SUPPLEMENT TO "MECHANISM DESIGN WITH AMBIGUOUS COMMUNICATION DEVICES"

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This supplement discusses in detail a number of extensions and issues raised in the manuscript "Mechanism Design With Ambiguous Communication Devices."

S.1. decision-theoretic issues

This section revisits the introductory example (Section 2) and provides an in-depth discussion of decision-theoretic issues.

Consider an urn containing 90 balls. Each ball is marked with either (θ, ω), (θ, ω'), (θ', ω), or (θ', ω'). There are 60 balls marked with θ and 30 balls marked with θ'. Moreover, there are only two possible compositions of the urn. With the first composition, all balls marked with θ (resp., θ') are also marked with ω (resp., ω'). With the second, all balls marked with θ (resp., θ') are also marked with ω' (resp., ω).

A ball is drawn from the urn at random. The decision maker is offered two bets, A and B. The bet A gives x if the ball is marked with θ and y if the ball is marked with θ', while the bet B gives y if the ball is marked with θ and x if the ball is marked with θ'. Prior to choosing a bet, the decision maker can observe whether the ball is marked with ω or ω'. The decision problem is represented in Figure S.1; the first (resp., second) line corresponds to prizes in state θ (resp., θ').

The decision maker is player 1 of type θ in our mechanism design problem. The state space represents the possible types of player 2 and messages player 1 can receive from the communication device as constructed in the main text. Moreover, the possible composition of the urn respects the prior belief of player 1 as well as the ambiguity in the communication device. Finally, conditional on player 1 of type θ expecting player 2 to tell the truth, the bet A corresponds to player 1 telling the truth, while the bet B corresponds to lying. Formally, the state space is $\{\theta, \theta'\} \times \{\omega, \omega'\}$ and the set of prior beliefs of the decision maker is $\{(2/3, 0, 0, 1/3), (0, 2/3, 1/3, 0)\}$. We maintain the assumption of multiple prior preferences and assume prior-by-prior updating (full Bayesian updating).

Clearly, upon learning whether the ball is marked with ω or ω', the decision maker (weakly) prefers A over B.[1]

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[1]A slight modification of the example would give strict preferences. For instance, assume that the first composition of the urn has 60 balls marked with (θ, ω), 15 balls with (θ', ω), and 15 balls
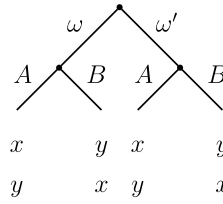
FIGURE S.1.—The decision problem.

Consider now the ex ante plans AA, AB, BA, and BB, where the first (resp., second) letter corresponds to the choice of bet conditional on ω (resp., ω'). For instance, the plan AB prescribes the choice of A if ω is revealed and the choice of B if ω' is revealed. Assume that the decision maker evaluates the plan AB by "reducing" it to the bet giving x if the ball is (θ, ω) or (θ', ω') and y if the ball is (θ', ω) or (θ, ω'). Similarly, for the other plans.

We have that the decision maker strictly prefers the plan BB to the plan AA, the plan AA to the plans AB and BA, and is indifferent between the plans AB and BA.

To sum up, we have that conditional on either ω or ω', the decision maker strictly prefers A to B, but ex ante, he strictly prefers the plan BB to AA. The decision maker's preferences are *dynamically inconsistent* and our construction precisely exploits this fact.

We briefly comment on this fundamental aspect of our analysis and refer the reader to the special issue of *Economics and Philosophy* (2009) for an in-depth discussion and further references. Dynamic consistency and Bayesian updating are intimately related to Savage's sure-thing principle, and ambiguity-sensitive preferences generally entail a violation of the sure-thing principle. Consequently, if one wants to analyze ambiguity-sensitive preferences, then either dynamic consistency or full Bayesian updating must be relaxed, at least to some extent. The approach we follow in this paper is to relax the assumption of dynamic consistency. To analyze dynamic games with dynamically inconsistent preferences, we assume that players are *consistent planners*, that is, at every information set a player is active, he chooses the best strategy given his opponents' strategies and the strategies he will actually follow (for more on consistent planning, see Strotz (1955) and Siniscalchi (2011)).

An alternative approach would be to maintain a form of dynamic consistency and to relax the assumption of full Bayesian updating. Hanany and Klibanoff (2007) provided such an alternative for the multiple-prior preferences. Without entering into details, their approach would require to update the prior

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with (θ', ω'), while the second composition has 60 balls marked with (θ, ω'), 15 balls with (θ', ω), and 15 balls with (θ', ω'). Conditional on either ω or ω', the set of posteriors is $\{(0, 1), (4/5, 1/5)\}$ and, thus, the decision maker strictly prefers A to B.

$(2/3, 0, 0, 1/3)$ upon observing ω and the prior $(0, 2/3, 1/3, 0)$ upon observing ω', so that the plan BB remains conditionally optimal.[2] Thus, according to their updating rule, the set of priors to be updated depends on the conditioning events (and, more generally, on the set of feasible plans and the unconditionally optimal plan considered). Whether one likes this feature or not, this is a logical implication of relaxing consequentialism so as to maintain dynamic consistency. We refer the reader to Siniscalchi (2009, 2011) and Al-Najjar and Weinstein (2009) for more on this issue. Furthermore, we hasten to stress that a violation of dynamic consistency as defined in Hanany and Klibanoff (2007, axiom **DC**, p. 268) is *not* a necessary condition for our results to hold. Indeed, we can modify the example so that the decision maker strictly prefers BB to AA and, conditional on either ω or ω', is indifferent between A and B. This does not violate axiom **DC** of Hanany and Klibanoff and yet f remains implementable by an ambiguous mechanism as constructed above (an example showing this is available upon request).

Yet another alternative approach is to maintain consequentialism and (a form of) dynamic consistency, but to limit the possible attitude toward ambiguity. For instance, Epstein and Schneider (2003) provided a condition on the set of priors, called rectangularity, that guarantees the absence of preference reversals. In our example, their approach would require the set of priors to be

$$\{(2/3, 0, 0, 1/3), (0, 2/3, 1/3, 0), (1/3, 0, 0, 2/3), (0, 1/3, 2/3, 0)\}.[3]$$

Importantly to us, regardless of the strengths and weaknesses of those approaches, ambiguous mechanisms can implement social choice functions that are not incentive compatible with respect to prior beliefs only if (a form of) dynamic inconsistency is assumed.

S.2. MIXED STRATEGIES

For simplicity, the paper has focused on pure strategies. This is not without loss of generality. Indeed, players with multiple-prior preferences can be indifferent between two pure strategies and yet strictly prefer a mixture of the two pure strategies over either one of them; this follows from the axiom of uncertainty aversion.

We now explain how to extend the analysis to mixed strategies.[4] A mixed strategy is a mapping $\sigma_i : (\{\emptyset\} \cup H_i) \times \Theta_i \to \Delta(\widehat{\Omega}_i) \times \Delta(M_i)$ such that

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[2]We assume that the feasible set of plans is AA, AB, BA, and BB.

[3]With this set of priors, the social choice function f is implementable by a classical (unambiguous) direct mechanism. Alternatively, this follows from Epstein and Schneider's definition of dynamic consistency, which says that if A is conditionally preferred to B, conditional on both events ω and ω', then AA is unconditionally preferred to BB.

[4]More accurately, we consider behavioral strategies. We adopt the terminology of mixed strategies to simplify the exposition.

$\sigma_i(\{\emptyset\}, \theta_i) \in \Delta(\widehat{\Omega}_i)$ and $\sigma_i(h_i, \theta_i) \in \Delta(M_i)$ for all θ_i, for all h_i. It is straightforward to modify the definition of a consistent planning equilibrium to allow for mixed strategies.

The history $(\widehat{\omega}_i, \omega_i)$ has *positive probability* under $(\sigma_i^*, \sigma_{-i}^*)$, if there exist a type θ_i', a probability system λ', and a prior p_i' such that $\sigma_i^*(\{\emptyset\}, \theta_i)[\widehat{\omega}_i] > 0$ and

$$\sum_{\theta_{-i}, \widehat{\omega}_{-i}, \omega_{-i}} \lambda'(\widehat{\omega}_i, \widehat{\omega}_{-i})\big[(\omega_i, \omega_{-i})\big] \sigma_{-i}^*(\{\emptyset\}, \theta_{-i})[\widehat{\omega}_{-i}] p_i'[\theta_{-i}] > 0.$$

The history $(h_i)_i \in \bigtimes_{i \in N} H_i$ has positive probability if each h_i has positive probability. The definition of implementation becomes the following:

DEFINITION S.1: The ambiguous mechanism $\langle\langle (\widehat{\Omega}_i, \Omega_i)_{i \in N}, \Lambda\rangle, \langle (M_i)_{i \in N}, g\rangle\rangle$ (partially) implements the social choice function f if there exists an equilibrium $(\sigma^*, \Pi^{H, \Theta})$ such that

$$g(m_i, m_{-i}) = f(\theta_i, \theta_{-i})$$

for all (m_i, m_{-i}) such that $\sigma_i^*(\theta_i, h_i)[m_i] \sigma_{-i}^*(\theta_{-i}, h_{-i})[m_{-i}] > 0$, for all (θ_i, θ_{-i}), for all (h_i, h_{-i}) with positive probability.

Lastly, the definition of incentive compatibility is extended as follows:

DEFINITION S.2: A social choice function f is incentive compatible for player $i \in N$ if there exists a nonempty set of beliefs $\Pi_i \subseteq \Delta(\Theta_{-i})$ such that, for all $\theta_i \in \Theta_i$,

$$\min_{\pi_i \in \Pi_i} \sum_{\theta_{-i} \in \Theta_{-i}} u_i\big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}\big) \pi_i[\theta_{-i}]$$

$$\geq \min_{\pi_i \in \Pi_i} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\theta_i' \in \Theta_i} \sigma_i(\theta_i)[\theta_i'] u_i\big(f(\theta_i', \theta_{-i}), \theta_i, \theta_{-i}\big) \pi_i[\theta_{-i}]$$

for all $\sigma_i(\theta_i) \in \Delta(\Theta_i)$. The social choice function f is incentive compatible if it is incentive compatible for each player $i \in N$.

All our analysis remains valid with these modifications.

To see this, let us revisit the proof of Theorem 1. Clearly, the sufficiency part remains valid without any modification. As for the necessary part, let us consider a history (h_i, h_{-i}) with positive probability under $(\sigma_i^*, \sigma_{-i}^*)$, where $(\sigma_i^*, \sigma_{-i}^*)$ is the equilibrium implementing f.

Consider player i of type θ_i. By definition of an equilibrium, we have that

$$
\min_{\pi_i(h_i, \theta_i) \in \Pi_i^{H, \Theta}(h_i, \theta_i)} \sum_{\theta_{-i}, h_{-i}, m_{-i}} \sigma_i^*(h_i, \theta_i)[m_i] \sigma_{-i}^*(h_{-i}, \theta_{-i})[m_{-i}]
$$
$$
\times u_i\big(g(m_i, m_{-i}), \theta_i, \theta_{-i}\big) \pi_i(h_i, \theta_i)\big[(h_{-i}, \theta_{-i})\big]
$$
$$
\geq \min_{\pi_i(h_i, \theta_i) \in \Pi_i^{H, \Theta}(h_i, \theta_i)} \sum_{\theta_{-i}, h_{-i}, m_{-i}} \sigma_i(h_i, \theta_i)[m_i] \sigma_{-i}^*(h_{-i}, \theta_{-i})[m_{-i}]
$$
$$
\times u_i\big(g(m_i, m_{-i}), \theta_i, \theta_{-i}\big) \pi_i(h_i, \theta_i)\big[(h_{-i}, \theta_{-i})\big]
$$

for all $\sigma_i(h_i, \theta_i) \in \Delta(M_i)$. In particular, this is true for all strategies $\bar{\sigma}_i(h_i, \theta_i)$ constructed as follows. Fix any $\sigma_i(\theta_i) \in \Delta(\Theta_i)$ and define $\bar{\sigma}_i(h_i, \theta_i) \in \Delta(M_i)$ by $\bar{\sigma}_i(h_i, \theta_i)[m_i] = \sum_{\theta_i'} \sigma_i(\theta_i)[\theta_i'] \sigma_i^*(h_i, \theta_i')[m_i]$, for all m_i. In words, under $\bar{\sigma}_i(h_i, \theta_i)$, at the history h_i, player i of type θ_i draws a fictitious type θ_i' with probability $\sigma_i(\theta_i)[\theta_i']$ and reports a message m_i as if his true type was θ_i'.

As in the main text, we have that if $\pi_i(h_i, \theta_i)[(h_{-i}, \theta_{-i})] > 0$, then h_{-i} has positive probability. Since the definition of implementation requires that $g(m_i, m_{-i}) = f(\theta_i, \theta_{-i})$ for all (m_i, m_{-i}) such that $\sigma_i^*(h_i, \theta_i)[m_i] \sigma_{-i}^*(h_{-i}, \theta_{-i})[m_{-i}] > 0$, for all (θ_i, θ_{-i}), for all histories (h_i, h_{-i}) with positive probability, we must therefore have that

$$
\min_{\pi_i(h_i, \theta_i) \in \Pi_i^{H, \Theta}(h_i, \theta_i)} \sum_{\theta_{-i}, h_{-i}} u_i\big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}\big) \pi_i(h_i, \theta_i)\big[(h_{-i}, \theta_{-i})\big]
$$
$$
\geq \min_{\pi_i(h_i, \theta_i) \in \Pi_i^{H, \Theta}(h_i, \theta_i)} \sum_{\theta_{-i}, h_{-i}, \theta_i'} \sigma_i(\theta_i)\big[\theta_i'\big] u_i\big(f(\theta_i', \theta_{-i}), \theta_i, \theta_{-i}\big)
$$
$$
\times \pi_i(h_i, \theta_i)\big[(h_{-i}, \theta_{-i})\big]
$$

for all $\sigma_i(\theta_i) \in \Delta(\Theta_i)$, where we have used the strategy $\bar{\sigma}_i(h_i, \theta_i) \in \Delta(M_i)$.

The rest of the proof is as in the main text. Theorem 2 is independent of whether mixed strategies are considered or not and, thus, remains valid.

A closing remark is in order. Notice that the social choice function f in the introductory example of the paper is not implementable when mixed strategies are considered. It is profitable for a player to randomize uniformly between θ and θ'. The profitability from mixed deviations depends on the class of mechanisms considered, however. Indeed, we show in Section S.5 that we can implement f with the use of an *extensive-form allocation mechanism*, which "nullifies" the benefit from mixed strategy deviations. Whether such general mechanisms can always be used to nullify the benefit from mixed strategy deviation remains an important open question for future research.

S.3. MULTIPLE ROUNDS OF COMMUNICATION

Another feature of the paper is the restriction to a single round of communication. We now consider an extension of ambiguous mechanisms to allow for multiple rounds of communication. The main result is that this does not affect the set of implementable social choice functions.

Given an allocation mechanism $\langle M, g \rangle$, we define the mediated extension of $\langle M, g \rangle$ as a mechanism in which $T < \infty$ stages of mediated communication are allowed before $\langle M, g \rangle$ is played. More precisely, there are $T+1$ stages. At stage $t \in \{1, \ldots, T\}$, players communicate through the ambiguous communication device $\langle (\widehat{\Omega}_{i,t}, \Omega_{i,t})_{i \in N}, \Lambda_t \rangle$, that is, each player i sends a message $\widehat{\omega}_{i,t} \in \widehat{\Omega}_{i,t}$ to, and receives a message $\omega_{i,t} \in \Omega_{i,t}$ from, the device. At stage $T + 1$, which is the allocation stage, players send a message $m_i \in M_i$ and the designer implements an alternative according to g. As in the paper, (i) communication is private and simultaneous, and (ii) the alternative implemented at the allocation stage depends only on the messages reported at that stage. The only difference is that communication now involves multiple rounds. We call such a mechanism a (multiple-round-communication) *M-R-C ambiguous mechanism*.

PROPOSITION S.1: *If the social choice function f is implementable by a M-R-C ambiguous mechanism $\langle \langle (\widehat{\Omega}_{i,t}, \Omega_{i,t})_{i \in N}, \Lambda_t \rangle_{t=1,\ldots,T}, \langle (M_i)_{i \in N}, g \rangle \rangle$, then f is implementable by a two-stage ambiguous mechanism, $\langle \langle (\widehat{\Omega}_i, \Omega_i)_{i \in N} \rangle, \langle (M_i)_{i \in N}, g \rangle \rangle$.*

PROOF: Suppose that the social choice function f is implementable by the M-R-C ambiguous mechanism $\langle \langle (\widehat{\Omega}_{i,t}, \Omega_{i,t})_{i \in N}, \Lambda_t \rangle_{t=1,\ldots,T}, \langle (M_i)_{i \in N}, g \rangle \rangle$, and let $(s^*, \Pi^{H,\Theta})$ be an equilibrium implementing f.

Consider the two-stage mechanism:

$$\left\langle \left\langle \left(\Theta_i, \bigtimes_t (\widehat{\Omega}_{i,t} \times \Omega_{i,t}) \right)_{i \in N}, \Lambda \right\rangle, \langle (M_i)_{i \in N}, g \rangle \right\rangle,$$

with Λ constructed as follows. For each $(\lambda_1, \ldots, \lambda_T) \in \Lambda_1 \times \cdots \times \Lambda_T$, we associate the communication system λ defined by

$$\lambda(\theta)\big[(\widehat{\omega}_t, \omega_t)_{t=1}^T\big] := s^*(\emptyset, \theta)[\widehat{\omega}_1]\lambda_1(\widehat{\omega}_1)[\omega_1] \times \cdots$$
$$\times s^*\big((\widehat{\omega}_t, \omega_t)_{t=1}^{\tau-1}, \theta\big)[\widehat{\omega}_\tau]\lambda_\tau(\widehat{\omega}_\tau)[\omega_\tau] \times \cdots$$
$$\times s^*\big((\widehat{\omega}_t, \omega_t)_{t=1}^{T-1}, \theta\big)[\widehat{\omega}_T]\lambda_T(\widehat{\omega}_T)[\omega_T].$$

In words, during the communication stage of the two-stage mechanism, the set of messages sent is Θ_i, the set of messages received is $\bigtimes_t (\widehat{\Omega}_{i,t} \times \Omega_{i,t})$ (i.e., all possible sequences of messages sent and received under the M-R-C mechanism), and the probability of receiving the profile of messages $(\widehat{\omega}_t, \omega_t)_{t=1}^T$ conditional on the type profile θ being the message sent is the probability of the

history $(\widehat{\omega}_t, \omega_t)_{t=1}^T$ under s^* when the type profile is θ. Let H^{**} be the set of histories of the two-stage mechanism.

Let s^{**} be the strategy profile defined by $s_i^{**}(\{\emptyset\}, \theta_i) = \theta_i$ at the initial history $\{\emptyset\}$, and $s_i^{**}((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i) = s_i^*((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i)$, for all histories $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$, for all types θ_i, for all player i. Note that $(\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T$ corresponds to a history h_i^{T+1} of the original M-R-C mechanism. We now argue that since the equilibrium $(s^*, \Pi^{H,\Theta})$ implements f, there exist assessments $\Pi^{H^{**},\Theta}$ such that $(s^{**}, \Pi^{H^{**},\Theta})$ implements f in the two-stage mechanism.

First, by definition, the history $h_i^{T+1} = (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T$ has positive probability under s^* in the M-R-C mechanism if there exist a type profile $(\theta_i^*, \theta_{-i}^*)$ and a collection of probability systems $(\lambda_t)_t$ such that

$$\sum_{(\widehat{\omega}_{-i,t}, \omega_{-i,t})_t} \lambda(\theta_i^*, \theta_{-i}^*)\big[\big((\widehat{\omega}_{i,t}, \widehat{\omega}_{-i,t}), (\omega_{i,t}, \omega_{-i,t})\big)_{t=1}^T\big] > 0.$$

It follows that the history $(\theta_i^*, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ has positive probability under s^{**}. Conversely, if $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ has positive probability under s^{**}, then the history $h_i^{T+1} = (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T$ has positive probability under s^* in the M-R-C mechanism.

Second, we derive the beliefs of player i of type θ_i at histories with positive probabilities. (At histories with zero probability, the beliefs are arbitrary.) So, suppose that the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ has positive probability under s^{**} and consider the event $[((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T)), \theta_{-i}]$. (Remember that player i has beliefs about the private histories *and* types of his opponents.) By construction of s^{**}, it must be that

$$\pi_i\big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i\big)\big[\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T)\big), \theta_{-i}\big] = 0,$$

if $\widehat{\theta}_{-i} \neq \theta_{-i}$ for all beliefs $\pi_i((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i)$ at the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$. Similarly, by construction of s^{**}, the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ has positive probability if and only if player i's type is $\widehat{\theta}_i$. Therefore, we have that the beliefs of player i of type $\widehat{\theta}_i$ at the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ are equal to $\Pi_i^{H,\Theta}((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \widehat{\theta}_i)$, the beliefs player i of type $\widehat{\theta}_i$ has in the M-R-C mechanism at the history $(\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T$ under s^*, that is,

$$\pi_i\big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \widehat{\theta}_i\big)\big[\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T)\big), \widehat{\theta}_{-i}\big]$$
$$= \pi_i\big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T, \widehat{\theta}_i\big)\big[(\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T, \widehat{\theta}_{-i}\big]$$

for all $\widehat{\theta}$, for all $(\widehat{\omega}_t, \omega_t)_t$, for all π_i. Alternatively, if player i's type θ_i is different from $\widehat{\theta}_i$, then the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ has still positive probability under s^{**} according to our definition, but Bayes's rule does not apply for player i of type θ_i. Thus, we can define the beliefs of player i of type θ_i at the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ arbitrarily. Assume that they are $\Pi_i^{H,\Theta}((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T, \theta_i)$.

Third, since for every history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ corresponds a history h_i^{T+1} in the M-R-C mechanism, the strategy $s_i^{**}((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \cdot)$ must be optimal at the history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$. To prove this claim, suppose by contradiction that there exist a history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$, a type θ_i, and a profitable deviation m_i at that history, that is,

$$
\min_{\pi_i((\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T, \theta_i) \in \Pi_i^{H,\Theta}((\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T, \theta_i)} \Big(\sum u_i \Big(g \Big(m_i,
$$

$$
s_{-i}^* \big((\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T, \theta_{-i} \big) \Big), \theta_i, \theta_{-i} \Big)
$$

$$
\times \pi_i \big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T, \theta_i \big) \big[(\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T, \theta_{-i} \big] \Big)
$$

$$
= \min_{\pi_i((\widehat{\theta}_i,(\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T), \theta_i) \in \Pi_i^{H**,\Theta}((\widehat{\theta}_i,(\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T), \theta_i)} \Big(\sum u_i \Big(g \Big(m_i,
$$

$$
s_{-i}^{**} \big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i} \big) \Big), \theta_i, \theta_{-i} \Big)
$$

$$
\times \pi_i \big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i \big) \big[(\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i} \big] \Big)
$$

$$
> \min_{\pi_i((\widehat{\theta}_i,(\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T), \theta_i) \in \Pi_i^{H**,\Theta}((\widehat{\theta}_i,(\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T), \theta_i)} \Big(\sum u_i \Big(g \Big(s_i^{**} \big((\widehat{\theta}_i,
$$

$$
(\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i \big), s_{-i}^{**} \big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i} \big) \Big), \theta_i, \theta_{-i} \Big)
$$

$$
\times \pi_i \big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i \big) \big[(\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i} \big] \Big)
$$

$$
= \min_{\pi_i((\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T, \theta_i) \in \Pi_i^{H,\Theta}((\widehat{\omega}_{i,t},\omega_{i,t})_{t=1}^T, \theta_i)} \Big(\sum u_i \Big(g \Big(s_i^* \big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T, \theta_i \big),
$$

$$
s_{-i}^* \big((\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T, \theta_{-i} \big) \Big), \theta_i, \theta_{-i} \Big)
$$

$$
\times \pi_i \big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T, \theta_i \big) \big[(\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T, \theta_{-i} \big] \Big),
$$

where the equalities follow from the construction of s^{**} and the assessments. This contradicts the fact that s^* is a consistent planning equilibrium.

Fourth, consider the communication stage of the two-stage mechanism. By construction, if player i of type θ_i truthfully reports θ_i at the initial history, the expected payoff is

$$
\min_{p_i \in P_i} \sum_{\theta_{-i}} u_i \big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i} \big) p_i[\theta_{-i}].
$$

Alternatively, suppose that player i of type θ_i announces $\widehat{\theta}_i$ and follows the plan $(\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T \mapsto m_i((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ in the second period (to simplify notation, we do not condition on θ_i and $\widehat{\theta}_i$). The expected payoff to player i is then

$$
\min_{p_i \in P_i, \lambda \in \Lambda} \sum_{(\theta_{-i}, \widehat{\theta}_{-i}), ((\widehat{\omega}_{i,t}, \widehat{\omega}_{-i,t}), (\omega_{i,t}, \omega_{-i,t}))_{t=1}^T} \Big(u_i\Big(g\Big(m_i\big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T\big),
$$
$$
s_{-i}^{**}\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i}\big)\Big), \theta_i, \theta_{-i}\Big)
$$
$$
\times \lambda(\widehat{\theta}_i, \widehat{\theta}_{-i})\big[\big((\widehat{\omega}_{i,t}, \widehat{\omega}_{-i,t}), (\omega_{i,t}, \omega_{-i,t})\big)_{t=1}^T\big]
$$
$$
\times s_{-i}^{**}(\{\emptyset\}, \theta_{-i})[\widehat{\theta}_{-i}] p_i[\theta_{-i}]\Big).
$$

For the deviation to be profitable, there must exist a message m_i, a history $(\widehat{\theta}, (\widehat{\omega}_t, \omega_t)_{t=1}^T)$ with positive probability, that is, $\lambda(\widehat{\theta}_i, \widehat{\theta}_{-i})[((\widehat{\omega}_{i,t}, \widehat{\omega}_{-i,t}), (\omega_{i,t}, \omega_{-i,t}))_{t=1}^T] > 0$ such that

$$
\min_{p_i \in P_i} \sum_{\theta_{-i}} u_i\Big(g\Big(m_i\big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T\big),
$$
$$
s_{-i}^{**}\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i}\big)\Big), \theta_i, \theta_{-i}\Big) p_i[\theta_{-i}]
$$
$$
> \min_{p_i \in P_i} \sum_{\theta_{-i}} u_i\big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}\big) p_i[\theta_{-i}].
$$

Also, from the definition of a consistent planning equilibrium, it must be the case that $m_i((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$ coincides with s_i^{**} following the private history $(\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T)$, for all types θ_i. Therefore, we must have that

$$
\min_{p_i \in P_i} \sum_{\theta_{-i}} u_i\Big(g\Big(s_i^{**}\big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i\big),
$$
$$
s_{-i}^{**}\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i}\big)\Big), \theta_i, \theta_{-i}\Big) p_i[\theta_{-i}]
$$
$$
> \min_{p_i \in P_i} \sum_{\theta_{-i}} u_i\big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}\big) p_i[\theta_{-i}].
$$

However, since the history $(\widehat{\theta}, (\widehat{\omega}_t, \omega_t)_{t=1}^T))$ has positive probability (and so has the history $(\widehat{\omega}_t, \omega_t)_{t=1}^T$ in the M-R-C mechanism), we have that

$$
g\big(s_i^{**}\big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i\big), s_{-i}^{**}\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i}\big)\big)
$$
$$
= g\big(s_i^*\big((\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T, \theta_i\big), s_{-i}^*\big((\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T, \theta_{-i}\big)\big) = f(\theta_i, \theta_{-i})
$$

for all (θ_i, θ_{-i}) (since f is implemented at *all* histories with positive probability) and, consequently,

$$\min_{p_i \in P_i} \sum_{(\theta_{-i}} u_i \Big(g\Big(s_i^{**}\big((\widehat{\theta}_i, (\widehat{\omega}_{i,t}, \omega_{i,t})_{t=1}^T), \theta_i\big),$$

$$s_{-i}^{**}\big((\widehat{\theta}_{-i}, (\widehat{\omega}_{-i,t}, \omega_{-i,t})_{t=1}^T), \theta_{-i}\big)\Big), \theta_i, \theta_{-i}\Big) p_i[\theta_{-i}]$$

$$= \min_{p_i \in P_i} \sum_{\theta_{-i}} u_i\big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}\big) p_i[\theta_{-i}],$$

a contradiction with the previous inequality. Therefore, player i of type θ_i has no profitable deviation at the initial history of the two-stage mechanism.
Q.E.D.

Since multiple prior preferences with full Bayesian updating exhibit dynamic inconsistency, it might appear puzzling that T rounds of communication are equivalent to a single round. After all, the single round of communication may give the players the power to commit to certain (communication) strategies that they are unable to carry out in the mechanism with multiple rounds of communication. To understand Proposition S.1, it is important to bear in mind that our definition of implementation requires social choice functions to be implemented for every history of messages sent and received that occurs with positive probability, irrespective of whether the history is truthful or not. In other words, if a type θ_i communicates *as if* type $\widehat{\theta}_i$, the resulting communication should give rise to posterior beliefs such that the social choice function is still correctly implemented at the allocation stage. And since the social choice function is correctly implemented in any case, type θ_i does not gain by communicating as if type $\widehat{\theta}_i$. Put differently, the power to commit—that the two-stage mechanism provides—does not provide any *real* benefit.

It is important to note that the equivalence between single and multiple rounds of communication would not be true if the definition of implementation required the social choice function to be implemented only at "truthful histories." (See Section S.6 for more on this issue.)

S.4. LEMMA 1

As mentioned in the main paper, condition 1 of Theorem 2, instead of being stated in terms of a set of probability systems to generate the posteriors regardless of the message received, can be stated instead in terms of a single probability system that generates the required posteriors over all messages; this is the equivalence result stated as Lemma 1. We now provide the proof of this result.

LEMMA 1: *The following statements are equivalent*:

1. *There exist a set of messages Ω_i^*, a probability system $\bar{\lambda}_i : \Theta_i \times \Theta_{-i} \to \Delta(\Omega_i)$, and a finite partition $\{\Phi_i^1, \ldots, \Phi_i^k, \ldots, \Phi_i^{|K_i|}\}$ of $\Theta_i \times \Omega_i^*$ such that, for all $k \in K_i$,*

$$\bigcup_{p_i \in P_i} \bigcup_{(\theta_i, \omega_i) \in \Phi_i^k} \{\zeta_i(p_i, \theta_i, \omega_i, \bar{\lambda}_i)\} = \Pi_i^k.$$

2. *There exist a set of messages Ω_i, a set of probability systems Λ_i, and a finite partition $\{\Phi_i^1, \ldots, \Phi_i^k, \ldots, \Phi_i^{|K_i|}\}$ of $\Theta_i \times \Omega_i$ such that, for all $k \in K_i$, for all $(\theta_i, \omega_i) \in \Phi_i^k$,*

$$\bigcup_{p_i \in P_i} \bigcup_{\lambda_i \in \Lambda_i} \{\zeta_i(p_i, \theta_i, \omega_i, \lambda_i)\} = \Pi_i^k.$$

PROOF: (1) $\Rightarrow$ (2). Let $\Omega_i = \Omega_i^*$. For each $k \in K_i$, consider a cyclic permutation $\rho^k : \Phi_i^k \to \Phi_i^k$ and write $\rho_{\Theta_i}^k(\theta_i, \omega_i)$ (resp., $\rho_{\Omega_i}^k(\theta_i, \omega_i)$) for the projection of $\rho^k(\theta_i, \omega_i)$ onto Θ_i (resp., Ω_i). Note that $\bigcup_{\rho^k : \text{cyclic permutation}} \{\rho^k(\theta_i, \omega_i)\} = \Phi_i^k$. Let $\rho : \Theta_i \times \Omega_i \to \Theta_i \times \Omega_i$ be the permutation obtained from the permutations $(\rho^k)_{k \in K_i}$, that is, $\rho(\theta_i, \omega_i) := \rho^k(\theta_i, \omega_i)$ if $(\theta_i, \omega_i) \in \Phi_i^k$. Define λ_i^ρ by $\lambda_i^\rho(\theta_i, \theta_{-i})[\omega_i] := \bar{\lambda}_i(\rho_{\Theta_i}(\theta_i, \omega_i), \theta_{-i})[\rho_{\Omega_i}(\theta_i, \omega_i)]$ and let Λ_i be the union of $\{\lambda_i^\rho\}$ over all permutations ρ such that each ρ^k defining ρ is a cyclic permutation.

For each $k \in K_i$, for each $(\theta_i, \omega_i) \in \Phi_i^k$, we then have that

$$\bigcup_{p_i \in P_i} \bigcup_{\lambda_i^\rho \in \Lambda_i} \{\zeta_i(p_i, \theta_i, \omega_i, \lambda_i^\rho)\} = \bigcup_{p_i \in P_i} \bigcup_{(\theta_i, \omega_i) \in \Phi_i^k} \{\zeta_i(p_i, \rho(\theta_i, \omega_i), \bar{\lambda}_i)\}$$

$$= \Pi_i^k.$$

(2) $\Rightarrow$ (1). Let $\Omega_i^* = \Omega_i \times \Lambda_i$ and define $\bar{\lambda}_i$ as

$$\bar{\lambda}_i(\theta_i, \theta_{-i})\big[(\omega_i, \lambda_i)\big] = \frac{1}{|\Lambda_i|} \lambda_i(\theta_i, \theta_{-i})[\omega_i]$$

for all (θ_i, θ_{-i}). Clearly, the posterior belief upon receiving the message (ω_i, λ_i) is the same as the posterior belief upon receiving the message ω_i when the communication device is λ_i, that is, $\zeta_i(p_i, \theta_i, (\omega_i, \lambda_i), \bar{\lambda}_i) = \zeta_i(p_i, \theta_i, \omega_i, \lambda_i)$ for all $(\theta_i, \omega_i) \in \Theta_i \times \Omega_i$.   *Q.E.D.*

Remark that condition (B) together with the nonemptyness of each Π_i^k imply that, for each (θ_i, ω_i), there exist θ_{-i} and λ_i such that $\lambda_i(\theta_i, \theta_{-i})[\omega_i] > 0$. To the contrary, assume that there exists (θ_i, ω_i) such that, for all θ_{-i} and all λ_i, $\lambda_i(\theta_i, \theta_{-i})[\omega_i] = 0$. It follows that for all p_i, $\sum_{\theta_{-i}} \lambda_i(\theta_i, \theta_{-i})[\omega_i] p_i[\theta_{-i}] = 0$ and, therefore, $\bigcup_{p_i \in P_i} \bigcup_{\lambda_i \in \Lambda_i} \{\zeta_i(p_i, \theta_i, \omega_i, \lambda_i)\} = \emptyset \neq \Pi_i^k$, a contradiction. Therefore, each (θ_i, ω_i) has positive probability under $\bar{\lambda}_i$. This fact is used in the proof of Theorem 2.

S.5. A MULTISTAGE ALLOCATION MECHANISM

Yet another essential feature of the class of mechanisms we consider is that the allocation mechanism is static. We now consider a simple example to show how a multistage allocation mechanism can expand the set of implementable social choice functions.

There are two players, labeled 1 and 2, two types θ and θ' for each player, and two alternatives x and y. Types are private information. We assume that players have multiple-prior preferences (Gilboa and Schmeidler ([1989])) with P_i the set of priors of player $i \in \{1, 2\}$ and u_i his utility function. Suppose that $u_1(x, \theta) = 1$, $u_1(y, \theta) = 0$, that player 1 of type θ' and player 2 of both types are indifferent between all alternatives. Assume $P_i = \{p_i\} = \{1/3\}$.[5]

The designer aims at (partially) implementing the social choice function f defined by $f(\theta, \theta) = x$, $f(\theta, \theta') = y$, $f(\theta', \theta) = y$, and $f(\theta', \theta') = x$.

Suppose now that the players can commit to playing behavioral mixed strategies. In that case, the social choice function f is not implementable by an ambiguous mechanism. Regardless of his beliefs, player 1 can guarantee a payoff of $1/2$ by mixing uniformly between θ and θ'. So, to satisfy the incentive compatibility constraints (IC), we need to find a finite collection of finite beliefs' sets $(\Pi_i^k)_k$ such that $\min \Pi_i^k \geq 1/2$ for each k (i.e., we need that $\min_{\pi_i^k \in \Pi_i^k} 1 \pi_i^k + 0(1 - \pi_i^k) \geq 1/2$). However, to generate the sets Π_i^k, we also need that the prior belief $1/3$ belongs to the convex hull of $\bigcup_k \Pi_i^k$, which is impossible (see condition (iii) in Theorem 2).

Yet, we claim that the social choice function is implementable by a more general ambiguous mechanism. The mechanism has three stages. In the first stage, player 2 reports either θ or θ' to the designer. Following player 2's report, the designer sends either ω or ω' to player 1. There are two possible probability systems, λ and λ'. The first probability system λ is fully specified by $\lambda(\omega|\theta) = 1$ and $\lambda(\omega'|\theta') = 1$, while $\lambda'(\omega'|\theta) = 1$ and $\lambda'(\omega|\theta') = 1$ fully specify the second probability system. Player 1 is not active in the first stage. In the second stage, player 1 reports either θ or θ' to the designer. If player 1 reports θ, the designer implements $f(\theta, \theta)$ (resp., $f(\theta, \theta')$) if player 2 reported θ (resp., θ') in the first stage. Alternatively, if player 1 reports θ', the mechanism moves to the third and final stage. In the third stage, player 1 has again to report θ or θ'. If player 1 reports θ', the designer implements $f(\theta', \theta)$ (resp., $f(\theta', \theta')$) if player 2 reported θ (resp., θ') in the first stage. Alternatively, if player 1 reports θ, the designer implements y, regardless of player 2's report. (Player 2 is not active at the second and third stage.) The distinctive feature of this mechanism is the multistage allocation mechanism. See Figure S.2 for a graphical illustration.

---

[5]The convention being p_i is the probability that the other player's type is θ. All our arguments remain valid if $P_i = [\underline{p}_i, \overline{p}_i]$ with $\underline{p}_i < 1 - \overline{p}_i$. Alternatively, if $\underline{p}_i \geq 1 - \overline{p}_i$, f is implementable with a classical direct mechanism, so that there is no need for nontrivial ambiguous mechanisms.
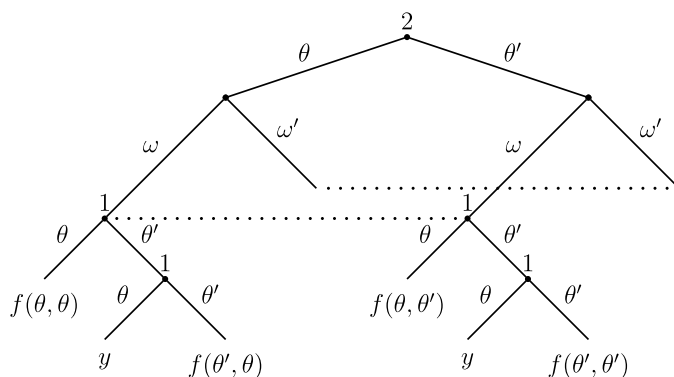
FIGURE S.2.—The mechanism.

We now argue that both players have an incentive to truthfully reveal their types at all stages. Since player 1 of type θ' and player 2 of either type are indifferent between all alternatives, they clearly have an incentive to truthfully reveal their types. So, let us focus on player 1 of type θ. Consider the history (ω, θ'), that is, player 1 has received the message ω from the designer at the first stage and has reported θ' at the second stage. By construction of the ambiguous communication device, player 1's set of beliefs is $\{0, 1\}$, that is, he believes that player 2 is either of type θ with probability 1 or of type θ' with probability 1. At (ω, θ'), player 1 is indifferent between reporting θ, which guarantees a payoff of zero, and reporting θ'. Moreover, no mixture between θ and θ' is strictly preferred to reporting θ.[6] Consequently, it is optimal for player 1 of type θ to truthfully report θ at the third stage following the history (ω, θ'). Let us move to the history (ω). At (ω), player 1' set of beliefs is $\{0, 1\}$ and so he is indifferent between reporting θ and any mixing between θ and θ' (conditional on reporting θ at the third stage and, thus, obtaining y for sure). So, it is optimal for player 1 of type θ to truthfully report θ at ω. A similar argument holds at ω', so that f is indeed implementable by the constructed mechanism.

S.6. A WEAKER NOTION OF IMPLEMENTATION

Our notion of implementation requires the social choice function to be implemented at all histories having positive probability under the equilibrium strategies.[7] This section presents a weaker definition of equilibrium.

---

[6]This follows from the *c*-independence of the multiple prior preferences (Gilboa and Schmeidler (1989)).

[7]As noted in the discussion at the end of Section S.3, this was crucial in establishing why a mechanism with one round of communication can implement any social choice function that a mechanism with $T > 1$ rounds of communication is able to implement.

We say that the history $((\widehat{\omega}_i, \omega_i), (\widehat{\omega}_{-i}, \omega_{-i}))$ has *positive probability under s^* at* (θ_i, θ_{-i}) if there exists $\lambda \in \Lambda$ such that

$$\lambda\big(s_i^*(\{\emptyset\}, \theta_i), s_{-i}^*(\{\emptyset\}, \theta_{-i})\big)\big[(\omega_i, \omega_{-i})\big] > 0,$$

and $(s_i^*(\{\emptyset\}, \theta_i), s_{-i}^*(\{\emptyset\}, \theta_{-i})) = (\widehat{\omega}_i, \widehat{\omega}_{-i})$.

Consider now the following weaker definition of implementation:

DEFINITION S.3: The ambiguous mechanism $\langle\langle(\widehat{\Omega}_i, \Omega_i)_{i\in N}, \Lambda\rangle, \langle(M_i)_{i\in N}, g\rangle\rangle$ (partially) implements the social choice function f if there exists a pure equilibrium $(s^*, \Pi^{H,\Theta})$ such that

$$g\big(s_i^*(h_i, \theta_i), s_{-i}^*(h_{-i}, \theta_{-i})\big) = f(\theta_i, \theta_{-i})$$

for all (h_i, h_{-i}) having positive probability under s^* at (θ_i, θ_{-i}), for all (θ_i, θ_{-i}).

Intuitively, this weaker notion requires f to be implemented only at "truthful" histories. We refer the reader to the main text for a discussion of this weaker concept. We now provide a characterization of the implementable social choice functions (in this weaker sense).

Suppose the social choice function f is implementable by the mechanism

$$\langle\langle(\widehat{\Omega}_i, \Omega_i)_{i\in N}, \Lambda\rangle, \langle(M_i)_{i\in N}, g\rangle\rangle,$$

and let s^* be an implementing equilibrium.

Denote $\tilde{f}: H_i \times \Theta_i \times H_{-i} \times \Theta_{-i} \to X$ the social choice function defined by

$$\tilde{f}(h_i, \theta_i, h_{-i}, \theta_{-i}) := g\big(s_i^*(h_i, \theta_i), s_{-i}^*(h_{-i}, \theta_{-i})\big)$$

for all $(h_i, \theta_i, h_{-i}, \theta_{-i})$.

By definition of an equilibrium, for all histories h_i of player i consistent with s_i^* (i.e., on-path histories), we must have that

$$\min_{\pi_i(h_i, \theta_i)\in\Pi_i^{H,\Theta}(h_i, \theta_i)} \sum_{(\theta_{-i}, h_{-i})} \pi_i(h_i, \theta_i)[h_{-i}, \theta_{-i}]$$
$$\times u_i\big(g\big(s_i^*(h_i, \theta_i), s_{-i}^*(h_{-i}, \theta_{-i})\big), \theta_i, \theta_{-i}\big)$$
$$\geq \min_{\pi_i(h_i, \theta_i)\in\Pi_i^{H,\Theta}(h_i, \theta_i)} \sum_{(\theta_{-i}, h_{-i})} \pi_i(h_i, \theta_i)[h_{-i}, \theta_{-i}]$$
$$\times u_i\big(g\big(s_i^*(h_i', \theta_i'), s_{-i}^*(h_{-i}, \theta_{-i})\big), \theta_i, \theta_{-i}\big)$$

for all (h_i', θ_i'), for all i. Hence, the social choice function $\tilde{f}$ must be incentive compatible: player i of type (h_i, θ_i) must have an incentive to truthfully reveal his private information. Note that this is true in particular for all histories of

the form $(s_i^*(\{\emptyset\}, \theta_i'), \omega_i)$, that is, when player i of type θ_i reports as if his type is θ_i' at the communication stage.

Moreover, since the mechanism implements f, we must have that $\tilde{f}(h_i, \theta_i, h_{-i}, \theta_{-i}) = f(\theta_i, \theta_{-i})$ for all histories (h_i, h_{-i}) having positive probabilities under s^* at (θ_i, θ_{-i}), that is, for all histories $(\widehat{\omega}_i, \omega_i)_i$ such that $\widehat{\omega}_i = s_i^*(\{\emptyset\}, \theta_i)$ for all i and $\lambda((\widehat{\omega}_i, \widehat{\omega}_{-i}))[(\omega_{-i}, \omega_{-i})] > 0$ for some $\lambda \in \Lambda$.

From the definition of an equilibrium, we must also have that for all i, for all θ_i, for all (θ_i', θ_i''),

$$
\min_{\lambda \in \Lambda, p_i \in P_i} \sum_{\theta_{-i}, (\omega_i, \omega_{-i})} \Big(u_i \Big(\tilde{f}\big((s_i^*(\theta_i', \{\emptyset\}), \omega_i), \theta_i'',
$$

$$
\big(s_{-i}^*(\theta_{-i}, \{\emptyset\}), \omega_{-i} \big) \big), \theta_{-i} \Big), \theta_i, \theta_{-i} \Big)
$$

$$
\times \lambda \Big(\big(s_i^*(\theta_i', \{\emptyset\}), s_{-i}^*(\theta_{-i}, \{\emptyset\}) \big) \Big) [(\omega_i, \omega_{-i})] \, p_i[\theta_{-i}]
$$

$$
\leq \min_{\lambda \in \Lambda, p_i \in P_i} \sum_{\theta_{-i}, (\omega_i, \omega_{-i})} \Big(u_i \Big(\tilde{f}\big((s_i^*(\theta_i, \{\emptyset\}), \omega_i), \theta_i,
$$

$$
\big(s_{-i}^*(\theta_{-i}, \{\emptyset\}), \omega_{-i} \big) \big), \theta_{-i} \Big), \theta_i, \theta_{-i} \Big)
$$

$$
\times \lambda \Big(\big(s_i^*(\theta_i, \{\emptyset\}), s_{-i}^*(\theta_{-i}, \{\emptyset\}) \big) \Big) [(\omega_i, \omega_{-i})] \, p_i[\theta_{-i}]
$$

$$
= \min_{p_i \in P_i} \sum_{\theta_{-i}} u_i\big(f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}\big) \, p_i[\theta_{-i}].
$$

This inequality states that player i of type θ_i has an incentive to communicate as type θ_i at the initial communication stage. In other words, it is not profitable for player i of type θ_i to communicate as type θ_i' at the initial communication stage (thus to generate histories of the form $(s_i^*(\theta_i', \{\emptyset\}), \omega_i)$) and to pretend to be type θ_i'' at the allocation stage.

To sum up, implementation of a social choice function f would imply that there exist a set of transition probabilities Λ, a set of messages $(\Omega_i)_i$, a social choice function $f^* : \bigtimes_i (\Theta_i \times \Omega_i) \times \bigtimes_i \Theta_i \to X$, belief sets $\Pi_i^{H, \Theta}$ such that f^* is incentive compatible at the allocation stage (with the updated posteriors), $f^*(h_i, \theta_i, h_{-i}, \theta_{-i}) = f(\theta_i, \theta_{-i})$ for all truthful histories (h_i, h_{-i}), and f^* is also incentive compatible at the initial stage.[8]

Conversely, we can repeat the arguments in the main text to show that one can implement any f for which there are a set of transition probabilities Λ, a set of messages $(\Omega_i)_i$, and a social choice function f^* that satisfy the properties above.

---

[8] f^* is the restriction of $\tilde{f}$ to the histories $(\widehat{\omega}_i, \omega_i)_i$ of the form $\widehat{\omega}_i = s_i^*(\{\emptyset\}, \theta_i')$ for some θ_i', for all i, and $\lambda((\widehat{\omega}_i, \widehat{\omega}_{-i}))[(\omega_i, \omega_{-i})] > 0$ for some $\lambda \in \Lambda$.

As a final remark, note that the stronger definition in the paper requires $f^*(h_i, \theta_i, h_{-i}, \theta_{-i}) = f(\theta_i, \theta_{-i})$ for all histories (h_i, h_{-i}) (truthful and untruthful), so that the incentive compatibility constraints are trivially satisfied at the initial stage, and f^* is essentially independent of the private histories. (With the stronger definition of implementation, the players are indifferent between truthful and untruthful reporting to the communication device at the communication stage; the social choice function is implemented at the allocation stage in any case.)

S.7. CONTINUUM

In economic applications, it is sometimes convenient to assume that sets of types, alternatives, messages, etc., are subsets of complete separable metric spaces. This section generalizes our results to the environments frequently found in applications. It is important to stress that we do not aim at the most general assumptions; rather, we aim at extending our results to the environments frequently encountered in applications.

So, assume that the space of alternatives X is a subset of a complete separable metric space, for example, X is a subset of $\mathbb{R}^{n+\ell}$ in allocation problems (transfers and quantities of the ℓ goods). For each i, we assume that Θ_i is a closed and bounded interval, which we assume to be $[0, 1]$. We assume the set of priors P_i consist of priors p_i having continuous and strictly positive densities. For any prior p_i, let p_i^d be the associated density. As usual, strategies are assumed to be measurable maps.

Let $\lambda : \times_i \widehat{\Omega}_i \to \Delta(\times_i \Omega_i)$ be a probability kernel, that is, $\lambda(\widehat{\omega})$ is a probability measure on the product algebra $\mathbb{B}_{(\times_i \Omega_i)}$ for each $\widehat{\omega}$, and $\lambda(\cdot)[E_\omega]$ is measurable for each event E_ω in the product algebra $\mathbb{B}_{(\times_i \Omega_i)}$. Let $\lambda_i : \times_i \widehat{\Omega}_i \to \Delta(\Omega_i)$ be the marginal probability kernel, that is, for each $\widehat{\omega} \in \widehat{\Omega}$ and $E_{\omega_i} \in \mathbb{B}_{\Omega_i}$, $\lambda_i(\widehat{\omega})[E_{\omega_i}] := \lambda(\widehat{\omega})[E_{\omega_i} \times \Omega_{-i}]$.

An important technical requirement is that players have well-defined conditional probabilities for all possible histories of messages sent and received $(\widehat{\omega}_i, \omega_i)$. If λ is a regular conditional probability, then these conditional probabilities are well-defined. So, to guarantee the existence of regular conditional probabilities, we assume that the sets of messages $(\widehat{\Omega}_i, \Omega_i)$ are compact subsets of complete and separable metric spaces. (See Faden (1985).) With these technical preliminaries completed, we now consider the results in the paper.

Theorem 1 goes through with only minor modifications. Given an ambiguous mechanism and for any pure strategy equilibrium, "history with positive probability" means "for all histories in the support of one of the measures over histories induced by the equilibrium and the mechanism." To be more precise, fix an equilibrium s^*, a probability system λ, and a prior p_i^d. We define the mea-

sure over Ω_i induced by λ, p_i, and s^* as follows. For each E_{ω_i} and $\widehat{\omega}_i$, consider the integral

$$\mu_{s^*, \lambda, p_i^d}(\widehat{\omega}_i)[E_{\omega_i}] := \int_{\Theta_{-i}} \lambda\big(\widehat{\omega}_i, s^*_{-i}(\{\emptyset\}, \theta_{-i})\big)[E_{\omega_i} \times \Omega_{-i}] p_i^d(\theta_{-i}) \, d\theta_{-i}.$$

The integral is well-defined (since the map $\widehat{\omega}_{-i} \mapsto \lambda(\omega_i, \cdot)[E_{\omega_i} \times \Omega_{-i}]$ is measurable and bounded, s^*_{-i} is measurable and $\int_{\Theta_{-i}} p_i^d(\theta_{-i}) \, d\theta_{-i} = 1 < +\infty$), and $\mu_{s^*, \lambda, p_i^d}(\widehat{\omega}_i)$ is clearly a probability measure on $(\Omega_i, \mathbb{B}_{\Omega_i})$ for each $\widehat{\omega}_i$. The history $(\widehat{\omega}_i, \omega_i)$ has "positive probability" under s^* if there exist θ_i, λ, and p_i^d such that $s_i^*(\{\emptyset\}, \theta_i) = \widehat{\omega}_i$ and ω_i is in the support of the measure $\mu_{s^*, \lambda, p_i^d}(\widehat{\omega}_i)$ over Ω_i, where we define the support as the closure of $\{\omega_i \in \Omega_i : \omega_i \in N_{\omega_i} \implies \mu_{s^*, \lambda, p_i^d}(\widehat{\omega}_i)[N_{\omega_i}] > 0$; N_{ω_i} an open set$\}$. Naturally, with an infinite type space, the partition $\{\Phi_i^k\}$ cannot be assumed to be finite.

Theorem 2 is more delicate. First, we need to verify that the equivalence result (i.e., Lemma 1) still holds in this more general setting. Second, we need to find the appropriate generalization of condition (2).

For simplicity, we assume throughout that $\Phi_i = \{\Theta_i \times \Omega_i\}$ (i.e., the partition has the single element $\Theta_i \times \Omega_i$). (The arguments extend straightforwardly to nontrivial partitions, as in the main text.) In what follows, we restrict the probability kernel λ to admit positive and continuous densities λ^d.[9]

Let us suppose first that Lemma 1 holds (we verify later that it does) and consider the generalization of condition (2) of Theorem 2. Consider the set of priors P_i and the set of posteriors Π_i and write P_i^d and Π_i^d for the corresponding set of densities. In that case, λ_i has the density λ_i^d given by $\lambda_i^d(\theta)[\omega_i] := \int_{\Omega_{-i}} \lambda^d(\theta)[(\omega_i, \omega_{-i})] \, d\omega_{-i}$ and $\zeta_i(p_i, \theta_i, \omega_i, \lambda_i)$ has the density given by

$$\zeta_i^d\big(p_i^d, \theta_i, \omega_i, \lambda_i^d\big)[\theta_{-i}] = \frac{\lambda_i^d(\theta_i, \theta_{-i})[\omega_i] p_i^d(\theta_{-i})}{\int_{\Theta_{-i}} \lambda_i^d(\theta_i, \theta_{-i})[\omega_i] p_i^d(\tilde{\theta}_{-i}) \, d\tilde{\theta}_{-i}}.$$

The second part of Theorem 2 reads then: There exist a finite measure space $(A_i, \mathbb{B}_{A_i})$ and measurable functions $\mu : A_i \times P_i^d \to \mathbb{R}_{++}$ and $\kappa : A_i \times P_i^d \to \Pi_i^d$ such that

(i) The function κ is surjective.
(ii) For each α, for each $(p_i^d, \tilde{p}_i^d)$,

$$\mu\big(\alpha, p_i^d\big)\big(\kappa(\alpha, p_i^d)/p_i^d\big) = \mu\big(\alpha, \tilde{p}_i^d\big)\big(\kappa(\alpha, \tilde{p}_i^d)/\tilde{p}_i^d\big).$$

(iii) The function $\mu(\cdot, p_i^d)$ is a positive density for each p_i^d, and $p_i^d = \int \mu(\alpha, p_i^d)\kappa(\alpha, p_i^d) \, d\alpha$ for each p_i^d.

[9]For an example without this restriction, see next section.

The proof is almost identical to the proof in the main text and left to the reader. In a nutshell, probability densities operate as regular probabilities. For instance, we clearly have that

$$p_i^d(\theta_{-i}) = \int_{\Theta_i \times \Omega_i} \left(\int_{\Theta_{-i}} \lambda_i^d(\theta_i, \theta_{-i})[\omega_i] p_i^d(\theta_{-i}) \, d\theta_{-i} \right)$$
$$\times \zeta_i^d \big(p_i^d, \theta_i, \omega_i, \lambda_i^d \big)[\theta_{-i}] \, d(\theta_i, \omega_i)$$

for all θ_{-i}, which is essentially condition (iii) above. It is routine to replicate the other arguments in the proof of Theorem 2.

The remaining task is to verify the validity of Lemma 1. The essence of the argument consists in showing that there exists a family of measurable bijections on Φ_i such that the set of posteriors obtained from a *collection* of probability kernels Λ_i^d *regardless of the messages received* is equal to the set of posteriors obtained from a *unique* probability kernel $\bar{\lambda}_i^d$ by conditioning over *all messages*, that is,

$$\bigcup_{p_i^d \in P_i^d} \bigcup_{(\theta_i, \omega_i) \in \Theta_i \times \Omega_i} \big\{ \zeta_i^d \big(p_i^d, \theta_i, \omega_i, \bar{\lambda}_i^d \big) \big\} = \bigcup_{p_i^d \in P_i^d} \bigcup_{\lambda_i^d \in \Lambda_i^d} \big\{ \zeta_i^d \big(p_i^d, \theta_i, \omega_i, \lambda_i^d \big) \big\}$$

for all (θ_i, ω_i). See Section S.4 for a more formal statement.

Fix $\bar{\lambda}_i$ as in part (1) of Lemma 1 and consider its associated density, $\bar{\lambda}_i^d$. We now construct a collection of probability systems Λ_i^d such that the above equality is satisfied. To that end, for any pair $(\omega_i', \omega_i'') \in \Omega_i \times \Omega_i$, define the function $g_{(\omega_i', \omega_i'')} : \Omega_i \to \Omega_i$ as

$$g_{(\omega_i', \omega_i'')}(\omega_i) = \begin{cases} \omega_i'' & \text{if } \omega_i = \omega_i', \\ \omega_i' & \text{if } \omega_i = \omega_i'', \\ \omega_i & \text{if } \omega_i \in \Omega_i \setminus \big\{ \omega_i', \omega_i'' \big\}. \end{cases}$$

The function $g_{(\omega_i', \omega_i'')}$ is a transposition of ω_i' and ω_i'' and, hence, a bijection from Ω_i to Ω_i. It is also a measurable function. To see this, consider any Borel set $E \in \mathbb{B}_{\Omega_i}$ and note that $g_{(\omega_i', \omega_i'')}^{-1}(E) = E$ if $(\omega_i', \omega_i'') \in E \times E$ and $(\omega_i', \omega_i'') \in (\mathbb{B}_{\Omega_i} \setminus E) \times (\mathbb{B}_{\Omega_i} \setminus E)$, while $g_{(\omega_i', \omega_i'')}^{-1}(E) = (E \setminus \{\omega_i'\}) \cup \{\omega_i''\}$ if $\omega_i' \in E$ and $\omega_i'' \notin E$ (resp., $g_{(\omega_i', \omega_i'')}^{-1}(E) = (E \setminus \{\omega_i''\}) \cup \{\omega_i'\}$ if $\omega_i'' \in E$ and $\omega_i' \notin E$). Since singletons are closed sets in metric spaces, $\Omega_i \setminus \{\omega_i'\}$ is open. Hence, $E \cap (\Omega_i \setminus \{\omega_i'\}) = E \setminus \{\omega_i'\}$ is a Borel set and so is $(E \setminus \{\omega_i'\}) \cup \{\omega_i''\}$. Similar arguments show that $(E \setminus \{\omega_i''\}) \cup \{\omega_i'\}$ is a Borel set.

For each $\omega_i' \in \Omega_i$, we also have

$$\bigcup_{\omega_i'' \in \Omega_i} \big\{ g_{(\omega_i', \omega_i'')}^{-1}(\omega_i') \big\} = \Omega_i.$$

For any (θ_i, θ_{-i}), define the function $\lambda^d_{i,(\omega'_i,\omega''_i)}(\theta_i, \theta_{-i}): \Omega_i \to \mathbb{R}$ by $\lambda^d_{i,(\omega'_i,\omega''_i)}(\theta_i, \theta_{-i})[\omega_i] := \bar{\lambda}^d_i(\theta_i, \theta_{-i})[g^{-1}_{(\omega'_i,\omega''_i)}(\omega_i))]$, for each ω_i. The function takes positive values and integrates to the unity, since it agrees with the density $\bar{\lambda}^d_i(\theta_i, \theta_{-i})$ almost everywhere, hence it is a density.

It follows that

$$\zeta^d_i\big(p^d_i, (\theta_i, \omega_i), \lambda^d_{i,(\omega'_i,\omega''_i)}\big)[\theta_{-i}]$$

$$= \frac{p^d_i(\theta_{-i})\bar{\lambda}^d_i(\theta_i, \theta_{-i})[g^{-1}_{(\omega'_i,\omega''_i)}(\omega_i)]}{\displaystyle\int_{\tilde{\theta}_{-i}} p^d_i(\tilde{\theta}_{-i})\bar{\lambda}^d_i(\theta_i, \tilde{\theta}_{-i})[g^{-1}_{(\omega'_i,\omega''_i)}(\omega_i)]\, d\tilde{\theta}_{-i}}$$

for all θ_{-i}, where $\lambda^d_{i,(\omega'_i,\omega''_i)}$ is the density kernel obtained from the construction above. We therefore have

$$\zeta^d_i\big(p^d_i, \theta_i, \omega_i, \lambda^d_{i,(\omega'_i,\omega''_i)}\big)[\theta_{-i}]$$

$$= \begin{cases} \zeta^d_i\big(p^d_i, \theta_i, \omega_i, \bar{\lambda}^d_i\big)[\theta_{-i}] & \text{if } \omega_i \neq \omega'_i, \omega''_i, \\ \zeta^d_i\big(p^d_i, \theta_i, \omega'_i, \bar{\lambda}^d_i\big)[\theta_{-i}] & \text{if } \omega_i = \omega''_i, \\ \zeta^d_i\big(p^d_i, \theta_i, \omega''_i, \bar{\lambda}^d_i\big)[\theta_{-i}] & \text{if } \omega_i = \omega'_i. \end{cases}$$

Clearly, $\zeta^d_i(p^d_i, \theta_i, \omega_i, \lambda^d_{i,(\omega'_i,\omega''_i)})$ is a density. Finally, taking the union over all the transpositions gives the desired result.

Conversely, fix a collection Λ^d_i as in part (2) of Lemma 1. We need to modify the construction of $\bar{\lambda}^d_i$, since Λ^d_i is not assumed to be finite. We *assume* that the ambiguous mechanism that can implement the social choice function f is of such a form that there exists a measure ν on Λ (and, thus, on Λ_i) having continuous positive density ν^d. With this assumption, we have that

$$\bar{\lambda}^d_i\big(\theta_i, \theta^d_{-i}\big)\big[(\omega_i, \lambda^d_i)\big] = \lambda^d_i(\theta_i, \theta_{-i})[\omega_i]\nu^d\big(\lambda^d_i\big).$$

Naturally, more general measure-theoretic constructions are possible, but that is beyond the scope of this supplement.

S.8. A PUBLIC GOOD EXAMPLE

We now consider a simple version of the public goods problem; in particular, we consider a continuum type space to illustrate some of the points made in the last section. There are two agents 1 and 2 with quasi-linear preferences. The project, if built gives gross utility of 1 to each. Utility is 0 if the project is not built. Each agent i has a privately known cost θ_i of building the project

with $\theta_i \in [0, 2]$.[10] The common prior on θ_i is given by some distribution F with density $f > 0$.

The set of physical allocations is given by $X = \{1, 2, \frac{1}{2}.1 \otimes \frac{1}{2}.2, 0\}$ with x denoting a generic element. The physical allocation $x = i$ corresponds to agent i building the project, $x = \frac{1}{2}.1 \otimes \frac{1}{2}.2$ corresponds to the construction of the project being allocated randomly to either agent with probability 1/2, and $x = 0$ corresponds to the project not being built. Denote $t_i \in \mathbb{R}$ the transfer received by player i from player j; t_i can be negative.

The Social Choice Function

The social choice function maps profile of costs (θ_1, θ_2) to a physical allocation $x(\theta_1, \theta_2)$ and transfers $(t_1(\theta_1, \theta_2), t_2(\theta_1, \theta_2))$. The designer aims at implementing the efficient, egalitarian (envy-free), and budget balanced allocation. Formally, for $i, j = 1, 2, i \neq j$, the physical allocation is

$$x(\theta_i, \theta_j) = \begin{cases} i & \text{if } \theta_i < \theta_j, \\ j & \text{if } \theta_i > \theta_j, \\ \frac{1}{2}.i \otimes \frac{1}{2}.j & \text{if } \theta_i = \theta_j. \end{cases}$$

The transfer t_i is

$$t_i(\theta_i, \theta_j) = \begin{cases} -\dfrac{\theta_j}{2} & \text{if } x = j, \\ \dfrac{\theta_i}{2} & \text{if } x = i, \\ 0 & \text{if } x = \frac{1}{2}.i \otimes \frac{1}{2}.j, \end{cases}$$

with $t_j(\theta_j, \theta_i) = -t_i(\theta_i, \theta_j)$ for all (θ_i, θ_j) to have ex post budget balance. Note that the net utility of each agent is $1 - \frac{1}{2}\min(\theta_1, \theta_2)$ at every state (θ_1, θ_2).

It is straightforward to check that the social choice function is not incentive compatible and hence is not implementable by using any classical mechanism. It is also easy to check that the social choice function is incentive compatible with respect to beliefs Π_i, where Π_i is (the convex hull of) $\mathcal{D}$, the set of all Dirac measures on $[0, 2]$.[11] We leave these as exercises for the reader and in the rest of the section show the construction of the required posterior belief set Π_i through the use of a suitable ambiguous communication device.

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[10]Since total utility when the project is built is 2, it is more convenient to have the type space as $[0, 2]$ rather than $[0, 1]$ and then to scale the cost.

[11]For any $c \in [0, 2]$, the Dirac measure μ_c is such that, for any Borel measurable subset $B \subseteq [0, 2]$, $\mu_c[B] = 1$ if $c \in B$ and $\mu_c[B] = 0$ for $c \notin B$. Then, $\mathcal{D} = \bigcup_c \mu_c$.

Constructing the Ambiguous Communication Device

Let the message spaces[12] $\widehat{\Omega}_i = \Omega_i = [0, 2]$. Let $A = [0, 2]$ be the index set such that Λ_i is given by $\Lambda_i = \bigcup_{\alpha \in A} \{\lambda_i^\alpha\}_{\alpha \in A}$. We have $\lambda^\alpha = \times_{i=1,2} \lambda_i^\alpha$.

Consider first the probability system $\lambda_i^0(\widehat{\omega}_i, \widehat{\omega}_{-i})[\omega_i]$ where

$$\lambda_i^0(\widehat{\omega}_i, \widehat{\omega}_j)[\omega_i] = \begin{cases} 1 & \text{if } \omega_i = \widehat{\omega}_j, \\ 0 & \text{otherwise.} \end{cases}$$

For any number $w \in [0, 2]$ and $y \in [0, 2]$, define the number $z = w + y$ (mod 2) as $z = w + y$ if $w + y \le 2$ and $z = w + y - 2$ if $w + y > 2$.

Consider now the following bijections. For any $\alpha \in (0, 2]$, define $\rho_\alpha : (0, 2] \to (0, 2]$ as

$$\rho_\alpha(\omega_i) = \omega_i + \alpha \quad (\text{mod } 2).$$

Define λ_i^α as

$$\lambda_i^\alpha(\widehat{\omega}_i, \widehat{\omega}_{-i})[\omega_i] = \lambda_i^0(\widehat{\omega}_i, \widehat{\omega}_{-i})\big[\rho_\alpha(\omega_i)\big].$$

Consider any message $\omega_i \in (0, 2]$ received by agent i (upon having sent a message $\widehat{\omega}_i$, which may not be equal to θ_i, the true cost of agent i). Assuming agent j is sending message truthfully to the ambiguous communication device, for *every* number $c \in (0, 2]$, there exists a λ_i^α, such that the posterior beliefs of agent i is

$$\zeta_{\theta_i}\big(F, \widehat{\omega}_i, \omega_i, \lambda_i^\alpha\big)[\theta_j] = \begin{cases} 1 & \text{if } \theta_j = c, \\ 0 & \text{otherwise.} \end{cases}$$

On receiving message $\omega_i = 0$, posterior belief of player i is, however, not $\mathcal{D}$. In fact, it is a singleton; i believes with certainty that $\theta_j = 0$. This, however, does not create any problem regarding implementation since there is no gainful deviation from truthful reporting of cost at the allocation stage if an agent believes with certainty that the other agent's cost is zero.

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[12]For this application, it suffices to have a message space that is a closed bounded interval. As shown below, we use a different construction than in Section S.7 for the bijection on Ω_i that is more similar in spirit to the cyclic permutations used in the main text. Note that we could have used the same construction as in Section S.7 in the main text also; the only difference would have been that it would have required a greater number of bijections than is used in the cyclic permutation.

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