := \# A_{-i}(I-1)(U_i + r_i)$ and $A_{-i} := \prod_{j \neq i} A_j$.

PROOF: Lemma D is immediately obtained from the recursive structure. *Q.E.D.*

Now, take any $\varepsilon > 0$ and any $0 \leq \bar{\delta} < 1$.[24] Then define $\hat{\eta} := \varepsilon(1-\bar{\delta})/6\# A(I-1)$.[25] For $\hat{\eta}$, take prior beliefs ρ_* from Proposition 1. Next, fix any (u, δ) such that $0 \leq \delta_i \leq \bar{\delta}$ for all i. Then take $\hat{r}(\hat{\eta}, u, \delta)$ from Proposition 1 and define $\bar{r}(\varepsilon, \bar{\delta}, u, \delta) := \min[1, \hat{r}(\hat{\eta}, u, \delta)]$. Finally, fix any r such that $0 < r_i \leq \bar{r}(\varepsilon, \bar{\delta}, u, \delta)$ for all i. Then it is sufficient to verify that $\sigma^*_{h_T}$ satisfies the definition of ε-NE for any large T. Take any $\hat{T}$ such that $2D_i\delta_i^{\hat{T}}/(1-\delta_i) \leq \varepsilon/3$ for all i. From Proposition 1, each player i eventually makes $\hat{\eta}$-accurate predictions regarding all her opponents' actions in the next $\hat{T}$ periods. From these and Lemma D, it follows that for any large T and any i,

$$\left|V_i^v(\sigma^*_{i,h_T}, \sigma^*_{-i,h_T}) - \bar{V}_i^v(\sigma^*_{-i,h_T})\right|$$
$$\leq \left|V_i^v(\sigma^*_{i,h_T}, \sigma^*_{-i,h_T}) - V_i^v(\sigma^*_{i,h_T}, \rho^i_{*,h_T})\right| + \left|\bar{V}_i^v(\rho^i_{*,h_T}) - \bar{V}_i^v(\sigma^*_{-i,h_T})\right|$$
$$\leq 2D_i \sum_{t=1}^{\infty} \delta_i^{t-1} \max_{h \in H_{t-1}} \max_{j \neq i} \left\|\sigma^*_{j,h_T}(h) - \rho^i_{*j,h_T}(h)\right\|$$
$$\leq 2D_i \left[\frac{1-\delta_i^{\hat{T}}}{1-\delta_i}\hat{\eta} + \frac{\delta_i^{\hat{T}}}{1-\delta_i}\right]$$

[24] I need to impose the upper bound $\bar{\delta}$ on the discount factors because even if I make arbitrarily small η-errors of predictions of ρ^i_* at each period, these small errors may accumulate over time so that if the discount factor δ_i is taken to be sufficiently close to 1, player i's payoff $V_i^v(\sigma^*_{h_T})$ may be bounded away from the maximum payoff $\bar{V}_i^v(\sigma^*_{-i,h_T})$. Therefore, $\sigma^*_{h_T}$ may not satisfy the definition of ANE.

[25] Note that $\hat{\eta} \leq \varepsilon(1-\bar{\delta})/3\# A_{-i}(I-1)(U_i+1)$ for all i because $U_i \leq 1$ for all i.

$$\leq \left(1 - \delta_i^{\hat{T}}\right)\frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon$$
$$\leq \varepsilon.$$

Therefore, $\sigma^*_{h_T}$ satisfies the definition of ε-NE for any large T.

REFERENCE

NOGUCHI, Y. (2015): "Merging With a Set of Probability Measures: A Characterization," *Theoretical Economics* (forthcoming). [2]

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